Convergence analysis of a family of 14-node brick elements *

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Abstract

In this paper, we will give a convergence analysis for a family of 14-node elements which was proposed by I. M. Smith and D. J. Kidger[Int. J. Numer. Meth. Engng., 35:1263–1275, 1992]. The 14 DOFs are taken as the values at the eight vertices and the six face-centroids. For second-order elliptic problems, we will show that among all the Smith-Kidger 14-node elements, Type 1, Type 2 and Type 5 elements provide optimal-order convergent solutions while Type 6 element gives one-order lower convergent solutions. Motivated by our proof, we also find that the order of convergence of the Type 6 14-node nonconforming element improves to be optimal if we change the DOFs into the values at the eight vertices and the integration values on the six faces. We also show that Type 1, Type 2 and Type 5 keep the optimal-order convergence if the integral DOFs on the six faces are adopted.

Keywords: Nonconforming element; Brick element; 14-node element; Second-order elliptic problem; Smith-Kidger element

1 Introduction

Among many three-dimensional brick elements, there have been well-known simplest conforming elements such as the trilinear element, the 27-node element and seredipity

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elements. For the nonconforming case, Rannacher-Turek [7] presented the rotated trilinear elements with the two types of 6 DOFs: the face-centroid values type and the face integrals type. Douglas-Santos-Sheen-Ye [2] then modified the element of Rannacher-Turek such that the face-centroid values type and the face integrals type are identical, that is, the element fulfills the mean value property "the face-centroid value = the face average integral". Later Park-Sheen presented a P_1-nonconforming finite element on cubic meshes which has only 4 DOFs [6]. Wilson also defined a linear-order nonconforming brick element [1, 15] with 11 DOFs whose polynomial space consists of trilinear polynomials plus $\{1-x_1^2, 1-x_2^2, 1-x_3^2\}$ on $\widehat{\mathbf{K}} = [-1,1]^3$ (see [1, Page 217, Remark 4.2.3]). All these three dimensional elements are of $O(h)$ convergence rate in energy norm.

In the direction of obtaining higher-order convergent nonconforming elements, Smith and Kidger [12] successfully developed three-dimensional brick elements of 14 DOFs by adding additional polynomials to P_2. They investigated six most possible 14 DOFs elements systematically considering the Pascal pyramid, and concluded that their Type 1 (as well as Type 2) and Type 6 elements are successful ones. The additional polynomial space for Type 1 element is the span of the four nonsymmetric cubic polynomials $\{x_1x_2x_3, x_1^2x_2, x_2^2x_3, x_3^2x_1\}$ while that for Type 6 element is the span of $\{x_1x_2x_3, x_1x_2^2x_3^2, x_1^2x_2x_3^2, x_1^2x_2^2x_3\}$. Only recently a new nonconforming brick element of 14 DOFs with quadratic and cubic convergence in the energy and L_2 norms, respectively, is introduced by Meng, Sheen, Luo, and Kim [5], which has the same type of DOFs but has only cubic polynomials added to P_2. And then, the authors compared these 14-node elements numerically, see [4]. Numerical tests show that at least for second-order elliptic problems Meng-Sheen-Luo-Kim and some of Smith-Kidger elements are convergent with optimal order or with lower order.

A convergence analysis for Meng-Sheen-Luo-Kim element was reported in [5] and is fairly easy because it satisfies the patch test of Irons [3], which implies that a successful P_k-nonconforming element needs to satisfy that on each interface the jump of adjacent polynomials be orthogonal to P_{k-1} polynomials on the interface. Unfortunately, it was found in mathematics that the patch test is neither necessary nor sufficient, see [9] and the references therein. As shown in this paper, the Smith-Kidge elements can only pass a lower order patch test or can not pass it, but give optimal order convergence from our numerical results or lower convergence order. Thus, the convergence analysis for Smith-Kidger element seems to be quite different and complex. For the convergence of the nonconforming element which fail to pass the patch test, see the works of Stummel, Shi, *etc.* [14, 8, 10, 11].

In this paper, we will provide a convergence analysis for Smith-Kidger elements for second-order elliptic problems. We show that although the patch test fails, Type 1, 2 and 5 Smith-Kidger elements are of optimal convergence order, while Type 6 element loses one order of accuracy. Furthermore, we also present a new brick element with the same DOFs, which is also convergent in optimal orders. Finally, if the value at the eight vertices and the integration values over six faces are taken as the DOFs, then we can show that Type 1, 2, 5, and 6 elements and the proposed new element can get optimal convergence order, which implies that Type 6 element improves one order of accuracy.

The paper is organized as follows. In Section 2, we will introduce Smith-Kidger elements and give the basis functions firstly. In Section 3, we define an interpolation operator and present our convergence analysis for Type 1 Smith-Kidger element. In Section 4, we will analyze the other elements and present the corresponding error estimates very briefly. In Section 5, a new 14-node brick element is proposed. Finally, in Section 6, we conclude our results.

2 The quadratic nonconforming brick elements

Let $\widehat{\mathbf{K}} = [-1,1]^3$ and denote the vertices and face-centroids by $V_j, 1 \le j \le 8$, and $M_k, 1 \le k \le 6$, respectively. (see Fig. 1)

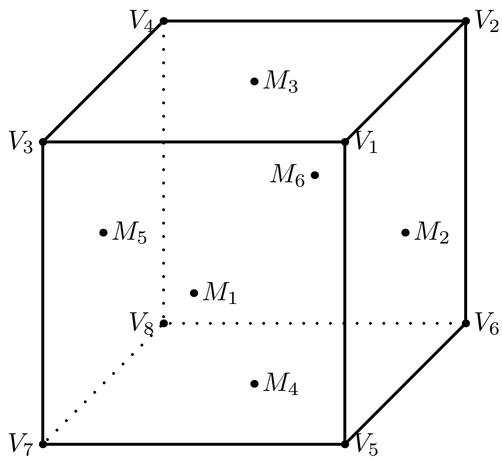

Figure 1: V_j denotes the vertices, $j = 1, 2, \ldots, 8$, and M_k denotes the face-centroids, $k = 1, 2, \ldots, 6$.

Smith and Kidger [12] defined the following six 14-node elements:

$$\widehat{\mathbb{P}}_{SK}^{(1)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{x}_1\widehat{x}_2\widehat{x}_3, \widehat{x}_1^2\widehat{x}_2, \widehat{x}_2^2\widehat{x}_3, \widehat{x}_3^2\widehat{x}_1\}, \tag{1a}$$
$$\widehat{\mathbb{P}}_{SK}^{(2)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{x}_1\widehat{x}_2\widehat{x}_3, \widehat{x}_1\widehat{x}_2^2, \widehat{x}_2\widehat{x}_3^2, \widehat{x}_3\widehat{x}_1^2\}, \tag{1b}$$
$$\widehat{\mathbb{P}}_{SK}^{(3)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{x}_1\widehat{x}_2\widehat{x}_3, \widehat{x}_1^3, \widehat{x}_2^3, \widehat{x}_3^3\}, \tag{1c}$$
$$\widehat{\mathbb{P}}_{SK}^{(4)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{x}_1\widehat{x}_2\widehat{x}_3, \widehat{x}_1^2\widehat{x}_2\widehat{x}_3, \widehat{x}_1\widehat{x}_2^2\widehat{x}_3, \widehat{x}_1\widehat{x}_2\widehat{x}_3^2\}, \tag{1d}$$
$$\widehat{\mathbb{P}}_{SK}^{(5)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{x}_1\widehat{x}_2\widehat{x}_3, \widehat{x}_1^2\widehat{x}_2 + \widehat{x}_1\widehat{x}_2^2, \widehat{x}_2^2\widehat{x}_3 + \widehat{x}_2\widehat{x}_3^2, \widehat{x}_3^2\widehat{x}_1 + \widehat{x}_3\widehat{x}_1^2\}, \tag{1e}$$
$$\widehat{\mathbb{P}}_{SK}^{(6)} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{x}_1\widehat{x}_2\widehat{x}_3, \widehat{x}_1\widehat{x}_2^2\widehat{x}_3^2, \widehat{x}_1^2\widehat{x}_2\widehat{x}_3^2, \widehat{x}_1^2\widehat{x}_2^2\widehat{x}_3\}, \tag{1f}$$

whose DOFs are the function values at the eight vertices and the six face-centroids. They reported that Type 3 element fails and is inadmissible. We also remark that Type 4 element is also inadmissible since $(\widehat{x}_1^2 - 1)\widehat{x}_2\widehat{x}_3 \in \widehat{\mathbb{P}}_{SK}^{(4)}$ vanishes at all these points. In [4], we observe that Type 1 (and 2) and Type 5 elements give optimal convergence results both in L^2- and H^1-norms at least for second-order elliptic problems, while Type 6 element loses one order of accuracy in each norm.

In what follows, we will give an error estimate for Type 1 Smith-Kidger element in detail; error estimates for the other types can be obtained completely analogously and thus are stated very briefly.

To begin with, denote by $V_m, m = 1, \cdots, 8$, the eight vertices $(j,k,l), |j| = |k| = |l| = 1, j,k,l \in \mathbb{Z}$, and and by $V_m, m = 1, \cdots, 6$, the six face-centroids $(j,k,l), |j|+|k|+|l| = 1, j,k,l \in \mathbb{Z}$. The basis functions corresponding to the eight vertices $V_j, j = 1, \cdots, 8$, are denoted by $\widehat{\phi}^V_{V_j}$, and those corresponding to the six face-centroids by $\widehat{\phi}^F_{M_j}, j = 1, \cdots, 6$.

To describe the brick elements in a uniform fashion, set

$$\begin{aligned} r_0(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \widehat{x}_1\widehat{x}_2\widehat{x}_3,\ r_1(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \widehat{x}_1\widehat{x}_3^2, \\ r_2(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \widehat{x}_2\widehat{x}_1^2,\ r_3(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \widehat{x}_3\widehat{x}_2^2, \end{aligned} \tag{2}$$

so that

$$\widehat{\mathbb{P}}^{(1)}_{SK} = P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{r_0(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3),\ r_1(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3),\ r_2(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3),\ r_3(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)\}.$$

Then the basis functions for Type 1 Smith-Kidger elements are given as follows:
for $(j,k,l) = V_m, m = 1, \cdots, 8$,

$$\begin{aligned} \widehat{\phi}^V_{(j,k,l)}(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \frac{1}{16}\left[-1 + \widehat{x}_1^2 + \widehat{x}_2^2 + \widehat{x}_3^2)\right] + \frac{1}{8}\left[jk\widehat{x}_1\widehat{x}_2 + jl\widehat{x}_1\widehat{x}_3 + kl\widehat{x}_2\widehat{x}_3 \right. \\ \left. + jklr_0(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) + r(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)\right], \end{aligned} \tag{3}$$

and, for $(j,k,l) = M_m, m = 1, \cdots, 6$,

$$\widehat{\phi}^F_{(j,k,l)}(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \frac{1}{4} + \frac{1}{2}\ell(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) + \frac{1}{4}q(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) - \frac{1}{2}r(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3), \tag{4}$$

where the linear, quadratic, and the remaining higher-order terms are defined by

$$\ell(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = j\widehat{x}_1 + k\widehat{x}_2 + l\widehat{x}_3, \tag{5a}$$

$$q(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = -(\widehat{x}_1^2 + \widehat{x}_2^2 + \widehat{x}_3^2) + 2(j\widehat{x}_1^2 + k\widehat{x}_2^2 + l\widehat{x}_3^2), \tag{5b}$$

$$r(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = jr_1(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) + kr_2(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) + lr_3(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3). \tag{5c}$$

Assume that $\Omega \in \mathbb{R}^3$ is a parallelepiped domain with boundary Γ. Let $(\mathscr{T}_h)_{h>0}$ be a regular family of triangulations of Ω into parallelepipeds $\mathbf{K}_j, j = 1, 2, \ldots, N_{\mathbf{K}}$, where $h = \max_{\mathbf{K}\in\mathscr{T}_h} h_{\mathbf{K}}$ with $h_{\mathbf{K}} = \mathrm{diam}(\mathbf{K})$. For each $\mathbf{K} \in \mathscr{T}_h$, let $T_{\mathbf{K}} : \widehat{\mathbf{K}} \to \mathbb{R}^3$ be an invertible affine mapping such that

$$\mathbf{K} = T_{\mathbf{K}}(\widehat{\mathbf{K}}),$$

and set $\phi_{\mathbf{K}} = \widehat{\phi} \circ T_{\mathbf{K}}^{-1} : \mathbf{K} \to \mathbb{R}$ for all $\widehat{\phi} \in \widehat{\mathbb{P}}^{(1)}_{SK}$, whose collection will be designated by

$$\mathbb{P}_{\mathbf{K}} = \mathrm{Span}\{\phi_{\mathbf{K}} \mid \widehat{\phi} \in \widehat{\mathbb{P}}^{(1)}_{SK}\}.$$

Let N_V and N_F denote the numbers of vertices and faces, respectively. Then set

$$\mathscr{V}_h = \{V_1, V_2, \cdots, V_{N_V} : \quad \text{the set of all vertices of } \mathbf{K} \in \mathscr{T}_h\},$$

$$
\begin{aligned}
\mathscr{F}_h &= \{F_1, F_2, \cdots, F_{N_F} : \quad \text{the set of all faces of } \mathbf{K} \in \mathscr{T}_h\},\\
\mathscr{F}_h^i &= \{F_1, F_2, \cdots, F_{N_F^i} : \quad \text{the set of all interior faces of } \mathbf{K} \in \mathscr{T}_h\},\\
\mathscr{M}_h &= \{M_1, M_2, \cdots, M_{N_F} : \quad \text{the set of all face-centroids on } \mathscr{F}_h\},\\
\mathscr{F}_h^{(1)} &= \{F \in \mathscr{F}_h : \quad \text{the set of all faces with outward normal } (\pm 1, 0, 0)\},\\
\mathscr{F}_h^{(2)} &= \{F \in \mathscr{F}_h : \quad \text{the set of all faces with outward normal } (0, \pm 1, 0)\},\\
\mathscr{F}_h^{(3)} &= \{F \in \mathscr{F}_h : \quad \text{the set of all faces with outward normal } (0, 0, \pm 1)\}.
\end{aligned}
$$

Obviously we have $\mathscr{F}_h = \mathscr{F}_h^{(1)} \cup \mathscr{F}_h^{(2)} \cup \mathscr{F}_h^{(3)}$.

We consider the following nonconforming finite element spaces:

$$
\begin{aligned}
\mathscr{N}\mathscr{C}^h &= \{\phi : \Omega \to \mathbb{R} |\ \phi|_{\mathbf{K}} \in \mathbb{P}_{\mathbf{K}} \forall \mathbf{K} \in \mathscr{T}_h, \phi \text{ is continuous at all } V_j \in \mathscr{V}_h, M_k \in \mathscr{M}_h\},\\
\mathscr{N}\mathscr{C}_0^h &= \{\phi \in \mathscr{N}\mathscr{C}^h |\ \phi(V) = 0\ \forall V_j \in \mathscr{V}_h \cap \Gamma \text{ and } \phi(M) = 0\ \forall M_k \in \mathscr{M}_h \cap \Gamma\}.
\end{aligned}
$$

3 The interpolation operator and convergence analysis

In this section we will define an interpolation operator and analyze convergence in the case of Dirichlet boundary value problems. The case of Neumann boundary value problem is quite similar and the results will be omitted.

3.1 The second order elliptic problem

Denote by $(\cdot, \cdot)$ the $L^2(\Omega)$ inner product and by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, which is an extension of the duality paring between $L^2(\Omega)$ and itself. By $\|\cdot\|_k$ and $|\cdot|_k$ we adopt the standard notations for the norm and seminorm for the Sobolev spaces $H^k(\Omega)$. Consider then the following Dirichlet boundary value problem:

$$
\begin{aligned}
-\Delta u &= f, \quad \Omega, && (6a)\\
u &= 0, \quad \Gamma, && (6b)
\end{aligned}
$$

with $f \in H^1(\Omega)$. We will further assume that the coefficients are sufficiently smooth and that the elliptic problem (6) has an $H^3(\Omega)$-regular solution such that $\|u\|_3 \le C\|f\|_1$. The weak problem is then given as usual: find $u \in H_0^1(\Omega)$ such that

$$
a(u, v) = \langle f, v \rangle, \quad v \in H_0^1(\Omega), \tag{7}
$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the bilinear form defined by $a(u, v) = (\nabla u, \nabla v)$ for all $u, v \in H_0^1(\Omega)$. The nonconforming Galerkin method for Problem (6) states as follows: find $u_h \in \mathscr{N}\mathscr{C}_0^h$ such that

$$
a_h(u_h, v_h) = \langle f, v_h \rangle, \quad v_h \in \mathscr{N}\mathscr{C}_0^h, \tag{8}
$$

where

$$a_h(u,v) = \sum_{\mathbf{K}\in\mathscr{T}_h} a_{\mathbf{K}}(u,v),$$

with $a_{\mathbf{K}}$ being the restriction of a to $\mathbf{K}$.

Notice that in order to have point values defined properly we need to recall the following Sobolev embedding theorem

$$W^{m,p}(\Omega) \longrightarrow C^0(\bar{\Omega}),\ \text{if}\ \frac{1}{p} - \frac{m}{d} < 0.$$

Thus we should have $p > \frac{3}{m}$. For a given cube $\mathbf{K} \in \mathscr{T}_h$, define the local interpolation operator $\Pi_{\mathbf{K}} : W^{1,p}(\mathbf{K}) \cap H_0^1(\Omega) \longrightarrow \widehat{\mathbb{P}}_{SK}^{(1)},\ p > 3$, by

$$\Pi_{\mathbf{K}}\phi(V_i) = \phi(V_i), \quad \Pi_{\mathbf{K}}\phi(M_j) = \phi(M_j)$$

for all vertices V_i and face-centroids M_j of $\mathbf{K}$. The global interpolation operator Π_h: $W^{1,p}(\Omega) \cap H_0^1(\Omega) \to \mathscr{N}\mathscr{C}_0^h$ is then defined through the local interpolation operator $\Pi_{\mathbf{K}}$ by $\Pi_h|_{\mathbf{K}} = \Pi_{\mathbf{K}}$ for all $\mathbf{K} \in \mathscr{T}_h$. Since Π_h preserves P_2 for all $\mathbf{K} \in \mathscr{T}_h$, it follows from the Bramble-Hilbert Lemma that

$$\begin{aligned} \sum_{\mathbf{K}\in\mathscr{T}_h} \|\phi - \Pi_h\phi\|_{0,\mathbf{K}} + h \sum_{\mathbf{K}\in\mathscr{T}_h} \|\phi - \Pi_h\phi\|_{1,\mathbf{K}} \le Ch^k\|\phi\|_k, \\ \phi \in W^{k,p}(\Omega) \cap H_0^1(\Omega), 1 \le k \le 3. \end{aligned} \tag{9}$$

We now consider the energy-norm error estimate and first consider the following Strang lemma [13].

Lemma 1. *Let $u \in H^1(\Omega)$ and $u_h \in \mathscr{N}\mathscr{C}_0^h$ be the solutions of Eq.* (7) *and Eq.* (8), *respectively. Then*

$$\|u - u_h\|_h \le C\Big\{ \inf_{v_h\in\mathscr{N}\mathscr{C}_0^h} \|u - v_h\|_h + \sup_{w_h\in\mathscr{N}\mathscr{C}_0^h} \frac{|a_h(u,w_h) - \langle f, w_h\rangle\,|}{\|w_h\|_h} \Big\}. \tag{10}$$

Here, and in what follows, $\|\cdot\|_h$ denotes the usual broken energy norm such that $\|v\|_h = \sqrt{a_h(v,v)}$.

Due to (9), the first term in the right side of (10) is bounded by

$$\inf_{v_h\in\mathscr{N}\mathscr{C}_0^h} \|u - v_h\|_h \le \|u - \Pi_h u\|_h \le Ch^s|u|_{s+1},\ 1 < s \le 2. \tag{11}$$

Denote by f_{jk} the trace of $f|_{\mathbf{K}_j}$ on $F_{jk} = \partial\mathbf{K}_j \cup \partial\mathbf{K}_k$ if it is nonempty. Similarly, the face F_{jk} will designate the boundary of K_j common with that of K_k.

Now let us bound the second term of the right side of (10) which denotes the consistency error. For a given cube $\mathbf{K} \in \mathscr{T}_h$, denote by $F_K^{x_1+}$ and $F_K^{x_1-}$ the face of $\mathbf{K}$ with outward normal $(1,0,0)$ and $(-1,0,0)$, respectively. Similarly, we denote the other faces by $F_K^{x_2+}, F_K^{x_2-}, F_K^{x_3+}$, and $F_K^{x_3-}$ so that $\partial\mathbf{K} = \{F_K^{x_1+}, F_K^{x_1-}, F_K^{x_2+}, F_K^{x_2-}, F_K^{x_3+}, F_K^{x_3-}\}$. Thus, integrating by parts elementwise, we have

$$\begin{aligned} a_h(u,w_h) - \langle f, w_h\rangle &= \sum_{\mathbf{K}\in\mathscr{T}_h} \left\langle \frac{\partial u}{\partial \boldsymbol{\nu}}, w_h \right\rangle_{\partial\mathbf{K}} \\ &= \sum_{\mathbf{K}\in\mathscr{T}_h} \int_{F_K^{x_1+}\cup F_K^{x_1-}} \frac{\partial u}{\partial \boldsymbol{\nu}} w_h \,\mathrm{d}s + \sum_{\mathbf{K}\in\mathscr{T}_h} \int_{F_K^{x_2+}\cup F_K^{x_2-}} \frac{\partial u}{\partial \boldsymbol{\nu}} w_h \,\mathrm{d}s \\ &\quad + \sum_{\mathbf{K}\in\mathscr{T}_h} \int_{F_K^{x_3+}\cup F_K^{x_3-}} \frac{\partial u}{\partial \boldsymbol{\nu}} w_h \,\mathrm{d}s =: E_1 + E_2 + E_3, \end{aligned} \tag{12}$$

where $\boldsymbol{\nu}$ is the unit outward normal to $\mathbf{K}$.

Before proceeding, we need the following lemma.

Lemma 2. *By F_k denote the face containing the centroid M_k and by $V_j^{F_k}, j = 1,2,3,4$, denote the vertices on the face F_k. If $p \in \widehat{\mathbb{P}}_{SK}^{(1)}$, $p(V_j^{F_k}) = 0, j = 1,2,3,4$, and $p(M_k) = 0$, then*

$$\int_{F_k} p(x_1,x_2,x_3)\mathrm{d}s = 0, \quad k = 1,2,\ldots,6. \tag{13}$$

Proof. Without loss of generality, we assume that $M_1 = (1,0,0)$. In this case, we have $p \in \widehat{\mathbb{P}}_{SK}^{(1)}$ and $p(1,\pm1,\pm1) = p(1,0,0) = 0$. We need to prove that

$$\int_{F_1} p(1,x_2,x_3)\,\mathrm{d}x_2\mathrm{d}x_3 = 0. \tag{14}$$

It follows from $p(1,\pm1,\pm1) = 0$ that

$$p(1,x_2,x_3) = l_1(x_2,x_3)(x_2^2-1) + l_2(x_2,x_3)(x_3^2-1), \quad l_j \in P_1(\mathbb{R}^2),\ j = 1,2. \tag{15}$$

Set

$$l_j(x_2,x_3) = a_j x_2 + b_j x_3 + c_j, \quad j = 1,2.$$

Then $p(1,0,0) = 0$ implies that $c_1 + c_2 = 0$, which reduces (15) to

$$p(1,x_2,x_3) = (a_1x_2 + b_1x_3)(x_2^2-1) + (a_2x_2 + b_2x_3)(x_3^2-1) + c_1(x_2^2 - x_3^2).$$

Since

$$\widehat{\mathbb{P}}_{SK}^{(1)}|_{x_1=1} = \mathrm{Span}\{1, x_2, x_3, x_2^2, x_2x_3, x_3^2, x_2^2x_3\}, \tag{16}$$

invoking $p \in \widehat{\mathbb{P}}_{SK}^{(1)}$, we have

$$a_1 = a_2 = b_2 = 0,$$

which leads to

$$p(1,x_2,x_3) = b_1x_3(x_2^2-1) + c_1(x_2^2 - x_3^2). \tag{17}$$

It follows from (17) that (13) holds. This completes the proof. □

This lemma implies that Type 1 element can pass lower order patch test (test functions are in P_0 not P_1), which will lead to a convergence solution for the second order elliptic problems, but the convergence order is not optimal. To bound E_1, E_2, E_3, we also need some interpolation operators.

3.2 Some interpolation and projection operators

For the reference element $\widehat{\mathbf{K}} = [-1,1]\times[-1,1]\times[-1,1]$, consider the interpolation problem on the face of $F_{\widehat{\mathbf{K}}}^{x_1+}$: the interpolation points are $(1,1,1)$, $(1,1,-1)$, $(1,-1,1)$, $(1,-1,-1)$, and $(1,0,0)$, which are the four vertices and the centroid of $F_{\widehat{\mathbf{K}}}^{x_1+}$, with the interpolation space $\widehat{Q}_1^*(F_{\widehat{\mathbf{K}}}^{x_1+})$, where

$$\widehat{Q}_1^*(F_{\widehat{\mathbf{K}}}^{x_1+}) := \mathrm{Span}\{1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_2\widehat{x}_3, \widehat{x}_3^2\} \subset \widehat{\mathbb{P}}_{SK}^{(1)}|_{\widehat{x}_1=1}$$

is an enriched bilinear polynomial space on the face $F_{\widehat{\mathbf{K}}}^{x_1+}$ (see (16)).

The above interpolation problem has a solution by using the bubble function

$$b(\widehat{x}_2,\widehat{x}_3) = 1-\widehat{x}_3^2,$$

and the standard bilinear interpolation basis functions

$$\begin{aligned}
q_1(\widehat{x}_2,\widehat{x}_3) &= \frac{1}{4}(1+\widehat{x}_2)(1+\widehat{x}_3), \quad q_2(\widehat{x}_2,\widehat{x}_3) = \frac{1}{4}(1-\widehat{x}_2)(1+\widehat{x}_3),\\
q_3(\widehat{x}_2,\widehat{x}_3) &= \frac{1}{4}(1-\widehat{x}_2)(1-\widehat{x}_3), \quad q_4(\widehat{x}_2,\widehat{x}_3) = \frac{1}{4}(1+\widehat{x}_2)(1-\widehat{x}_3),
\end{aligned}$$

as follows:

$$\widehat{\varphi}_j = q_j - \frac{1}{4}b,\ j=1,\cdots,4; \quad \widehat{\varphi}_5 = b.$$

Thus for a continuous function f defined on the face $F_{\widehat{\mathbf{K}}}^{x_1+}$, the interpolation polynomial is given by

$$\widehat{\mathscr{I}_F^{x_1+}} f = f(1,1,1)\widehat{\varphi}_1 + f(1,-1,1)\widehat{\varphi}_2 + f(1,-1,-1)\widehat{\varphi}_3 + f(1,1,-1)\widehat{\varphi}_4 + f(1,0,0)\widehat{\varphi}_5. \tag{18}$$

And then we can also define the interpolation operator on the opposite face with the same space and denote it by $\widehat{\mathscr{I}_F^{x_1-}}$. Similarly, define the interpolation operators on the other faces of $\widehat{\mathbf{K}}$ with the corresponding spaces:

$$\begin{aligned}
\widehat{Q}_1^*(F_{\widehat{\mathbf{K}}}^{x_2}) &:= \mathrm{Span}\{1, \widehat{x}_3, \widehat{x}_1, \widehat{x}_3\widehat{x}_1, \widehat{x}_1^2\} \subset \widehat{\mathbb{P}}_{SK}^{(1)}|_{\widehat{x}_2=\pm1},\\
\widehat{Q}_1^*(F_{\widehat{\mathbf{K}}}^{x_3}) &:= \mathrm{Span}\{1, \widehat{x}_1, \widehat{x}_2, \widehat{x}_1\widehat{x}_2, \widehat{x}_2^2\} \subset \widehat{\mathbb{P}}_{SK}^{(1)}|_{\widehat{x}_3=\pm1}
\end{aligned}$$

and denote them by $\widehat{\mathscr{I}_F^{x_2+}}, \widehat{\mathscr{I}_F^{x_2-}}, \widehat{\mathscr{I}_F^{x_3+}}, \widehat{\mathscr{I}_F^{x_3-}}$, respectively.

For a given $\mathbf{K} \in \mathscr{T}_h$, we can define the interpolation operator $\mathscr{I}_F^{x_i\pm}$ by $\mathscr{I}_F^{x_i\pm} = \widehat{\mathscr{I}_F^{x_i\pm}} \circ T_{\mathbf{K}}^{-1}$. Notice that $\left[\mathscr{I}_F^{x_i+} w_h\right]_F = 0$ for all interior faces F for every $w_h \in \mathscr{N}\mathscr{C}_0^h$. Moreover, the above interpolation operators preserve linear polynomials on each surface as stated in the following lemma.

Lemma 3. *$\mathscr{I}_F^{x_i\pm}$ map $w_h \in \mathscr{N}\mathscr{C}_0^h$ such that their images across interior faces are continuous for all interior faces F. Moreover, they preserve bilinear polynomials on faces.*

Moreover, the above interpolation has the following interesting property:

Lemma 4. *For all* $w_h \in \mathscr{N}\mathscr{C}_0^h$ *and* $\mathbf{K} \in \mathscr{T}_h$,

$$w_h|_{F_K^{x_i+}} - \mathscr{I}_F^{x_i+}(w_h|_{\mathbf{K}}) = w_h|_{F_K^{x_i-}} - \mathscr{I}_F^{x_i-}(w_h|_{\mathbf{K}}), \quad i = 1, 2, 3. \tag{19}$$

Proof. We only prove the case of $i = 1$ in Eq. (19) which suffices to prove the statement on the reference element $\widehat{\mathbf{K}}$:

$$\widehat{w}_h|_{\widehat{F}_{\widehat{\mathbf{K}}}^{x_1+}} - \widehat{\mathscr{I}}_{\widehat{F}}^{x_1+}(\widehat{w}_h) = \widehat{w}_h|_{\widehat{F}_{\widehat{\mathbf{K}}}^{x_1-}} - \widehat{\mathscr{I}}_{\widehat{F}}^{x_1-}(\widehat{w}_h) \quad \forall \widehat{w}_h \in \widehat{\mathbb{P}}_{SK}^{(1)}. \tag{20}$$

Due to the interpolation property, $\widehat{w}_h|_{\widehat{F}_{\widehat{\mathbf{K}}}^{x_1+}} - \widehat{\mathscr{I}}_{\widehat{F}}^{x_1+}(\widehat{w}_h) = 0$ for all $\widehat{w}_h \in \widehat{Q}_1^*(F_{\widehat{\mathbf{K}}}^{x_1+})$, and the same is true if x_1^+ is replaced by x_1^-. Thus, it suffices to show that (19) holds for all $\widehat{w}_h \in \widehat{\mathbb{P}}_{SK}^{(1)}|_{\widehat{x}_1=1} \setminus \widehat{Q}_1^*(F_{\widehat{\mathbf{K}}}^{x_1+})$, which is nothing but $\mathrm{Span}\{\widehat{x}_2^2, \widehat{x}_2^2\widehat{x}_3\}$. Since both $\widehat{x}_2^2$ and $\widehat{x}_2^2\widehat{x}_3$ are independent of $\widehat{x}_1$, it is obvious that (20) holds for each of them. This proves the lemma. $\square$

Define $RQ = \mathrm{Span}\{1, \widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_1^2 - \widehat{x}_2^2, \widehat{x}_1^2 - \widehat{x}_3^2\}$. For the reference element $\widehat{\mathbf{K}}$, let $R_{\widehat{\mathbf{K}}} : H^2(\widehat{\mathbf{K}}) \to RQ$ be an interpolation operator defined by

$$R_{\widehat{\mathbf{K}}}\widehat{\phi}(\widehat{M_j}) = \widehat{\phi}(\widehat{M_j}), j = 1, \ldots, 6$$

for all $\widehat{\phi} \in H^2(\widehat{\mathbf{K}})$. It is exactly the so-called rotation element. Obviously,

$$R_{\widehat{\mathbf{K}}}\widehat{\phi} = \sum_{i=1}^{6} \widehat{\phi}(\widehat{M_i})\widehat{\psi}_i(\widehat{x}, x_2, x_3)$$

where

$$\begin{cases} \widehat{\psi}_i = \frac{1}{6}\left(1 + 3\widehat{x}_i + \sum_{1\le j\le 3, j\ne i}(\widehat{x}_i^2 - \widehat{x}_j^2)\right), \\ \widehat{\psi}_{7-i} = \frac{1}{6}\left(1 - 3\widehat{x}_i + \sum_{1\le j\le 3, j\ne i}(\widehat{x}_i^2 - \widehat{x}_j^2)\right), \end{cases} \quad i = 1, 2, 3.$$

It is easy to notice that for $i = 1, 2, \ldots, 6$

$$\widehat{\psi}_i|_{\widehat{x}_j=1} = \widehat{\psi}_i|_{\widehat{x}_j=-1}, \quad \text{if } j \ne i \text{ and } j \ne 7 - i$$

and for $i = 1, 2, 3$

$$\widehat{\psi}_i|_{\widehat{x}_i=1} = \widehat{\psi}_{7-i}|_{\widehat{x}_i=-1} = 1 - \frac{1}{6}\sum_{1\le j\le 3, j\ne i}\widehat{x}_j^2,$$
$$\widehat{\psi}_i|_{\widehat{x}_i=-1} = \widehat{\psi}_{7-i}|_{\widehat{x}_i=1} = -\frac{1}{6}\sum_{1\le j\le 3, j\ne i}\widehat{x}_j^2.$$

Thus we have

$$R_{\widehat{\mathbf{K}}}\widehat{\phi}|_{\widehat{x}_j=1} \quad = \quad \sum_{i=1}^{6}\widehat{\phi}(\widehat{M_i})\widehat{\psi}_i|_{x_j=1}$$

$$
\begin{aligned}
&= \sum_{1\le i\le 6, i\ne j, i\ne 7-j} \widehat{\phi}(\widehat{M}_i)\widehat{\psi}_i|_{x_j=1} + \widehat{\phi}(\widehat{M}_j)\left(1-\frac{1}{6}\sum_{1\le i\le 3, i\ne j}\widehat{x}_i^2\right)\\
&\quad -\widehat{\phi}(\widehat{M}_{7-j})\left(\frac{1}{6}\sum_{1\le i\le 3, i\ne j}\widehat{x}_i^2\right)\\
&:= \Theta(\widehat{\mathbf{K}}, \widehat{\phi}, \{\widehat{x}_1, \widehat{x}_2, \widehat{x}_3\}\setminus \widehat{x}_j) + \widehat{\phi}(\widehat{M}_j),
\end{aligned}
$$

and

$$
\begin{aligned}
R_{\widehat{\mathbf{K}}}\widehat{\phi}|_{\widehat{x}_j=-1} &= \sum_{i=1}^{6}\widehat{\phi}(\widehat{M}_i)\widehat{\psi}_i|_{x_j=-1}\\
&= \sum_{1\le i\le 6, i\ne j, i\ne 7-j} \widehat{\phi}(\widehat{M}_i)\widehat{\psi}_i|_{x_j=1} - \widehat{\phi}(\widehat{M}_j)\left(\frac{1}{6}\sum_{1\le i\le 3, i\ne j}\widehat{x}_i^2\right)\\
&\quad -\widehat{\phi}(\widehat{M}_{7-j})\left(1-\frac{1}{6}\sum_{1\le i\le 3, i\ne j}\widehat{x}_i^2\right)\\
&= \Theta(\widehat{\mathbf{K}}, \widehat{\phi}, \{\widehat{x}_1, \widehat{x}_2, \widehat{x}_3\}\setminus \widehat{x}_j) + \widehat{\phi}(\widehat{M}_{7-j}),
\end{aligned}
$$

For any given $\mathbf{K} \in \mathscr{T}_h$, we can define the interpolation operator $R_{\mathbf{K}} := R_{\widehat{\mathbf{K}}} \cdot T_K^{-1}$. Denote by $M_K^{x_j+}$ and $M_K^{x_j-}$ the centroids of the faces $F_K^{x_j+}$ and $F_K^{x_j-}$, respectively. Then for any $\phi \in H^2(\mathbf{K})$, we have

$$
\begin{aligned}
R_{\mathbf{K}}\phi|_{F_K^{x_j+}} &= \Theta(\mathbf{K}, \phi, \{x_1, x_2, x_3\}\setminus x_j) + \phi(M_K^{x_j+}),\\
R_{\mathbf{K}}\phi|_{F_K^{x_j-}} &= \Theta(\mathbf{K}, \phi, \{x_1, x_2, x_3\}\setminus x_j) + \phi(M_K^{x_j-}).
\end{aligned}
$$

3.3 The error estimates

Turn to bound $|E_1| + |E_2| + |E_3|$ in (12). Below, we will give an estimate of $|E_1|$ in detail while similar estimates of $|E_2|$ and $|E_3|$ will be omitted.

It is easy to notice that for any $F \in \partial\mathbf{K}' \cap \partial\mathbf{K}'' \cap \mathscr{F}_h^{(1)} \ne \emptyset$ and $w \in \mathscr{N}\mathscr{C}_0^h$, we have

$$
\begin{aligned}
&\int_F \nabla u\big(w|_{\mathbf{K}'} - w|_{\mathbf{K}''}\big)\mathrm{d}s\\
=& \int_F \nabla u\big((w|_{\mathbf{K}'} - \mathscr{I}_F^{x_1}(w)) - (w|_{\mathbf{K}''} - \mathscr{I}_F^{x_1}(w))\big)\mathrm{d}s\\
=& \int_F \big(\nabla u - M_F(\nabla u)\big)\big((w|_{\mathbf{K}'} - \mathscr{I}_F^{x_1}(w)) - (w|_{\mathbf{K}''} - \mathscr{I}_F^{x_1}(w))\big)\mathrm{d}s
\end{aligned}
$$

where $M_F(\nabla u)$ denotes the value of ∇u at the centroid of F. The second equality holds due to the orthogonality (13). Hence we have

$$
E_1 = \sum_{\mathbf{K}\in\mathscr{T}_h}\sum_{i=1}^{3}\int_{F_K^{x_1+}\cup F_K^{x_1-}} \frac{\partial u}{\partial x_i} w\nu_i \mathrm{d}s
$$

$$
\begin{aligned}
&= \sum_{\mathbf{K}\in\mathscr{T}_h}\sum_{i=1}^{3}\left(\int_{F_K^{x_1+}}\frac{\partial u}{\partial x_i}(w-\mathscr{I}_F^{x_1+}(w))\nu_i \mathrm{d}s+\int_{F_K^{x_1-}}\frac{\partial u}{\partial x_i}(w-\mathscr{I}_F^{x_1-}(w))\nu_i \mathrm{d}s\right)\\
&= \sum_{\mathbf{K}\in\mathscr{T}_h}\sum_{i=1}^{3}\left(\int_{F_K^{x_1+}}\left(\frac{\partial u}{\partial x_i}-M_{F_K^{x_1+}}(\frac{\partial u}{\partial x_i})\right)(w-\mathscr{I}_F^{x_1+}(w))\nu_i \mathrm{d}s\right.\\
&\quad\left.+\int_{F_K^{x_1-}}\left(\frac{\partial u}{\partial x_i}-M_{F_K^{x_1-}}(\frac{\partial u}{\partial x_i})\right)(w-\mathscr{I}_F^{x_1-}(w))\nu_i \mathrm{d}s\right),
\end{aligned}
$$

where $\boldsymbol{\nu}=(\nu_1,\nu_2,\nu_3)^T$ is the outward normal derivative of F. Thus due to (19), we arrive at

$$
\begin{aligned}
E_1 &= \sum_{\mathbf{K}\in\mathscr{T}_h}\sum_{i=1}^{3}\left(\int_{F_K^{x_1+}}\left(\frac{\partial u}{\partial x_i}-\Theta(\mathbf{K},\frac{\partial u}{\partial x_i},x_2,x_3)-M_{F_K^{x_1+}}(\frac{\partial u}{\partial x_i})\right)(w-\mathscr{I}_F^{x_1+}(w))\nu_i \mathrm{d}s\right.\\
&\quad\left.+\int_{F_K^{x_1-}}\left(\frac{\partial u}{\partial x_i}-\Theta(\mathbf{K},\frac{\partial u}{\partial x_i},x_2,x_3)-M_{F_K^{x_1-}}(\frac{\partial u}{\partial x_i})\right)(w-\mathscr{I}_F^{x_1-}(w))\nu_i \mathrm{d}s\right)\\
&= \sum_{\mathbf{K}\in\mathscr{T}_h}\sum_{i=1}^{3}\left(\int_{F_K^{x_1+}}\left(\frac{\partial u}{\partial x_i}-R_{\mathbf{K}}\frac{\partial u}{\partial x_i}\right)(w-\mathscr{I}_F^{x_1+}(w))\nu_i \mathrm{d}s\right.\\
&\quad\left.+\int_{F_K^{x_1-}}\left(\frac{\partial u}{\partial x_i}-R_{\mathbf{K}}\frac{\partial u}{\partial x_i}\right)(w-\mathscr{I}_F^{x_1-}(w))\nu_i \mathrm{d}s\right).
\end{aligned}
$$

Since $R_{\mathbf{K}}$ and $\mathscr{I}_F^{x}$ preserves $P_1(\mathbf{K})$ and $P_1(F_K^{x_1})$, respectively, it follows from trace theorem and Cauchy-Schwartz inequality, we get

$$|E_1|\le Ch^2||w||_h||u||_{H^3(\Omega)}.$$

Similarly, we also have

$$|E_2|\le Ch^2||w||_h||u||_{H^3(\Omega)},\ |E_3|\le Ch^2||w||_h||u||_{H^3(\Omega)}.$$

Hence

$$\sup_{w\in\mathscr{N}\mathscr{C}_0^h}\frac{|a_h(u,w)-\langle f,w\rangle|}{\|w\|_h}=\sup_{w\in\mathscr{N}\mathscr{C}_0^h}\frac{|E_1+E_2+E_3|}{||w||_h}\le Ch^2||u||_{H^3(\Omega)}.$$

By collecting the above results, we get the following energy-norm error estimate.

Theorem 1. *Let $u\in H^3(\Omega)\cap H_0^1(\Omega)$ and $u_h\in\mathscr{N}\mathscr{C}_0^h$ satisfy (7) and (8), respectively. Then we have the energy norm error estimate:*

$$||u-u_h||_h\le Ch^2||u||_3.$$

By a standard Aubin-Nitsche duality argument, an $L_2(\Omega)$-error estimate can be easily obtained.

Theorem 2. *Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ and $u_h \in \mathscr{N}\mathscr{C}_0^h$ be the solution of (7) and (8), respectively. Then we have*

$$\|u - u_h\|_0 \le Ch^3 \|u\|_3.$$

Proof. Let $\eta_h = \Pi_h u - u_h \in \mathscr{N}\mathscr{C}_0^h$ and consider the dual problem:

$$\begin{aligned} -\Delta\psi &= \eta_h, \quad \Omega, && (21a)\\ \psi &= 0, \quad \partial\Omega. && (21b)\end{aligned}$$

Since $\eta_h \in L^2(\Omega)$, the elliptic regularity implies that $\|\psi\|_2 \le C\|\eta_h\|$. An application of (9) to the triangle inequality makes us to prove only $\|\eta_h\|_h \le Ch^2\|u\|_3$.

First, we have from Theorem 1 and (9) that

$$\|\eta_h\|_h \le \|u - u_h\|_h + \|u - \Pi_h u\|_h \le Ch^2\|u\|_3. \tag{22}$$

Following the arguments in the derivation of the energy estimate, we have

$$\begin{aligned} \|\eta_h\|^2 &= -\sum_{\mathbf{K}\in\mathscr{T}_h} (\eta_h, \Delta\psi)_{\mathbf{K}} \\ &= \sum_{\mathbf{K}\in\mathscr{T}_h} (\nabla\eta_h, \nabla\psi)_{\mathbf{K}} - \sum_{\mathbf{K}\in\mathscr{T}_h} \langle \eta_h, \boldsymbol{\nu}\cdot\nabla\psi\rangle_{\partial\mathbf{K}} \\ &= a_h(\eta_h, \psi) - \sum_{\mathbf{K}\in\mathscr{T}_h} \langle \eta_h - \mathscr{I}_F\eta_h, \boldsymbol{\nu}\cdot\nabla\psi\rangle_{\partial\mathbf{K}}, \\ &= a_h(\eta_h, \psi) - \sum_{\mathbf{K}\in\mathscr{T}_h} \langle \eta_h - \mathscr{I}_F\eta_h, \boldsymbol{\nu}\cdot(\nabla\psi - M_F(\nabla\psi))\rangle_{\partial\mathbf{K}}, \end{aligned}$$

where $M_F(\nabla\psi)$ denotes the value of $\nabla\psi$ at the centroid of F. Hence, invoking the elliptic regularity and (22)

$$\begin{aligned} \|\eta_h\|^2 &\le |a_h(\eta_h, \psi)| + \left[\sum_{\mathbf{K}\in\mathscr{T}_h} |\eta_h - \mathscr{I}_F\eta_h|^2_{0,\partial\mathbf{K}}\right]^{\frac12} \left[\sum_{\mathbf{K}\in\mathscr{T}_h} |\boldsymbol{\nu}\cdot(\nabla\psi - M_F(\nabla\psi))|^2_{0,\partial\mathbf{K}}\right]^{\frac12} \\ &\le |a_h(\eta_h, \psi)| + Ch^{\frac12}\|\eta_h\|_h h^{\frac12}\|\psi\|_2 \\ &\le |a_h(\eta_h, \psi)| + Ch^3\|u\|_h\ \|\eta_h\|. \end{aligned} \tag{23}$$

Thus it remains to bound $|a_h(\eta_h, \psi)|$. For this, write

$$a_h(\eta_h, \psi) = a_h(\eta_h, \psi - \Pi_h\psi) + a_h(\Pi_h u - u, \Pi_h\psi) + a_h(u - u_h, \Pi_h\psi). \tag{24}$$

The first term in (24) is bounded as follows:

$$\begin{aligned} |a_h(\eta_h, \psi - \Pi_h\psi)| &\le C\|\eta_h\|_h\|\psi - \Pi_h\psi\|_h \\ &\le Ch^2\|u\|_3 h\|\psi\|_2 \le Ch^3\|u\|_3 h\|\eta_h\|. \end{aligned} \tag{25}$$

Since the second term in (24) can be decomposed by

$$a_h(\Pi_h u - u, \Pi_h\psi) = \sum_{\mathbf{K}\in\mathscr{T}_h} (\Pi_h u - u, -\Delta(\Pi_h\psi))_{\mathbf{K}}$$

$$
\begin{aligned}
&\quad + \sum_{\mathbf{K}\in\mathscr{T}_h} \langle \Pi_h u - u, \boldsymbol{\nu}\cdot\nabla\Pi_h\psi)\rangle_{\partial\mathbf{K}} \\
&= \sum_{\mathbf{K}\in\mathscr{T}_h} (\Pi_h u - u, -\Delta\Pi_h\psi)_{\mathbf{K}} \\
&\quad + \sum_{\mathbf{K}\in\mathscr{T}_h} \langle \Pi_h u - u, \boldsymbol{\nu}\cdot(\nabla\Pi_h\psi - M_F(\nabla\Pi_h\psi))\rangle_{\partial\mathbf{K}},
\end{aligned}
$$

it can be estimated as follows:

$$
|a_h(\Pi_h u - u, \Pi_h\psi)| \le Ch^3\|u\|_3\|\psi\|_2 + Ch^{\frac{3}{2}}\|u\|_3 h^{\frac{1}{2}}\|\psi\|_2 \le Ch^3\|u\|_3\|\eta_h\|. \tag{26}
$$

The third term in (24) is bounded in the same fashion as

$$
|a_h(u - \Pi_h u, \Pi_h\psi)| \le Ch^3\|u\|_3\|\eta_h\|. \tag{27}
$$

Collecting (25)–(27) and plugging into (24) combined with (23), one sees that the theorem follows by deviding both sides by $\|\eta_h\|$. □

4 Error estimates of other Smith-Kidger elements

In this section we claim that the approximation of the solutions of the second-order elliptic problem with Type 2 and 5 Smith-Kidger elements is also convergent in optimal order. In these case, it is easy to check that the orthogonality in Lemma 2 holds. The difference during the proof lies in the construction of the interpolation operator. For the second type element, the interpolation of $\mathscr{I}_F^{x_1}$, $\mathscr{I}_F^{x_2}$, $\mathscr{I}_F^{x_3}$ should be $\mathrm{Span}\{1, x_2, x_3, x_2x_3, x_2^2\}$, $\mathrm{Span}\{1, x_1, x_3, x_1x_3, x_3^2\}$ and $\mathrm{Span}\{1, x_1, x_2, x_1x_2, x_1^2\}$, respectively. And for the fifth type element, the corresponding interpolation spaces should be taken as $\mathrm{Span}\{1, x_2, x_3, x_2x_3, x_3^2 + x_2^2\}$, $\mathrm{Span}\{1, x_1, x_3, x_1x_3, x_1^2 + x_3^2\}$ and $\mathrm{Span}\{1, x_1, x_2, x_1x_2, x_1^2 + x_2^2\}$, respectively.

For Type 6 element, the orthogonality in Lemma 2 does not hold, but the Eq. (19) hods. Thus, we have

$$
\begin{aligned}
|E_1| &= \left| \sum_{\mathbf{K}\in\mathscr{T}_h} \left(\int_{F_K^{x_1+}} \frac{\partial u}{\partial\boldsymbol{\nu}}(w - \mathscr{I}_F^{x_1+}(w))\,\mathrm{d}s + \int_{F_K^{x_1-}} \frac{\partial u}{\partial\boldsymbol{\nu}}(w - \mathscr{I}_F^{x_1-}(w))\,\mathrm{d}s \right) \right| \\
&= \left| \sum_{\mathbf{K}\in\mathscr{T}_h} \left(\int_{F_K^{x+}} \left(\frac{\partial u}{\partial\boldsymbol{\nu}} - P_{\mathbf{K}}^0(\frac{\partial u}{\partial\boldsymbol{\nu}}) \right) (w - \mathscr{I}_F^{x_1+}(w))\,\mathrm{d}s \right. \right. \\
&\quad \left. \left. + \int_{F_K^{x_1-}} \left(\frac{\partial u}{\partial\boldsymbol{\nu}} - P_{\mathbf{K}}^0(\frac{\partial u}{\partial\boldsymbol{\nu}}) \right) (w - \mathscr{I}_F^{x_1-}(w))\,\mathrm{d}s \right) \right| \\
&\le Ch||u||_2||w||_h,
\end{aligned}
$$

where

$$
P_{\mathbf{K}}^0\left(\frac{\partial u}{\partial\boldsymbol{\nu}}\right) = \frac{1}{|\mathbf{K}|}\int_{\mathbf{K}} \frac{\partial u}{\partial\boldsymbol{\nu}}\,\mathrm{d}s,
$$

and $|\mathbf{K}| = \int_{\mathbf{K}} \mathrm{d}s$. By a similar derivation, we will get

Theorem 3. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy (7) and u_h be the solution of (8) with the sixth type element. Then we have the energy norm error estimate:*

$$||u - u_h||_h \leq Ch||u||_2,$$
$$||u - u_h||_0 \leq Ch^2\|u\|_2.$$

5 A new 14-node brick element

In this section, we present a new element with 14-node. The degrees of freedom are the same with those in Smith-Kidger element and Meng-Sheen-Luo-Kim element. But the shape function space is taken as $P_2 \oplus \mathrm{Span}\{x_1x_2x_3, x_1(x_2^2+x_3^2), x_2(x_1^2+x_3^2), x_3(x_1^2+x_2^2)\}$. Denote the corresponding higher-degree polynomials to those in (2) as follows:

$$\begin{aligned} r_0(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3) = \widehat{x}_1\widehat{x}_2\widehat{x}_3,\ r_1(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3) = \frac{1}{2}\widehat{x}_1(\widehat{x}_2^2+\widehat{x}_3^2),\\ r_2(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3) = \frac{1}{2}\widehat{x}_2(\widehat{x}_1^2+\widehat{x}_3^2),\ r_3(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3) = \frac{1}{2}\widehat{x}_3(\widehat{x}_1^2+\widehat{x}_2^2). \end{aligned} \tag{28}$$

Then, again equipped with the 8 vertex values plus 6 face integrals DOFs, the basis functions corresponding to the vertices and face-integrals are given exactly same as the formulae (3), (4), and (5).

In order to analyze convergence, we need to verify the orthogonality in Lemma 2 and Eq. (20). The orthogonality can be checked directly as in the proof in Lemma 2. In order to check Eq. (20), it is enough to define the corresponding interpolation spaces as $\mathrm{Span}\{1, x_2, x_3, x_2x_3, x_3^2+x_2^2\}$, $\mathrm{Span}\{1, x_1, x_3, x_1x_3, x_1^2+x_3^2\}$ and $\mathrm{Span}\{1, x_1, x_2, x_1x_2, x_1^2+x_2^2\}$, respectively. Thus, by following the same argument as in the previous sections, we also get optimal convergence for the second-order elliptic problems. That is, in this case, Theorems 1 and 2 hold.

6 Further remarks and conclusions

In this paper, we have proved that for second-order elliptic problems, the Smith-Kidger element of type 1, 2 and 5 can obtain optimal convergence order both in energy norm and $L_2(\Omega)$ norm, while the sixth type element loses one order of accuracy in each norm. In the proof, the key points lie in that they have weak orthogonality (Lemma 2) and satisfy Eq. (20). In [5]. We also proposed another kind of DOFs, that is, the values at the eight vertices and the integration values over six faces. Indeed, it is easy to check that Type 1, 2, 5 and the new element presented in this paper give optimal convergence orders for second-order elliptic problems. Besides, we can show that if the face-centroid values DOFs are replaced by the face integrals DOFs, Type 6 element also are of optimal-order convergence owing to a weak orthogonality, thus improving one order accuracy.

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