On barrier option pricing by Erlangization in a regime-switching model with jumps

Griselda Deelstra*
Department of Mathematics, Université libre de Bruxelles, Belgium
Guy Latouche†
Faculty of Sciences, Université libre de Bruxelles, Belgium
Matthieu Simon‡
School of Mathematics and Statistics, University of Melbourne, Australia

Abstract

We consider the risk-neutral pricing of vanilla, digital and down-and-out call options when the underlying asset price evolves like the exponential of a Markov-modulated Brownian motion (MMBM) with two-sided phase-type jumps. The price of such options is intimately related to the first passage properties of the MMBM. To analyse these first passages, we randomize the time horizon using Erlang distributions with suitable parameters and apply matrix-analytic methods. This provides us with closed form approximations of the options prices, with a very high precision, as shown by several numerical illustrations. In particular, we consider an example in which the phase-type jump distribution is constructed in such a way that it mimics fat tails.

KEYWORDS: Markov-modulated Brownian motion; phase-type jumps; Erlangization; matrix-analytic methods; option pricing; barrier options; regime-switching.

*Postal address: CP210, boulevard du Triomphe, 1050 Bruxelles (Belgium). E-mail to: Griselda.Deelstra(at)ulb.ac.be

†Postal address: CP212, boulevard du Triomphe, 1050 Bruxelles (Belgium). E-mail to: Guy.Latouche(at)ulb.ac.be

‡Postal address: School of Mathematics and Statistics, University of Melbourne, Victoria 3010 (Australia). E-mail to: matthieus(at)unimelb.edu.au

1 Introduction

Regime-switching frameworks have been extensively used in the literature for studying various financial problems; one may model business cycles in the market by means of a Markov process whose states represent the different regimes of the economy. More precisely, whenever the state of the underlying Markov process (i.e. the regime in the market) changes, the model parameters are modulated according to the new state. When concentrating on option pricing, the price of the risky asset involved is often modelled as the exponential of a Markov-modulated Brownian motion (MMBM). We refer to Elliott *et al.* [12] for a good overview of the literature and financial applications of regime-switching models.

The main focus of this paper is the pricing of path-dependent options like digital options and down-and-out call options in a Markov modulated Brownian motion framework in the presence of two-sided phase-type jumps. This regime-switching Lévy model possesses nice features as a security-price model, because it includes the short-run behaviour captured by the jump-diffusion component and the long-run market cycle by the Markov chain component. Furthermore, this rich structure remains analytically tractable when one studies the first passage time problem, i.e. the first time a stochastic process crosses a constant upper or lower threshold. Therefore, the model turns out to be very useful for other applications as well, such as credit risk issues and derivatives; these applications however are beyond the scope of this paper.

Many numerical techniques have been developed to price exotic options in a regime-switching environment, although they mainly concentrate upon a Markov modulated Black & Scholes model. We present here a short, incomplete overview only, and refer to the references within the papers mentioned below.

In a geometric Brownian motion (GBM) model with regime-switching, Boyle and Draviam [6] derive coupled Black & Scholes-type partial differential equations that govern the dynamics of several exotic options like European, Asian and lookback options. Hieber and Scherer [19] present in a Markov-switching Black & Scholes framework an efficient algorithm for the pricing of barrier options, named the Brownian bridge algorithm. Elliott *et al.* [14] use both probabilistic and partial differential equation (PDE) approaches to price barrier options in a Markovian Black & Scholes economy. Chan and Zhu [9] present an explicit analytic solution in infinite series form for the price of a European-style barrier option in a two-state regime-switching Black & Scholes economy. Kim *et al.* [24] study a regime-switching model by using PDE techniques for calculating individual and joint default probabilities in a Markov modulated GBM model.

A very useful method to derive barrier option prices is based upon first passage times. In a regime-switching model, this first passage time problem can be reduced

to a quadratic matrix equation, the solutions of which are called matrix Wiener-Hopf factors (see e.g. Rogers [28] or Asmussen [1]) and which generally have to be solved numerically. However, in the case of 2 or 3 regimes or in the case of a zero drift term, the matrix Wiener-Hopf factorization can be derived analytically (see e.g. Hieber [17]) and Hieber [18] studies in such a regime-switching Black & Scholes setting the valuation of digital, barrier and lookback options.

Jiang and Pistorius [21] consider the problem of pricing a perpetual American put option in an exponential regime-switching phase-type jump diffusion framework. Their solution is based upon the Laplace transform of first passage times and follows from a fluid embedding technique, a matrix Wiener-Hopf factorization result and a Laplace inversion.

Kijima and Siu [22] study the first passage time problem under a regime-switching double exponential jump-diffusion process. Following the fluidization technique, as explained in Jiang and Pistorius [21] for instance, they first turn the original model into an augmented regime-switching diffusion model whose sample paths are continuous and then use probabilistic arguments to formulate the first passage time problem as a solution to a system of linear equations. Therefore, in the case of a regime-switching double exponential jump-diffusion process, their methodology is based upon solving a system of linear equations and the numerical inversion of the Laplace transform.

Kim *et al.* [23] provide an iterative algorithm for solving the matrix equations of Jiang and Pistorius [21] with complex parameters and obtain in this way the corresponding Laplace transform of the first passage times. They use some numerical inversion algorithms such as the Euler method to invert this Laplace transform. As an application, they compute the prices of defaultable bonds under a structural model with regime-switching and double phase-type jumps.

As far as we know, there are no papers in the literature providing numerical algorithms for obtaining (approximate) prices of path-dependent options like digital options and down-and-out call options when the risky asset is modelled as the exponential of a Markov-modulated Brownian motion with phase-type jumps. This papers intends to fill this gap by concentrating upon approximations obtained by Erlangization. This randomization method was originally developed in Carr [8] for the valuation of American put options. Whereas the technique is known in risk theory as Erlangization (see e.g. Asmussen and Albrecher [2], Ch. IX.8), in finance, it is also referred to as the "Canadization" method, see e.g. Mijatović *et al.* [25] for an overview. The method has already been used for both American-type and barrier option pricing in a no regime-switching framework, see e.g. Avram *et al.* [3] and Boyarchenko and Levendorskiĭ [5].

When using an Erlangization of the MMBM in order to obtain an approximation of the prices, one replaces the maturity date T by an Erlang random variable q. The

expectation of q is chosen equal to T and its variance may be taken arbitrarily close to zero, so that the approximation may be made as precise as desired. Our analysis then relies on the use of matrix analytic methods to obtain explicit expressions for different quantities related to the path properties of the MMBM up to time q, from which the approximating option prices follow. Compared to other existing methods, this approach does not require the inversion of Laplace (or Fourier) transforms. Moreover, by choosing a number of Erlangization intervals large enough, the obtained precision turns out to be very high, notwithstanding a relatively short calculation time.

In some circumstances, the restriction to phase-type jumps may be seen as a drawback of the model. This is due to the exponential asymptotic decay of their density functions that might not fit the observed reality well enough. However, it is possible to construct phase-type distributions which mimic fat tail behavior. We illustrate this with a numerical example in which the jumps distribution is inspired from Robert and Le Boudec [27].

This article is organized as follows: Section 2 introduces the regime-switching model and presents the path-dependent options we will study. The Erlangization technique is introduced in Section 3. Section 4 gives details on the first passage transform and Section 5 concentrates upon the transient distribution. The main theoretical results on the pricing of path-dependent options under regime-switching in the presence of phase-type jumps are summarized in Section 6. Finally, Section 7 presents several numerical illustrations.

2 Barrier option pricing framework

2.1 MMBMs with phase-type jumps

An MMBM with two-sided phase-type jumps may be seen as a stochastic process (X, φ, ξ) where the level X is modulated by two Markov processes φ and ξ:

- The process φ governs the phase transitions in the absence of jumps. It is defined on a finite phase space $\mathcal{S}_\sigma$. When $\varphi = j$, the level X evolves like a Brownian motion with drift $d_j \in \mathbb{R}$ and variance $\sigma_j^2 > 0$.
- The process ξ determines the size of the jumps. The upward jumps have phase-type distribution represented by a generator R_+ on a state space $\mathcal{S}_+$, and the downward jumps have phase-type distribution represented by a generator R_- on the state space $\mathcal{S}_-$.

When $\varphi = j \in \mathcal{S}_\sigma$, two kinds of transitions are possible: instantaneous transitions from j to a different state $v \in \mathcal{S}_\sigma$ at a rate Q_{jv}, or jumps; the rate at which a jump

occurs and the process ξ begins in state $k \in \mathcal{S}_+ \cup \mathcal{S}_-$ is equal to W_{jk}. When the jump process ξ is in state $k \in \mathcal{S}_+ \cup \mathcal{S}_-$, V_{kj} is the rate at which the jump terminates and the phase $j \in \mathcal{S}_\sigma$ is chosen to regain control over the fluid level.

We introduce the matrices W, V and R, decomposed as follows:

$$W = \begin{bmatrix} W_{\sigma+} & W_{\sigma-} \end{bmatrix}, \quad V = \begin{bmatrix} V_{+\sigma} \\ V_{-\sigma} \end{bmatrix}, \quad R = \begin{bmatrix} R_+ & 0 \\ 0 & R_- \end{bmatrix}.$$

In particular, the matrices W and V contain the transition rates W_{jk} and V_{kj} ($j \in \mathcal{S}_\sigma$ and $k \in \mathcal{S}_+ \cup \mathcal{S}_-$). The rates Q_{jv} ($j \neq v \in \mathcal{S}_\sigma$) are collected in the subgenerator matrix Q, whose diagonal elements are determined such that $[Q\ W]\mathbf{1} = \mathbf{0}$, where $\mathbf{1}$ and $\mathbf{0}$ are column vectors with each component equal to 1 and 0, respectively. Finally, we define $D = \mathrm{diag}(d_j)_{j\in\mathcal{S}_\sigma}$ and $\Sigma = \mathrm{diag}(\sigma_j)_{j\in\mathcal{S}_\sigma}$.

In other words, (X, φ, ξ) can be seen as a Markov-modulated Lévy process with two-sided phase-type jumps, in which the jumps can (but are not forced to) trigger a phase transition. When $\varphi = j \in \mathcal{S}_\sigma$, the continuous part of X is a Brownian motion with drift d_j and variance σ_j^2. An upward jump occurs at rate $(W_{\sigma+}\mathbf{1})_j$ and a downward jump occurs at rate $(W_{\sigma-}\mathbf{1})_j$. If J_j^+ and J_j^- represent the absolute size of an upward and downward jump that occurred in phase j, then for all $k \in \mathcal{S}_\sigma$ and $x \geq 0$,

$$\mathbb{P}\left(J_j^+ \in dx, \varphi = k \text{ after the jump}\right) = \frac{1}{(W_{\sigma+}\mathbf{1})_j}\left(W_{\sigma+}e^{R_+x}V_{+\sigma}\right)_{jk} dx,$$
$$\mathbb{P}\left(J_j^- \in dx, \varphi = k \text{ after the jump}\right) = \frac{1}{(W_{\sigma-}\mathbf{1})_j}\left(W_{\sigma-}e^{R_-x}V_{-\sigma}\right)_{jk} dx.$$

2.2 Risk-neutral option pricing

In this paper, we study the pricing of path-dependent options in the case where the underlying risky asset prices are determined by the following dynamics

$$S(t) = S_0\, e^{X(t)},$$

where $S_0 \in \mathbb{R}_+$ is fixed and (X, φ, ξ) is a MMBM with two-sided phase-type jumps as described in Section 2.1. We assume that $X(0) = 0$ and that $\varphi(0)$ has initial distribution $\boldsymbol{\alpha} \in \mathbb{R}^{|\mathcal{S}_\sigma|}$, where $|\mathcal{S}_\sigma|$ stands for the cardinal number of $\mathcal{S}_\sigma$. The interest rate process $\{r(t)\}$ depends on the phase occupied by φ in the following way

$$r(t) = \sum_{j\in\mathcal{S}_\sigma} r_j \mathbb{1}_{\varphi(t)=j},$$

with the coefficients r_j being constant. The integrated interest rate process $\{C(t)\}$ is defined by

$$C(t) = \int_0^t r(s)\, ds. \tag{2.1}$$

We assume that the model is defined under a risk neutral measure. Note that the risk neutral measure associated to a regime switching model is not unique in general. When starting from the real-world probability measure, one of the most common approaches is to use the regime switching random Esscher transform to determine a risk neutral measure. This transform has the advantages of preserving the (Markov-modulated) Lévy nature of the model and of minimising the conditional relative entropy with respect to the historical measure (see e.g. Elliott *et al.* [12] for details).

To ensure that the model is defined under a risk neutral measure, we impose the following constraint on the parameters:

Lemma 2.1. *If the relation*

$$\left(D - \Theta + \frac{1}{2}\Sigma^2 + Q - W_{\sigma+}(I + R_+)^{-1}V_{+\sigma} + W_{\sigma-}(I - R_-)^{-1}V_{-\sigma}\right)\mathbf{1} = \mathbf{0} \tag{2.2}$$

holds, where $\Theta = diag(r_j)_{j\in\mathcal{S}_\sigma}$*, then the model is defined under a risk neutral measure.*

Proof. We look for a condition on the parameters which implies that the process $\{e^{-C(t)}S(t)\}$ is a martingale under a risk-neutral measure $\mathbb{Q}$, which boils down to finding a sufficient condition such that

$$\mathbb{E}\left[e^{-C(t)+X(t)}\right] = 1 \qquad \forall t \geq 0, \tag{2.3}$$

where $\mathbb{E}[\cdot]$ is the expectation under $\mathbb{Q}$. It is easy to show that

$$\mathbb{E}\left[e^{-C(t)+X(t)}\right] = \boldsymbol{\alpha}\mathbb{E}\left[e^{Bt}\right]\mathbf{1}$$

where $B = D - \Theta + \frac{1}{2}\Sigma^2 + Q - W_{\sigma+}(I + R_+)^{-1}V_{+\sigma} + W_{\sigma-}(I - R_-)^{-1}V_{-\sigma}$. If $B\mathbf{1} = \mathbf{0}$, then $\boldsymbol{\alpha}\mathbb{E}\left[e^{Bt}\right]\mathbf{1} = 1$ for all t and thus (2.3) is satisfied. This yields the announced condition. □

In the sequel, we assume that (2.2) holds and that the matrix $I + R_+$ is invertible. First, we consider a vanilla call option with maturity date T and strike price K. Its price at time 0 is

$$P_{\mathrm{V}}(S_0, K, T) = \mathbb{E}\left[e^{-C(T)}\left(S(T) - K\right)_+\right]. \tag{2.4}$$

We also examine a simple digital option that pays one unit of currency if $S(t)$ goes below a fixed level $B < S_0$ before T. Denoting the first passage time of the process $\{S(t)\}$ to the level B by τ_B, the price at time 0 of this option is

$$P_{\mathrm{DG}}(S_0, B, T) = \mathbb{E}\left[e^{-C(T)}\, \mathbb{1}_{\tau_B<T}\right]. \tag{2.5}$$

Finally, we study down-and-out call (DOC) options, which pay the same amount as a vanilla option, namely $(S(T) - K)_+$, under the constraint that the value of the process $\{S(t)\}$ does not go below a fixed barrier $B < S_0$ before T. Its price at time 0 is

$$P_{\mathrm{DOC}}(S_0, B, K, T) = \mathbb{E}\left[e^{-C(T)}\, (S(T) - K)_+\, \mathbb{1}_{\tau_B>T}\right]. \tag{2.6}$$

To determine approximations of these prices (2.4), (2.5) and (2.6), we replace the maturity T by a random variable $q \sim \mathrm{Erlang}(N, \frac{N}{T})$ where $N \in \mathbb{N}_0$. The expectation of q equals T and its variance T^2/N goes to zero as N goes to infinity. So, for large values of N, the variable q provides a good approximation of T. Observe that if $N = 1$, Erlangization gives us the Laplace transform of the prices with respect to the maturity: if $q \sim \mathrm{Exp}(\mu)$ and $P(t)$ denotes one of the three expectations (2.4), (2.5) and (2.6) with maturity t,

$$P(q) = \mu \int_0^\infty e^{-\mu T} P(T)\, dT.$$

The advantage of dealing with q instead of T is that we can partition the time horizon into N successive periods of exponential duration with parameter $\mu = N/T$, which will be called *Erlangization intervals* in the sequel. The behaviour of (X, φ, ξ) up to time T is approximated by the behaviour of an absorbing MMBM (Y, ζ) that we present in detail in the next section.

3 Erlangization

Using a combined approach of fluidization (see e.g. [21], [22] or [28]) and of Erlangization (see e.g. [2], Ch. IX.8), we define as follows a very useful approximating absorbing MMBM without jumps (Y, ζ): ζ is a bivariate Markov process $\zeta(t) = (\phi(t), e(t))$ where $\phi \in \mathcal{S}_\sigma \cup \mathcal{S}_+ \cup \mathcal{S}_-$ indicates the phase and $e \in \{1, \cdots, N\}$ refers to the Erlangization interval. When ζ has gone through all the Erlangisation intervals, it is sent to an absorbing state $\star$ and the MMBM (Y, ζ) ceases to evolve. We denote by $\tau_\star = \inf\{t > 0 \,|\, \zeta(t) = \star\}$ the absorbing time of (Y, ζ). The state space of ζ is therefore

$$\mathcal{E} = \{\star\} \cup \mathcal{E}_\sigma^{(1)} \cup \cdots \cup \mathcal{E}_\sigma^{(N)} \cup \mathcal{E}_+^{(1)} \cup \cdots \cup \mathcal{E}_+^{(N)} \cup \mathcal{E}_-^{(1)} \cup \cdots \cup \mathcal{E}_-^{(N)}, \tag{3.1}$$

where $\star$ is the absorbing state and where for $1 \leq k \leq N$,

$$\begin{aligned}\mathcal{E}_\sigma^{(k)} &= \{(j,k) \,|\, j \in \mathcal{S}_\sigma\},\\ \mathcal{E}_+^{(k)} &= \{(j,k) \,|\, j \in \mathcal{S}_+\},\\ \mathcal{E}_-^{(k)} &= \{(j,k) \,|\, j \in \mathcal{S}_-\}.\end{aligned}$$

Using the notation $\mathcal{E}_\sigma = \mathcal{E}_\sigma^{(1)} \cup \cdots \cup \mathcal{E}_\sigma^{(N)}$, $\mathcal{E}_+ = \mathcal{E}_+^{(1)} \cup \cdots \cup \mathcal{E}_+^{(N)}$ and $\mathcal{E}_- = \mathcal{E}_-^{(1)} \cup \cdots \cup \mathcal{E}_-^{(N)}$, and the ordering $\mathcal{E} = \{\star\} \cup \mathcal{E}_\sigma \cup \mathcal{E}_+ \cup \mathcal{E}_-$, we define the generator G of ζ as follows

$$G = \left[\begin{array}{c|ccc} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \boldsymbol{v} & & & \\ \mathbf{0} & & M & \\ \mathbf{0} & & & \end{array}\right] \tag{3.2}$$

with

$$M = \begin{bmatrix} I \otimes (Q - \mu I) + J \otimes \mu I & I \otimes W_{\sigma+} & I \otimes W_{\sigma-} \\ I \otimes V_{+\sigma} & I \otimes R_+ & 0 \\ I \otimes V_{-\sigma} & 0 & I \otimes R_- \end{bmatrix}, \tag{3.3}$$

where $\otimes$ is the Kronecker product, and $\boldsymbol{v}$ and J are the vector of $|\mathcal{E}_\sigma|$ components and the $N \times N$ matrix:

$$\boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mu\mathbf{1} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Informally stated, the process (Y, ζ) evolves until the ringing of an Erlang distributed clock that only ticks when $\phi \in \mathcal{S}_\sigma$, and then it is absorbed. The jumps of (X, φ, ξ) are replaced in (Y, ζ) by a linear variation of the level, at absolute rate 1 during an interval with length determined by the size of the corresponding jump in (X, φ, ξ). During these intervals, there is no change of Erlangization interval in (3.3), that is, the clock in (Y, ζ) is stopped. As a consequence, the replacement of jumps by linear variation does not affect the state of the process at the time of absorption: $(Y(\tau_\star), \phi(\tau_\star))$ is equal to $(X(q), \varphi(q))$ in distribution. More generally, when defining

$$\mathcal{T}(t) = \int_0^t \mathbb{1}_{\phi(s) \in \mathcal{S}_\sigma} \, ds,$$

the equality $[Y(\mathcal{T}^{-1}(t)), \zeta(\mathcal{T}^{-1}(t))] = [X(t), \varphi(t)]$ holds in distribution for all $t < q$. In particular, the levels crossed by Y on $[0, \tau_\star[$ are the same as the ones crossed by

X on $[0, q[$: for any level b, denoting by $\tau_b(X)$ and $\tau_b(Y)$ the first time b is crossed by X and Y respectively, we have that $\mathbb{P}\left(\tau_b(X) < q\right) = \mathbb{P}\left(\tau_b(Y) < \tau_\star\right)$ when X and Y start from the same level in the same phase.

4 First passage transforms

We start by introducing the process $\gamma(\boldsymbol{\theta}; t)$ as the following weighted occupation time of (Y, ζ) in the phases of $\mathcal{S}_\sigma$ up to time t

$$\gamma(\boldsymbol{\theta}; t) = \sum_{j \in \mathcal{S}_\sigma} \theta_j \int_0^t \mathbb{1}_{\phi(u)=j}\, du,$$

for general weights θ_j summarized in the vector $\boldsymbol{\theta}$. Note that when $\theta_j = r_j$ for all $j \in \mathcal{S}_\sigma$, then $\gamma(\boldsymbol{\theta}; q)$ is an approximation of $C(T)$, the integrated interest rate process until T defined in (2.1). Let us now define the first passage time $\tau = \inf\{t > 0 \,|\, Y(t) = 0\}$. We are interested in the first passage transform matrices Ψ and e^{Ux} such that

$$\Psi_{(i,k);(j,l)} = \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau)} \mathbb{1}_{\zeta(\tau)=(j,l)} \,|\, Y(0) = 0, \zeta(0) = (i,k)\right] \tag{4.1}$$

for $k, l \in \{1, \ldots, N\}$, $i \in \mathcal{S}_\sigma \cup \mathcal{S}_+$, $j \in \mathcal{S}_\sigma \cup \mathcal{S}_-$, and

$$\left(e^{Ux}\right)_{(i,k);(j,l)} = \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau)} \mathbb{1}_{\zeta(\tau)=(j,l)} \,|\, Y(0) = x, \zeta(0) = (i,k)\right] \tag{4.2}$$

for $k, l \in \{1, \ldots, N\}$, $i, j \in \mathcal{S}_\sigma \cup \mathcal{S}_-$ and $x \geq 0$. In view of the state space decomposition (3.1), the matrices Ψ and U have the structure

$$\Psi = \left[\begin{array}{cccc|cccc}
I_\sigma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & I_\sigma & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & I_\sigma & 0 & 0 & \cdots & 0 \\
\hline
\Psi_{+\sigma}^{(1)} & \Psi_{+\sigma}^{(2)} & \cdots & \Psi_{+\sigma}^{(N)} & \Psi_{+-}^{(1)} & \Psi_{+-}^{(2)} & \cdots & \Psi_{+-}^{(N)} \\
0 & \Psi_{+\sigma}^{(1)} & \cdots & \Psi_{+\sigma}^{(N-1)} & 0 & \Psi_{+-}^{(1)} & \cdots & \Psi_{+-}^{(N-1)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \Psi_{+\sigma}^{(1)} & 0 & 0 & \cdots & \Psi_{+-}^{(1)}
\end{array}\right]$$

and

$$U = \left[\begin{array}{cccc|cccc} U_{\sigma\sigma}^{(1)} & U_{\sigma\sigma}^{(2)} & \cdots & U_{\sigma\sigma}^{(N)} & U_{\sigma-}^{(1)} & U_{\sigma-}^{(2)} & \cdots & U_{\sigma-}^{(N)} \\ 0 & U_{\sigma\sigma}^{(1)} & \cdots & U_{\sigma\sigma}^{(N-1)} & 0 & U_{\sigma-}^{(1)} & \cdots & U_{\sigma-}^{(N-1)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U_{\sigma\sigma}^{(1)} & 0 & 0 & \cdots & U_{\sigma-}^{(1)} \\ \hline U_{-\sigma}^{(1)} & U_{-\sigma}^{(2)} & \cdots & U_{-\sigma}^{(N)} & U_{--}^{(1)} & U_{--}^{(2)} & \cdots & U_{--}^{(N)} \\ 0 & U_{-\sigma}^{(1)} & \cdots & U_{-\sigma}^{(N-1)} & 0 & U_{--}^{(1)} & \cdots & U_{--}^{(N-1)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U_{-\sigma}^{(1)} & 0 & 0 & \cdots & U_{--}^{(1)} \end{array}\right].$$

Note that the four corners of Ψ and U are block-Toeplitz submatrices, and we make use of this special structure below. We also introduce the notation

$$\begin{aligned} \Psi^{(1)} = \begin{bmatrix} I_\sigma & 0 \\ \Psi_{+\sigma}^{(1)} & \Psi_{+-}^{(1)} \end{bmatrix}, \quad \Psi^{(k)} = \begin{bmatrix} 0 & 0 \\ \Psi_{+\sigma}^{(k)} & \Psi_{+-}^{(k)} \end{bmatrix} \ (k = 2, \ldots, N), \\ U^{(k)} = \begin{bmatrix} U_{\sigma\sigma}^{(k)} & U_{\sigma-}^{(k)} \\ U_{-\sigma}^{(k)} & U_{--}^{(k)} \end{bmatrix} \ (k = 1, \ldots, N). \end{aligned} \tag{4.3}$$

We deal in the sequel with different combinations of sub-matrices and for that reason, we introduce the subsets $\mathcal{S}_d = \mathcal{S}_\sigma \cup \mathcal{S}_-$, and $\mathcal{S}_u = \mathcal{S}_\sigma \cup \mathcal{S}_+$.

Proposition 4.1. *The matrices $\Psi^{(1)}$ and $U^{(1)}$ satisfy the equation*

$$\begin{aligned} &\frac{1}{2}\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \left(U^{(1)}\right)^2 + \begin{bmatrix} D & & \\ & I_+ & \\ & & -I_- \end{bmatrix} \begin{bmatrix} I_\sigma & 0 \\ \Psi_{+\sigma}^{(1)} & \Psi_{+-}^{(1)} \\ 0 & I_- \end{bmatrix} U^{(1)} \\ &\quad + \begin{bmatrix} Q - \mu I_\sigma - \Theta & W_{\sigma+} & W_{\sigma-} \\ V_{+\sigma} & R_+ & 0 \\ V_{-\sigma} & 0 & R_- \end{bmatrix} \begin{bmatrix} I_\sigma & 0 \\ \Psi_{+\sigma}^{(1)} & \Psi_{+-}^{(1)} \\ 0 & I_- \end{bmatrix} = 0, \end{aligned} \tag{4.4}$$

where $\Theta = diag(\boldsymbol{\theta})$. For $k \geq 2$, $U_{-d}^{(k)} = 0$. The matrix $Z^{(2)} = \begin{bmatrix} U_{\sigma d}^{(2)} \\ \Psi_{+d}^{(2)} \end{bmatrix}$ is the unique solution of the Sylvester equation

$$\begin{bmatrix} U_{\sigma\sigma}^{(1)} + 2\Sigma^{-2} D & 2\Sigma^{-2} W_{\sigma+} \\ \Psi_{+\sigma}^{(1)} & R_+ \end{bmatrix} Z^{(2)} + Z^{(2)} U^{(1)} + \begin{bmatrix} 2\mu\Sigma^{-2} & 0 \\ 0 & 0 \end{bmatrix} = 0, \tag{4.5}$$

and finally, for $k \geq 3$, the matrix $Z^{(k)} = \begin{bmatrix} U^{(k)}_{\sigma d} \\ \Psi^{(k)}_{+d} \end{bmatrix}$ is the unique solution of the Sylvester equation

$$\begin{bmatrix} U^{(1)}_{\sigma\sigma} + 2\Sigma^{-2}D & 2\Sigma^{-2}W_{\sigma+} \\ \Psi^{(1)}_{+\sigma} & R_+ \end{bmatrix} Z^{(k)} + Z^{(k)}U^{(1)} + \sum_{l=2}^{k-1} Z^{(l)}_{u\sigma} U^{(k-l+1)}_{\sigma d} = 0. \tag{4.6}$$

Proof. Before its absorption, the process (Y, ζ) behaves as a classical MMBM with generator M in (3.3) and drift and variance matrices

$$C = \begin{bmatrix} I_\sigma \otimes D & & \\ & I_+ & \\ & & -I_- \end{bmatrix}, \qquad S = \begin{bmatrix} I_\sigma \otimes \Sigma^2 & & \\ & 0 & \\ & & 0 \end{bmatrix}.$$

It follows from Ivanovs [20, Equation (2.2)] that

$$\frac{1}{2} S\Pi U^2 + C\Pi U + M\Pi = 0,$$

where $\Pi = \begin{bmatrix} \Psi \\ [0 \;\; I_-] \end{bmatrix}$. The announced result can then be obtained by re-writing this system according to the block decomposition (4.3). □

Various numerical procedures are available in the literature to solve (4.4) and obtain $\Psi^{(1)}$ and $U^{(1)}$ numerically, see for instance Asmussen [1], Breuer [7] and Nguyen and Poloni [26]. Once the matrices $\Psi^{(1)}$ and $U^{(1)}$ are known, the other blocks of Ψ and U are easily obtained by solving the linear equations (4.5) and (4.6), which can be made numerically in a very efficient way (see e.g. Gardiner *et al.* [15]).

Let $\tilde{U}$ be the matrix obtained by permuting the rows and columns of U so that the states are arranged in lexicographic order with respect to the Erlangization intervals:

$$\tilde{U} = \begin{bmatrix} U^{(1)} & U^{(2)} & U^{(3)} & \cdots & U^{(N)} \\ 0 & U^{(1)} & U^{(2)} & \cdots & U^{(N-1)} \\ 0 & 0 & U^{(1)} & \cdots & U^{(N-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & U^{(1)} \end{bmatrix},$$

In the following, we will use the blocks $E^{(k)}(x)$ that constitute the matrix

$$E(x) = e^{\tilde{U}x} = \begin{bmatrix} E^{(1)}(x) & E^{(2)}(x) & E^{(3)}(x) & \cdots & E^{(N)}(x) \\ 0 & E^{(1)}(x) & E^{(2)}(x) & \cdots & E^{(N-1)}(x) \\ 0 & 0 & E^{(1)}(x) & \cdots & E^{(N-2)}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & E^{(1)}(x) \end{bmatrix}.$$

These blocks are given by

$$E_{ij}^{(k)}(x) = \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau)}\mathbb{1}_{\zeta(\tau)=(j,k)} \,|\, Y(0)=x, \zeta(0)=(i,1)\right]$$

for $k \in \{1,\cdots,N\}$, $i,j \in \mathcal{S}_\sigma \cup \mathcal{S}_-$ and $x \geq 0$. They may be computed numerically in an efficient way by using the block-triangular structure of $\tilde{U}$ (see e.g. Bini *et al.* [4]).

We also define the matrices Ψ^* and U^* related to first passage times to *higher* levels and defined in a manner similar to Ψ and U:

$$\Psi^*_{(i,k);(j,l)} = \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau)}\mathbb{1}_{\zeta(\tau)=(j,l)} \,|\, Y(0)=0, \zeta(0)=(i,k)\right] \tag{4.7}$$

for $k,l \in \{1,\cdots,N\}$, $i \in \mathcal{S}_\sigma \cup \mathcal{S}_-$, $j \in \mathcal{S}_\sigma \cup \mathcal{S}_+$, and

$$\left(e^{U^*x}\right)_{(i,k);(j,l)} = \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau)}\mathbb{1}_{\zeta(\tau)=(j,l)} \,|\, Y(0)=-x, \zeta(0)=(i,k)\right] \tag{4.8}$$

for $k,l \in \{1,\cdots,N\}$, $i,j \in \mathcal{S}_\sigma \cup \mathcal{S}_+$ and $x \geq 0$. These matrices have a block-structure similar to Ψ and U:

$$\begin{gathered}\Psi^{*(1)} = \begin{bmatrix} I & 0 \\ \Psi^{*(1)}_{-\sigma} & \Psi^{*(1)}_{-+} \end{bmatrix}, \quad \Psi^{*(k)} = \begin{bmatrix} 0 & 0 \\ \Psi^{*(k)}_{-\sigma} & \Psi^{*(k)}_{-+} \end{bmatrix} \ (k=2,...,N), \\ U^{*(k)} = \begin{bmatrix} U^{*(k)}_{\sigma\sigma} & U^{*(k)}_{\sigma+} \\ U^{*(k)}_{+\sigma} & U^{*(k)}_{++} \end{bmatrix} \ (k=1,...,N)\end{gathered} \tag{4.9}$$

and these blocks can be derived from analogous equations as those stated in Proposition 4.1. As for the exponential of U, we use the blocks $E^{*(k)}(x)$ constituted by the elements of e^{U^*x}:

$$E_{ij}^{*(k)}(x) = \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau)}\mathbb{1}_{\zeta(\tau)=(j,k)} \,|\, Y(0)=-x, \zeta(0)=(i,1)\right]$$

for $k \in \{1,\cdots,N\}$, $i,j \in \mathcal{S}_\sigma \cup \mathcal{S}_+$ and $x \geq 0$.

5 Transient distribution

In this section, we analyse the time $\tau_\star$ of absorption of (Y,ζ). Inspired by the results of Latouche and Dendievel [11] in the absence of jumps and for $\boldsymbol{\theta} = \mathbf{0}$, we introduce the vectors $\boldsymbol{F}_k(y,x)$ such that

$$\left(\boldsymbol{F}_k(y,x)\right)_i = \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)}\mathbb{1}_{Y(\tau_\star)\leq x} \,|\, Y(0)=y, \zeta(0)=(i,N-k+1)\right], \tag{5.1}$$

for $k \in \{1, \cdots, N\}$ and $i \in \mathcal{S}_\sigma \cup \mathcal{S}_+ \cup \mathcal{S}_-$, and the vectors $\boldsymbol{g}_k$, $\boldsymbol{h}_k$ and $\boldsymbol{\nu}_k$ such that

$$\begin{aligned}
(\boldsymbol{g}_k)_i &= \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)}\mathbb{1}_{Y(\tau_\star)\leq 0} \,|\, Y(0)=0, \zeta(0)=(i, N-k+1)\right], \\
(\boldsymbol{h}_k)_i &= \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)}\mathbb{1}_{Y(\tau_\star)\geq 0} \,|\, Y(0)=0, \zeta(0)=(i, N-k+1)\right], \\
(\boldsymbol{\nu}_k)_i &= \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)} \,|\, \zeta(0)=(i, N-k+1)\right].
\end{aligned}$$

Note that the vectors $\boldsymbol{\nu}_k$ do not depend on $Y(0)$ since $\gamma(\boldsymbol{\theta};\tau_\star)$ does not depend on $Y(0)$. A first proposition expresses the vectors $\boldsymbol{F}_k(y,x)$ in function of these vectors $\boldsymbol{g}_k$, $\boldsymbol{h}_k$ and $\boldsymbol{\nu}_k$:

Proposition 5.1. *For $x \leq y$,*

$$\boldsymbol{F}_{k,d}(y,x) = \sum_{l=1}^{k} E^{(l)}(y-x)\boldsymbol{g}_{k-l+1,d}, \tag{5.2}$$

$$\boldsymbol{F}_{k,u}(y,x) = \sum_{l=1}^{k} \Psi^{(l)}\boldsymbol{F}_{k-l+1,d}(y,x). \tag{5.3}$$

For $x \geq y$,

$$\boldsymbol{F}_{k,u}(y,x) = \boldsymbol{\nu}_{k,u} - \sum_{l=1}^{k} E^{*(l)}(x-y)\boldsymbol{h}_{k-l+1,u}, \tag{5.4}$$

$$\boldsymbol{F}_{k,d}(y,x) = \boldsymbol{\nu}_{k,d} - \sum_{l=1}^{k}\sum_{m=1}^{k-l+1} \Psi^{*(l)}E^{*(m)}(x-y)\boldsymbol{h}_{k-l-m+2,u}. \tag{5.5}$$

Proof. Equation (5.2) is proved as follows: starting from level y in $\mathcal{S}_u$ with k remaining Erlangization intervals before absorption, the process Y must reach the level x in l intervals, for $1 \leq l \leq k$. This justifies the factor $E^{(l)}(y-x)$. Next, starting from x, the process must be below x at the end of the $k-l+1$ remaining intervals, which yields the factor $\boldsymbol{g}_{k-l+1,d}$.

Equation (5.3) is obtained in a similar way: starting from level y in $\mathcal{S}_+$ when there remain k Erlangization intervals before absorption, the process Y must first return to y in $\mathcal{S}_d$, and then must be below x at the end of the remaining Erlangization intervals.

To prove (5.4) and (5.5), it suffices to remark that

$$\begin{aligned}
&\mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)}\mathbb{1}_{Y(\tau_\star)\leq x} \,|\, Y(0)=y, \zeta(0)=(i, N-k+1)\right] \\
&\qquad = \boldsymbol{\nu}_k - \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)}\mathbb{1}_{Y(\tau_\star)\geq x} \,|\, Y(0)=y, \zeta(0)=(i, N-k+1)\right],
\end{aligned}$$

and to use a similar arguments as above to determine the second term in the right-hand side. □

Next, we determine the vectors $\boldsymbol{\nu}_k$:

Proposition 5.2. *For* $1 \leq k \leq N$,

$$\boldsymbol{\nu}_{k,\sigma} = \mu^k \left[\Theta + \mu I - Q - W_{\sigma+}(-R_+)^{-1}V_{+\sigma} - W_{\sigma-}(-R_-)^{-1}V_{-\sigma}\right]^{-k} \mathbf{1}, \tag{5.6}$$

$$\boldsymbol{\nu}_{k,+} = (-R_+)^{-1}V_{+\sigma}\,\boldsymbol{\nu}_{k,\sigma}, \tag{5.7}$$

$$\boldsymbol{\nu}_{k,-} = (-R_-)^{-1}V_{-\sigma}\,\boldsymbol{\nu}_{k,\sigma}. \tag{5.8}$$

Proof. The vector $\boldsymbol{\nu}_{k,\sigma}$ contains the transform of the occupation times $\gamma(\boldsymbol{\theta};\tau_\star)$ of (Y,ζ) starting from $\mathcal{S}_\sigma$ with k remaining Erlangization intervals. Equivalently, $\boldsymbol{\nu}_{k,\sigma}$ contains the transform of the occupation times $\tilde{\gamma}(\boldsymbol{\theta};t_k)$ up to time $t_k \sim \text{Erlang}(k,\mu)$ for the Markov process $\tilde{\varphi}$, obtained by considering ϕ in the local time of the phase of $\mathcal{S}_\sigma$. The generator of this process is

$$L = Q + W_{\sigma+}(-R_+)^{-1}V_{+\sigma} + W_{\sigma-}(-R_-)^{-1}V_{-\sigma},$$

and, for $t \geq 0$ fixed, it is well-known that

$$\mathbb{E}\left[e^{-\tilde{\gamma}(\boldsymbol{\theta};t)}\right] = e^{(L-\Theta)t}\mathbf{1},$$

see e.g. Elliott and Osakwe [13, Proposition 2]. So, by conditioning on t_k, we have

$$\begin{aligned}\boldsymbol{\nu}_{k,\sigma} &= \int_0^\infty e^{(L-\Theta)t}\mu^k e^{-\mu t}\frac{t^{k-1}}{(k-1)!}\mathbf{1}\,dt \\ &= \mu^k \int_0^\infty \frac{t^{k-1}}{(k-1)!}e^{-(\Theta+\mu I-L)t}\mathbf{1}\,dt. \end{aligned} \tag{5.9}$$

For $k=1$, the equality (5.9) coincides with (5.6). For $k>1$, integrating by parts in (5.9), one easily finds the recursion

$$\boldsymbol{\nu}_{k,\sigma} = \mu(\Theta + \mu I - L)^{-1}\boldsymbol{\nu}_{k-1,\sigma},$$

which yields (5.6). The relations (5.7) and (5.8) are immediate since by definition, there is no contribution made to $\gamma(\boldsymbol{\theta};\tau_\star)$ when the phase process is in $\mathcal{S}_+$ and $\mathcal{S}_-$. □

Let us now turn to the vectors $\boldsymbol{g}_k$. The Brownian trajectories make the analysis of these vectors more complicated than in [11], and we need to proceed by a limiting argument. Before stating the results, we introduce the functions $\chi_{r_1,\ldots,r_h}$, for integers $r_1,\ldots,r_h$, equal to zero if $r_1 = r_2 = \cdots = r_h = 1$ and equal to one otherwise. We first concentrate upon the vectors $\boldsymbol{g}_{k,d}$.

Proposition 5.3. *For $1 \leq k \leq N$,*

$$\boldsymbol{g}_{k,d} = \Omega^{-1}\boldsymbol{\omega}_k \tag{5.10}$$

where

$$\begin{aligned}
\Omega_{\sigma d} &= \left(U^{*(1)}_{\sigma u} + U^{(1)}_{\sigma d}\Psi^{*(1)}\right)\Psi^{(1)},\\
\Omega_{-d} &= [0 \;\; I_-] - \Psi^{*(1)}_{-u}\Psi^{(1)},
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\omega}_{k,\sigma} &= \sum_{l=1}^{k}\left(U^{*(l)}_{\sigma u}\boldsymbol{\nu}_{k-l+1,u} + \sum_{m=1}^{k-l+1} U^{(l)}_{\sigma d}\Psi^{*(m)}\boldsymbol{\nu}_{k-l-m+2,u}\right)\\
&\quad - \sum_{l=1}^{k} U^{(l)}_{\sigma d}\boldsymbol{\nu}_{k-l+1,d} - \sum_{l=1}^{k}\sum_{m=1}^{k-l+1} U^{*(l)}_{\sigma u}\Psi^{(m)}\boldsymbol{g}_{k-l-m+2,d}\,\chi_{l,m}\\
&\quad - \sum_{l=1}^{k}\sum_{m=1}^{k-l+1}\sum_{n=1}^{k-l-m+2} U^{(l)}_{\sigma d}\Psi^{*(m)}\Psi^{(n)}\boldsymbol{g}_{k-l-m-n+3,d}\,\chi_{l,m,n},
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\omega}_{k,-} &= \boldsymbol{\nu}_{k,-} - \sum_{l=1}^{k}\Psi^{*(l)}_{-u}\boldsymbol{\nu}_{k-l+1,u}\\
&\quad + \sum_{l=1}^{k}\sum_{m=1}^{k-l+1}\Psi^{*(l)}_{-u}\Psi^{(m)}\boldsymbol{g}_{k-l-m+2,d}\,\chi_{l,m}.
\end{aligned}$$

Proof. The vector $\boldsymbol{g}_{k,-}$ satisfies the relation

$$\boldsymbol{g}_{k,-} = \boldsymbol{\nu}_{k,-} - \sum_{l=1}^{k}\Psi^{*(l)}_{-u}\boldsymbol{\nu}_{k-l+1,u} + \sum_{l=1}^{k}\sum_{m=1}^{k-l+1}\Psi^{*(l)}_{-u}\Psi^{(m)}\boldsymbol{g}_{k-l-m+2,d}.$$

Indeed, the term $(\boldsymbol{\nu}_{k,-} - \sum_{l=1}^{k}\Psi^{*(l)}_{-u}\boldsymbol{\nu}_{k-l+1,u})$ is the probability that, starting from zero, the process never comes back to zero thereafter. In the third term, $\Psi^{*(l)}_{-u}\Psi^{(m)}$ is the probability that, starting from zero, the process comes back to zero while there remain $k-l-m+2$ Erlangization intervals before the absorption. This yields

$$\Omega_{-d}\,\boldsymbol{g}_{k,d} = \boldsymbol{\omega}_{k,-}. \tag{5.11}$$

To obtain an expression for $\boldsymbol{g}_{k,\sigma}$, we fix $\varepsilon > 0$ and define the vectors $\boldsymbol{g}_{k,d}(\varepsilon)$ analogously to $\boldsymbol{g}_{k,d}$ but with the additional constraint that when level zero is reached before absorption, the process must also attain the level $-\varepsilon$ before the absorption:

$$\boldsymbol{g}_{k,i}(\varepsilon) = \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)}\mathbb{1}_{Y(\tau_\star)\leq 0}\,\mathbb{1}_{\delta_{-\varepsilon}>\delta_0}\,|\,Y(0)=0,\zeta(0)=(i,N-k+1)\right]$$

for $i \in \mathcal{S}_d$, where δ_y is the last passage time to level y before $\tau_\star$. When the process starts in $\mathcal{S}_\sigma$, the vectors $\boldsymbol{g}_{k,\sigma}(\varepsilon)$ are determined by

$$\boldsymbol{g}_{k,\sigma}(\varepsilon) = \sum_{l=1}^{k} E^{(l)}_{\sigma d}(\varepsilon) \left(\boldsymbol{\nu}_{k-l+1,d} - \sum_{m=1}^{k-l+1} \sum_{n=1}^{k-l-m+2} \Psi^{*(m)} E^{*(n)}(\varepsilon) \boldsymbol{\nu}_{k-l-m-n+3,u} \right) + \sum_{l=1}^{k} \sum_{m=1}^{k-l+1} \sum_{n=1}^{k-l-m+2} \sum_{p=1}^{k-l-m-n+3} E^{(l)}_{\sigma d}(\varepsilon) \Psi^{*(m)} E^{*(n)}(\varepsilon) \Psi^{(p)} \boldsymbol{g}_{k-l-m-n-p+4,d}(\varepsilon).$$

Indeed, the first term in the right-hand side covers the event that, when starting from zero, the process reaches $-\varepsilon$ and never returns to zero thereafter. In the second term, the matrices $E^{(l)}_{\sigma d}(\varepsilon)\Psi^{*(m)}E^{*(n)}(\varepsilon)\Psi^{(p)}$ cover the event that, when starting from zero, the process reaches $-\varepsilon$ and attains zero before $\tau_\star$. Reorganizing the last equality leads to

$$\begin{aligned} &\left([I_\sigma \ 0] - E^{(1)}_{\sigma d}(\varepsilon) \Psi^{*(1)} E^{*(1)}(\varepsilon) \Psi^{(1)} \right) \boldsymbol{g}_{k,d}(\varepsilon) \\ &= \sum_{l=1}^{k} E^{(l)}_{\sigma d}(\varepsilon) \left(\boldsymbol{\nu}_{k-l+1,d} - \sum_{m=1}^{k-l+1} \sum_{n=1}^{k-l-m+2} \Psi^{*(m)} E^{*(n)}(\varepsilon) \boldsymbol{\nu}_{k-l-m-n+3,u} \right) \\ &\quad + \sum_{l=1}^{k} \sum_{m=1}^{k-l+1} \sum_{n=1}^{k-l-m+2} \sum_{p=1}^{k-l-m-n+3} E^{(l)}_{\sigma d}(\varepsilon) \Psi^{*(m)} E^{*(n)}(\varepsilon) \Psi^{(p)} \boldsymbol{g}_{k-l-m-n-p+4,d}(\varepsilon)\, \chi_{l,m,n,p}. \end{aligned} \tag{5.12}$$

Now, since $E(\varepsilon) = e^{\tilde{U}\varepsilon}$ and $E^*(\varepsilon) = e^{\tilde{U}^*\varepsilon}$, the following relations clearly hold

$$\begin{aligned} E^{(k)}(\varepsilon) &= I\delta_{k,1} + \varepsilon U^{(k)} + o(\varepsilon), \\ E^{*(k)}(\varepsilon) &= I\delta_{k,1} + \varepsilon U^{*(k)} + o(\varepsilon). \end{aligned} \tag{5.13}$$

Using (4.3), (4.9) and the relations (5.13), one notices that

$$[I_\sigma \ 0] - E^{(1)}_{\sigma d}(\varepsilon) \Psi^{*(1)} E^{*(1)}(\varepsilon) \Psi^{(1)} = -\varepsilon \Omega_{\sigma d} + o(\varepsilon), \tag{5.14}$$

as well as that the right-hand side of (5.12) is equal to $-\varepsilon \boldsymbol{\omega}_{k,\sigma} + o(\varepsilon)$. Therefore, equality (5.12) may be rewritten as

$$\Omega_{\sigma d}\, \boldsymbol{g}_{k,d}(\varepsilon) = \boldsymbol{\omega}_{k,\sigma} + o(\varepsilon). \tag{5.15}$$

As $\lim_{\varepsilon \to 0^+} \boldsymbol{g}_{k,d}(\varepsilon) = \boldsymbol{g}_{k,d}$ for all k, taking the limit in (5.15) and combining the result with (5.11) yield the announced result (5.10). □

Finally, we determine the vectors $\boldsymbol{g}_{k,+}$ and $\boldsymbol{h}_k$.

Proposition 5.4. *For* $1 \leq k \leq N$,

$$\boldsymbol{g}_{k,+} = \sum_{l=1}^{k} \Psi_{+d}^{(l)} \boldsymbol{g}_{k-l+1,d}, \tag{5.16}$$

$$\boldsymbol{h}_k = \boldsymbol{\nu}_k - \boldsymbol{g}_k. \tag{5.17}$$

Proof. The first result (5.16) is easily obtained by considering that when starting from the level zero in $\mathcal{S}_+$, the process must come back to zero from above after l intervals, say. Next it must be below the level zero at the end of the $k-l+1$ remaining intervals.

The second result (5.17) follows immediately from the definition of $\boldsymbol{h}_k$:

$$\begin{aligned} \boldsymbol{h}_k &= \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)} \mathbb{1}_{Y(\tau_\star)\geq 0} \,|\, Y(0)=0, \zeta(0)=(.,N-k+1)\right] \\ &= \boldsymbol{\nu}_k - \mathbb{E}\left[e^{-\gamma(\boldsymbol{\theta};\tau_\star)} \mathbb{1}_{Y(\tau_\star)\leq 0} \,|\, Y(0)=0, \zeta(0)=(.,N-k+1)\right]. \end{aligned}$$

□

We conclude from the different propositions in this section that the vectors $\boldsymbol{F}_k(y,x)$, defined in equation (5.1), are completely determined. In the following section, we apply the different results to option pricing.

6 Option pricing by Erlangization

In this section, we use the Erlangization method to obtain approximations for the prices (2.4), (2.5) and (2.6) of digital, vanilla and down-and-out call options in the settings of a regime-switching model with two-sided phase-type jumps. The approximation consists in replacing the maturity date T of the options with $q \sim \text{Erlang}(N, \frac{N}{T})$. The integrated interest rate $C(T)$ defined in (2.1) then is approximated by $C(q)$, which is equal to the variable $\gamma(\boldsymbol{r};\tau_\star)$ defined for the Erlangized MMBM (Y,ζ) when started in the first Erlangization period, and where r_j represents the constant interest rate in phase $j \in \mathcal{S}_\sigma$.

6.1 Digital options

Proposition 6.1. *The price at time* 0 *of a digital option as defined in* (2.5)*, with initial price* S_0*, barrier level* $B < S_0$ *and maturity* $q \sim$ *Erlang*$(N, \frac{N}{T})$ *is given by*

$$P_{DG}(S_0, B, q) = \boldsymbol{\alpha} \sum_{l=1}^{N} E_{\sigma d}^{(l)}(s_0 - b)\, \boldsymbol{\nu}_{N-l+1,d}, \tag{6.1}$$

where $s_0 = \ln(S_0)$ *and* $b = \ln(B)$.

Proof. Recall that $\tau_\star$ is the absorbing time of (Y,ζ) and that τ is the first passage time below the level zero by Y. Applying the Erlangization technique, we have

$$P_{\mathrm{DG}}(S_0,B,q)=\sum_{i\in\mathcal{S}_\sigma}\alpha_i\,\mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)}\mathbb{1}_{\tau<\tau_\star}\,|\,Y(0)=s_0-b,\zeta(0)=(i,1)\right].$$

In order to have a non-zero value for the indicator function in this expectation, the process must reach the level zero before $\tau_\star$. Conditioning on the number of intervals needed to achieve this first passage, leads to the following observation

$$\begin{aligned}&\mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)}\mathbb{1}_{\tau<\tau_\star}\,|\,Y(0)=s_0-b,\zeta(0)=(i,1)\right]\\&\qquad=\sum_{l=1}^{N}\sum_{j\in\mathcal{S}_d}\left(E^{(l)}(s_0-b)\right)_{ij}\mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)}\,|\,\zeta(0)=(j,l)\right],\end{aligned}$$

which proves (6.1). □

6.2 Vanilla call options

In this section and the next, we use the vectors $\boldsymbol{\Pi}(S_0,K,k)$ defined as follows:

$$\Pi_j(S_0,K,k)=\mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)}\left(e^{Y(\tau_\star)}-K\right)_+\,|\,Y(0)=s_0,\zeta(0)=(j,N-k+1)\right]\quad(6.2)$$

for $j\in\mathcal{S}_\sigma\cup\mathcal{S}_+\cup\mathcal{S}_-$, where $s_0=\ln(S_0)$ as before. It gives the price of a vanilla option when the risky asset is assumed to be the exponential of (Y,ζ), starting in phase j with k remaining Erlangization intervals before the maturity date. So, the price of a vanilla option as defined in (2.4) with maturity q is given by

$$P_{\mathrm{V}}(S_0,K,q)=\boldsymbol{\alpha}\boldsymbol{\Pi}_\sigma(S_0,K,N).\quad(6.3)$$

The next proposition shows that the values of $\Pi_j(S_0,K,k)$ depend on the starting phase j and on the sign of S_0-K. To present the formulae in a compact way, we use the notation $A_{(1,l)}$ for the l-th block in the first (block) line of A (e.g. $\tilde{U}_{(1,l)}=U^{(l)}$).

Proposition 6.2. *Let $\kappa=\ln(K)$. If $S_0\geq K$,*

$$\begin{aligned}&\boldsymbol{\Pi}_d(S_0,K,k)\\&=(S_0-K)\left(\boldsymbol{g}_{k,d}+\sum_{l=1}^{k}\Psi^{*(l)}\boldsymbol{h}_{k-l+1,u}\right)+K\sum_{l=1}^{k}\left((I-\tilde{U})^{-1}e^{\tilde{U}(s_0-\kappa)}\right)_{(1,l)}\boldsymbol{g}_{k-l+1,d}\\&\quad-S_0\sum_{l=1}^{k}\left((I-\tilde{U})^{-1}\right)_{(1,l)}\boldsymbol{g}_{k-l+1,d}-S_0\sum_{l=1}^{k}\sum_{m=1}^{k-l+1}\Psi^{*(l)}\left((I+\tilde{U}^*)^{-1}\right)_{(1,m)}\boldsymbol{h}_{k-l-m+2,u}\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{\Pi}_u(S_0,K,k)\\
&\quad=(S_0-K)\left(\boldsymbol{h}_{k,u}+\sum_{l=1}^{k}\Psi^{(l)}\boldsymbol{g}_{k-l+1,d}\right)+K\sum_{l=1}^{k}\sum_{m=1}^{k-l+1}\Psi^{(l)}\left((I-\tilde{U})^{-1}e^{\tilde{U}(s_0-\kappa)}\right)_{(1,m)}\boldsymbol{g}_{k-l-m+2,d}\\
&\qquad-S_0\sum_{l=1}^{k}\left((I+\tilde{U}^*)^{-1}\right)_{(1,l)}\boldsymbol{h}_{k-l+1,u}-S_0\sum_{l=1}^{k}\sum_{m=1}^{k-l+1}\Psi^{(l)}\left((I-\tilde{U})^{-1}\right)_{(1,m)}\boldsymbol{g}_{k-l-m+2,d}.
\end{aligned}$$

If $S_0<K$,

$$\begin{aligned}
\boldsymbol{\Pi}_u(S_0,K,k)&=\sum_{l=1}^{k}E^{*(l)}(\kappa-s_0)\boldsymbol{\Pi}_u(K,K,k-l+1),\\
\boldsymbol{\Pi}_d(S_0,K,k)&=\sum_{l=1}^{k}\Psi^{*(l)}\boldsymbol{\Pi}_u(S_0,K,k-l+1).
\end{aligned}$$

Proof. Let us begin with the case $S_0\geq K$ and let us denote by $\boldsymbol{f}(u,x)$ the conditional density vector of $(\gamma(\boldsymbol{r};\tau_\star),Y(\tau_\star))$ given the initial phase. Then

$$\begin{aligned}
\boldsymbol{\Pi}_d(S_0,K,k)&=\int_0^\infty\int_\kappa^\infty e^{-u}(e^x-e^\kappa)\boldsymbol{f}(u,x)\,dx\,du\\
&=\int_\kappa^\infty(e^x-e^\kappa)\left(\int_0^\infty e^{-u}\boldsymbol{f}(u,x)\,du\right)dx\\
&=\int_\kappa^\infty(e^x-e^\kappa)\frac{d}{dx}\boldsymbol{F}_{k,d}(s_0,x)\,dx.
\end{aligned}$$

We use Proposition 5.1 and split the last integral into two parts to obtain

$$\boldsymbol{\Pi}_d(S_0,K,k)=\int_\kappa^{s_0}(e^x-e^\kappa)\frac{d}{dx}\boldsymbol{F}_{k,d}(s_0,x)\,dx+\int_{s_0}^\infty(e^x-e^\kappa)\frac{d}{dx}\boldsymbol{F}_{k,d}(s_0,x)\,dx. \quad (6.4)$$

Using (5.2), the first integral above may be rewritten as

$$\begin{aligned}
&\int_\kappa^{s_0}(e^x-e^\kappa)\frac{d}{dx}\boldsymbol{F}_{k,d}(s_0,x)\,dx\\
&\qquad=\sum_{l=1}^{k}\left(\int_\kappa^{s_0}e^x\frac{d}{dx}E^{(l)}(s_0-x)\,dx-K\int_\kappa^{s_0}\frac{d}{dx}E^{(l)}(s_0-x)\,dx\right)\boldsymbol{g}_{k-l+1,d}.
\end{aligned}$$

Since

$$\int_\kappa^{s_0}\frac{d}{dx}E^{(l)}(s_0-x)\,dx=I\delta_{l,1}-E^{(l)}(s_0-\kappa)$$

and

$$\begin{aligned}\int_\kappa^{s_0} e^x \frac{d}{dx} E^{(l)}(s_0 - x)\, dx &= \left(- \int_\kappa^{s_0} e^x e^{\tilde{U}(s_0 - x)} \tilde{U}\, dx \right)_{(1,l)} \\ &= \left(-e^{s_0} \int_0^{s_0 - \kappa} e^{-(I - \tilde{U})x} \tilde{U}\, dx \right)_{(1,l)} \\ &= \left(-e^{s_0} (I - \tilde{U})^{-1} \tilde{U} (I - e^{-(I-\tilde{U})(s_0 - \kappa)}) \right)_{(1,l)} \\ &= \left((I - (I - \tilde{U})^{-1})(S_0 I - K e^{\tilde{U}(s_0 - \kappa)}) \right)_{(1,l)},\end{aligned}$$

we find

$$\begin{aligned}\int_\kappa^{s_0} (e^x - e^\kappa) \frac{d}{dx} \boldsymbol{F}_{k,d}(s_0, x)\, dx = \sum_{l=1}^{k} &\left((S_0 - K) I \delta_{l,1} - S_0 \left((I - \tilde{U})^{-1} \right)_{(1,l)} \right) \boldsymbol{g}_{k-l+1,d} \\ &+ K \sum_{l=1}^{k} \left((I - \tilde{U})^{-1} e^{\tilde{U}(s_0 - \kappa)} \right)_{(1,l)} \boldsymbol{g}_{k-l+1,d}.\end{aligned}$$

The second integral in the right hand side of (6.4) can be treated analogously by using (5.5). The expression for $\boldsymbol{\Pi}_u(S_0, K, k)$ follows by a similar argument.

When $S_0 < K$, the process Y starts at level s_0 and must reach level κ before the absorption. Assuming that κ is reached in phase j and that l is the number of Erlangization intervals used to achieve the first passage to κ, one then further concentrates upon the expectation of the discounted payoff starting from the level κ and with $N - k + l$ remaining Erlangization intervals:

$$\mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)} \left(e^{Y(\tau_\star)} - K \right)_+ \,|\, Y(0) = \kappa, \zeta(0) = (j, N - k + l) \right],$$

which leads to the announced formulae. □

6.3 Down-and-out call options

Proposition 6.3. *The price $P_{DOC}(S_0, B, K, q)$ of a down-and-out call option as defined in (2.6) with barrier $B < S_0$ and maturity date q is given by*

$$P_{DOC}(S_0, B, K, q) = P_V(S_0, K, q) - \boldsymbol{\alpha} \sum_{l=1}^{N} E_{\sigma d}^{(l)}(s_0 - b) \boldsymbol{\Pi}_d(B, K, N - l + 1), \quad (6.5)$$

where $P_V(S_0, K, q)$ and the vectors $\boldsymbol{\Pi}_d(B, K, N - l + 1)$ are given in (6.3) and in Proposition 6.2, respectively.

Proof. Applying the same Erlangization methodology as above and denoting the first passage time of Y to level $b < Y(0) = s_0$ by τ_b, we have

$$\begin{aligned} P_{\mathrm{DOC}}(S_0, B, K, q) &= \sum_{i \in \mathcal{S}_\sigma} \alpha_i \, \mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)} \left(e^{Y(\tau_\star)} - K \right)_+ \mathbb{1}_{\tau_b > \tau_\star} \,|\, Y(0) = s_0, \zeta(0) = (i,1) \right] \\ &= \sum_{i \in \mathcal{S}_\sigma} \alpha_i \, \mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)} \left(e^{Y(\tau_\star)} - K \right)_+ \,|\, Y(0) = s_0, \zeta(0) = (i,1) \right] \\ &\quad - \sum_{i \in \mathcal{S}_\sigma} \alpha_i \, \mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)} \left(e^{Y(\tau_\star)} - K \right)_+ \mathbb{1}_{\tau_b < \tau_\star} \,|\, Y(0) = s_0, \zeta(0) = (i,1) \right]. \end{aligned}$$

In the right hand side of the last equality, the first expectation equals the price of a vanilla option given in (6.3). To compute the last expectation, we condition on the number of Erlangization intervals needed to achieve the first passage to level b in order to meet the condition in the indicator function. This leads to the following equality

$$\begin{aligned} &\mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)} \left(e^{Y(\tau_\star)} - K \right)_+ \mathbb{1}_{\tau_b < \tau_\star} \,|\, Y(0) = s_0, \zeta(0) = (i,1) \right] \\ &\quad = \sum_{j \in \mathcal{S}_d} \sum_{l=1}^{N} \left(E^{(l)}(s_0 - b) \right)_{ij} \mathbb{E}\left[e^{-\gamma(\boldsymbol{r};\tau_\star)} \left(e^{Y(\tau_\star)} - K \right)_+ \,|\, Y(0) = b, \zeta(0) = (j,l) \right], \end{aligned}$$

which yields the announced result. □

The accuracy of the price approximations above can be made as high as desired by increasing the value of N. We notice, however, that the dimension of the matrices in the pricing formulas increases when N increases, which implies a higher computational cost. In the first two examples in the next section, we compare our approximations with results from the literature and we illustrate how moderate values of N already yield a good accuracy.

7 Numerical illustrations

Example 1. To show the precision of the option prices obtained by our approximations, we start with a simple example without jumps, taken from the literature. Here, (X, φ) is characterized by the matrices

$$Q = \begin{bmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \qquad D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \tag{7.1}$$

	$N = 150$	$N = 200$	$N = 400$	$N = 500$	**MC**
$S(0) = 98$	8.0788	8.0800	8.0818	8.0822	**8.0840** $\pm$ 2.4e$-$03
$S(0) = 100$	9.3339	9.3352	9.3372	9.3376	**9.3387** $\pm$ 2.5e$-$03
$S(0) = 102$	10.6786	10.6799	10.6820	10.6824	**10.6836** $\pm$ 2.6e$-$03

Table 1: Approximations of the price of a vanilla call option in the MMBM framework with two phases, when $r_1 = r_2 = 0.05$, $T = 1$, $K = 100$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$ and $q_1 = q_2 = 0.5$. The column "MC" refers to the Monte Carlo results in [18, Table 2].

	$N = 10$	$N = 20$	$N = 30$	$N = 40$	$N = 50$	**MC**
$K=B=0.6$	0.4177	0.4177	0.4177	0.4177	0.4177	**0.4177** $\pm$ 5.6e$-$06
$K=B=0.8$	0.2217	0.2219	0.2219	0.2219	0.2219	**0.2220** $\pm$ 7.2e$-$05
$K=B=0.9$	0.1183	0.1185	0.1185	0.1185	0.1186	**0.1187** $\pm$ 1.7e$-$04

Table 2: Approximations of the price of a DOC option in the MMBM framework with two phases, when $S(0) = 1$, $r_1 = r_2 = 0.03$, $T = 1$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $q_1 = 0.8$ and $q_2 = 0.6$. The column "MC" refers to the Monte Carlo results in [19, Table 2].

and with initial vector $\boldsymbol{\alpha} = [1\ 0]$. The interest rate is equal for both regimes and denoted by r. In order to ensure that the model is defined under a risk neutral measure, the drift parameters are determined by $d_1 = r - \frac{1}{2}\sigma_1^2$ and $d_2 = r - \frac{1}{2}\sigma_2^2$ to satisfy the constraint (2.2).

In Table 1, we show the approximated price of a vanilla call option when $r = 0.05$, $T = 1$, $K = 100$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$ and $q_1 = q_2 = 0.5$, for different values of N and $S(0)$. This parameter set has been used in Boyle and Draviam [6] as well as in Hieber [18]. The last column contains Monte-Carlo estimates together with the intervals, which are taken from Table 2 in [18]. We notice that the approximations based on Erlangization fall inside the Monte-Carlo intervals for $N = 400$. If a precision of two digits after the decimal point is enough, then prices can be obtained by using approximately 150 Erlangization intervals. In Table 2, we show the approximated price of a DOC option when $S(0) = 1$, $r = 0.03$, $T = 1$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $q_1 = 0.8$, $q_2 = 0.6$ and $B = K$, for different values of K and $S(0)$. The prices in the column "MC" are taken from the last column in Table 2 of [19] and have been obtained through Monte Carlo simulations based upon both antithetic variables and control variables. In this case, we observe that the prices obtained with as little as N=20 Erlangization intervals are already very precise, in comparison with the benchmark of the Monte Carlo results.

	$N = 100$	$N = 150$	$N = 200$	$N = 300$	**MC**
Digital	0.3644	0.3648	0.3650	0.3651	0.3647 ± 0.0009
Vanilla	19.412	19.417	19.420	19.422	19.43 ± 0.04
DOC	18.219	18.226	18.229	18.232	18.26 ± 0.04

Table 3: Approximated prices of digital, vanilla and DOC options in the MMBM framework with two phases and exponential jumps, when $q_1 = 1$, $q_2 = 4$, $\sigma_1 = 0.15$, $\sigma_2 = 0.3$, $\lambda = 10$, $\mu = 5$, $T = 1$, $K = 90$, $B = 80$, $S_0 = 100$ and $r_1 = r_2 = 0.03$. The column "MC" refers to the Monte Carlo results obtained with 10^6 simulations.

Example 2. As in Example 1, the value of the asset evolves like the exponential of an MMBM with two phases but, in addition, jumps occur whenever the phase changes. When considering several assets at the same time, this kind of jumps are called synchronous jumps and they are interesting in economic modelling as well as in calibration issues, see e.g. Hainaut and Colwell [16] and Chourdakis [10].

We interpret the first phase as a normal regime of the market, while the second phase represents a bad regime of the market with a higher variance. A transition from phase 2 to phase 1 is accompanied by an upward jump of X, which is exponentially distributed with parameter λ. A transition from phase 1 to phase 2 is accompanied by a downward jump of X, which is exponentially distributed with parameter μ. The parameters of this example are

$$Q = \begin{bmatrix} -q_1 & 0 \\ 0 & -q_2 \end{bmatrix}, \qquad W_+ = \begin{bmatrix} 0 \\ q_2 \end{bmatrix}, \qquad W_- = \begin{bmatrix} q_1 \\ 0 \end{bmatrix},$$

together with the matrices Σ and D in (7.1), the initial vector $\boldsymbol{\alpha} = [1\ 0]$ and $R_+ = -\lambda$, $V_+ = [\,\lambda\ \ 0\,]$, $R_- = -\mu$, $V_- = [\,0\ \ \mu\,]$. The parameters d_1 and d_2 are fixed so as to satisfy the risk neutral condition (2.2).

For the numerical experiments, we set $q_1 = 1$, $q_2 = 4$, $\sigma_1 = 0.15$, $\sigma_2 = 0.3$, $\lambda = 10$, $\mu = 5$, $T = 1$, $K = 90$, $B = 80$, $S_0 = 100$ and $r_1 = r_2 = 0.03$. In Table 3, we show the prices obtained for increasing values of N, and in the last column, the prices we obtained by performing Monte-Carlo estimates with 10^6 simulations (and discretization interval of length 0.001). Again, we observe good agreement between our approximation and Monte-Carlo simulations, even for moderate values of N.

Example 3. In this last example, we use phase-type distributions which mimic fat tail behaviours. The numerical derivation of the approximate option prices follows from the straightforward implementation of the closed form approximations obtained in the previous section.

The model is the same as in Example 2, except that the downward jumps happening together with a transition from phase 1 to phase 2 are now defined by a more general phase-type distributed random variable $J \sim PH(\boldsymbol{e}_1, A)$, to take into account the fact that downward jumps are usually larger than upward jumps, especially in periods of crisis. We use a subgenerator A which is inspired by Robert and Le Boudec [27] and has the following form:

$$A = \begin{bmatrix} -(c+s_a) & (1/a) & (1/a)^2 & (1/a)^3 & \cdots & (1/a)^{n-1} \\ b/a & -b/a & 0 & 0 & \cdots & 0 \\ (b/a)^2 & 0 & -(b/a)^2 & 0 & \cdots & 0 \\ (b/a)^3 & 0 & 0 & -(b/a)^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ (b/a)^{n-1} & 0 & 0 & 0 & \cdots & -(b/a)^{n-1} \end{bmatrix}, \tag{7.2}$$

with $n \in \mathbb{N}_0$, $a > 1$, $a > b$, $b, c > 0$ and $s_a = \frac{1}{a} + \frac{1}{a^2} + \cdots + \frac{1}{a^{n-1}}$. The other parameters are the same as in Example 2, except for W_-, R_- and V_- which are given by

$$W_- = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad R_- = A, \quad V_- = [\,\mathbf{0} \;\; -A\mathbf{1}\,].$$

The values of d_1 and d_2 are chosen in order to satisfy the risk neutral condition (2.2).

The jump size J is equal to the time before absorption in the process represented by (7.2) and has the following interpretation: the process starts in phase one and has two possibilities for the next step. With probability $c/(c+s_a)$ it goes to the absorbing state. With the complementary probability $s_a/(c+s_a)$ it begins an excursion, that is, a visit in one of the states $2, 3, ..., n$ before returning to state 1. The duration of an excursion is exponentially distributed with parameter $(b/a)^k$ if the state $k+1$ is chosen ($k = 1, 2, ..., n-1$), which happens with a probability $\pi_k = (1/s_a)(1/a)^k$. The higher the state k where an excursion starts, the smaller the probability π_k, but the longer the duration of this excursion. It is clear from this interpretation that the size J of the downward jumps is equal in distribution to

$$J = \sum_{i=1}^{M-1} \left(\boldsymbol{\tau}^{(i)} + \boldsymbol{\tau}_1^{(i+1)} \right) + \boldsymbol{\tau}_1^{(1)}$$

where $M \sim \text{geom}(c/(c+s_a))$ is the number of visits to state 1 before absorption, $\boldsymbol{\tau}_1^{(i)}$ is the duration of the i-th visit to state 1 and $\boldsymbol{\tau}^{(i)}$ is the duration of the i-th excursion.

The mean and variance of J are easily shown to be determined by

$$\mathbb{E}[J] = \frac{1}{c}\sum_{l=0}^{n-1}\left(\frac{1}{b}\right)^l,$$
$$\mathbb{V}ar[J] = \mathbb{E}[J]^2 + \frac{2}{c}\sum_{l=1}^{n-1}\left(\frac{a}{b^2}\right)^l.$$

It is interesting to note that the expectation of J does not depend on a and that $\mathbb{V}ar[J]$ is an unbounded, increasing function in a. We may therefore fix the mean of J for certain n, b and c and then choose a variance as large as desired by taking a high enough.

For the numerical experiments, we set $q_1 = 1$, $q_2 = 4$, $\sigma_1 = 0.15$, $\sigma_2 = 0.3$, $\lambda = 10$, $T = 1$, $K = 90$, $B = 80$, $S_0 = 100$ and $r_1 = r_2 = 0.03$. The results presented in Tables 4 and 5 are obtained by using $N = 100$ Erlangization intervals since several tests showed that augmenting N did no further change the approximations in a significant way. Next, we have chosen several sets of values for n, b and c, in such a way that for all cases $\mathbb{E}[J] = 0.2$ but that the values for the variance range from 0.2 to 1.3e+06. We take $n = 8$, $b = 3$ and $c = 7.5$ in Table 4, and $n = 7$, $b = 0.75$ and $c = 97.39$ in Table 5. Finally, as $\mathbb{E}[J]$ does not vary with a, we compute the different option prices for various values of a, letting a increase so that the variance of J increases.

In these numerical experiments, we observe that as the variance increases, the prices of digital options have the tendency to remain relatively stable, while the prices for vanilla and down-and-out call options tend to decrease.

As mentioned before, this example was particularly chosen to show that phase-type distributions can have important tails. The graphs in Figure 1 show the tails for the distribution of J for parameters taken from Table 4, with $a = 80$, and from Table 5, with $a = 11$, respectively.

Some Pareto-distributions are plotted as well, for comparison. The graphs show that for the purpose of asset price modelling, this kind of phase-type distributions clearly may have enough mass in their tails. These observations seem further promising and useful in other domains where more explicit heavy tails are necessary, like for e.g. catastrophic option pricing or insurance linked issues.

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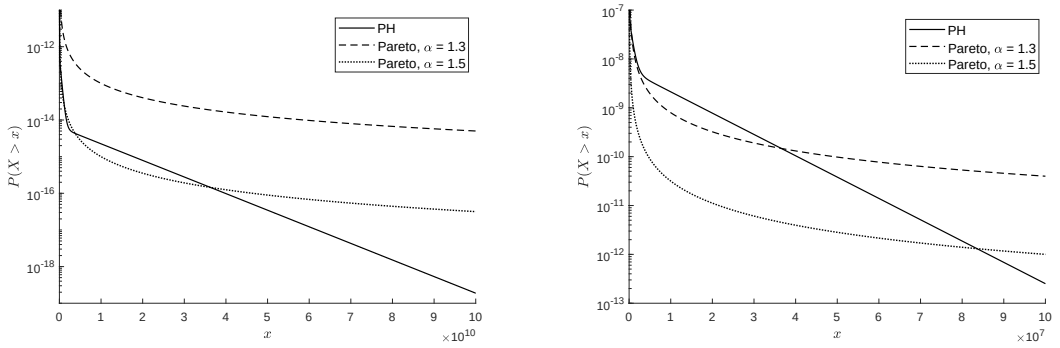

Figure 1: Tail of the PH distributions compared with the tail of two Pareto distributions. Left: $n = 8$, $b = 3$, $c = 7.5$ and $a = 80$. Right: $n = 7$, $b = 0.75$, $c = 97.39$ and $a = 11$.

	$a = 6$	$a = 10$	$a = 17$	$a = 25$	$a = 40$	$a = 70$	$a = 80$
$\mathbb{V}ar[J]$	0.54	3	48	531	11786	526900	1.3e+06
Digital	0.3293	0.3301	0.3309	0.3313	0.3318	0.3321	0.3322
Vanilla	18.66	18.35	18.14	18.04	17.95	17.89	17.88
DOC	17.55	17.24	17.03	16.93	16.84	16.78	16.77

Table 4: Approximated prices of digital, vanilla and DOC options in the case where $n = 8$, $b = 3$ and $c = 7.5$. The mean $\mathbb{E}[J] = 0.2$ does not vary with a.

	$a = 0.8$	$a = 1.15$	$a = 2$	$a = 3$	$a = 5$	$a = 10$	$a = 11$
$\mathbb{V}ar[J]$	0.54	2.94	57.7	582	11414	687000	1.2e+06
Digital	0.2516	0.2436	0.2481	0.2493	0.2500	0.2505	0.2505
Vanilla	19.50	17.34	16.28	16.08	15.97	15.91	15.90
DOC	18.85	16.67	15.58	15.38	15.27	15.20	15.20

Table 5: Approximated prices of digital, vanilla and DOC options in the case where $n = 7$, $b = 0.75$ and $c = 97.39$. The mean $\mathbb{E}[J] = 0.2$ does not vary with a.

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