On the construction of trivariate near-best quasi-interpolants based on C2 quartic splines on type-6 tetrahedral partitions

On the construction of trivariate near-best quasi-interpolants based on C^2 quartic splines on type-6 tetrahedral partitions

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Abstract

The construction of new quasi-interpolants (QIs) having optimal approximation order and small infinity norm and based on a trivariate C^2 quartic box spline is addressed in this paper. These quasi-interpolants, called near-best QIs, are obtained in order to be exact on the space of cubic polynomials and to minimize an upper bound of their infinity norm which depends on a finite number of free parameters in a tetrahedral sequence defining the coefficients of the QIs. We show that this problem has always a unique solution, which is explicitly given. We also prove that the sequence of the resulting near-best quasi-interpolants converges in the infinity norm to the Schoenberg operator.

Key words: Trivariate box spline, Type-6 tetrahedral partition, Tetrahedral sequences, Near-best quasi-interpolation
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1 Introduction

The construction of appropriate non-discrete models from given discrete volume data is an important problem in many applications, such as scientific visualization, computer graphics, medical imaging, numerical simulation, etc.

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Classical approaches are based on trivariate tensor-product polynomial splines. If we require a certain smoothness along the coordinate axes, such splines can be of high coordinate degree, that can create unwanted oscillations and often require (approximate) derivative data at certain prescribed points. These reasons raise the natural problem of constructing alternative smooth spline models, that use only data values on the volumetric grid and simultaneously approximate smooth functions as well as their derivatives. Moreover, in order to avoid unwanted oscillations, it is desirable that polynomial sections have total degree, instead of the coordinate degree, that is typical of tensor product schemes.

Therefore, in the literature, alternative smooth spline models using only data values on the volumetric grid and of total degree have been proposed. A first possible approach, beyond the classical tensor product scheme, is represented by blending sums of univariate and bivariate C^1 quadratic spline quasi-interpolants (see e.g. [18,22,26]). Other methods based on trivariate C^1 splines of total degree have been proposed, in [16,29] and [25] on type-6 tetrahedral partitions, in [23] on truncated octahedral partitions, in [27,28,30] on Powell-Sabin (Worsey-Piper) split, and in [24] by using quadratic trivariate super splines on uniform tetrahedral partitions. Furthermore, higher smoothness C^2 has been considered in [10–12,18,20], where the reconstruction of volume data is provided in the space of C^2 quartic splines.

The aim of this paper is to continue the investigation of such kind of trivariate spaces of total degree four and smoothness C^2, with approximation order four. In particular, we propose the construction of a new general family of quasi-interpolants (abbr. QIs) on $\mathbb{R}^3$, called of near-best type, motivated by the good results obtained by this method in the univariate and bivariate settings (see [1–6,13,18,19,21]).

Moreover, we recall that a fundamental property of quasi-interpolants is that they do not require the solution of huge systems of linear equations, as occurs in the construction of interpolating operators, and this is very important in the 3D setting.

In particular, this general family of operators is constructed by imposing the exactness on the space $\mathbb{P}_3$ of trivariate polynomials of total degree at most three and by minimizing an upper bound for the operator infinity norm.

Such a technique has been partially used in [12] for the construction of QIs on a bounded domain based on C^2 quartic splines. Since the main goal is to deal with functions defined on a bounded domain, it is necessary to construct coefficient functionals associated with boundary generators (i.e. generators with supports not completely inside the domain), so that the functionals involve data points inside or on the boundary of the domain. Therefore, they propose to minimize an upper bound for the infinity norm of the operator, by using a technique that takes into account both the value of such an upper bound and the position of the data points.

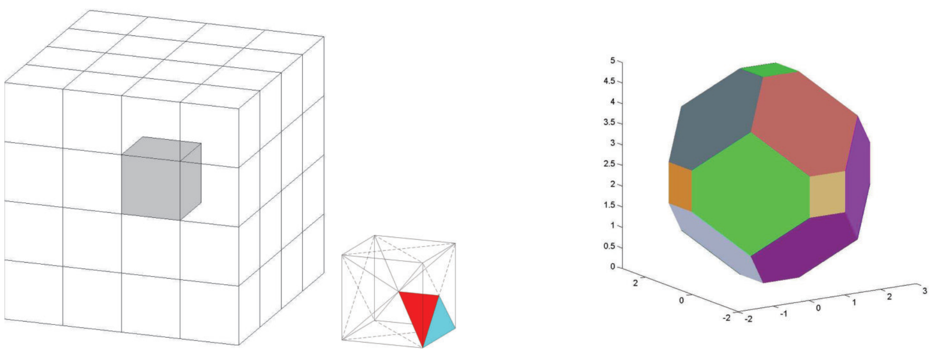
The main difference with respect to the present paper is that in this paper we propose and analyse the general construction of near-best QIs in the whole space $\mathbb{R}^3$.

The paper is organized as follows. In Section 2, we recall definitions and properties of the space of C^2 quartic splines on type-6 tetrahedral partitions. In Section 3, we explain in details the construction of near-best QIs. They are obtained by solving a minimization problem that admits always a unique solution. We provide norm and error estimates. In Section 4 we provide some results concerning the performances of the near-best QIs when the degree of the involved box spline increases. Finally, a section devoted to conclusions is included.

2 On the space of trivariate C^2 quartic splines

In this section we study the spline space generated by the integer translates of a trivariate C^2 quartic box spline specified by a set of seven directions.

We consider the box spline proposed in [17], that is a box spline whose direction vectors form a cube and its four diagonals, thus $\mathbb{R}^3$ is cut into a symmetric regular arrangement of tetrahedra called type-6 tetrahedral partition (see Fig. 1 (a)).



(a) (b)

Fig. 1. (a) The uniform type-6 tetrahedral partition and (b) the support of the seven directional box spline

Following [17], we consider the set of seven direction vectors of $\mathbb{Z}^3$ and spanning $\mathbb{R}^3$

$$X = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

defined by

$$\begin{gathered} e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1), \quad e_4 = (1,1,1), \\ e_5 = (-1,1,1), \quad e_6 = (1,-1,1), \quad e_7 = (-1,-1,1). \end{gathered} \tag{2.1}$$

According to [7, Chap. 11] and [8, Chap. 1], since the set X has seven elements and the domain is $\mathbb{R}^3$, the box spline $B(\cdot) = B(\cdot|X)$ is of degree four. The continuity of the resulting box spline depends on the determination of the number d, such that $d+1$ is the minimal number of directions that needs to be removed from X to obtain a reduced set that does not span $\mathbb{R}^3$: then one deduces that the continuity class is C^{d-1}. In our case $d = 3$, thus the polynomial pieces defined over each tetrahedron are of degree four and they are joined with C^2 smoothness.

The support of the C^2 quartic box spline B is the truncated rhombic dodecahedron centered at the point $(\frac{1}{2}, \frac{1}{2}, \frac{5}{2})$ and contained in the cube $[-2,3] \times [-2,3] \times [0,5]$, see Fig. 1($b$). Its projections on the coordinate planes are the octagonal supports of the bivariate C^2 quartic box spline with the following set of directions of $\mathbb{R}^2$: $\{(1,0); (0,1); (1,1); (1,1); (-1,1); (-1,1)\}$.

Now we consider the space $\mathcal{S}(X)$ spanned by the integer translates of the box spline B

$$\mathcal{S}(X) = \left\{ s = \sum_{\alpha \in \mathbb{Z}^3} c_\alpha B(\cdot - \alpha), \quad c_\alpha \in \mathbb{R} \right\}.$$

This space is in general a subspace of the whole space $\mathcal{S}_4^2(\mathbb{R}^3)$ of all C^2 quartic splines. Moreover, it is well-known that $\mathbb{P}_3 \subset \mathcal{S}(X)$ and $\mathbb{P}_4 \not\subset \mathcal{S}(X)$.

We introduce the scaled spline space $\mathcal{S}_h(X)$ associated with $\mathcal{S}(X)$ ([8, Chap. 3])

$$\mathcal{S}_h(X) = \sigma_h(\mathcal{S}(X)) = \{\sigma_h s : \ s \in \mathcal{S}(X)\},$$

which is defined by means of the scaling operator σ_h, $h > 0$

$$\sigma_h f : x \mapsto f\left(\frac{x}{h}\right). \tag{2.2}$$

Thus, $\mathcal{S}_h(X)$ is the spline space defined on the refined lattice $h\mathbb{Z}^3$.

We also recall [8, Chap. 3] that the approximation power of $\mathcal{S}(X)$ is the largest r for which

$$\operatorname{dist}(f, \mathcal{S}_h) = O(h^r)$$

for all sufficiently smooth f, with the distance measured in the $L_p(\Omega)$-norm $(1 \le p \le \infty)$. In our case we get $r = 4$.

3 Near best quasi-interpolants in $\mathcal{S}(X)$

A quasi-interpolant

$$Q : \mathcal{F} \to \mathcal{S}(X)$$

based on the box spline B is a linear operator defined on some functional space $\mathcal{F}$ by an expression of the form

$$Qf = \sum_{\alpha \in \mathbb{Z}^3} \lambda_\alpha(f) B_\alpha$$

with $B_\alpha(x) = B_{(\alpha_1,\alpha_2,\alpha_3)}(x_1, x_2, x_3) = B(x_1-\alpha_1+\frac{1}{2}, x_2-\alpha_2+\frac{1}{2}, x_3-\alpha_3+\frac{5}{2})$ and $\lambda_\alpha(f)$ linear combinations of values of f at specific points. We also introduce the scaled quasi-interpolation operator

$$Q_h f(x) = \sigma_h Q \sigma_{1/h} f(x), \tag{3.1}$$

where σ_h is defined by (2.2).

In order to define the coefficient functionals $\lambda_\alpha(f)$, firstly, for any $n \geq 1$, let Λ_n be the octahedron with vertices $(\pm n, 0, 0)$, $(0, \pm n, 0)$, $(0, 0, \pm n)$. We denote by Λ_n^* the set of points $\Lambda_n \cap \mathbb{Z}^3$ (see Fig. 2 for $n = 2$, 3 and 4).

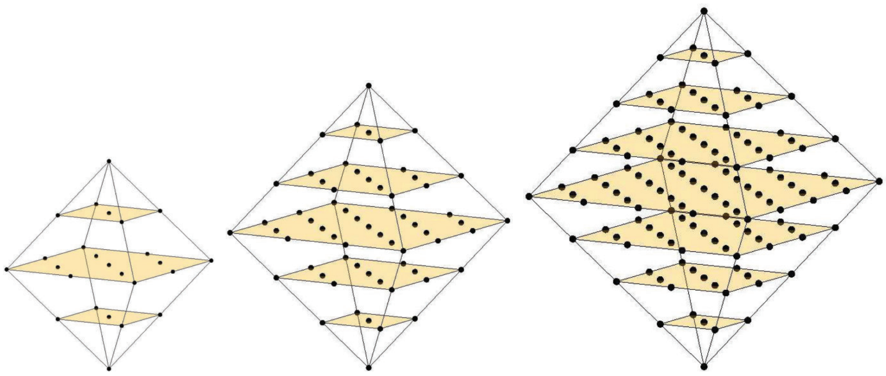


Fig. 2. From left to right, the octahedron Λ^n with the corresponding points Λ_n^*, for $n = 2$, $n = 3$ and $n = 4$.

The choice of octahedron shapes can be considered a 3D generalization of the rhombic one used in [1,5,6,13] for the construction of 2D near-best QIs.

Then, associated with the tetrahedral sequence Λ_n^* we define the coefficient of B_α as

$$\lambda_{n,\alpha}(f) = \sum_{\beta \in \Lambda_n^*} c_\beta f(\alpha + \beta) \tag{3.2}$$

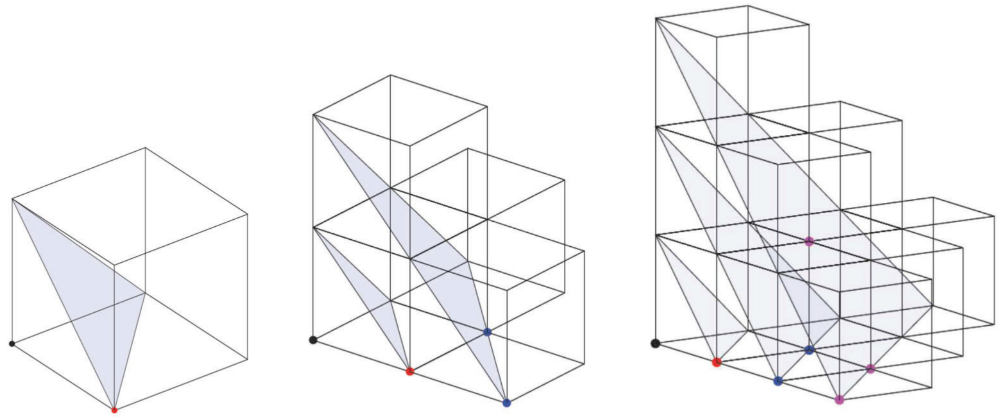


Fig. 3. From left to right, sequences of points associated with parameters in Λ_1^*, Λ_2^* and Λ_3^*.

and the corresponding operator as

$$Q_n f = \sum_{\alpha\in\mathbb{Z}^3} \lambda_{n,\alpha}(f) B_\alpha \tag{3.3}$$

Thanks to the symmetry of Λ_n, it is sufficient to know $\ell(n) < \operatorname{card}\Lambda_n^*$ points in Λ_n^* and the position of the other ones can be obtained by symmetry. When $n = 0$, the unique coefficient is $c_{(0,0,0)}$, and $\ell(0) = 1$. Let $n = 1$. The coefficients c_β involved in Λ_1^* are derived by symmetry from the parameters $c_{(0,0,0)}$ in Λ_0^* and $c_{(1,0,0)}$, and then $\ell(1) = 2$. The last one is the unique point in Λ_1^* lying in the plane of equation $x + y + z = 1$ that satisfies the inequalities $x \geq y \geq z$ (see Fig. 3). When $n = 2$, there are $\ell(2) = 4$ parameters, derived from the ones in Λ_1^* and those in Λ_2^* satisfying equation $x + y + z = 2$ as well as the inequalities $x \geq y \geq z$, namely $c_{(2,0,0)}$ and $c_{(1,1,0)}$ are added. Fig. 3 shows also the case $n = 3$, while the cases $n = 4$ and $n = 5$ are illustrated in Fig. 4.

For $\ell(n)$ we have the following explicit expression (see [9]):

$$\ell(n) = \left\lfloor \frac{1}{72}(n+5)\left(2n^2 + 5n + 5\right) \right\rfloor, \quad n \geq 1,$$

where $\lfloor x \rfloor$ stands for the integer part of x. It is also known that $\ell(n)$ is the integer number closer to $\frac{1}{72}(n+3)(2n^2 + 15n + 1)$, $n \geq 0$ (see the label A181120 in The On-line Encyclopedia of Integer Sequences, https://oeis.org/). The sequence $\ell(n)$ is obtained as partial sums of the sequence of general term $\operatorname{round}\left(\frac{(n+3)^2}{12}\right)$, where $\operatorname{round}(x)$ denotes the integer number closer to x.

After some algebra, we have obtained that

$$\ell(n) = \frac{1}{2}(t+1)(2(2t+1)(3t+1) - t + s(s+1+6t))$$

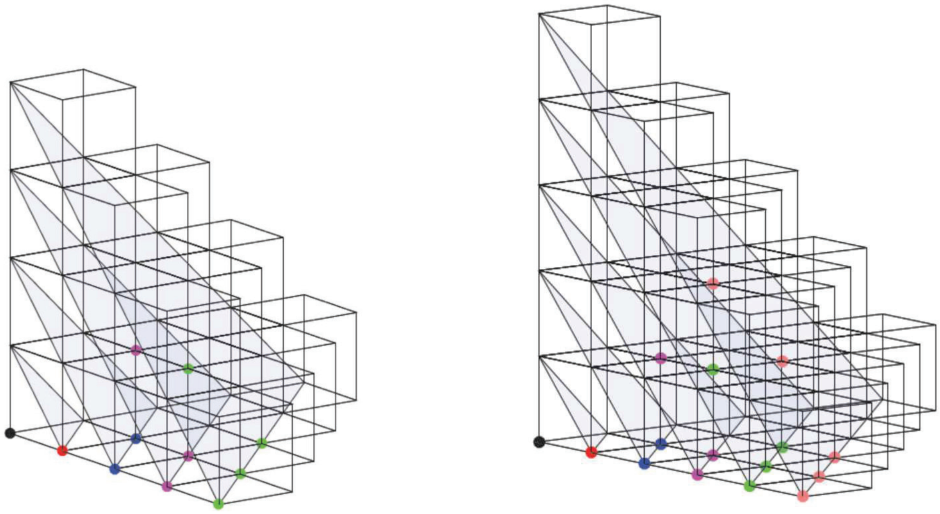


Fig. 4. From left to right, sequences of points associated with parameters in Λ_4^* and Λ_5^*.

if $n = 6t + s$, $s = 0, \ldots, 5$.

From (3.2), it is clear that, for $\|f\|_\infty \leq 1$, $|\lambda_\alpha(f)| \leq \|\mathbf{c}\|_1$, where $\mathbf{c}$ is the vector with components c_β, $\beta \in \Lambda_n^*$. Therefore, since the scaled translates of B form a partition of unity [7], we deduce immediately

$$\|Q_n\|_\infty \leq \|c\|_1.$$

Now, we can try to find a solution of the minimization problem

$$\min\left\{\|c\|_1 : c \in \mathbb{R}^{\operatorname{card}(\Lambda_n^*)}, Ac = b\right\}, \tag{3.4}$$

where $Ac = b$ is the linear system expressing that Q_n is exact on $\mathbb{P}_3$. In our case, in order to obtain such a system, we require that, for $f \in \mathbb{P}_3$, each coefficient functional coincides with the corresponding one of the differential quasi-interpolating operator [20] exact on $\mathbb{P}_3$

$$\widehat{Q}f = \sum_{\alpha \in \mathbb{Z}^3} \left(I - \frac{5}{24}\Delta + \frac{3}{128}\Delta^2\right) f(\alpha) B_\alpha.$$

By using the monomial expansion of $\mathbb{P}_3$ (see [20], Table 1) we have to impose twenty conditions:

$$\left(I - \frac{5}{24}\Delta + \frac{3}{128}\Delta^2\right) p(0,0,0) = \sum_{\beta \in \Lambda_n^*} c_\beta p(\beta)$$

for all $p \in \mathbb{P}_3$. Due to the symmetries, there only remain the following two equations:

$$1 = \sum_{\beta\in\Lambda_n^*} c_\beta, \qquad -\frac{5}{12} = \sum_{\beta\in\Lambda_n^*} \beta_1^2 c_\beta, \tag{3.5}$$

with $\beta = (\beta_1, \beta_2, \beta_3)$. They provide the linear system involved in (3.4). Since the number of unknowns in problem (3.4) has been reduced from card(Λ_n^*) to $\ell(n)$ by imposing the symmetries of the octahedron Λ_n to Λ_n^*, we can express the equations above in terms of the $\ell(n)$ free parameters. They can be decomposed into eight different types. To describe them, let $I_r := \{r, -r\}$, $r \in \mathbb{N}$, and $I_0 := \{0\}$, and let $\mathcal{P}_3(r_1, r_2, r_3)$ be the 3! permutations of the different numbers r_1, r_2 and r_3. Thus, they are

- $c_{(0,0,0)}$, associated with the point $(0, 0, 0)$;
- $c_{(i,0,0)}$, $1 \le i \le n$, related to the 6 points $(\pm i, 0, 0)$, $(0, \pm i, 0)$ and $(0, 0, \pm i)$;
- $c_{(i,i,0)}$, $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, corresponding to the 12 sites in $I_i \times I_i \times I_0$, $I_i \times I_0 \times I_i$ or $I_0 \times I_i \times I_i$;
- $c_{(i,j,0)}$, $1 \le j \le \left\lfloor \frac{n-1}{2} \right\rfloor$ and $j+1 \le i \le n-j$, linked to the 24 points in $\cup_{\sigma\in\mathcal{P}_3(i,j,0)} \left(I_{\sigma(i)} \times I_{\sigma(j)} \times I_{\sigma(0)}\right)$;
- $c_{(i,i,i)}$, $1 \le i \le \left\lfloor \frac{n}{3} \right\rfloor$, related to the 8 points in $I_i \times I_i \times I_i$;
- $c_{(i,j,j)}$, $1 \le j \le \left\lfloor \frac{n-1}{3} \right\rfloor$ and $j+1 \le i \le n-2j$, corresponding to the 24 evaluation points in $I_i \times I_j \times I_j$, $I_j \times I_i \times I_j$ or $I_j \times I_j \times I_i$;
- $c_{(i,i,j)}$, $1 \le j \le \left\lfloor \frac{n-2}{3} \right\rfloor$ and $j+1 \le i \le \left\lfloor \frac{n-j}{2} \right\rfloor$, associated to the 24 sites in $I_i \times I_i \times I_j$, $I_i \times I_j \times I_i$ or $I_j \times I_i \times I_i$;
- $c_{(i,j,k)}$, $1 \le k \le \left\lfloor \frac{n}{3} \right\rfloor - 1$, $k+1 \le j \le \left\lfloor \frac{n-1-k}{2} \right\rfloor$ and $j+1 \le i \le n-j-k$, associated with the 48 points in $\cup_{\sigma\in\mathcal{P}_3(i,j,k)} \left(I_{\sigma(i)} \times I_{\sigma(j)} \times I_{\sigma(k)}\right)$.

Therefore, equations (3.5) can be expressed as follows:

$$\begin{aligned}1 = {} & c_{(0,0,0)} + 6\sum_{i=1}^{n} c_{(i,0,0)} + 12\sum_{i=1}^{\lfloor \frac{n}{2}\rfloor} c_{(i,i,0)} + 24\sum_{j=1}^{\lfloor \frac{n-1}{2}\rfloor}\sum_{i=j+1}^{n-j} c_{(i,j,0)} + 8\sum_{i=1}^{\lfloor \frac{n}{3}\rfloor} c_{(i,i,i)} \\ & + 24\sum_{j=1}^{\lfloor \frac{n-1}{3}\rfloor}\sum_{i=j+1}^{n-2j} c_{(i,j,j)} + 24\sum_{j=1}^{\lfloor \frac{n-2}{3}\rfloor}\sum_{i=j+1}^{\lfloor \frac{n-j}{2}\rfloor} c_{(i,i,j)} + 48\sum_{k=1}^{\lfloor \frac{n}{3}\rfloor-1}\sum_{j=k+1}^{\lfloor \frac{n-1-k}{2}\rfloor}\sum_{i=j+1}^{n-j-k} c_{(i,j,k)},\end{aligned} \tag{3.6}$$

$$\begin{aligned}-\frac{5}{24} = {} & \sum_{i=1}^{n} i^2 c_{(i,0,0)} + 4\sum_{i=1}^{\lfloor \frac{n}{2}\rfloor} i^2 c_{(i,i,0)} + 4\sum_{j=1}^{\lfloor \frac{n-1}{2}\rfloor}\sum_{i=j+1}^{n-j} \left(i^2+j^2\right) c_{(i,j,0)} + 4\sum_{i=1}^{\lfloor \frac{n}{3}\rfloor} i^2 c_{(i,i,i)} \\ & + 4\sum_{j=1}^{\lfloor \frac{n-1}{3}\rfloor}\sum_{i=j+1}^{n-2j} \left(i^2+2j^2\right) c_{(i,j,j)} + 4\sum_{j=1}^{\lfloor \frac{n-2}{3}\rfloor}\sum_{i=j+1}^{\lfloor \frac{n-j}{2}\rfloor} \left(2i^2+j^2\right) c_{(i,i,j)} \\ & + 4\sum_{k=1}^{\lfloor \frac{n}{3}\rfloor-1}\sum_{j=k+1}^{\lfloor \frac{n-1-k}{2}\rfloor}\sum_{i=j+1}^{n-j-k} \left(i^2+j^2+k^2\right) c_{(i,j,k)}.\end{aligned} \tag{3.7}$$

If $\mathbf{c}^*$ is a solution of the minimization problem (3.4), then the associated QI Q_n^* defined by (3.2) and (3.3) is called near-best quasi-interpolant.

The objective function of our minimization problem has the following expression:

$$\begin{aligned}\|c\|_1 = {} & |c_{(0,0,0)}| + 6\sum_{i=1}^{n} |c_{(i,0,0)}| + 12 \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} |c_{(i,i,0)}| + 24 \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i=j+1}^{n-j} |c_{(i,j,0)}| \\ & + 8 \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor} |c_{(i,i,i)}| + 24 \sum_{j=1}^{\lfloor \frac{n-1}{3} \rfloor} \sum_{i=j+1}^{n-2j} |c_{(i,j,j)}| + 24 \sum_{j=1}^{\lfloor \frac{n-2}{3} \rfloor} \sum_{i=j+1}^{\lfloor \frac{n-j}{2} \rfloor} |c_{(i,i,j)}| \\ & + 48 \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor -1} \sum_{j=k+1}^{\lfloor \frac{n-1-k}{2} \rfloor} \sum_{i=j+1}^{n-j-k} |c_{(i,j,k)}|.\end{aligned}$$

The existence of at least a solution to problem (3.4) is guaranteed because of the objective function is a polyhedral function. Next result proves that in fact there is a unique solution.

Theorem 1 *The unique solution of the minimization problem* (3.4) *is the vector* $\mathbf{c}^*$ *such that the only values different from zero are*

$$c^*_{(0,0,0)} = 1 + \frac{5}{(2n)^2} \quad \textit{and} \quad c^*_{(n,0,0)} = -\frac{5}{6(2n)^2}.$$

PROOF. $\|\mathbf{c}\|_1$ is a function of $\ell(n)$ variables of seven different types: $c_{(i,0,0)}$, $0 \le i \le n$; $c_{(i,i,0)}$, $1 \le i \le \lfloor \frac{n}{2} \rfloor$; $c_{(i,j,0)}$, $1 \le j \le \lfloor \frac{n-1}{2} \rfloor$, $j+1 \le i \le n-j$; $c_{(i,i,i)}$, $1 \le i \le \lfloor \frac{n}{3} \rfloor$; $c_{(i,j,j)}$, $1 \le j \le \lfloor \frac{n-1}{3} \rfloor$, $j+1 \le i \le n-2j$; $c_{(i,i,j)}$, $1 \le j \le \lfloor \frac{n-2}{3} \rfloor$, $j+1 \le i \le \lfloor \frac{n-j}{2} \rfloor$, and $c_{(i,j,k)}$, $1 \le k \le \lfloor \frac{n}{3} \rfloor - 1$, $k+1 \le j \le \lfloor \frac{n-1-k}{2} \rfloor$, $j+1 \le i \le n-j-k$.

The exactness of Q_n on $\mathbb{P}_3$ is equivalent to equations (3.5), so we can express $c_{(0,0,0)}$ and $c_{(n,0,0)}$ in terms of the other variables c_β, $\beta \in \Lambda_n^* \backslash \{(0,0,0), (n,0,0)\} =: \widetilde{\Lambda}_n^*$. Therefore, minimizing $\|c\|_1$ under the linear constraints given in (3.5) becomes equivalent to minimizing in $\mathbb{R}^{\ell(d)-2}$ a polyhedral convex function depending on the variables c_β, $\beta \in \widetilde{\Lambda}_n^*$. Let ω be this function and let c_β, $\beta \in \widetilde{\Lambda}_n^*$.

Denote by $\bar{\omega}_\beta(c_\beta)$ the restriction of ω obtained by replacing its variables by zero except c_β. We will prove that this univariate function $\bar{\omega}_\beta$ attains its minimum value uniquely at $0 \in \mathbb{R}$.

Assume for example that $\beta = (i,0,0)$, $1 \le i \le n$, with $n \ge n_{(i,0,0)} := 2$. By cancelling all the variables different from $c_{(0,0,0)}$, $c_{(n,0,0)}$ and $c_{(i,0,0)}$ in (3.6)-(3.7), then these equations become

$$c_{(0,0,0)}+6c_{(i,0,0)}+6c_{(n,0,0)}=1, \quad 2i^2c_{(i,0,0)}+2n^2c_{(n,0,0)}=-\frac{5}{12}.$$

Then, the expressions of $c_{(0,0,0)}$ and $c_{(n,0,0)}$ in terms of $c_{(i,0,0)}$ are given by

$$c_{(0,0,0)}=1+\frac{5}{4n^2}-6\left(1-\frac{i^2}{n^2}\right)c_{(i,0,0)}, \quad c_{(n,0,0)}=-\frac{5}{24n^2}-\left(\frac{i}{n}\right)^2c_{(i,0,0)}.$$

Thus, $\bar{\omega}_{(i,0,0)}\left(c_{(i,0,0)}\right)$ takes the following expression

$$\begin{aligned}\bar{\omega}_{(i,0,0)}\left(c_{(i,0,0)}\right) &= \left|c_{(0,0,0)}\right|+6\left|c_{(i,0,0)}\right|+6\left|c_{(n,0,0)}\right| \\ &= \left|1+\frac{5}{4n^2}-6\left(1-\frac{i^2}{n^2}\right)c_{(i,0,0)}\right|+6|c_{(i,0,0)}|+6\left|\frac{5}{24n^2}+\left(\frac{i}{n}\right)^2c_{(i,0,0)}\right|.\end{aligned}$$

It is a piecewise linear function on the intervals of the real line induced by the partition $x_1^{(i,0,0)}:=-\frac{5}{24i^2}<0<x_2^{(i,0,0)}:=\frac{5+4n^2}{24(n^2-i^2)}$. Explicitly,

$$\bar{\omega}_{(i,0,0)}\left(c_{(i,0,0)}\right)=\begin{cases}1+p_1^{(i,0,0)}c_{(i,0,0)}, & c_{(i,0,0)}<x_1^{(i,0,0)},\\ 1+\frac{5}{2n^2}+p_2^{(i,0,0)}c_{(i,0,0)}, & x_1^{(i,0,0)}\leq c_{(i,0,0)}<0,\\ 1+\frac{5}{2n^2}+p_3^{(i,0,0)}c_{(i,0,0)}, & 0\leq c_{(i,0,0)}<x_2^{(i,0,0)},\\ -1+p_4^{(i,0,0)}c_{(i,0,0)}, & x_2^{(i,0,0)}\leq c_{(i,0,0)},\end{cases}$$

with

$$p_1^{(i,0,0)}=-12,\ p_2^{(i,0,0)}=-12\left(1-\frac{i^2}{n^2}\right),\ p_3^{(i,0,0)}=12\frac{i^2}{n^2},\ p_4^{(i,0,0)}=12.$$

These slopes satisfy the inequalities

$$p_1^{(i,0,0)}<p_2^{(i,0,0)}<p_3^{(i,0,0)}<p_4^{(i,0,0)}.$$

Therefore, $\bar{\omega}_{(i,0,0)}$ is a strictly convex function that attains its minimum value at 0.

A similar technique can be applied for each of the other variables in (3.5). Table 1 provides for every variable c_β, $\beta=(i,i,0)$, $(i,j,0)$, (i,i,i), (i,j,j), (i,i,j) and (i,j,k), the value n_β of n from which the results shown are valid, as well as the values x_1^β and x_2^β that give rise to the intervals $I_1^\beta:=\left(-\infty,x_1^\beta\right)$, $I_2^\beta:=\left[x_1^\beta,0\right)$, $I_3^\beta:=\left[0,x_2^\beta\right)$ and $I_4^\beta:=\left[x_2^\beta,+\infty\right)$ in which the restrictions of $\overline{\omega}_\beta$ are polynomials of degree less than or equal to one. The slopes p_i^β, $1\leq i\leq 4$, of those restrictions of $\overline{\omega}_\beta$ to I_i^β, $1\leq i\leq 4$, are shown in Table 2. In all these cases the univariate objective function is strictly convex attaining its minimum value at zero.

Table 1
The real line is decomposed into four intervals associated with the values $x_1^\beta < 0 < x_2^\beta$. On each interval the reduced objective function is a polynomial of degree less than or equal to one. Those values depend on n, and these expressions are true for $n \geq n_\beta$.

β	i,j,k	n_β	x_1^β	x_2^β
$(i,i,0)$	$1 \leq i \leq \lfloor \frac{n}{2} \rfloor$	2	$-\frac{5}{96i^2}$	$\frac{5+4n^2}{48(n^2-2i^2)}$
$(i,j,0)$	$\begin{cases} 1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor \\ j+1 \leq i \leq n-j \end{cases}$	3	$-\frac{5}{96(i^2+j^2)}$	$\frac{5+24n^2}{96(n^2-i^2-j^2)}$
(i,i,i)	$1 \leq i \leq \lfloor \frac{n}{3} \rfloor$	3	$-\frac{5}{96i^2}$	$\frac{5+4n^2}{32(n^2-3i^2)}$
(i,j,j)	$\begin{cases} 1 \leq j \leq \lfloor \frac{n-1}{3} \rfloor \\ j+1 \leq i \leq n-2j \end{cases}$	4	$-\frac{5}{96(i^2+2j^2)}$	$\frac{5+4n^2}{96(n^2-i^2-2j^2)}$
(i,i,j)	$\begin{cases} 1 \leq j \leq \lfloor \frac{n-2}{3} \rfloor \\ j+1 \leq i \leq \left\lfloor \frac{n-j}{2} \right\rfloor \end{cases}$	5	$-\frac{5}{96(2i^2+j^2)}$	$\frac{5+4n^2}{96(n^2-2i^2-j^2)}$
(i,j,k)	$\begin{cases} 1 \leq k \leq \lfloor \frac{n}{3} \rfloor - 1 \\ k+1 \leq j \leq \lfloor \frac{n-1-k}{2} \rfloor \\ j+1 \leq i \leq n-j-k \end{cases}$	6	$-\frac{5}{96(i^2+j^2+k^2)}$	$-\frac{5+4n^2}{96(i^2+j^2+k^2-2n^2)}$

Some values of n are not included in Tables 1 and 2. A direct calculation in each case shows that the resulting univariate function also attains the minimum at zero.

Consequently, we conclude that the convex function $\|\mathbf{c}\|_1$ attains its global minimum uniquely at $0 \in \mathbb{R}^{\ell(n)-2}$. $\square$

We remark that in case $n = 2$ we obtain the quasi-interpolant $\tilde{Q}^2$ proposed in [20]. Finally, it is easy to prove the following proposition concerning the unique solution of the minimization problem (3.4).

Proposition 2 *For all $n \geq 1$, the infinity norm of the near-best quasi interpolant Q_n^* associated with the unique solution of problem* (3.4) *and given by*

$$Q_n^* f = \sum_{\alpha \in \mathbb{Z}^3} \left(\left(1 + \frac{5}{(2n)^2} \right) f(\alpha) - \frac{5}{6(2n)^2} \sum_{\ell=1}^{3} f(\alpha \pm n e_\ell) \right) B_\alpha$$

Table 2
On each interval I_i^β, $1 \le i \le 4$, the slope p_i^β of the restriction of the reduced objective function is given. Except in the last case $\beta = (i,j,k)$, it holds that $p_1^\beta < p_2^\beta < p_3^\beta < p_4^\beta$. When $\beta = (i,j,k)$, $p_1^\beta \le p_2^\beta < p_3^\beta < p_4^\beta$.

	p_1^β	p_2^β	p_3^β	p_4^β
β	$\left(-\infty, x_1^\beta\right)$	$\left[x_1^\beta, 0\right)$	$\left[0, x_2^\beta\right)$	$\left[x_2^\beta, +\infty\right)$
$(i,i,0)$	-24	$24\left(\frac{2i^2}{n^2}-1\right)$	$48\frac{i^2}{n^2}$	24
$(i,j,0)$	-48	$\frac{i^2+j^2-n^2}{n^2}$	$48\frac{i^2+j^2}{n^2}$	48
(i,i,i)	-8	$16\left(3\frac{i^2}{n^2}-1\right)$	$48\frac{i^2}{n^2}$	16
(i,j,j)	-48	$48\frac{i^2+2j^2-n^2}{n^2}$	$48\frac{i^2+2j^2}{n^2}$	48
(i,i,j)	-48	$48\frac{2i^2+j^2-n^2}{n^2}$	$48\frac{2i^2+j^2}{n^2}$	48
(i,j,k)	-72	$24\frac{2\left(i^2+j^2+k^2\right)-3n^2}{n^2}$	$24\frac{2\left(i^2+j^2+k^2\right)-n^2}{n^2}$	72

satisfies the inequality

$$\|Q_n^*\|_\infty \le 1 + \frac{5}{2n^2}.$$

It becomes an equality for enough large n. Moreover, the sequence $(Q_n^*)_{n\ge1}$ converges in the infinity norm to the Schoenberg's operator.

PROOF. Let $f \in C\left(\mathbb{R}^3\right)$ such that $\|f\|_\infty \le 1$. Then,

$$\begin{aligned}
|Q_n^* f| &\le \sum_{\alpha\in\mathbb{Z}^3}\left(\left(1+\frac{5}{(2n)^2}\right)|f(\alpha)| + \frac{5}{6(2n)^2}\sum_{\ell=1}^{3}|f(\alpha \pm n e_\ell)|\right)B_\alpha,\\
&\le \|f\|_\infty \sum_{\alpha\in\mathbb{Z}^3}\left(\left(1+\frac{5}{(2n)^2}\right) + 6\frac{5}{6(2n)^2}\right)B_\alpha,\\
&\le 1 + \frac{5}{2n^2},
\end{aligned}$$

where e_ℓ, $\ell = 1,2,3$ are defined in (2.1). Hence, $\|Q_n^*\|_\infty \le 1 + \frac{5}{2n^2}$. For a large enough value n, it holds

$$\|Q_n^*\|_\infty \le 1 + \frac{5}{2n^2}.$$

On the other hand, since the Schoenberg's operator S is defined as

$$Sf = \sum_{\alpha\in\mathbb{Z}^3} f(\alpha)\, B_\alpha,$$

we obtain

$$Q_n^* f - Sf = \sum_{\alpha\in\mathbb{Z}^3} \left(\frac{5}{(2n)^2} f(\alpha) - \frac{5}{6(2n)^2} \sum_{\ell=1}^{3} f(\alpha \pm ne_\ell) \right) B_\alpha.$$

Therefore

$$|Q_n^* f - Sf| \le \|f\|_\infty \sum_{i\in\mathbb{Z}^2} \frac{5}{2n^2} B_\alpha \le \frac{5}{2n^2}.$$

Then, we conclude that $\|Q_n^* - S\|_\infty \le \frac{5}{2n^2}$, i.e. Q_n^* converges to S when $n \longrightarrow +\infty$. $\square$

Finally, standard results in approximation theory [8, Chap.3] allow us to immediately deduce the theorem below, for which we need the following notations:

- let H be a compact set, then, for any function $f \in C(H)$, we denote by $\|f\|_H = \sup\{|f(x_1, x_2, x_3)| : (x_1, x_2, x_3) \in H\}$ the infinity norm of f;
- $D^\beta = D^{\beta_1\beta_2\beta_3} = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \partial x_3^{\beta_3}}$;
- $|f|_{r,B} = \max_{|\beta|=r} \left\| D^\beta f \right\|_B$;
- let $\Lambda_{n,(u_1,u_2,u_3)}$, $(u_1, u_2, u_3) \in \mathbb{Z}^3$, be the octahedron with vertices $(u_1 \pm n, u_2, u_3)$, $(u_1, u_2 \pm n, u_3)$, $(u_1, u_2, u_3 \pm n)$;
- let T be a tetrahedron included in the subcube centered at the point $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then we set $\Omega_T^n = \bigcup_{u_1=\alpha_1-2}^{\alpha_1+2} \bigcup_{u_2=\alpha_2-2}^{\alpha_2+2} \bigcup_{u_3=\alpha_3-2}^{\alpha_3+2} \Lambda_{n,(u_1,u_2,u_3)}$.

Theorem 3 *Given a tetrahedron T, let $f \in C^4(\Omega_T^n)$, $n \ge 1$ and $|\gamma| = 0, 1, 2, 3, 4$. Then there exist constants $K_{|\gamma|} > 0$, independent on h, such that*

$$\left\| D^\gamma (f - Q_{n,h}^* f) \right\|_T \le K_{|\gamma|} h^{4-|\gamma|} |f|_{4,\Omega_T^n},$$

where $Q_{n,h}^*$ is the scaled quasi-interpolation operator Q_n^* defined by (3.1).

A global version of this result follows by taking the maximum over all tetrahedra T.

4 Numerical results

In order to illustrate the theoretical results, in this section we present some numerical tests obtained by a computational procedure developed in a Matlab environment. For the evaluation of box splines we can refer to [14], where an algorithm, that uses the Bernstein-Bézier form of the box spline, is proposed.

We approximate the following functions:

(1) the smooth trivariate test function of Franke type

$$f_1(x,y,z) = \frac{1}{2}e^{-10((x-\frac{1}{4})^2+(y-\frac{1}{4})^2)} + \frac{3}{4}e^{-16((x-\frac{1}{2})^2+(y-\frac{1}{4})^2+(z-\frac{1}{4})^2)}$$
$$\frac{1}{2}e^{-10((x-\frac{3}{4})^2+(y-\frac{1}{8})^2+(z-\frac{1}{2})^2)} - \frac{1}{4}e^{-20((x-\frac{3}{4})^2+(y-\frac{3}{4})^2)},$$

on the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^3$;

(2) $f_2(x,y,z) = \frac{1}{9}\tanh(9(z-x-y)+1)$, on the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^3$;

(3) the Marschner-Lobb function [15]

$$f_3(x,y,z) = \frac{1}{2(1+\beta_1)}\left(1 - \sin\frac{\pi z}{2} + \beta_1\left(1+\cos\left(2\pi\beta_2\cos\left(\frac{\pi\sqrt{x^2+y^2}}{2}\right)\right)\right)\right)$$

with $\beta_1 = \frac{1}{4}$ and $\beta_2 = 6$, on the cube $[-1,1]^3$. This function is extremely oscillating and therefore it represents a difficult test for any efficient three-dimensional reconstruction method;

(4) $f_4(x,y,z) = \dfrac{\pi y e^{xy}}{40(e-2)}\sin\pi z$, on the cube $[0,1]^3$.

For each test function, defined on the cube $[a,b]^3$, we compute the scaled quasi-interpolants $Q^*_{n,h}f$, $n = 1,\ldots,5$, with $h = (b-a)/N$ and $N = 16, 32, 64, 128$. Then, using a $139 \times 139 \times 139$ uniform three-dimensional grid G of points in the domain $[a,b]^3$, we compute the maximum absolute errors

$$E_n f = \max_{(u,v,w)\in G} |f(u,v,w) - Q^*_{n,h}f(u,v,w)|,$$

for increasing values of N, see Table 3. In the table we also report an estimate of the approximation order, $r_n f$, obtained by the logarithm to base two of the ratio between two consecutive errors.

We can notice that the theoretical results are confirmed. Moreover, we can observe that there is a deterioration of the performances, increasing the value of n. Indeed, we have shown that the near-best quasi-interpolating operator Q^*_n converges to the Schoenberg operator for $n \to \infty$, and then it may inherit all Schoenberg operator's properties, in particular the approximation order 2.

5 Conclusions

We have dealt with the construction of trivariate C^2 quartic quasi-interpolating splines in the space spanned by the integer translates of the 7-direction box spline to satisfy the following requirements: (a) the quasi-interpolation operators reproduce the trivariate polynomials in $\mathbb{P}_3$; and (b) they have small infinity norms. The coefficients of the quasi-interpolants are linear combinations

Table 3
Maximum absolute errors and numerical convergence orders.

N	E_1f_1	r_1f_1	E_2f_1	r_2f_1	E_3f_1	r_3f_1	E_4f_1	r_4f_1	E_5f_1	r_5f_1
16	6.13e-03	-	1.11e-02	-	1.82e-02	-	2.63e-02	-	3.42e-02	-
32	4.22e-04	3.86	7.97e-04	3.80	1.40e-03	3.70	2.19e-03	3.58	3.14e-03	3.44
64	2.71e-05	3.96	5.17e-05	3.95	9.24e-05	3.92	1.49e-04	3.88	2.19e-04	3.84
128	1.69e-06	4.00	3.25e-06	3.99	5.85e-06	3.98	9.46e-06	3.97	1.41e-05	3.96
N	E_1f_2	r_1f_2	E_2f_2	r_2f_2	E_3f_2	r_3f_2	E_4f_2	r_4f_2	E_5f_2	r_5f_2
16	4.95e-03	-	6.16e-03	-	7.72e-03	-	9.13e-03	-	1.02e-02	-
32	5.77e-04	3.10	8.23e-04	2.90	1.17e-03	2.72	1.57e-03	2.54	1.95e-03	2.39
64	4.55e-05	3.67	6.89e-05	3.58	1.06e-04	3.47	1.54e-04	3.35	2.10e-04	3.22
128	3.01e-06	3.92	4.66e-06	3.89	7.38e-06	3.84	1.11e-05	3.79	1.58e-05	3.73
N	E_1f_3	r_1f_3	E_2f_3	r_2f_3	E_3f_3	r_3f_3	E_4f_3	r_4f_3	E_5f_3	r_5f_3
16	1.97e-01	-	1.94e-01	-	1.84e-01	-	1.79e-01	-	1.77e-01	
32	1.34e-01	0.56	1.21e-01	0.68	1.20e-01	0.61	1.20e-01	0.57	1.20e-01	0.56
64	2.74e-02	2.29	4.34e-02	1.47	5.20e-02	1.21	5.19e-02	1.21	5.23e-02	1.20
128	2.59e-03	3.40	5.35e-03	3.02	8.96e-03	2.54	1.24e-02	2.06	1.49e-02	1.81
N	E_1f_4	r_1f_4	E_2f_4	r_2f_4	E_3f_4	r_3f_4	E_4f_4	r_4f_4	E_5f_4	r_5f_4
16	1.30e-05	-	3.72e-05	-	7.71e-05	-	1.32e-04	-	2.02e-04	-
32	8.17e-07	4.00	2.34e-06	3.99	4.87e-06	3.98	8.40e-06	3.98	1.29e-05	3.96
64	5.11e-08	4.00	1.46e-07	4.00	3.05e-07	4.00	5.27e-07	3.99	8.13e-07	3.99
128	3.20e-09	4.00	9.16e-09	4.00	1.91e-08	4.00	3.30e-08	4.00	5.09e-08	4.00

of the values of the function to be approximated at points in neighbourhoods of the support of the corresponding integer translates of the box spline, and the coefficients of those linear combinations depend on a subset of parameters. They are determined minimizing an upper bound of the infinity norm of the operator, subject to the linear equations providing the exactness on $\mathbb{P}_3$. We have proved that this minimization problem has a unique solution, which has been explicitly computed and provides the corresponding near-best quasi-interpolant. We have proved that the sequence of near-best quasi-interpolants converges in the infinity norm to the Schoenberg operator. Finally, we ha provide some results showing that the performances of the near-best QIs when the degree of the involved box spline increases.

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