Polynomial Reproduction of Multivariate Scalar Subdivision Schemes

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Abstract

A stationary subdivision scheme generates the full space of polynomials of degree up to k if and only if its mask satisfies sum rules of order $k+1$, or its symbol satisfies zero conditions of order $k+1$. This property is often called the polynomial reproduction property of the subdivision scheme. It is a well-known fact that this property is, in general, only necessary for the associated refinable function to have approximation order $k+1$.

In this paper we study a different polynomial reproduction property of multivariate scalar subdivision scheme with dilation matrix mI, $|m| \geq 2$. Namely, we are interested in capability of a subdivision scheme to reproduce in the limit exactly the same polynomials from which the data is sampled. The motivation for this paper are the results in [7] that state that such a reproduction property of degree k of the subdivision scheme is sufficient for having approximation order $k+1$.

Our main result yields simple algebraic conditions on the subdivision symbol for computing the exact degree of such polynomial reproduction and also for determining the associated parametrization. The parametrization determines the grid points to which the newly computed values are attached at each subdivision iteration to ensure the higher degree of polynomial reproduction. We illustrate our results with several examples.

Keyword: Subdivision schemes, polynomial reproduction, subdivision parametrization, approximation order

Introduction

Interest in subdivision schemes is motivated by their applications in computer graphics, computer aided geometric design, animation, wavelet and frame construction. Important properties of subdivision schemes such as convergence, regularity, polynomial generation, approximation order, etc., have been studied by several authors, see surveys [3, 4, 8, 9, 14, 15] and references therein. In this paper we would like to distinguish between the concepts of polynomial generation and polynomial reproduction of subdivision schemes. The so-called

polynomial generation of degree k is the capability of subdivision to generate the full space of polynomials of degree up to k. This property is equivalent to sum rules of order $k+1$ on the subdivision mask, or, equivalently, to zero conditions of order $k+1$ on the subdivision symbol, see e.g. [4, 12, 14]. Polynomial generation of degree k also implies that the associated refinable function has accuracy of order k [15], but is, in general, only necessary for the corresponding shift-invariant space to have approximation order $k+1$. This has been already observed for B-splines and box splines [1]. The so-called polynomial reproduction is the capability of subdivision schemes to produce in the limit exactly the same polynomials from which the data is sampled. The results in [7, Section 2.4] state that polynomial reproduction of degree k of convergent subdivision is sufficient for the associated shift-invariant space to have approximation order $k+1$. This motivates our interest in polynomial reproduction of subdivision schemes.

Our main goal is to derive simple algebraic conditions on subdivision symbol that allow us to determine the degree of its polynomial reproduction. Note that the concepts of polynomial reproduction and generation coincide in the case $k=0$, *i.e.* in the case of reproduction of constants. In the L_2 setting, polynomial generation of degree k is also sufficient for approximation order $k+1$, see e.g. [12]. For convergent interpolatory subdivision schemes, the concepts of polynomial generation and reproduction are equivalent and, thus, characterize the approximation power of the corresponding shift-invariant space, see [11]. We emphasize that there is a multitude of results on approximation order of a refinable function. Those results however are mostly derived from the properties of the associated shift-invariant space and not from the properties of the coefficients of the refinement equation - the subdivision mask, in the subdivision context. There are also data pre-processing techniques for achieving the optimal approximation order of a shift-invariant space associate with a convergent subdivision scheme, [7, 9, 13]. The polynomial reproduction of order k makes preprocessing unnecessary, which is undoubtedly an advantage.

In the univariate case, polynomial reproduction has been studied in [10] for binary primal and dual subdivision schemes and extended in [6] to univariate subdivision schemes of any a-rity. In [6] the authors provide unified algebraic conditions on the subdivision symbol for polynomial reproduction with no restrictions on the associated parametrization - the grid points to which the newly computed values are attached at each subdivision iteration. The results of our paper extend [6] to the multivariate setting for scalar subdivision with dilation matrix mI, $|m| \geq 2$. To the best of our knowledge our results are the first ones on multivariate polynomial reproduction of subdivision schemes. Our interest in the case of dilation matrix mI is motivated, e.g., by the bivariate $\sqrt{3}-$subdivision whose refinable function is also refinable with respect to dilation $-3I$ and iterated mask $a(z_1 z_2^{-2}, z_1^2 z_2^{-1}) \cdot a(z_1, z_2)$. There are several other expansive dilation matrices M which satisfy $M^n = mI$, and, thus such that our results are applicable.

The main result of our paper, Theorem 2.6, states that for a non-singular

subdivision scheme with finitely supported mask $\mathrm{a} = \{\mathrm{a}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$ and symbol $a(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}}\, \boldsymbol{z}^{\boldsymbol{\alpha}}$ the polynomial reproduction of order k is equivalent to

$$\big(D^{\mathbf{j}}a\big)(\mathbf{1}) = |m|^s \prod_{i=1}^{s} \prod_{\ell_i=0}^{j_i-1} (\tau_i - \ell_i) \quad \text{and} \quad \big(D^{\mathbf{j}}a\big)(\boldsymbol{\varepsilon}) = 0 \ \text{ for } \ \boldsymbol{\varepsilon} \in \Xi',\ \ |\mathbf{j}| \le k\,, \tag{0.1}$$

where Ξ' is a finite set of certain multi-indices and $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_s) \in \mathbb{R}^s$ appears in the parametrization associated with the subdivision scheme. The importance of condition (0.1) for $k = 1$ is that it allows us to identify the correct parametrization that for any non-singular or for even only convergent subdivision scheme guarantees at least the reproduction of linear polynomial. The parametrization determines the grid points to which the newly computed values are attached at each step of subdivision recursion to ensure the higher degree of polynomial reproduction of a scheme.

This paper is organized as follows. The first section sets the notation, provides the background on multivariate subdivision schemes stressing the difference between polynomial reproduction and polynomial generation. We also define sequence of parameter values associated with a subdivision scheme, *i.e.*, the subdivision parametrization. In section 2, we first provide algebraic tools for determining the correct parametrization needed to ensure reproduction of linear polynomials. We also give the necessary and sufficient conditions on the symbol of any non-singular subdivision scheme that guarantee polynomial reproduction up to a certain degree. Effect of a shift of the mask on the degree of polynomial reproduction is investigated in Section 3. There we also show that the concept of polynomial reproduction and polynomial generation are equivalent for convergent interpolatory schemes. Thus, reproducing the results in [11]. In section 3 we also provide the correct parametrization for box spline subdivision schemes together with several examples. The effect of the shifts of the box splines on the approximation order of the corresponding shift-invariant spaces has been already observed e.g. in [1].

1 Background and notation

1.1 Subdivision scheme

A scalar s-variate subdivision scheme with a *dilation matrix* mI, $|m| \ge 2$, is given by a scalar finitely supported sequence $\mathrm{a} = \{\mathrm{a}_{\boldsymbol{\alpha}} \in \mathbb{R},\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$, the so-called *mask*. The *subdivision operator* $\mathcal{S}_{\mathrm{a}}$ acting on data sequences $\mathrm{d} = \{\mathrm{d}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathbb{Z}^s\} \in \ell(\mathbb{Z}^s)$ is defined by

$$\big(\mathcal{S}_{\mathrm{a}}\mathrm{d}\big)_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}}\ \mathrm{d}_{\boldsymbol{\beta}}\,, \quad \boldsymbol{\alpha} \in \mathbb{Z}^s\,, \tag{1.1}$$

where $\ell(\mathbb{Z}^s)$ is the space of scalar sequences indexed by $\mathbb{Z}^s$. A *subdivision scheme* is the recursive algorithm given by

$$\mathrm{d}^{(r+1)} = \mathcal{S}_{\mathrm{a}}\mathrm{d}^{(r)} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}}\mathrm{d}_{\boldsymbol{\beta}}^{(r)}\,, \quad \mathrm{d}^{(0)} \in \ell(\mathbb{Z}^s), \quad r \in \mathbb{N}_0\,, \tag{1.2}$$

where $\mathbb{N}_0$ is the set of natural numbers including zero.

The *symbol* of a subdivision scheme is given by the Laurent polynomial

$$a(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}}\, \boldsymbol{z}^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} = (z_1,\ldots,z_s) \in (\mathbb{C}\setminus\{0\})^s\,,$$

where for $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_s) \in \mathbb{Z}^s$ we define $\boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2}\cdot\ldots\cdot z_s^{\alpha_s}$. Denoting with

$$\mathrm{E} = \{0,\ldots,|m|-1\}^s\,, \tag{1.3}$$

the set of representatives of $\mathbb{Z}^s/m\mathbb{Z}^s$ containing $\mathbf{0} = (0,0,\ldots,0)$, the $|m|^s$ *submasks* and their symbols $a_{\boldsymbol{e}}(\boldsymbol{z})$ are defined by

$$\{\mathrm{a}_{\boldsymbol{e}+m\boldsymbol{\alpha}},\ \boldsymbol{\alpha}\in\mathbb{Z}^s\} \tag{1.4}$$

and

$$a_{\boldsymbol{e}}(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{e}+m\boldsymbol{\alpha}}\, \boldsymbol{z}^{\boldsymbol{e}+m\boldsymbol{\alpha}}\,, \quad \boldsymbol{e}\in\mathrm{E}\,, \tag{1.5}$$

respectively. Then, we get the following decomposition of the mask symbol

$$a(\boldsymbol{z}) = \sum_{\boldsymbol{e}\in\mathrm{E}} a_{\boldsymbol{e}}(\boldsymbol{z})\,. \tag{1.6}$$

1.2 Sum rules and zero conditions

The *sum rules of order* 1 in terms of submasks read as follows

$$a_{\boldsymbol{e}}(\mathbf{1}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{e}+m\boldsymbol{\alpha}} = 1\,, \quad \boldsymbol{e}\in\mathrm{E}\,. \tag{1.7}$$

For $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_s) \in \mathbb{R}^s$ set $z_j = e^{-i\pi\xi_j}$, $j = 1,\ldots,s$, the set in (1.3) corresponds to

$$\Xi = \Xi_{\mathrm{E}} = \{\boldsymbol{\varepsilon} = e^{-i\frac{2\pi}{m}\boldsymbol{e}}\ :\ \boldsymbol{e}\in\mathrm{E}\}\,, \tag{1.8}$$

and contains $\mathbf{1} = (1,1,\ldots,1)$. The sum rules of order 1 take an equivalent form

$$a(\mathbf{1}) = |m|^s \quad\text{and}\quad a(\boldsymbol{\varepsilon}) = 0 \quad \text{for } \boldsymbol{\varepsilon}\in\Xi' := \Xi\setminus\{\mathbf{1}\}\,. \tag{1.9}$$

Following the notation in [5], we call Ξ' the zero set, and the conditions in (1.9) the *zero condition of order one (Condition Z_1)*. In the literature, both the conditions in (1.7) and their equivalent form in (1.9) are called the sum rules of order one. We also make use of the higher order sum rules, see [12] and references therein: The mask symbol $\mathrm{a}(\boldsymbol{z})$ is said to satisfy the *zero condition of order k (Condition Z_k)*, if

$$a(\mathbf{1}) = |m|^s \quad\text{and}\quad \big(D^{\mathbf{j}}a\big)(\boldsymbol{\varepsilon}) = 0 \quad \text{for} \quad \boldsymbol{\varepsilon}\in\Xi' \quad\text{and}\quad |\mathbf{j}| < k\,. \tag{1.10}$$

We denote by $D^{\mathbf{j}}$ the $\mathbf{j}$-th directional directional derivative.

1.3 Parametrization

Since most of the properties of a subdivision scheme, e.g. its convergence, smoothness or its support size, do not depend on the choice of the associated parameter values $\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}$, $\alpha \in \mathbb{Z}^s$, to which the data $\mathrm{d}_{\boldsymbol{\alpha}}^{(r)}$, $\alpha \in \mathbb{Z}^s$, generated by the r-th step of a subdivision recursion is attached, these are usually set to

$$\mathbf{t}_{\boldsymbol{\alpha}}^{(r)} := \frac{\boldsymbol{\alpha}}{m^r}, \quad \boldsymbol{\alpha} \in \mathbb{Z}^s, \quad r \geq 0. \tag{1.11}$$

We refer to the choice in (1.11) as *standard* parametrization. We show in section 2 that the capability of subdivision to reproduce polynomials does depend on the choice of the associated parameter values and the standard parametrization is not always the optimal one. As in [6] for the univariate case, the choice

$$\mathbf{t}_{\boldsymbol{\alpha}}^{(r)} := \mathbf{t}_{\mathbf{0}}^{(r)} + \frac{\boldsymbol{\alpha}}{m^r}, \quad \mathbf{t}_{\mathbf{0}}^{(r)} = \mathbf{t}_{\mathbf{0}}^{(r-1)} - \frac{\boldsymbol{\tau}}{m^r}, \quad \mathbf{t}_{\mathbf{0}}^{(0)} = 0, \quad \boldsymbol{\alpha} \in \mathbb{Z}^s, \quad r \geq 0, \tag{1.12}$$

with a suitable $\boldsymbol{\tau} \in \mathbb{R}^s$ turns out to be a better selection.
We call the sequence $\{\mathbf{t}^{(r)},\ k \geq 0\}$, with $\mathbf{t}^{(r)} = \{\mathbf{t}_{\boldsymbol{\alpha}}^{(r)},\ \alpha \in \mathbb{Z}^s\}$ the *sequence of parameter values associated with the subdivision scheme.*

1.4 Convergence and non-singularity of subdivision

Following [10], our definition of convergence depends on the parameter values associated with a given subdivision scheme. Since the subdivision process generates denser and denser sequences of data $\mathrm{d}^{(r)}$, $r \geq 0$, a notion of convergence can be established by using a sequence $\{F^{(r)}, r \geq 0\}$ of continuous functions $F^{(r)}$ that interpolate the data $\mathrm{d}^{(r)}$ at the parameter values $\{\mathbf{t}^{(r)},\ k \geq 0\}$ associated to the subdivision scheme, namely

$$F^{(r)}(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}) = \mathrm{d}_{\boldsymbol{\alpha}}^{(r)}, \qquad \boldsymbol{\alpha} \in \mathbb{Z}^s, \quad r \geq 0. \tag{1.13}$$

Definition 1.1. *If the sequence of continuous functions* $\{F^{(r)},\ r \geq 0\}$ *satisfying* (1.13) *converges, then we denote its limit by*

$$g_{\mathrm{d}} := \lim_{r \to \infty} F^{(r)},$$

and say that g_{d} *is the* limit function *of the associated subdivision scheme (1.2) for the initial data* $\mathrm{d}^{(0)} = \{\mathrm{d}_{\boldsymbol{\alpha}}^{(0)},\ \boldsymbol{\alpha} \in \mathbb{Z}^s\} \in \ell(\mathbb{Z}^s)$. *The limit function* $\phi := g_\delta$ *with the initial data*

$$\delta_\alpha = \begin{cases} 1, & \alpha = \mathbf{0}, \\ 0, & \textit{otherwise} \end{cases}, \quad \alpha \in \mathbb{Z}^s,$$

is called the basic limit function of this scheme, it is compactly supported and satisfies the refinement equation

$$\phi = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}} \phi(m \cdot - \boldsymbol{\alpha}) \tag{1.14}$$

with refinement coefficients given by the mask a.

Definition 1.2. *A subdivision scheme is called* non-singular*, if it is convergent, and* $g_{\mathrm{d}} = 0$ *if and only if* d *is the zero sequence, i.e.* $\mathrm{d}_{\boldsymbol{\alpha}} = 0$ *for all* $\boldsymbol{\alpha} \in \mathbb{Z}^s$*.*

Next we show that the notion of non-singular subdivision scheme is equivalent to the notion of linear independence of the integer shifts of its basic limit function.

Proposition 1.3. *A convergent subdivision scheme* S_{a} *is non-singular if and only if the integer translates of the solution* ϕ *of refinement equation (1.14) with coefficients satisfying* $a_{\boldsymbol{e}}(\mathbf{1}) = 1,\ \boldsymbol{e} \in \mathrm{E}$*, are linearly independent and form a partition of unity.*

Proof:
"$\Longrightarrow$:" Assume that S_{a} is convergent, then the basic limit function $\phi = S_{\mathrm{a}}^{\infty}\delta$ satisfies the refinement equation (1.14) and its integer shifts form a partition of unity. If the convergent subdivision scheme S_{a} is non-singular, then for any starting sequence $\mathrm{d} \in \ell(\mathbb{Z}^s)$ we have

$$S_{\mathrm{a}}^{\infty}\mathrm{d} = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^s} \mathrm{d}_{\boldsymbol{\alpha}}\phi(\cdot - \boldsymbol{\alpha}) = 0$$

if and only if d is the zero sequence.
"$\Longleftarrow$:" Assume that the integer translates of the solution ϕ of refinement equation (1.14) are linearly independent and form a partition of unity. This implies that the subdivision scheme S_{a} associated with the symbol $a(\boldsymbol{z})$ is convergent, see [8, Lemma 2.3]. It is also then non-singular, otherwise one easily gets a contradiction to the assumption on linear independence of the translates of ϕ. ■

1.5 Polynomial generation versus polynomial reproduction

We denote by Π_k the space of multivariate polynomial of total degree $k \in \mathbb{N}_0$.

Definition 1.4 (Polynomial generation)**.** *A convergent stationary subdivision scheme* S_{a} generates *polynomials up to degree* d_G *(is* Π_{d_G}*-generating) if for any polynomial* $\pi \in \Pi_{d_G}$ *there exists some initial data* $\mathrm{q}^{(0)} \in \ell_\infty(\mathbb{Z}^s)$ *such that* $S_{\mathrm{a}}^{\infty}\mathrm{q}^{(0)} = \pi$*. Moreover, the initial data* $\mathrm{q}^{(0)}$ *is sampled from a polynomial* $\widetilde{\pi} \in \Pi_{d_G}$ *with the same leading coefficients as* $\pi \in \Pi_{d_G}$*.*

Note that the assumptions on the properties of $\widetilde{\pi}$ in the above Definition are justified by [7, Lemma 2.1]. Note also that polynomial generation is also studied in [4].

We continue with a slightly different notion, the notion of polynomial reproduction which requires a specific choice of starting sequences of a polynomial limit. The concepts of polynomial reproduction and generation coincide for $d_G = d_R = 0$.

Definition 1.5 (Polynomial reproduction)**.** *A convergent subdivision scheme* S_{a} *with parameter values* $\{\mathbf{t}^{(r)},\ r \geq 0\}$ *is* reproducing *polynomials up to degree* d_R *(is* Π_{d_R}*-reproducing) if for any polynomial* $\pi \in \Pi_{d_R}$ *and for the initial data* $\mathrm{p}^{(0)} = \{\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(0)}),\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$ *the limit of the subdivision satisfies* $S_{\mathrm{a}}^{\infty}\mathrm{p}^{(0)} = \pi$*.*

Another important property of subdivision is the so-called step-wise polynomial reproduction, we make use of it in Section 2.

Definition 1.6 (Step-wise polynomial reproduction)**.** *A convergent subdivision scheme* S_{a} *with parameter values* $\{\mathbf{t}^{(r)},\ r \geq 0\}$ *is* step-wise polynomial reproducing *up to degree* k *(is step-wise* Π_k*-reproducing) if for any polynomial* $\pi \in \Pi_k$ *and for the data* $\mathrm{d}^{(r)} = \{\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}),\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$

$$\mathrm{d}^{(r+1)} = S_{\mathrm{a}}\mathrm{d}^{(r)} \quad \textit{or, equivalently,} \quad \pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}) = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}}\, \pi(\mathbf{t}_{\boldsymbol{\beta}}^{(r)}), \quad \boldsymbol{\alpha} \in \mathbb{Z}^s\,. \tag{1.15}$$

The next proposition shows that for a non-singular subdivision scheme the concepts of polynomial reproduction and step-wise polynomial reproduction are equivalent.

Proposition 1.7. *A non-singular, subdivision scheme* S_{a} *is* step-wise polynomial reproducing *up to degree* k *if and only if it is* polynomial reproducing *up to degree* k*.*

Proof: "$\Longrightarrow$:" For any polynomial $\pi \in \Pi_k$, let $\mathrm{d}^{(0)} := \{\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(0)}),\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$. If the subdivision scheme S_{a} is step-wise Π_k-reproducing, then the sequence $\{F^{(r)}, r \geq 0\}$ of continuous functions $F^{(r)}$ satisfying (1.13) with $\mathrm{d}_{\boldsymbol{\alpha}}^{(r)} = \pi\left(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}\right)$, $\alpha \in \mathbb{Z}^s$, converges uniformly to π as $r \to \infty$, since the distance between the grid points $m^{-r}\mathbb{Z}^s$ goes to zero.
"$\Longleftarrow$:" Let us assume next that the subdivision scheme S_{a} is Π_k*-reproducing.* Let $r \geq 0$. On the one hand, applying the subdivision scheme to the data $\mathrm{d}^{(r)} = \{\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}),\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$ we obtain,

$$S_{\mathrm{a}}^{\infty}\mathrm{d}^{(r)} = S_{\mathrm{a}}^{\infty}\mathrm{d}^{(r+1)} = \pi, \quad \mathrm{d}^{(r+1)} = S_{\mathrm{a}}\mathrm{d}^{(r)}.$$

On the other hand, for the sequence $\mathrm{p}^{(r+1)} := \{\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}),\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$ we also have

$$S_{\mathrm{a}}^{\infty}\mathrm{p}^{(r+1)} = \pi\,.$$

Therefore, by the linearity of the operator S_a it follows

$$S_{\mathrm{a}}^{\infty}\left(\mathrm{d}^{(r+1)} - \mathrm{p}^{(r+1)}\right) = 0$$

and, thus, $\mathrm{p}^{(r+1)} = \mathrm{d}^{(r+1)} = S_{\mathrm{a}}\mathrm{d}^{(r)}$ due to the assumption of non-singularity. Thus, the claim follows. ■

2 Algebraic condition for polynomial generation and reproduction

In this section, for a non-singular subdivision scheme, we determine the value of $\boldsymbol{\tau} \in \mathbb{R}^s$ in (1.12) that guarantees the polynomial reproduction of linear polynomials, see Proposition 2.3. In Theorem 2.6, we then provide algebraic conditions on $a(\boldsymbol{z})$ for checking the reproduction of polynomials of higher degree. These algebraic conditions depend on the previously obtained value of $\boldsymbol{\tau}$. In the case of only convergent schemes see Corollaries 2.4 and 2.7. We start by defining the tensor product polynomial of degree $|\mathbf{j}|$, $\mathbf{j} \in \mathbb{N}_0^s$, given by

$$q_{\mathbf{0}}(\boldsymbol{z}) := 1, \quad q_{\mathbf{j}}(z_1, \ldots, z_s) := \prod_{i=1}^{s} \prod_{\ell_i=0}^{j_i-1} (z_i - \ell_i), \quad \mathbf{j} = (j_1, \ldots, j_s). \tag{2.1}$$

The following auxiliary proposition states known results on polynomial generation that we make use of in this section.

Proposition 2.1. *Let S_{a} be a convergent subdivision scheme.*

(i) The subdivision scheme S_{a} reproduces constant sequences or, equivalently, its symbol $a(\boldsymbol{z})$ satisfies (1.7), if and only if $a(\boldsymbol{z})$ satisfies (1.9).

(ii) Let $k \in \mathbb{N}$. The symbol $a(\boldsymbol{z})$ satisfies condition Z_k if and only if

$$\big(D^{\mathbf{j}} a_{\boldsymbol{e}}\big)(\mathbf{1}) = |m|^{-s} D^{\mathbf{j}} a(\mathbf{1}), \quad \textit{for} \quad \boldsymbol{e} \in \mathrm{E}, \quad \mathbf{j} \in \mathbb{N}_0^s, \quad |\mathbf{j}| < k.$$

(iii) Let $k \in \mathbb{N}$. The symbol $a(\boldsymbol{z})$ satisfies Condition Z_k if and only if

$$\sum_{\boldsymbol{\beta} \in \mathbb{Z}^2} q_{\mathbf{j}}(\boldsymbol{\alpha} - m\boldsymbol{\beta}) \mathrm{a}_{\boldsymbol{\alpha} - m\boldsymbol{\beta}} = |m|^{-s} D^{\mathbf{j}} a(\mathbf{1}), \quad \alpha \in \mathbb{Z}^s, \quad \mathbf{j} \in \mathbb{N}_0^s, \quad |\mathbf{j}| < k. \tag{2.2}$$

Proof: The proof of (i) for dilation matrix $2I$ follows from the definition of the operator S_{a} and [4, Section 6] and in the case of dilation matrix mI, $|m| > 2$, is in [13, Lemma 3.3]. The proof of (ii): Let $\mathbf{j} \in \mathbb{N}_0^s$, $|\mathbf{j}| < k$. Since $a_{\boldsymbol{e}}(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^s} \mathrm{a}_{\boldsymbol{e} - m\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{e} - m\boldsymbol{\alpha}}$, its $\mathbf{j}$-th derivative satisfies

$$\big(D^{\mathbf{j}} a_{\boldsymbol{e}}\big)(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^s} q_{\mathbf{j}}(\boldsymbol{e} - m\boldsymbol{\alpha}) \mathrm{a}_{\boldsymbol{e} - m\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{e} - m\boldsymbol{\alpha} - \mathbf{j}}, \quad \boldsymbol{z} \in (\mathbb{C} \setminus \{0\})^s. \tag{2.3}$$

Next, due to $a(\boldsymbol{z}) = \sum_{\boldsymbol{e} \in \mathrm{E}} a_{\boldsymbol{e}}(\boldsymbol{z})$, we have

$$\begin{aligned} \big(D^{\mathbf{j}} a\big)(\boldsymbol{\varepsilon}) &= \sum_{\boldsymbol{e} \in \mathrm{E}} \big(D^{\mathbf{j}} a_{\boldsymbol{e}}\big)(\boldsymbol{\varepsilon}) = \sum_{\boldsymbol{e} \in \mathrm{E}} \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^s} q_{\mathbf{j}}(\boldsymbol{e} - m\boldsymbol{\alpha}) \mathrm{a}_{\boldsymbol{e} - m\boldsymbol{\alpha}} \boldsymbol{\varepsilon}^{\boldsymbol{e} - m\boldsymbol{\alpha} - \mathbf{j}} \\ &= \sum_{\boldsymbol{e} \in \mathrm{E}} \boldsymbol{\varepsilon}^{\boldsymbol{e} - \mathbf{j}} \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^s} q_{\mathbf{j}}(\boldsymbol{e} - m\boldsymbol{\alpha}) \mathrm{a}_{\boldsymbol{e} - m\boldsymbol{\alpha}} = \sum_{\boldsymbol{e} \in \mathrm{E}} \boldsymbol{\varepsilon}^{\boldsymbol{e} - \mathbf{j}} \big(D^{\mathbf{j}} a_{\boldsymbol{e}}\big)(\mathbf{1}) \end{aligned}$$

for $\boldsymbol{\varepsilon} \in \Xi$. The rows of the matrix of this linear system are given by $(\boldsymbol{\varepsilon}^{\boldsymbol{e}-\mathbf{j}})_{\boldsymbol{e}\in \mathrm{E}}$. This matrix is invertible and Condition Z_k, due to

$$\sum_{\boldsymbol{e}\in \mathrm{E}} \boldsymbol{\varepsilon}^{\boldsymbol{e}-\mathbf{j}} = \begin{cases} |m|^s, & \boldsymbol{\varepsilon} = 1, \\ 0, & \text{otherwise}, \end{cases}$$

is equivalent to

$$\sum_{\boldsymbol{e}\in \mathrm{E}} D^{\mathbf{j}} a_{\boldsymbol{e}}(\mathbf{1}) = D^{\mathbf{j}} a(\mathbf{1}), \quad D^{\mathbf{j}} a_{\boldsymbol{e}}(\mathbf{1}) = D^{\mathbf{j}} a_{\widetilde{\boldsymbol{e}}}(\mathbf{1}), \quad \boldsymbol{e}, \widetilde{\boldsymbol{e}} \in \mathrm{E}, \quad \boldsymbol{e} \neq \widetilde{\boldsymbol{e}}.$$

Proof of (iii): For any $\boldsymbol{\alpha} \in \mathbb{Z}^s$ there exists $\boldsymbol{e} \in \mathrm{E}$ and $\boldsymbol{\beta} \in \mathbb{Z}^s$ such that $\boldsymbol{\alpha} = \boldsymbol{e} + m\beta$. Thus,

$$D^{\mathbf{j}} a_{\boldsymbol{e}}(\mathbf{1}) = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} q_{\mathbf{j}}(\boldsymbol{\alpha} - m\boldsymbol{\beta}) \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}}, \quad \boldsymbol{e} \in E\,.$$

The claim follows by (ii). ■

Remark 2.2. *Note that the proof of (ii) is also implied, for example, by [2, Theorem 3.7].*

The next result provides a simple algebraic conditions for determining $\boldsymbol{\tau} \in \mathbb{R}^s$, which appears in (1.12) and guarantees the reproduction of linear polynomials.

Proposition 2.3. *Let S_{a} be a non-singular subdivision scheme that generates linear polynomials, i.e. its symbol satisfies Condition Z_1. Then S_{a} reproduces linear polynomials if and only if its parameter values are given by (1.12) with $\boldsymbol{\tau} = |m|^{-s} \left(D^{\boldsymbol{\epsilon}_1} a(\mathbf{1}), \ldots, D^{\boldsymbol{\epsilon}_s} a(\mathbf{1})\right)$.*

Proof: According to Proposition 1.7 for non-singular subdivision schemes, polynomial reproduction is equivalent to step-wise polynomial reproduction. Moreover, any convergent subdivision scheme reproduces the constants, hence it is sufficient to prove the claim for polynomials of the form $\pi(x_1, \ldots, x_s) = x_j$, $j = 1, \ldots, s$. Let $r \in \mathbb{N}_0$ and set $\mathrm{d}^{(r)}_{\boldsymbol{\alpha}} = \pi(\mathbf{t}^{(r)}_{\boldsymbol{\alpha}})$, $\boldsymbol{\alpha} \in \mathbb{Z}^s$, with $\pi(\boldsymbol{x}) = x_j$. Then for any $\boldsymbol{\alpha} \in \mathbb{Z}^s$ and $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s)$ we get

$$\begin{aligned}
\mathrm{d}^{(r+1)}_{m\boldsymbol{\alpha}+\boldsymbol{e}} &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{m(\boldsymbol{\alpha}-\boldsymbol{\beta})+\boldsymbol{e}} \mathrm{d}^{(r)}_{\boldsymbol{\beta}} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{m\boldsymbol{\beta}+\boldsymbol{e}} \mathrm{d}^{(r)}_{\boldsymbol{\alpha}-\boldsymbol{\beta}} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{m\boldsymbol{\beta}+\boldsymbol{e}} \left(t^{(r)}_{\mathbf{0},j} + \frac{\alpha_j - \beta_j}{m^r} \right) \\
&= \underbrace{\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{m\boldsymbol{\beta}+\boldsymbol{e}} \left(t^{(r)}_{\mathbf{0},j} + \frac{m\alpha_j + e_j}{m^{r+1}} \right)}_{a_{\boldsymbol{e}}(\mathbf{1})} - \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{m\boldsymbol{\beta}+\boldsymbol{e}} \frac{m\beta_j + e_j}{m^{r+1}} \\
&= \left(t^{(r)}_{\mathbf{0},j} + \frac{m\alpha_j + e_j}{m^{r+1}} \right) - \frac{D^{\boldsymbol{\epsilon}_j} a_{\boldsymbol{e}}(\mathbf{1})}{m^{r+1}} \\
&= \left(t^{(r)}_{\mathbf{0},j} + \frac{m\alpha_j + e_j}{m^{r+1}} \right) - \frac{D^{\boldsymbol{\epsilon}_j} a(\mathbf{1})}{|m|^s \cdot m^{r+1}},
\end{aligned}$$

where the last equality is due to Proposition 2.1 part (ii) for $\mathbf{j} = \boldsymbol{\epsilon}_j$. Thus, $\mathrm{d}^{(r+1)}_{m\boldsymbol{\alpha}+\boldsymbol{e}}$ is equal to

$$\pi(t^{(r+1)}_{m\boldsymbol{\alpha}+\boldsymbol{e}}) = t^{(r+1)}_{\mathbf{0},j} + \frac{m\alpha_j + e_j}{m^{r+1}} = t^{(r)}_{\mathbf{0},j} + \frac{m\alpha_j - \tau_j + e_j}{m^{r+1}}, \quad \alpha \in \mathbb{Z}^s,$$

if and only if $\tau_j = |m|^{-s} D^{\boldsymbol{\epsilon}_j} a(\mathbf{1})$. ■

Note that not all convergent subdivision schemes are non-singular, e.g. the subdivision scheme based on the four directional box spline symbol is not. For such schemes one can still determine a parametrization that ensures its polynomial generation property. The following result is a direct consequence of Proposition 2.3.

Corollary 2.4. *Let S_{a} be a convergent subdivision scheme that generates linear polynomials, i.e. its symbol satisfies Condition Z_1. Then S_{a} reproduces linear polynomials if its parameter values are given by (1.12) with*

$$\boldsymbol{\tau} = |m|^{-s} \left(D^{\boldsymbol{\epsilon}_1} a(\mathbf{1}), \ldots, D^{\boldsymbol{\epsilon}_s} a(\mathbf{1}) \right).$$

The following result is crucial for the proof of the main result, Theorem 2.6 of this section. It allows us to express the polynomial generation of S_{a} in terms of the properties of its symbol.

Proposition 2.5. *Let $k \in \mathbb{N}$, $\boldsymbol{\tau} \in \mathbb{R}^s$ and $q_{\mathbf{j}}$ as in (2.1). A subdivision symbol $a(\boldsymbol{z})$ satisfies*

$$\big(D^{\mathbf{j}} a\big)(\mathbf{1}) = |m|^s q_{\mathbf{j}}(\boldsymbol{\tau}), \quad \big(D^{\mathbf{j}} a\big)(\boldsymbol{\varepsilon}) = 0 \quad \boldsymbol{\varepsilon} \in \Xi', \quad \mathbf{j} \in \mathbb{N}_0^s, \quad |\mathbf{j}| \le k, \tag{2.4}$$

if and only if

$$\sum_{\boldsymbol{\beta} \in \mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha} - m\boldsymbol{\beta}} \boldsymbol{\beta}^{\mathbf{j}} = \left(\frac{\boldsymbol{\alpha} - \boldsymbol{\tau}}{m} \right)^{\mathbf{j}}, \quad \alpha \in \mathbb{Z}^s, \quad \mathbf{j} \in \mathbb{N}_0^s, \quad |\mathbf{j}| \le k\,. \tag{2.5}$$

Proof: First, note that due to Proposition 2.1 the conditions in (2.4) are equivalent to

$$q_{\mathbf{j}}(\boldsymbol{\tau}) = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^s} q_{\mathbf{j}}(\boldsymbol{\alpha} - m\boldsymbol{\beta}) \mathrm{a}_{\boldsymbol{\alpha} - m\boldsymbol{\beta}}, \quad \mathbf{j} \in \mathbb{N}_0^s, \quad |\mathbf{j}| \le k\,. \tag{2.6}$$

Using this equivalent formulation, we prove the proposition by induction on k. For $k = 0$ the claim is true since for any $\boldsymbol{\tau} \in \mathbb{R}^s$ we get

$$q_{\mathbf{0}}(\boldsymbol{\tau}) = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha} - m\boldsymbol{\beta}} = \left(\frac{\boldsymbol{\alpha} - \boldsymbol{\tau}}{m} \right)^{\mathbf{0}} = 1.$$

Next, we assume that the claim is true for all $|\mathbf{j}| \le k-1$ and prove it for $\mathbf{j} \in \mathbb{N}_0^s$ with $|\mathbf{j}| = k$. To this purpose, we write the polynomial $q_{\mathbf{j}}$ in $\boldsymbol{x}$ of (total) degree $|\mathbf{j}| = k$ as

$$q_{\mathbf{j}}(\boldsymbol{\alpha} - m\boldsymbol{x}) = \sum_{\boldsymbol{\ell} \in \mathbb{N}_0^s,\ |\boldsymbol{\ell}| \le k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \boldsymbol{x}^{\boldsymbol{\ell}}, \quad \boldsymbol{x} \in \mathbb{R}^s, \quad c_{\mathbf{j},\boldsymbol{\alpha},\mathbf{j}} \neq 0\,. \tag{2.7}$$

Therefore, using the induction assumption and by (2.6) and (2.7) we have

$$\begin{aligned} q_{\mathbf{j}}(\boldsymbol{\tau}) &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|\le k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} \boldsymbol{\beta}^{\boldsymbol{\ell}} \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|=k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} \boldsymbol{\beta}^{\boldsymbol{\ell}} + \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|\le k-1} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} \boldsymbol{\beta}^{\boldsymbol{\ell}} \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|=k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} \boldsymbol{\beta}^{\boldsymbol{\ell}} + \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|\le k-1} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m}\right)^{\boldsymbol{\ell}} \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|=k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \left(\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} \boldsymbol{\beta}^{\boldsymbol{\ell}} - \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m}\right)^{\boldsymbol{\ell}} \right) + q_{\mathbf{j}}(\boldsymbol{\tau})\,. \end{aligned}$$

The last equality is due to the fact that

$$q_{\mathbf{j}}(\boldsymbol{\tau}) = q_{\mathbf{j}}\left(\boldsymbol{\alpha} - m\cdot\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m}\right) = \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|\le k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m}\right)^{\boldsymbol{\ell}}.$$

Hence, due to $c_{\mathbf{j},\boldsymbol{\alpha},\mathbf{j}} \neq 0$, the above identity holds if and only if

$$\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} \boldsymbol{\beta}^{\mathbf{j}} - \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m}\right)^{\mathbf{j}} = 0, \quad \text{for } \mathbf{j}\in\mathbb{N}_0^s, \quad |\mathbf{j}| = k,$$

and the claim follows. ■

We are now ready to prove the main results of this paper.

Theorem 2.6. *Let $k \in \mathbb{N}_0$. A non-singular subdivision scheme with symbol $a(\boldsymbol{z})$ and associated parametrization in (1.12) with some $\boldsymbol{\tau} \in \mathbb{R}^s$ reproduces polynomials of degree up to k if and only if*

$$\big(D^{\mathbf{j}} a\big)(\mathbf{1}) = |m|^s q_{\mathbf{j}}(\boldsymbol{\tau}) \quad \textit{and} \quad \big(D^{\mathbf{j}} a\big)(\boldsymbol{\varepsilon}) = 0 \quad \textit{for} \quad \boldsymbol{\varepsilon}\in\Xi', \quad |\mathbf{j}| \le k\,.$$

Proof: The proof is by induction on k. In the case $k = 0$ the claim follows by part (i) of Proposition 2.1. By Proposition 1.7 it suffices to prove the result for the stepwise polynomial reproduction.

"$\Longleftarrow$:" We write any polynomial π of degree k as $\pi(\boldsymbol{x}) = \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s, |\boldsymbol{\ell}|=k} c_{\boldsymbol{\ell}} \boldsymbol{x}^{\boldsymbol{\ell}} + \widetilde{\pi}(\boldsymbol{x})$ with $\widetilde{\pi} \in \Pi_{k-1}$. Let $r \ge 0$. We show that the sequence

$$\mathrm{d}^{(r)} = \left\{ \pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}) = \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s, |\boldsymbol{\ell}|=k} c_{\boldsymbol{\ell}} \left(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}\right)^{\boldsymbol{\ell}} + \widetilde{\pi}(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}),\ \boldsymbol{\alpha}\in\mathbb{Z}^s \right\}$$

satisfies $\mathrm{d}^{(r+1)} = S_{\mathrm{a}}\mathrm{d}^{(r)} = \{\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}),\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$. In fact, due to the induction assumption, by (1.12) and Proposition 2.5, we have

$$\begin{aligned}
\mathrm{d}_{\boldsymbol{\alpha}}^{(r+1)} &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}}\mathrm{d}_{\boldsymbol{\beta}}^{(r)} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,|\boldsymbol{\ell}|=k} c_{\boldsymbol{\ell}} \left(\mathbf{t}_{\mathbf{0}}^{(r)} + \frac{\boldsymbol{\beta}}{m^r}\right)^{\boldsymbol{\ell}} + \widetilde{\pi}(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}) \\
&= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,|\boldsymbol{\ell}|=k} c_{\boldsymbol{\ell}} \sum_{\mathbf{h}\le\boldsymbol{\ell}} \binom{\boldsymbol{\ell}}{\mathbf{h}} \left(\frac{\boldsymbol{\beta}}{m^r}\right)^{\mathbf{h}} \left(\mathbf{t}_{\mathbf{0}}^{(r)}\right)^{\mathbf{h}-\boldsymbol{\ell}} + \widetilde{\pi}(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}) \\
&= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,|\boldsymbol{\ell}|=k} c_{\boldsymbol{\ell}} \sum_{\mathbf{h}\le\boldsymbol{\ell}} \binom{\boldsymbol{\ell}}{\mathbf{h}} \left(\mathbf{t}_{\mathbf{0}}^{(r)}\right)^{\mathbf{h}-\boldsymbol{\ell}} \left(\frac{1}{m^r}\right)^{\mathbf{h}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \boldsymbol{\beta}^{\mathbf{h}} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} + \widetilde{\pi}(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}) \\
&= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,|\boldsymbol{\ell}|=k} c_{\boldsymbol{\ell}} \sum_{\mathbf{h}\le\boldsymbol{\ell}} \binom{\boldsymbol{\ell}}{\mathbf{h}} \left(\mathbf{t}_{\mathbf{0}}^{(r)}\right)^{\mathbf{h}-\boldsymbol{\ell}} \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m^{r+1}}\right)^{\mathbf{h}} + \widetilde{\pi}(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}) \\
&= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,|\boldsymbol{\ell}|=k} c_{\boldsymbol{\ell}} \left(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}\right)^{\boldsymbol{\ell}} + \widetilde{\pi}(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}) = \pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}), \quad \boldsymbol{\alpha} \in \mathbb{Z}^s.
\end{aligned}$$

The one but last equality is due to

$$\mathbf{t}_{\mathbf{0}}^{(r)} + \frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m^{r+1}} = \mathbf{t}_{\mathbf{0}}^{(r+1)} + \frac{\boldsymbol{\alpha}}{m^{r+1}} = \mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}.$$

"$\Longrightarrow$:" Let $\mathbf{j} \in \mathbb{N}_0^s$ be such that $|\mathbf{j}| = k$ with $j_i = k$ for some $i = 1, \ldots, s$. Let the polynomial $\pi(\boldsymbol{x}) = \boldsymbol{x}^{\mathbf{j}}$ and the sequence $\mathrm{d}^{(r)} = \{\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}),\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$. On the one hand, by similar arguments as above, we get

$$\mathrm{d}_{\boldsymbol{\alpha}}^{(r+1)} = \sum_{\mathbf{h}\le\mathbf{j}} \binom{\mathbf{j}}{\mathbf{h}} \left(\mathbf{t}_{\mathbf{0}}^{(r)}\right)^{\mathbf{h}-\mathbf{j}} \left(\frac{1}{m^r}\right)^{\mathbf{h}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \boldsymbol{\beta}^{\mathbf{h}} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}}.$$

On the other hand, the definition of $\mathbf{t}_{\mathbf{0}}^{(r+1)}$ yields

$$\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r+1)}) = \sum_{\mathbf{h}\le\mathbf{j}} \binom{\mathbf{j}}{\mathbf{h}} \left(\mathbf{t}_{\mathbf{0}}^{(r)}\right)^{\mathbf{h}-\mathbf{j}} \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m^{r+1}}\right)^{\mathbf{h}}.$$

The polynomial reproduction, *i.e.* $\mathrm{d}^{(r+1)} = S_{\mathrm{a}}\mathrm{d}^{(r)}$, implies that

$$\sum_{\mathbf{h}\le\mathbf{j}} \binom{\mathbf{j}}{\mathbf{h}} \left(\mathbf{t}_{\mathbf{0}}^{(r)}\right)^{\mathbf{h}-\mathbf{j}} \left(\frac{1}{m^r}\right)^{\mathbf{h}} \left[\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \boldsymbol{\beta}^{\mathbf{h}} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} - \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m}\right)^{\mathbf{h}}\right] = 0.$$

Thus, by induction for $\mathbf{h} = \mathbf{j}$ we have

$$\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \boldsymbol{\beta}^{\mathbf{j}} \mathrm{a}_{\boldsymbol{\alpha}-m\boldsymbol{\beta}} - \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\tau}}{m}\right)^{\mathbf{j}} = 0, \quad \alpha \in \mathbb{Z}^s.$$

The claim follows from Proposition 2.5. ■

For convergent schemes we readily get the following result, which is due to the fact that for convergent schemes the step-wise polynomial reproduction implies polynomial reproduction.

Corollary 2.7. *Let $k \in \mathbb{N}_0$. A convergent subdivision scheme with symbol $a(\boldsymbol{z})$ and associated parametrization in (1.12) with some $\boldsymbol{\tau} \in \mathbb{R}^s$ reproduces polynomials of degree up to k if*

$$\big(D^{\mathbf{j}}a\big)(\mathbf{1}) = |m|^s q_{\mathbf{j}}(\boldsymbol{\tau}) \quad \textit{and} \quad \big(D^{\mathbf{j}}a\big)(\boldsymbol{\varepsilon}) = 0 \quad \textit{for} \quad \boldsymbol{\varepsilon} \in \Xi', \quad |\mathbf{j}| \le k\,.$$

3 Applications and examples

It is natural to expect that any shift of the subdivision mask does not effect the polynomial reproduction properties of the corresponding scheme, which is confirmed by the next result.

Lemma 3.1. *A convergent subdivision scheme S_{a} with the symbol $a(\boldsymbol{z})$ reproduces polynomials up to degree k if and only if so does the shifted scheme $S_{\widetilde{\mathrm{a}}}$ with the symbol $\widetilde{a}(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{\alpha}} a(\boldsymbol{z})$, $\boldsymbol{\alpha} \in \mathbb{Z}^s$.*

Proof: By Proposition 2.3 and due to $a(\mathbf{1}) = m^s$, we get the following identity for the suitable $\boldsymbol{\tau}$ of S_{a} and $\widetilde{\boldsymbol{\tau}}$ of $S_{\widetilde{\mathrm{a}}}$

$$\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau} + \boldsymbol{\alpha}.$$

By Leibnitz differentiation formula and due to the fact that $D^{\boldsymbol{\ell}}\boldsymbol{z}^{\boldsymbol{\alpha}}$, $\boldsymbol{\ell} \in \mathbb{N}_0^s$, evaluated at $\mathbf{1}$ is equal to $q_{\boldsymbol{\ell}}(\boldsymbol{\alpha})$ in (2.1) we have

$$D^{\mathbf{j}}\widetilde{a}(\mathbf{1}) = \sum_{\boldsymbol{\ell} \in \mathbb{N}_0^s, \boldsymbol{\ell} \le \mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} q_{\boldsymbol{\ell}}(\boldsymbol{\alpha}) D^{\mathbf{j}-\boldsymbol{\ell}} a(\mathbf{1}).$$

Thus, by Corollary 2.7, to prove the claim we need to show that

$$q_{\mathbf{j}}(\boldsymbol{\tau} + \boldsymbol{\alpha}) = \sum_{\boldsymbol{\ell} \in \mathbb{N}_0^s, \boldsymbol{\ell} \le \mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} q_{\boldsymbol{\ell}}(\boldsymbol{\alpha}) q_{\mathbf{j}-\boldsymbol{\ell}}(\boldsymbol{\tau}).$$

By definition of $q_{\mathbf{j}}$ it is a tensor product polynomial, thus, it suffices to show that the following two univariate polynomials in $\tau \in \mathbb{R}$ are equal

$$\prod_{n=0}^{j-1} (\tau + \alpha - n) = \sum_{0 \le \ell \le j} \binom{j}{\ell} \prod_{i=0}^{\ell-1} (\tau - i) \prod_{t=0}^{j-\ell-1} (\alpha - t).$$

The claim follows by the one dimensional result [6, Corollary 5.1]. ■

3.1 Box splines

An $s-$variate box spline is given by its symbol

$$a_\Theta(\boldsymbol{z}) = 2^s \prod_{\theta\in\Theta} \frac{1+\boldsymbol{z}^\theta}{2}, \tag{3.1}$$

where θ runs through all the columns of the $s \times n$, rank s matrix $\Theta \in \mathbb{Z}^{s\times n}$ with $n \geq s$. It is well-known that the subdivision schemes associated with the symbols $a_\Theta(\boldsymbol{z})$ are convergent, if the matrix Θ is such that removing any column from Θ does not change its rank, see [1, p. 127]. Next results gives the correct parametrization for box spline subdivision schemes.

Lemma 3.2. *A subdivision scheme with the symbol* $\mathrm{a}_\Theta(\boldsymbol{z})$ *in* (3.1) *reproduces linear polynomials if its associated parameter values are as in* (1.12) *with*

$$\boldsymbol{\tau} = \frac{1}{2}\left(\sum_{\theta\in\Theta}\theta_1, \ldots, \sum_{\theta\in\Theta}\theta_s\right), \quad \theta = (\theta_1, \ldots, \theta_s).$$

Proof: The result follows from Corollary 2.4 and the simple fact that

$$D^{\epsilon_j} a_\Theta(\boldsymbol{z}) = 2^s \cdot \frac{1}{2} \cdot \sum_{\theta\in\Theta} \theta_j z^{\theta-\epsilon_j} \prod_{\substack{\tilde\theta\in\Theta \\ \tilde\theta\neq\theta}} \frac{1+\boldsymbol{z}^{\tilde\theta}}{2}, \quad z \in (\mathbb{C}\setminus\{0\})^s.$$

■

Remark 3.3. *In case* Θ *is unimodular, i.e. each* $s \times s$ *submatrix of* Θ *has determinant* 1 *or* -1*, the integer shifts of the corresponding box splines are linear independent and therefore the subdivision scheme associated with* $\mathrm{a}_\Theta(\boldsymbol{z})$*, if convergent, is non-singular due to Proposition 1.3. Hence, the results of Proposition 2.3 and of Theorem 2.6 hold.*

We consider an example of the 3−directional box splines with

$$\Theta = \begin{pmatrix} \underbrace{\begin{matrix}1 & \ldots & 1\\ 0 & \ldots & 0\end{matrix}}_{k \text{ times}} & \underbrace{\begin{matrix}0 & \ldots & 0\\ 1 & \ldots & 1\end{matrix}}_{\ell \text{ times}} & \underbrace{\begin{matrix}1 & \ldots & 1\\ 1 & \ldots & 1\end{matrix}}_{n \text{ times}} \end{pmatrix}$$

and the corresponding symbols $a_\Theta(z)$ are denoted by

$$B_{k,\ell,n}(z_1, z_2) = 4 \cdot \left(\frac{1+z_1}{2}\right)^k \left(\frac{1+z_2}{2}\right)^\ell \left(\frac{1+z_1z_2}{2}\right)^n, \quad k,\ell,n \in \mathbb{N}_0\,.$$

In the case $k = \ell = n = 2$, results in [5] imply that the degree of polynomial generation is $k = 4$. Now, to check the degree of polynomial reproduction we first use Lemma 3.2 to compute $\boldsymbol{\tau}$ that guarantees the reproduction of linear polynomials, *i.e.* $\boldsymbol{\tau} = \frac{1}{2}(k+n, \ell+n) = (2,2)$. Using Theorem 2.6 with this $\boldsymbol{\tau}$ we see that the scheme does not reproduce polynomials of degree $k = 2$, since $q_{1,1}(\boldsymbol{\tau}) = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = 4$, but $D^{(1,1)}B_{2,2,2}(\mathbf{1}) = 18$.

3.2 Three dimensional example

Three dimensional examples can also be considered. For example, it is easy to show that the subdivision scheme with the mask symbol

$$\begin{aligned} a(z_1,z_2,z_3) \;=\; & 2^3\Big[6z_1z_2z_3\left(\frac{1+z_1}{2}\right)^2\left(\frac{1+z_2}{2}\right)^2\left(\frac{1+z_3}{2}\right)^2\left(\frac{1+z_1z_2z_3}{2}\right)^2 - \\ & \frac{5}{4}z_1\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)^3\left(\frac{1+z_3}{2}\right)^3\left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\ & \frac{5}{4}z_2\left(\frac{1+z_1}{2}\right)^3\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_3}{2}\right)^3\left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\ & \frac{5}{4}z_3\left(\frac{1+z_1}{2}\right)^3\left(\frac{1+z_2}{2}\right)^3\left(\frac{1+z_3}{2}\right)\left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\ & \frac{5}{4}z_1z_2z_3\left(\frac{1+z_1}{2}\right)^3\left(\frac{1+z_2}{2}\right)^3\left(\frac{1+z_3}{2}\right)^3\left(\frac{1+z_1z_2z_3}{2}\right)\Big]. \end{aligned}$$

is convergent. Moreover, using Theorem 2.6 we get that for $\boldsymbol{\tau} = (3,3,3)$, thus, the scheme also reproduces linear polynomials. Since $D^{(2,0,0)}a(\mathbf{1}) = 46 \neq 8 \cdot q_{2,0,0}(\boldsymbol{\tau}) = 48$, the scheme is not reproducing polynomials of degree 2.

3.3 Interpolatory schemes

Exactly the same argument for interpolatory schemes as in [6, Corollary 5.3] extends to the multivariate case. Interpolatory schemes are such whose mask satisfies

$$\mathrm{a}_{\mathbf{0}} = 1, \quad \mathrm{a}_{m\boldsymbol{\alpha}} = 0, \quad \boldsymbol{\alpha} \in \mathbb{Z}^s\,,$$

and therefore, when convergent, with a limit function interpolating the initial data as well as all the data generated through the recursions.

Let us assume that an interpolatory scheme generates polynomials up to degree k. Due to the special structure of the symbol of interpolatory schemes

$$a(\boldsymbol{z}) = 1 + \sum_{\boldsymbol{e}\in E\setminus\{\mathbf{0}\}} a_{\boldsymbol{e}}(\boldsymbol{z})$$

we get $D^{\mathsf{j}}a(\mathbf{1}) = 0$ for all $\mathbf{j} \in \mathbb{N}_0^s$ with $|\mathbf{j}| \leq k$. Then, by Proposition 2.3, the suitable choice of $\boldsymbol{\tau}$ in (1.12) for reproduction of linear polynomials is $\boldsymbol{\tau} = \mathbf{0}$. Corollary 2.7 and the definition of $q_{\mathbf{j}}$ in (2.1) imply that the scheme also reproduces polynomials up to degree k with this $\boldsymbol{\tau}$. Thus, the following result holds and confirms that our results reproduce results in [11].

Proposition 3.4. *A convergent interpolatory scheme S_{a} reproduces polynomials up to degree k if and only if it generates polynomials of degree up to k.*

For the butterfly scheme with the symbol

$$a(z_1,z_2) = 4\cdot z_1^{-3}z_2^{-3}[7z_1z_2B_{2,2,2} - 2z_1B_{1,3,3} - 2z_2B_{3,1,3} - 2z_1z_2B_{3,3,1}]$$

we have $\boldsymbol{\tau} = (0,0)$, as expected. Since $a(z_1, z_2)$ satisfies sum rules of order 4 (see, again [5]), the subdivision scheme generates cubic polynomials, it also reproduces cubic polynomials by Corollary 2.7.

3.4 $\sqrt{3}-$subdivision

The approximating $\sqrt{3}-$subdivision scheme from [16] with the mask symbol

$$\begin{aligned} a(\boldsymbol{z}) &= \frac{1}{6}\left(z_1 z_2 + z_1^{-1} z_2^{-1} + z_1^{-1} z_2^2 + z_1^{-2} z_2 + z_1 z_2^{-2} + z_1^2 z_2^{-1}\right) \\ &+ \frac{1}{3}\left(z_1^{-1} + z_2 + z_1 z_2^{-1}\right) + \frac{1}{3}\left(z_2^{-1} + z_1 + z_1^{-1} z_2\right) \end{aligned}$$

satisfies sum rules at most of order 2. The associated dilation matrix $M = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$ satisfies $M^2 = -3I$ and the corresponding refinable function is also refinable with respect to the iterated mask $a(z_1 z_2^{-2}, z_1^2 z_2^{-1}) \cdot a(\boldsymbol{z})$. By Corollary 2.4, the corresponding scheme reproduces linear polynomials, if $\boldsymbol{\tau} = (0,0)$. Thus, associated refinable function has approximation order 2.

4 Conclusions

In this paper we give algebraic conditions on the symbol of a multivariate subdivision scheme with dilation matrix mI, $|m| \geq 2$, that allow us to determine the degree of polynomial reproduction of the scheme. These conditions also yield the correct parametrization for any convergent subdivision scheme to guarantee polynomial reproduction of degree at least 1. This is true in particular for subdivision schemes associated with box splines. The restriction of a dilation matrix of type mI and tensor product structure of the polynomial in (2.1) let us extend the univariate results in [6] easily to the multivariate setting. We believe that this paper is an important first step towards the investigation of polynomial reproduction of multivariate subdivision schemes with general dilation matrix, which is currently under investigation.

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