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Block Hybrid Method for the Numerical solution of Fourth order Boundary Value Problems

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ABSTRACT

A linear Multistep Block Hybrid Method with four off-grid points is presented for the direct approximation of the solution of fourth order Boundary Value Problems. Multiple Finite Difference formulas are derived and grouped into a unique block to form a numerical integrator to solve directly the fourth order problem, without the need to reduce it previously to a first-order system. The convergence of the proposed method is discussed. The superiority of this method over existing methods is established numerically considering different problems that have appeared in the literature.

1. Introduction

In this work, the numerical approximation to the solution of the 4th-order problem of the type

$$y^{(iv)} = f(x, y, y', y'', y'''), \quad a \le x \le b, \tag{1}$$

subject to the following boundary conditions

$$y(a) = \alpha_0, \qquad y(b) = \beta_0, \qquad y'(a) = \alpha_1, \qquad y'(b) = \beta_1, \tag{2}$$

where $a, b, \alpha_i, \beta_i, i = 0, 1$, are given real numbers, is considered. We assume that the function f is continuous in $[a, b] \times \mathbf{R}^4$ and verifies Lipschitz conditions on the variable $\mathbf{y} = (y, y', y'', y''')$, that is, it holds that for any $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{R}^4$ there exist constants $L_j > 0, j = 0, \ldots, 3$, such that

$$|f(x, \mathbf{y}_1) - f(x, \mathbf{y}_2)| \le \sum_{j=0}^{3} L_j |y_1^{(j)} - y_2^{(j)}|.$$

This assumption guarantees the existence and uniqueness of a solution for problem (1)–(2) in a certain subset of $[a, b] \times \mathbf{R}^4$ (see [1]). We also assume that f and its derivatives up to the third order with respect to the independent variable are differentiable, in order to address the convergence analysis of the method.

It is noteworthy to state here that the block hybrid method developed in this paper can be extended to solve (1) with any of the following boundary conditions

$$y^{(i)}(a) = \alpha_1, \qquad y^{(i)}(b) = \beta_1, \qquad y^{(j)}(a) = \alpha_2, \qquad y^{(j)}(b) = \beta_2, \qquad \text{with } 0 \le i < j \le 1, \ldots, 4,$$

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as can be seen in the numerical examples.

Problems arising from engineering and other sciences, just to mention but few have been modelled into higher order linear and nonlinear initial and boundary value problems. Fourth order problem finds its application in the static deflection of a uniform beam. The deflection curve of each set of boundary conditions is such that the beam with both ends embedded gives rise to Neumann and Dirichlet (first order) boundary conditions. A simply supported beam gives rise to Neumann and Dirichlet (second order) boundary conditions, and cantilever beam (left end embedded, right end free), gives rise to different conditions (see [2,3]). The theorem and proof for the general conditions concerning the existence and uniqueness of the solution of (1)–(2) have been given by Keller [4].

Numerical approaches for the solution of fourth order boundary value problems are numerous in the literature. Some of those methods intended for obtaining the approximate solution of fourth order BVPs include, but are not limited to, Variational Iteration Method (VIM) by Noor and Mohyud-Din [5,6], Quintic Spline method by Siddiqi and Akram [7], Spline-based methods by Kasi et al. [8], the Least Value Method by Huanmin and Minggen [9]. Other approaches are based on collocation methods, Variation of Parameter methods, Adomian Decomposition methods, Differential Transform Methods, just to mention a few (see [10,11] among others).

In this work, we consider linear multistep hybrid formulas with four intra-step grid points. These formulas are constructed using a collocation approach, and later are put together to form a Block Hybrid Method (see [12–15]). The derivation of the method is presented in the following section. In Section 3 the convergence analysis is presented. The numerical examples presented in Section 4 confirm the good performance of the new method over other approaches in literature.

2. Derivation of the method

This section describes the derivation of a continuous implicit four intra-step hybrid block method for approximating the solution of the BVP in (1)–(2). Consider the grid points on the interval of integration $[a,b]$, $\pi_N \equiv \{a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b\}$, with h the constant step size, $h = x_j - x_{j-1}$, $j = 1, 2, \ldots, N$. The method relies on the approximation of the exact solution $y(x)$ at the grid points of four adjacent subintervals and the corresponding intermediate points, by the polynomial $p(x)$ given by

$$y(x) \simeq p(x) = \sum_{i=0}^{12} \rho_i x^i \,, \tag{3}$$

which yields the successive derivative approximations, and in particular

$$y^{(iv)}(x) \simeq p^{(iv)}(x) = \sum_{i=4}^{12} \rho_i i(i-1)(i-2)(i-3)x^{i-4} \,. \tag{4}$$

Here, the ρ_i are real unknown coefficients to be determined. In order to simplify the derivation of the method we will consider a generic block interval, $[x_n, x_{n+4}]$, since the formulas thus obtained may be easily shifted to the all the successive blocks for $n = 0, 4, \ldots, N-4$. Thus, we consider the points $x_{n+\frac{j}{2}} = x_n + \frac{j}{2}h$, $j = 0, 1, \ldots, 8$, in the four-step scheme for approximating the solution on $[x_n,\ x_{n+4}]$. After considering the approximation in (3) applied to the points $x_n, x_{n+1}, x_{n+2}, x_{n+3}$, and the fourth derivative in (4) applied to the points $x_{n+\frac{j}{2}}$, $j = 0, 1, \ldots, 8$, we obtain a system of 13 equations with 13 unknowns (the ρ_i, $i = 0, 1, \ldots, 12$). This system may be written in matrix form as

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \ldots & x_n^{11} & x_n^{12} \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & \ldots & x_{n+1}^{11} & x_{n+1}^{12} \\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & \ldots & x_{n+2}^{11} & x_{n+2}^{12} \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & \ldots & x_{n+3}^{11} & x_{n+3}^{12} \\
0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & \ldots & 7920x_n^7 & 11880x_n^8 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{1}{2}} & 360x_{n+\frac{1}{2}}^2 & \ldots & 7920x_{n+\frac{1}{2}}^7 & 11880x_{n+\frac{1}{2}}^8 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x_{n+1}^2 & \ldots & 7920x_{n+1}^7 & 11880x_{n+1}^8 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{3}{2}} & 360x_{n+\frac{3}{2}}^2 & \ldots & 7920x_{n+\frac{3}{2}}^7 & 11880x_{n+\frac{3}{2}}^8 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & \ldots & 7920x_{n+2}^7 & 11880x_{n+2}^8 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{5}{2}} & 360x_{n+\frac{5}{2}}^2 & \ldots & 7920x_{n+\frac{5}{2}}^7 & 11880x_{n+\frac{5}{2}}^8 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+3} & 360x_{n+3}^2 & \ldots & 7920x_{n+3}^7 & 11880x_{n+3}^8 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{7}{2}} & 360x_{n+\frac{7}{2}}^2 & \ldots & 7920x_{n+\frac{7}{2}}^7 & 11880x_{n+\frac{7}{2}}^8 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+4} & 360x_{n+4}^2 & \ldots & 7920x_{n+4}^7 & 11880x_{n+4}^8
\end{pmatrix}
\begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \\ \rho_7 \\ \rho_8 \\ \rho_9 \\ \rho_{10} \\ \rho_{11} \\ \rho_{12} \end{pmatrix}
=
\begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \end{pmatrix}$$

where the approximate values are given by $y^{(j)}_{n+i} \simeq y^{(j)}(x_{n+i})$ and $f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i}, y'''_{n+i})$.

Solving the above system using a CAS like Mathematica we can obtain the values of the coefficients ρ_i, $i = 0, 1, \ldots, 12$, which are not worth to include here as they are cumbersome expressions. After simplification, the approximate values obtained through the polynomial in (3) adopt the form

$$y_{n+\frac{j}{2}} = \sum_{i=0}^{3} \alpha^0_{ij} y_{n+i} + h^4 \sum_{i=0}^{8} \beta^0_{ij} f_{n+i/2}, \quad j = 1, 3, 5, 7, 8. \tag{5}$$

where the α^0_{ij} and β^0_{ij} are the coefficients of the main formulas, which are given in Table 7 in the Appendix.

The additional methods are obtained by evaluating the successive derivatives of $p(x)$ to give

$$h^l y^{(l)}_{n+\frac{j}{2}} = \sum_{i=0}^{3} \alpha^l_{ij} y_{n+i} + h^4 \sum_{i=0}^{8} \beta^l_{ij} f_{n+i/2}, \quad l = 1, 2, 3, \ \ j = 0(1)8 \tag{6}$$

where the α^l_{ij} and β^l_{ij} are the coefficients appearing in Tables 8–10 in Appendix. The notations $y^{(l)}_{n+\frac{j}{2}}$ stand for the approximation of the lth order derivative of $y(x)$ at $x_{n+\frac{j}{2}}$.

In order to get the main formulas in (5), we evaluate $p(x)$ at the points $x = x_{n+\frac{i}{2}}$, $i = 1, 3, 5, 7, 8$, and after some simplifications, we obtain the formulas whose coefficients appear in Table 7 in Appendix.

Then, evaluating $p'(x)$ at the points $x = x_{n+\frac{i}{2}}$, $i = 0, 1, \ldots, 8$, we obtain the formulas for approximating the first derivatives whose coefficients are given in Table 8 in Appendix.

Similarly, evaluating $p''(x)$ at the points $x = x_{n+\frac{i}{2}}$, $i = 0, 1, \ldots, 8$, we obtain the formulas for approximating the second derivatives whose coefficients are in Table 9 in Appendix.

Finally, evaluating $p'''(x)$ at the points $x = x_{n+\frac{i}{2}}$, $i = 0, 1, \ldots, 8$, we obtain the formulas for approximating the third derivatives whose coefficients are given in Table 10 in Appendix.

All the formulas in (5)–(6) considered together form the block method, which will be named BHM for short. If we had to solve an IVP with this method, it should be applied sequentially on block of intervals of the form $[x_n, x_{n+4}]$, $n = 0, 4, N-4$, where N, the number of subintervals, must be a multiple of 4 in order to reach the final point $x_N = b$. But we want to use them to solve a BVP. In this case, we consider all the formulas in (5)–(6) for $n = 0(4)N-4$ at the same time. This results in a system of $8N$ equations, which altogether with the four boundary conditions leads to a system of $8N+4$ equations in the $8N+4$ unknowns $\left\{y_j, y'_j, y''_j, y'''_j\right\}$ for $j = 0(1/2)N$.

3. Analysis of the method

3.1. Local truncation error and order

Given a sufficiently differentiable function $z(x)$, the linear difference operators associated with the formulas in (5)–(6) are given as follows

$$\mathcal{L}^l_{j/2}[z(x);\ h] \equiv h^l z^{(l)}(x + \frac{j}{2}h) - \left[\sum_{i=0}^{3} \alpha^l_{ij} z(x + ih) + h^4 \sum_{i=0}^{8} \beta^l_{ij} z^{(iv)}(x + \frac{i}{2}h)\right] \tag{7}$$

for $l = 0$ and $j = 1, 3, 5, 7, 8$, or for $l = 1, 2, 3$ and $j = 0(1)8$.

The local truncation error of each of the formulas in (5)–(6) is the amount by which the exact solution of the ODE fails to satisfy the corresponding difference operator. Thus, after expanding (7) in Taylor series around x we get that each of the local truncation errors is of the form

$$\mathcal{L}^l_{j/2}[z(x);\ h] = C_0 z(x) + C_1 h z'(x) + C_2 h^2 z''(x) + \cdots + C_q h^q z^{(q)}(x) + O(h^{(q+1)}) \tag{8}$$

where the C_i's are constants. If we have that

$$C_0 = C_1 = C_2 = \cdots = C_{p+\mu-1} = 0, \qquad \text{and} \qquad C_{p+\mu} \neq 0$$

where μ is the order of the differential equation, then it is said that the formula is consistent of order p (see [16]), and $C_{p+\mu}$ is called the principal error constant. In this case it is $\mu = 4$ and the local truncation errors of the main formulas are given respectively by

$$\begin{aligned}
\mathcal{L}^0_{\frac{1}{2}}[y(x_n);\ h] &= \frac{41171875}{3313090972090368}\, y^{(12)}(x_n)h^{12} + O(h^{13})\\
\mathcal{L}^0_{\frac{3}{2}}[y(x_n);\ h] &= \frac{36671875}{1104363657363456}\, y^{(12)}(x_n)h^{12} + O(h^{13})\\
\mathcal{L}^0_{\frac{5}{2}}[y(x_n);\ h] &= \frac{-621296875}{3313090972090368}\, y^{(12)}(x_n)h^{12} + O(h^{13})\\
\mathcal{L}^0_{\frac{7}{2}}[y(x_n);\ h] &= \frac{225390625}{67614101471232}\, y^{(12)}(x_n)h^{12} + O(h^{13})\\
\mathcal{L}^0_{4}[y(x_n);\ h] &= \frac{2640625}{202215025152}\, y^{(12)}(x_n)h^{12} + O(h^{13})\,.
\end{aligned} \tag{9}$$

For the formulas in (6) the local truncation errors are similarly obtained. Consequently, from the above results, the order of the block method is $p = 8$.

3.2. Convergence analysis

We first assume that the following boundary conditions in (2)

$$y_0 = \alpha_0, \qquad y_N = \beta_0, \qquad y'_0 = \alpha_1, \qquad y'_N = \beta_1$$

are given. Thus, the vector of unknowns $\bar{Y}$ is given by

$$\begin{aligned}
\bar{Y} = \big(&y_{1/2}, y_1, y_{3/2}, \dots, y_{N-1/2}, y'_{1/2}, y'_1, y'_{3/2}, \dots, y'_{N-1/2},\\
&y''_0, y''_{1/2}, y''_1, y''_{3/2}, \dots, y''_N, y'''_0, y'''_{1/2}, y'''_1, y'''_{3/2}, \dots, y'''_N\big)^T\,.
\end{aligned}$$

This makes a total of $(2N-1)+(2N-1)+(2N+1)+(2N+1) = 8N$ unknowns.

On the other hand, we have five formulas in Table 7 which for $n = 0(4)N-4$ make a total of $5N/4$ formulas. Similarly, we have in Table 8 nine formulas. If we take $n = 0(4)N-4$ we obtain $9N/4$ formulas more. With the formulas in Tables 9 and 10 we proceed similarly, that is, we get $9N/4 + 9N/4$ formulas more. In this way, the total number of equations will be $5N/4 + 9N/4 + 9N/4 + 9N/4 = 8N$.

We have then, a system with $8N$ equations and $8N$ unknowns, whose solution provides a set of approximate values of the BVP. This system, obtained from the formulas corresponding to the coefficients in Tables 7–10, can be written compactly as

$$P\bar{Y} + h^4 Q\,\bar{F} + R = 0\,, \tag{10}$$

where

$$\begin{aligned}
\bar{F} = \big(&f_0, f_{1/2}, f_1, f_{3/2}, \dots, f_N, f'_0, f'_{1/2}, f'_1, f'_{3/2}, \dots, f'_N,\\
&f''_0, f''_{1/2}, f''_1, f''_{3/2}, \dots, f''_N, f'''_0, f'''_{1/2}, f'''_1, f'''_{3/2}, \dots, f'''_N\big)^T\,,
\end{aligned}$$

P, Q are constant matrices formed by the coefficients of the formulas, and R is a (8N)-vector containing the known values in the formulas, that is,

$$\begin{aligned}
R = \big(&\alpha^0_{01}y_0, \alpha^0_{03}y_0, \alpha^0_{05}y_0, \alpha^0_{07}y_0, \alpha^0_{08}y_0, 0, \dots, 0, -y_N,\\
&\alpha^1_{00}y_0 - hy'_0, \alpha^1_{01}y_0, \alpha^1_{02}y_0, \dots, \alpha^1_{08}y_0, 0, \dots, 0, -hy'_N,\\
&\alpha^2_{00}y_0, \alpha^2_{01}y_0, \alpha^2_{02}y_0, \dots, \alpha^2_{08}y_0, 0, \dots, 0,\\
&\alpha^3_{00}y_0, \alpha^3_{01}y_0, \alpha^3_{02}y_0, \dots, \alpha^3_{08}y_0, 0, \dots, 0\big)^T\,.
\end{aligned}$$

Specifically, P is a $(8N \times 8N)$- matrix of coefficients formed by submatrices, as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14}\\ P_{21} & P_{22} & P_{23} & P_{24}\\ P_{31} & P_{32} & P_{33} & P_{34}\\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}.$$

The submatrix P_{11} corresponds to the first $5N/4$ formulas, obtained from (5) for $n = 0(4)N-4$ (after passing all the terms to the right hand side). The first row of P_{11} is the coefficients of the first formula corresponding to the terms $y_{1/2}, y_1, y_{3/2}, \dots, y_{N-1/2}$. The second row of P_{11} is the coefficients of the second formula corresponding to the same terms

$y_{1/2}, y_1, y_{3/2}, \dots, y_{N-1/2}$, and so on. Thus, P_{11} is a matrix of size $5N/4 \times (2N-1)$ of the form

$$P_{11} = \begin{bmatrix} \dot{p}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \dot{p}_{22} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \dot{p}_{\frac{N}{4}\frac{N}{4}} \end{bmatrix}$$

where each of the $N/4$ submatrices $\dot{p}_{ii}$ is given by

$$\dot{p}_{ii} = \begin{bmatrix} \alpha^0_{01} & -1 & \alpha^0_{11} & 0 & \alpha^0_{21} & 0 & \alpha^0_{31} & 0 & 0 \\ \alpha^0_{03} & 0 & \alpha^0_{13} & -1 & \alpha^0_{23} & 0 & \alpha^0_{33} & 0 & 0 \\ \alpha^0_{05} & 0 & \alpha^0_{15} & 0 & \alpha^0_{25} & -1 & \alpha^0_{35} & 0 & 0 \\ \alpha^0_{07} & 0 & \alpha^0_{17} & 0 & \alpha^0_{27} & 0 & \alpha^0_{37} & -1 & 0 \\ \alpha^0_{08} & 0 & \alpha^0_{18} & 0 & \alpha^0_{28} & 0 & \alpha^0_{38} & 0 & -1 \end{bmatrix},$$

except for $\dot{p}_{11}$ where the first column is missing, and $\dot{p}_{\frac{N}{4}\frac{N}{4}}$ where the last column is missing. Furthermore, in the triangular arrangement of the $\dot{p}_{ii}$ in P_{11}, the first column of each $\dot{p}_{ii}$ is placed below the last column of each $\dot{p}_{i-1\,i-1}$ for $i = 2, 3, \dots, N/4$.

The submatrix P_{12} corresponds again to the first $5N/4$ formulas, obtained from Table 7 for $n = 0(4)N-4$ (after passing all the terms to the right hand side). The first row of P_{12} is formed by the coefficients of the first formula corresponding to the terms $y'_{1/2}, y'_1, y'_{3/2}, \dots, y'_{N-1/2}$. The second row of P_{21} is formed by the coefficients of the second formula corresponding to the terms $y'_{1/2}, y'_1, y'_{3/2}, \dots, y'_{N-1/2}$, and so on. Thus, P_{12} is a null matrix of size $5N/4 \times (2N-1)$.

The submatrix P_{13} contains the coefficients of the first $5N/4$ formulas corresponding to the terms $y''_0, y''_{1/2}, y''_1, y''_{3/2}, \dots, y''_N$. Thus, P_{13} is also a null matrix of size $5N/4 \times (2N+1)$.

The submatrix P_{14} contains the coefficients of the first $5N/4$ formulas corresponding to the terms $y'''_0, y'''_{1/2}, y'''_1, y'''_{3/2}, \dots, y'''_N$. Thus, P_{14} is also a null matrix of size $5N/4 \times (2N+1)$.

The submatrices $P_{2j}, j = 1, 2, 3, 4$, contain the coefficients of the next $9N/4$ formulas obtained from Table 8 for $n = 0(4)N-4$ and have respectively sizes of $9N/4 \times (2N-1)$, $9N/4 \times (2N-1)$, $9N/4 \times (2N+1)$ and $9N/4 \times (2N+1)$. The submatrix P_{21} contains the coefficients of these formulas corresponding to the terms $y_{1/2}, y_1, y_{3/2}, \dots, y_{N-1/2}$, and is written

$$P_{21} = \begin{bmatrix} \tilde{p}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{p}_{22} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{p}_{\frac{N}{4}\frac{N}{4}} \end{bmatrix}$$

where each of the $N/4$ submatrices $\tilde{p}_{ii}$ is given by

$$\tilde{p}_{ii} = \begin{bmatrix} \alpha^1_{00} & 0 & \alpha^1_{10} & 0 & \alpha^1_{20} & 0 & \alpha^1_{30} & 0 & 0 \\ \alpha^1_{01} & 0 & \alpha^1_{11} & 0 & \alpha^1_{21} & 0 & \alpha^1_{31} & 0 & 0 \\ \alpha^1_{02} & 0 & \alpha^1_{12} & 0 & \alpha^1_{22} & 0 & \alpha^1_{32} & 0 & 0 \\ \alpha^1_{03} & 0 & \alpha^1_{13} & 0 & \alpha^1_{23} & 0 & \alpha^1_{33} & 0 & 0 \\ \alpha^1_{04} & 0 & \alpha^1_{14} & 0 & \alpha^1_{24} & 0 & \alpha^1_{34} & 0 & 0 \\ \alpha^1_{05} & 0 & \alpha^1_{15} & 0 & \alpha^1_{25} & 0 & \alpha^1_{35} & 0 & 0 \\ \alpha^1_{06} & 0 & \alpha^1_{16} & 0 & \alpha^1_{26} & 0 & \alpha^1_{36} & 0 & 0 \\ \alpha^1_{07} & 0 & \alpha^1_{17} & 0 & \alpha^1_{27} & 0 & \alpha^1_{37} & 0 & 0 \\ \alpha^1_{08} & 0 & \alpha^1_{18} & 0 & \alpha^1_{28} & 0 & \alpha^1_{38} & 0 & 0 \end{bmatrix},$$

except for $\tilde{p}_{11}$ where the first column is missing, and $\tilde{p}_{\frac{N}{4}\frac{N}{4}}$ where the last column is missing. Furthermore, in the triangular arrangement of the submatrices $\tilde{p}_{ii}$ in P_{21}, the first column of each $\tilde{p}_{ii}$ is placed below the last column of each $\tilde{p}_{i-1\,i-1}$ for $i = 2, 3, \dots, N/4$.

The submatrix P_{22} contains the coefficients of these formulas corresponding to the terms $y'_{1/2}, y'_1, y'_{3/2}, \dots, y'_{N-1/2}$, and is written as

$$P_{22} = h \begin{bmatrix} \hat{p}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \hat{p}_{22} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \hat{p}_{\frac{N}{4}\frac{N}{4}} \end{bmatrix}$$

where each of the $N/4$ submatrices $\hat{p}_{ii}$ is given by

$$\hat{p}_{ii} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

except for $\hat{p}_{11}$ where the first column is missing, and $\hat{p}_{\frac{N}{4}\frac{N}{4}}$ where the last column is missing. Furthermore, in the triangular arrangement of the submatrices $\hat{p}_{ii}$ in P_{22}, the first column of each $\hat{p}_{ii}$ is placed below the last column of each $\hat{p}_{i-1\,i-1}$ for $i = 2, 3, \ldots, N/4$.

Finally, P_{23} and P_{24} are null matrices.

For the submatrices $P_{3j}, P_{4j}, j = 1, 2, 3, 4$, the situation is similar as the previous one. The submatrices P_{31} and P_{41} have a similar structure as that of matrix P_{21}, given as

$$P_{31} = \begin{bmatrix} \bar{p}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \bar{p}_{22} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \bar{p}_{\frac{N}{4}\frac{N}{4}} \end{bmatrix}$$

where each of the $N/4$ submatrices $\bar{p}_{ii}$ is given by

$$\bar{p}_{ii} = \begin{bmatrix} \alpha_{00}^2 & 0 & \alpha_{10}^2 & 0 & \alpha_{20}^2 & 0 & \alpha_{30}^2 & 0 & 0 \\ \alpha_{01}^2 & 0 & \alpha_{11}^2 & 0 & \alpha_{21}^2 & 0 & \alpha_{31}^2 & 0 & 0 \\ \alpha_{02}^2 & 0 & \alpha_{12}^2 & 0 & \alpha_{22}^2 & 0 & \alpha_{32}^2 & 0 & 0 \\ \alpha_{03}^2 & 0 & \alpha_{13}^2 & 0 & \alpha_{23}^2 & 0 & \alpha_{33}^2 & 0 & 0 \\ \alpha_{04}^2 & 0 & \alpha_{14}^2 & 0 & \alpha_{24}^2 & 0 & \alpha_{34}^2 & 0 & 0 \\ \alpha_{05}^2 & 0 & \alpha_{15}^2 & 0 & \alpha_{25}^2 & 0 & \alpha_{35}^2 & 0 & 0 \\ \alpha_{06}^2 & 0 & \alpha_{16}^2 & 0 & \alpha_{26}^2 & 0 & \alpha_{36}^2 & 0 & 0 \\ \alpha_{07}^2 & 0 & \alpha_{17}^2 & 0 & \alpha_{27}^2 & 0 & \alpha_{37}^2 & 0 & 0 \\ \alpha_{08}^2 & 0 & \alpha_{18}^2 & 0 & \alpha_{28}^2 & 0 & \alpha_{38}^2 & 0 & 0 \end{bmatrix},$$

and similarly,

$$P_{41} = \begin{bmatrix} \ddot{p}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddot{p}_{22} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \ddot{p}_{\frac{N}{4}\frac{N}{4}} \end{bmatrix},$$

where each of the $N/4$ submatrices $\ddot{p}_{ii}$ is given by

$$\ddot{p}_{ii} = \begin{bmatrix} \alpha_{00}^3 & 0 & \alpha_{10}^3 & 0 & \alpha_{20}^3 & 0 & \alpha_{30}^3 & 0 & 0 \\ \alpha_{01}^3 & 0 & \alpha_{11}^3 & 0 & \alpha_{21}^3 & 0 & \alpha_{31}^3 & 0 & 0 \\ \alpha_{02}^3 & 0 & \alpha_{12}^3 & 0 & \alpha_{22}^3 & 0 & \alpha_{32}^3 & 0 & 0 \\ \alpha_{03}^3 & 0 & \alpha_{13}^3 & 0 & \alpha_{23}^3 & 0 & \alpha_{33}^3 & 0 & 0 \\ \alpha_{04}^3 & 0 & \alpha_{14}^3 & 0 & \alpha_{24}^3 & 0 & \alpha_{34}^3 & 0 & 0 \\ \alpha_{05}^3 & 0 & \alpha_{15}^3 & 0 & \alpha_{25}^3 & 0 & \alpha_{35}^3 & 0 & 0 \\ \alpha_{06}^3 & 0 & \alpha_{16}^3 & 0 & \alpha_{26}^3 & 0 & \alpha_{36}^3 & 0 & 0 \\ \alpha_{07}^3 & 0 & \alpha_{17}^3 & 0 & \alpha_{27}^3 & 0 & \alpha_{37}^3 & 0 & 0 \\ \alpha_{08}^3 & 0 & \alpha_{18}^3 & 0 & \alpha_{28}^3 & 0 & \alpha_{38}^3 & 0 & 0 \end{bmatrix}$$

where there is valid a similar comment as for P_{21} concerning the placement of the submatrices $\bar{p}_{ii}$ and $\ddot{p}_{ii}$ in the triangular arrays.

The matrices P_{33} and P_{44} are similar to P_{22} after changing h by h^2 and h^3 respectively. Finally, $P_{32} = P_{34} = P_{42} = P_{43} = \mathbf{0}$.

Thus, the structure of matrix P is

$$P = \begin{bmatrix} P_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ P_{21} & P_{22} & \mathbf{0} & \mathbf{0} \\ P_{31} & \mathbf{0} & P_{33} & \mathbf{0} \\ P_{41} & \mathbf{0} & \mathbf{0} & P_{44} \end{bmatrix}.$$

On the other hand, matrix Q has size $8N \times (8N+4)$ and the entries correspond to the coefficients of the $8N$ equations in the terms of $\bar{F}$. It can be written as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix}$$

where the submatrices $Q_{1j}, j = 1, \ldots, 4$ have dimension $5N/4 \times (2N+1)$, while the rest of submatrices have dimension $9N/4 \times (2N+1)$. It is clear from the formulas corresponding to the coefficients in Tables 7–10 that the $Q_{ij}, i > 1, j = 1, \ldots, 4$, are null matrices, while

$$Q_{11} = \begin{bmatrix} \dot{q}_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \dot{q}_{22} & 0 & 0 & \cdots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \cdots & 0 & \dot{q}_{\frac{N}{4}\,\frac{N}{4}} \end{bmatrix},$$

where each of the $N/4$ submatrices $\dot{q}_{ii}$ is given by

$$\dot{q}_{ii} = \begin{bmatrix} \beta^0_{01} & \beta^0_{11} & \beta^0_{21} & \beta^0_{31} & \beta^0_{41} & \beta^0_{51} & \beta^0_{61} & \beta^0_{71} & \beta^0_{81} \\ \beta^0_{03} & \beta^0_{13} & \beta^0_{23} & \beta^0_{33} & \beta^0_{43} & \beta^0_{53} & \beta^0_{63} & \beta^0_{73} & \beta^0_{83} \\ \beta^0_{05} & \beta^0_{15} & \beta^0_{25} & \beta^0_{35} & \beta^0_{45} & \beta^0_{55} & \beta^0_{65} & \beta^0_{75} & \beta^0_{85} \\ \beta^0_{07} & \beta^0_{17} & \beta^0_{27} & \beta^0_{37} & \beta^0_{47} & \beta^0_{57} & \beta^0_{67} & \beta^0_{77} & \beta^0_{87} \\ \beta^0_{08} & \beta^0_{18} & \beta^0_{28} & \beta^0_{38} & \beta^0_{48} & \beta^0_{58} & \beta^0_{68} & \beta^0_{78} & \beta^0_{88} \end{bmatrix}.$$

The matrices $Q_{i1}, i = 2, 3, 4$, have a similar structure

$$Q_{i1} = \begin{bmatrix} \dot{q}^i_{11} & 0 & 0dot & 0 & \cdots & 0 \\ 0 & \dot{q}^i_{22} & 0 & 0 & \cdots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \cdots & 0 & \dot{q}^i_{\frac{N}{4}\,\frac{N}{4}} \end{bmatrix},$$

where each of the $N/4$ submatrices $\dot{q}^i_{jj}$ is given by

$$\dot{q}^i_{jj} = \begin{bmatrix} \beta^{i-1}_{00} & \beta^{i-1}_{10} & \beta^{i-1}_{20} & \beta^{i-1}_{30} & \beta^{i-1}_{40} & \beta^{i-1}_{50} & \beta^{i-1}_{60} & \beta^{i-1}_{70} & \beta^{i-1}_{80} \\ \beta^{i-1}_{01} & \beta^{i-1}_{11} & \beta^{i-1}_{21} & \beta^{i-1}_{31} & \beta^{i-1}_{41} & \beta^{i-1}_{51} & \beta^{i-1}_{61} & \beta^{i-1}_{71} & \beta^{i-1}_{81} \\ \beta^{i-1}_{02} & \beta^{i-1}_{12} & \beta^{i-1}_{22} & \beta^{i-1}_{32} & \beta^{i-1}_{42} & \beta^{i-1}_{52} & \beta^{i-1}_{62} & \beta^{i-1}_{72} & \beta^{i-1}_{82} \\ \beta^{i-1}_{03} & \beta^{i-1}_{13} & \beta^{i-1}_{23} & \beta^{i-1}_{33} & \beta^{i-1}_{43} & \beta^{i-1}_{53} & \beta^{i-1}_{63} & \beta^{i-1}_{73} & \beta^{i-1}_{83} \\ \beta^{i-1}_{04} & \beta^{i-1}_{14} & \beta^{i-1}_{24} & \beta^{i-1}_{34} & \beta^{i-1}_{44} & \beta^{i-1}_{54} & \beta^{i-1}_{64} & \beta^{i-1}_{74} & \beta^{i-1}_{84} \\ \beta^{i-1}_{05} & \beta^{i-1}_{15} & \beta^{i-1}_{25} & \beta^{i-1}_{35} & \beta^{i-1}_{45} & \beta^{i-1}_{55} & \beta^{i-1}_{65} & \beta^{i-1}_{75} & \beta^{i-1}_{85} \\ \beta^{i-1}_{06} & \beta^{i-1}_{16} & \beta^{i-1}_{26} & \beta^{i-1}_{36} & \beta^{i-1}_{46} & \beta^{i-1}_{56} & \beta^{i-1}_{66} & \beta^{i-1}_{76} & \beta^{i-1}_{86} \\ \beta^{i-1}_{07} & \beta^{i-1}_{17} & \beta^{i-1}_{27} & \beta^{i-1}_{37} & \beta^{i-1}_{47} & \beta^{i-1}_{57} & \beta^{i-1}_{67} & \beta^{i-1}_{77} & \beta^{i-1}_{87} \\ \beta^{i-1}_{08} & \beta^{i-1}_{18} & \beta^{i-1}_{28} & \beta^{i-1}_{38} & \beta^{i-1}_{48} & \beta^{i-1}_{58} & \beta^{i-1}_{68} & \beta^{i-1}_{78} & \beta^{i-1}_{88} \end{bmatrix}.$$

As before, in the triangular arrangement of the $\dot{q}_{ii}$ in Q_{11}, and the $\dot{q}^i_{jj}$ in Q_{i1} the first column of each $\dot{q}^i_{jj}$ is placed below the last column of each $\dot{q}^i_{j-1j-1}$ for $j = 2, \ldots, N/4$.

Now let Y be the vector of true values corresponding to the approximated values in $\bar{Y}$, that is,

$$\begin{aligned} Y = \big(& y(x_{1/2}), y(x_1), y(x_{3/2}), \ldots, y(x_{N-1/2}), y'(x_{1/2}), y'(x_1), y'(x_{3/2}), \ldots, y'(x_{N-1/2}), \\ & y''(x_0), y''(x_{1/2}), y''(x_1), y''(x_{3/2}), \ldots, y''(x_N), y'''(x_0), y'''(x_{1/2}), y'''(x_1), y'''(x_{3/2}), \ldots, y'''(x_N)\big)^T \end{aligned}$$

and let F be the vector of true values corresponding to the approximated values in $\bar{F}$, that is,

$$\begin{aligned} F = \big(& f(x_0, y(x_0), y'(x_0), y''(x_0), y'''(x_0)), \ldots, f(x_N, y(x_N), y'(x_N), y''(x_N), y'''(x_N)), \\ & \ldots, f'''(x_0, y(x_0), y'(x_0), y''(x_0), y'''(x_0)), \ldots, f'''(x_N, y(x_N), y'(x_N), y''(x_N), y'''(x_N))\big)^T . \end{aligned}$$

We have that for the true values it is

$$PY + h^4Q\ F + R = L(h) \tag{11}$$

where $L(h)$ is a (8N)-vector containing the local truncation errors of the formulas, whose terms are of order $O(h^{12})$ (see (9)).

If we denote the errors between the true values and the approximated ones by

$$e_i = y(x_i) - y_i, \qquad e'_i = y'(x_i) - y'_i, \qquad e''_i = y''(x_i) - y''_i, \qquad e'''_i = y'''(x_i) - y'''_i,$$

for $i = 0, 1/2, 1, 3/2, \ldots, N$, we consider the vector of errors as

$$E = \big(e_{1/2}, e_1, \ldots, e_{N-1/2}, e'_{1/2}, e'_1, \ldots, e'_{N-1/2},$$
$$e''_0, e''_{1/2}, e''_1, \ldots, e''_N, e'''_0, e'''_{1/2}, e'''_1, \ldots, e'''_N\big)^T.$$

We note that E is a $(8N)$-vector that contains the errors of the unknowns in $\bar{Y}$, while in view of the boundary conditions we have that

$$e_0 = y(x_0) - y_0 = 0, \qquad e_N = y(x_N) - y_N = 0, \qquad e'_0 = y'(x_0) - y'_0 = 0, \qquad e'_N = y'(x_N) - y'_N = 0. \tag{12}$$

Using the notations above we state the main result of this section in the following theorem.

Theorem 3.1. *Let Y and F be defined as above. Let $\bar{Y}$ be an approximation of the solution vector Y of the system formed by combining the methods corresponding to Table 7 through Table 10 for $n = 0(4)N-4$, and E the vector of errors as defined above, assuming that the exact solution $y(x)$ of the BVP in (1)–(2) verifies that $y(x) \in C^m[a,b]$ for m as large as necessary. Then, the BHM is a convergent method of eighth order, that is, $\|E\| \le K\,h^8$.*

Proof. The proof follows the guidelines in [17]. In order to make more understandable the following analysis, we will include in the matrix equations subscripts indicating the corresponding dimensions. Thus, the exact equation in (11) adopts the form

$$P_{8N\times 8N}Y_{8N} + h^4Q_{8N\times(8N+4)}\,F_{8N+4} + R_{8N} = L(h)_{8N}, \tag{13}$$

while the approximate equation in (10) is rewritten as

$$P_{8N\times 8N}\bar{Y}_{8N} + h^4Q_{8N\times(8N+4)}\,\bar{F}_{8N+4} + R_{8N} = 0\,. \tag{14}$$

On subtracting (14) from (13) we get

$$P_{8N\times 8N}\left(Y_{8N} - \bar{Y}_{8N}\right) + h^4Q_{8N\times(8N+4)}\left(F_{8N+4} - \bar{F}_{8N+4}\right) = L(h)_{8N},$$

that is,

$$P_{8N\times 8N}E_{8N} + h^4Q_{8N\times(8N+4)}\left(F - \bar{F}\right)_{8N+4} = L(h)_{8N}\,. \tag{15}$$

Assuming that f, f', f'', f''' are smooth enough, by using the Mean-Value Theorem we can write for $j = 0, 1, 2, 3$ and $i = 0(1/2)N$ that

$$\begin{aligned} &f^{(j)}(x_i, y(x_i), y'(x_i), y''(x_i), y'''(x_i)) - f^{(j)}(x_i, y_i, y'_i, y''_i, y'''_i) \\ &\quad = (y(x_i) - y_i)\,\frac{\partial f^{(j)}}{\partial y}\,(\xi_i^j) + \left(y'(x_i) - y'_i\right)\frac{\partial f^{(j)}}{\partial y'}\,(\xi_i^j) \\ &\qquad + \left(y''(x_i) - y''_i\right)\frac{\partial f^{(j)}}{\partial y''}\,(\xi_i^j) + \left(y'''(x_i) - y'''_i\right)\frac{\partial f^{(j)}}{\partial y'''}\,(\xi_i^j)\,, \end{aligned}$$

where for each i the $\xi_i^j, j = 0, 1, 2, 3$, are intermediate points on the line segment joining $(x_i, y_i, y'_i, y''_i, y'''_i)$ to $(x_i, y(x_i), y'(x_i), y''(x_i), y'''(x_i))$. Thus, the vector $F - \bar{F}$ may be expressed as

$$F - \bar{F} = U^F\bar{E}$$

where U^F is a matrix of size $(8N+4)\times(8N+4)$ and $\bar{E}$ is a vector with $8N+4$ terms containing all the errors,

$$\bar{E} = \big(e_0, e_{1/2}, e_1, \ldots, e_N, e'_0, e'_{1/2}, e'_1, \ldots, e'_N,$$
$$e''_0, e''_{1/2}, e''_1, \ldots, e''_N, e'''_0, e'''_{1/2}, e'''_1, \ldots, e'''_N\big)^T,$$

and

$$U^F = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{pmatrix}$$

whose entries U_{ij} are $(2N+1)\times(2N+1)$ diagonal matrices written as

$$U_{ij}=\begin{pmatrix} \frac{\partial f^{(i-1)}}{\partial y^{(j-1)}}\left(\xi_0^{i-1}\right) & 0 & \cdots & 0 & 0\\ 0 & \frac{\partial f^{(i-1)}}{\partial y^{(j-1)}}\left(\xi_{1/2}^{i-1}\right) & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \frac{\partial f^{(i-1)}}{\partial y^{(j-1)}}\left(\xi_{N-1/2}^{i-1}\right) & 0\\ 0 & 0 & \cdots & 0 & \frac{\partial f^{(i-1)}}{\partial y^{(j-1)}}\left(\xi_N^{i-1}\right) \end{pmatrix}.$$

Now, having in mind that some of the errors in $\bar{E}$ are null (see (12)) we can put

$$\left(F-\bar{F}\right)_{8N+4}=\left(U^F\right)_{(8N+4)\times(8N+4)}\bar{E}_{(8N+4)}=\left(\bar{U}^F\right)_{(8N+4)\times(8N)}E_{(8N)}$$

where $\bar{U}^F$ is obtained from U^F after eliminating the first and last columns in the matrices U_{i1}, U_{i2} for $i=1,\ldots,4$ (that is, we eliminate in U^F the columns in the positions $1, 2N+1, 2N+2$ and $4N+2$).

Then, the equation in (15) becomes

$$P_{8N\times 8N}\,E_{8N}+h^4Q_{8N\times(8N+4)}\left(\bar{U}^F\right)_{(8N+4)\times 8N}\,E_{8N}=L(h)_{8N} \tag{16}$$

which, regardless of sizes, can be written simply as

$$(P+h^4Q\;\bar{U}^F)E=L(h)\,. \tag{17}$$

Now, consider the matrix

$$\kappa=P+h^4Q\;\bar{U}^F \tag{18}$$

It is claimed that κ is invertible for sufficiently small h. First, we claim that P is invertible. To see this, since

$$P=[P_{ij}]=\begin{cases}\text{nonzero} & : \quad i=1,2,3,4, \quad j=1\\ \text{nonzero} & : \quad i=j=2,3,4\\ 0 & : \quad \text{otherwise}\end{cases}$$

then, P is a lower triangular matrix containing nonzero diagonal submatrices, therefore its determinant exists, hence it is nonsingular. It is known that a matrix with nonzero main diagonal is invertible, hence P^{-1} exists.

Now, (18) can be written as

$$|\kappa|=|P+h^4Q\;\bar{U}^F|=|P||I-C| \tag{19}$$

where $C=-h^4Q\;\bar{U}^FP^{-1}$, then $|\zeta I-C|=0$ is the characteristic polynomial of C, so that

$$|\zeta I-C|=(\zeta-\zeta_1)\cdots(\zeta-\zeta_{4N})$$

where ζ_i are eigenvalues of the matrix C. When $\zeta=1$, we have

$$|I-C|=(1-\zeta_1)\cdots(1-\zeta_{4N})$$

for $|I-C|\neq 0$, then each $\zeta_i\neq 0$. If $\hat{\zeta}_i$ is an eigenvalue of C, so is $h^4\zeta_i$, thus we need $h^4\zeta_i\neq 1$. So we choose h such that $h^4\notin\left\{\frac{1}{\hat{\zeta}_i}\middle|\hat{\zeta}_i \text{ are nonzero eigenvectors of } Q\bar{U}^FP^{-1}\right\}$. For such h, $|I-C|\neq 0$, so that

$$|\kappa|=|P||I-h^4Q\;\bar{U}^F|\neq 0 \tag{20}$$

hence κ is invertible. Then we have that

$$\begin{aligned}\kappa\;E &= L(h)\\ E &= \kappa^{-1}L(h)\\ \|E\| &= \|\kappa^{-1}L(h)\|\\ &\leq \|\kappa^{-1}\|\;\|L(h)\|\\ &= O(h^{-4})O(h^{12})\\ &\leq K\,h^8\,,\end{aligned}$$

which completes the proof. □

3.3. Existence and uniqueness of the discrete solution

The following result establishes the existence and uniqueness of the solution provided by the system of equations in (10).

Theorem 3.2. *Assuming that $f(x, \mathbf{y})$ verifies a Lipschitz condition on the variable $\mathbf{y} = (y, y', y'', y''')$, it holds that the system in (10) has a unique solution whenever $h < \dfrac{1}{L\,a\,(8N+4)^{1/2}}$, where $L = \max\limits_{\{i=0,1,2,3\}} \{L_i\}$, $a = \max\limits_{\{i=1,\dots,8N;j=1,\dots,8N+4\}} \{|A_{ij}|\}$, with $A = P^{-1}Q\big|_{h=1}$.*

Proof. Let us consider the function $G : \mathbb{R}^{8N} \longrightarrow \mathbb{R}^{8N}$ where the i-component of $G(\xi)$ is given by

$$(G(\xi))_i = (-P^{-1}R - h^4 P^{-1} Q \bar{F}(\xi))_i,$$

where $\xi = (\xi_1, \dots, \xi_{8N}) \in \mathbb{R}^{8N}$.

Note that for $\xi = \bar{Y}$ the system in (10) adopts the form $\xi = G(\xi)$, so that the existence and uniqueness of the solution of system (10) is equivalent to that of the equation $\xi = G(\xi)$.

We consider in $\mathbb{R}^{8N}$ the maximum norm $\|\xi\| = \max_{1\le i\le 8N} \{|\xi_i|\}$. We have that

$$\begin{aligned} |G(\xi)_i - G(\xi^*)_i| &= |h^4 \left[P^{-1}Q\left(\bar{F}(\xi) - \bar{F}(\xi^*)\right)\right]_i| \\ &\le h\,a \sum_{j=1}^{8N+4} L\,|\xi_j - \xi_j^*|\,, \end{aligned}$$

where $L = \max\limits_{\{i=0,1,2,3\}} \{L_i\}$, and $a = \max\limits_{\{i=1,\dots,8N;j=1,\dots,8N+4\}} \{|A_{ij}|\}$, being $A = P^{-1}Q\big|_{h=1}$.

Taking into account the above inequalities and using the Cauchy–Schwartz inequality we can put

$$\begin{aligned} \|G(\xi) - G(\xi^*)\| &= \max_{\{1\le i\le 8N\}} \{\|G(\xi)_i - G(\xi^*)_i\|\} \\ &\le h\,a\,L\,(8N+4)^{1/2}\,\|\xi - \xi^*\| = k\,\|\xi - \xi^*\| \end{aligned}$$

with $k = h\,a\,L\,(8N+4)^{1/2}$.

As long as $k < 1$ we will have that G is a contraction. Hence, by Banach's Fixed-Point Theorem the proof is completed. □

For solving the system in (10) we could use a Newton-type method, as they are probably the most widely used methods in applications (see [18–22]). As it is well-known, the classical Newton's method is quadratically convergent from good starting guesses provided that the Jacobian is nonsingular. There are many results from the Kantorovich's theorem [23] about the balls of convergence [24–26]. There are also modified Newton's methods which are convergent even if the Jacobian is singular [27]. Nevertheless, those are limited theoretical results and out of the balls one can get convergence too. From a practical point of view what is important is to choose an initial guess sufficiently close to the root. For the problem in hand we have considered as initial guesses averaged values of the variables given by

$$\begin{aligned} y_{j/2} &= \frac{(y(a) + j\frac{h}{2}y'(a)) + (y(b) - (2N-j)\frac{h}{2}y'(b))}{2}, \quad j = 1, \dots, 2N-1 \\ y'_{j/2} &= \frac{y'(a) + y'(b)}{2}, \quad j = 1, \dots, 2N-1 \\ y''_{j/2} &= \frac{y'(b) - y'(a)}{Nh}, \quad j = 0, \dots, 2N \\ y'''_{j/2} &= 1, \quad j = 0, \dots, 2N. \end{aligned}$$

The stopping criteria used in the Newton's method are

$$|\bar{Y}_n - \bar{Y}_{n-1}| < 10^{-10} \qquad \text{and} \qquad |F(\bar{Y}_n)| < 10^{-10},$$

or when the number of iterations exceeds 50.

3.4. Computational procedure

The proposed method is implemented in a block form. We solve the system given by all the formulas corresponding to Tables 7–10 simultaneously using the system *Mathematica*, enhanced by the feature ***NSolve[]*** for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature ***FindRoot[]***, as summarized in the algorithm below.

All codes were written in Mathematica 12.0 and run on a PC with an Intel i7 2.00 GHz CPU processor, 8 GB memory and 64-bit Windows 10 operating system.

Table 1
Comparison of the absolute errors obtained for Problem 1.

x	BHM (N = 8)	x	BHM (N = 20)	VIM [5]	HBVP [28]
0.000	0.00	0.1	9.83E−20	7.78E−8	4.45E−10
0.125	1.99E−16	0.2	3.25E−19	2.72E−7	5.54E−10
0.250	6.30E−16	0.3	5.85E−19	8.24E−7	8.95E−11
0.375	1.04E−15	0.4	7.98E−19	7.77E−7	2.03E−10
0.500	1.22E−15	0.5	9.00E−19	9.71E−7	3.32E−11
0.625	1.17E−15	0.6	8.63E−19	1.05E−6	1.53E−10
0.750	6.95E−16	0.7	6.84E−19	9.63E−7	9.48E−11
0.875	2.39E−16	0.8	4.12E−19	6.84E−7	5.18E−10
1.000	0.00	0.9	1.33E−19	2.71E−7	4.15E−10
CPU(s.)	0.0312		0.218	–	–

Table 2
Maximum absolute errors and ROC obtained for Problem 1.

N	Max Abs Err	ROC
4	5.152E−12	–
8	1.133E−14	8.82
16	3.772E−17	8.23
32	1.461E−19	8.01
64	5.809E−22	7.97
128	1.419E−25	7.98

Algorithm.

Data: a, b (integration interval), N (number of steps), y_a, y'_a, y_b, y'_b (boundary values), f
Result: *sol*, discrete approximate solution of the BVP (1)–(2)
Let $n = 0, 4, \ldots, N-4, \quad x_0 = a, x_N = b, \quad h = \frac{b-a}{N}$;
Let $y_0 = y_a, y'_0 = y'_a, y_N = y_b, y'_N = y'_b$;
Solve equations in (5)–(6) to get $\bar{Y}$
Let $sol = \{(x_i, y_i)\}_{i=0,1,2,\ldots,N}$.
End

4. Numerical examples

Here, some numerical examples are presented to show the accuracy of the new developed method, BHM. In the examples considered, we have calculated the absolute errors at different points, which were obtained as $err(x_i) = |y(x_i) - y_i|$. The computational time in seconds used by the proposed method is denoted as CPU(s.). It can be seen in the provided tables that the proposed method is very efficient.

Example 1. We consider the following nonlinear boundary value problem [28]

$$\begin{array}{l} y^{(iv)}(x) = \sin(x) + \sin^2(x) - (y''(x))^2, \quad x \in [0, 1], \\ y(0) = 0, \quad y'(0) = 1, \quad y(1) = \sin(1), \quad y'(1) = \cos(1) \end{array} \tag{21}$$

with solution $y(x) = \sin(x)$.

Table 1 shows the absolute errors obtained with the proposed method, the variational iteration method in [5] (which considers an approximating polynomial of degree 11), and the Hermite based collocation method in [28] (using an approximating polynomial of degree 8). Note that we have considered different number of mesh points with the BHM, $N = 8$ and $N = 20$ (as N must be an integer multiple of 4, $N = 20$ is the lowest value to get the same grid points as in [5] and [28]). Table 2 shows the maximum absolute errors at the grid points on the integration interval, and the approximate order of convergence, ROC, obtained through the formula

$$p \simeq -\log_2\left(\frac{\max_{\{i=1,\ldots,2N\}} |y(x_i) - y_i|}{\max_{\{i=1,\ldots,N\}} |y(x_i) - y_i|}\right),$$

showing the good agreement with the theoretical results.

Example 2. We consider the following boundary value problem [28]

$$\begin{array}{l} y^{(iv)}(x) = (1+c)y''(x) - cy(x) + \frac{1}{2}cx^2 - 1, \quad x \in [0, 1], \\ y(0) = 0, \qquad y'(0) = 0, \qquad y(1) = 1.5 + \sinh(1), \qquad y'(1) = 1 + \cos(1) \end{array} \tag{22}$$

Table 3
Comparison of the absolute errors obtained for Problem 2.

x	BHM*	HBVP*	BHM**	HBVP**	BHM***	HBVP***
0.1	1.09E−19	5.44E−10	9.72E−20	1.84E−10	7.52E−20	3.18E−9
0.2	3.64E−19	6.71E−10	3.19E−19	6.43E−9	2.41E−19	1.28E−8
0.3	6.57E−19	9.72E−11	5.74E−19	2.39E−9	4.27E−19	2.54E−8
0.4	9.00E−19	2.49E−10	7.84E−19	3.67E−9	5.81E−19	3.55E−8
0.5	1.02E−18	1.85E−11	8.91E−19	3.78E−9	6.61E−19	3.89E−8
0.6	9.85E−19	2.03E−10	8.63E−19	3.18E−9	6.47E−19	3.49E−8
0.7	7.86E−19	1.65E−10	6.94E−19	2.60E−9	5.30E−19	2.56E−8
0.8	4.76E−19	7.51E−10	4.25E−19	2.04E−9	3.33E−19	1.42E−8
0.9	1.55E−19	5.90E−10	1.40E−19	9.49E−10	1.13E−19	4.31E−9
CPU(s.)	0.140	–	0.125	–	0.156	–

*for $c = -1/2$.
**for $c = 5$.
***for $c = 20$.

Table 4
Comparison of the absolute errors obtained for Problem 3.

x	BHM	HBVP [28]
0.1	0.00E00	7.35E−16
0.2	1.39E−17	2.34E−15
0.3	2.78E17	4.11E−15
0.4	0.00E00	5.83E−15
0.5	0.00E00	5.99E−15
0.6	0.00E00	5.55E−15
0.7	0.00E00	5.21E−15
0.8	1.11E−16	3.10E−15
0.9	1.11E−16	5.55E−16
CPU(s.)	0.187	–

with solution $y(x) = 1 + \frac{1}{2}x^2 + \sinh(x)$. Here the solution is independent of c. The absolute errors for $c = -1/2, 5, 20$, obtained with the BHM approach taking $N = 20$ and with the method in [28], named as HBVP, which uses an approximating Hermite polynomial of degree 8, are presented in Table 3. We can see that the proposed method provides great accuracy.

Example 3. We consider the following nonlinear boundary value problem [28]

$$\begin{aligned} &y^{(iv)}(x) = y(x)^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, \quad x \in [0, 1], \\ &y(0) = 0, \qquad y'(0) = 0, \qquad y(1) = 1, \qquad y'(1) = 1 \end{aligned} \tag{23}$$

with exact solution $y(x) = x^5 - 2x^4 + 2x^2$.

Table 4 shows the errors obtained for Problem (23) using the BHM, and the HBVP method in [28] using an approximating Hermite polynomial of degree 6. The BHM was implemented for N=20. We note that for this problem the method BHM is exact for any appropriate value of N and the errors are in fact due to accumulated roundoff errors.

Example 4. We consider the following linear boundary value problem [8]

$$\begin{aligned} &y^{(iv)}(x) - y(x) = -4(2x\cos x + 3\sin x), \quad x \in [0, 1], \\ &y(0) = y(1) = 0, \qquad y''(0) = 0, \qquad y''(1) = 2\sin 1 + 4\cos 1. \end{aligned} \tag{24}$$

The problem has exact solution $y(x) = (x^2 - 1)\sin x$.

Table 5 shows the numerical results for Problem (24) with the BHM, and the Galerkin method that uses quintic splines in [8]. The maximum absolute error obtained by the method in [8] is 5.275×10^{-6} for $N = 10$ subintervals on the domain [0, 1], while the maximum error obtained with the BHM is 1.77719×10^{-10}, for $N = 4$ and 9.35329×10^{-13}, for $N = 8$ subintervals on the domain [0, 1]. It is worth noting that the number of subintervals N must be a multiple of 4 in order to reach the final point of the domain $x_N = 1$. For comparison to obtain the global errors at the grid points 0.1(0.1)0.9, we set $N = 20$.

Example 5. We consider the following nonlinear boundary value problem [8,29]

$$\begin{aligned} &y^{(iv)}(x) - 6e^{-4y(x)} = -12(1+x)^{-4}, \quad x \in [0, 1], \\ &y(0) = 0, \qquad y(1) = \ln 2, \qquad y'(0) = 1, \qquad y'(1) = 0.5. \end{aligned} \tag{25}$$

Table 5
Comparison of errors obtained for Problem 4.

x	BHM	Galerkin method [8]
0.1	2.08438E−16	2.16812E−06
0.2	3.98916E−16	3.88920E−06
0.3	5.54491E−16	5.21541E−06
0.4	6.60577E−16	5.21541E−06
0.5	7.05643E−16	5.21541E−06
0.6	6.83231E−16	5.27501E−06
0.7	5.91857E−16	4.58956E−06
0.8	4.37534E−16	2.98023E−06
0.9	2.32878E−16	1.38581E−06
CPU(s.)	0.127	–

Table 6
Comparison of errors obtained for Problem 5.

x	BHM (N = 8)	x	BHM (N = 8)	Galerkin method [8]	BM [29]
0	0.00	0.1	7.85372E−13	2.235174E−08	7.47213E−9
0.125	1.11E−9	0.2	2.11953E−12	7.599592E−07	3.38799E−8
0.250	3.39E−9	0.3	3.03829E−12	2.026558E−06	7.82674E−8
0.375	5.23E−9	0.4	3.37647E−12	2.413988E−06	1.41700E−7
0.500	5.65E−9	0.5	3.17196E−12	3.129244E−06	2.26096E−7
0.625	4.52E−9	0.6	2.57178E−12	4.917383E−06	3.19801E−7
0.750	2.61E−9	0.7	1.75371E−12	4.887581E−06	3.84959E−7
0.875	8.04E−10	0.8	9.17155E−13	3.099442E−06	3.80675E−
1.00	0.00	0.9	2.63345E−13	2.324581E−06	2.66007E−7
CPU(s.)	0.062		0.203	–	–

Table 7
Coefficients of main formulas for $y_{n+j/2}$.

j	α^0_{0j}	α^0_{1j}	α^0_{2j}	α^0_{3j}	β^0_{0j}	β^0_{1j}	β^0_{2j}	β^0_{3j}	β^0_{4j}	β^0_{5j}	β^0_{6j}	β^0_{7j}	β^0_{8j}
1	$\frac{5}{16}$	$\frac{15}{16}$	$\frac{-5}{16}$	$\frac{1}{16}$	$\frac{867593}{539240067072}$	$\frac{-533994833}{168512520960}$	$\frac{-45804085}{2853122048}$	$\frac{-259363987}{19258573824}$	$\frac{-472251067}{77034295296}$	$\frac{-17391}{356640256}$	$\frac{-96983981}{385171476480}$	$\frac{8713259}{134810016768}$	$\frac{111343}{14978890752}$
3	$\frac{-1}{16}$	$\frac{9}{16}$	$\frac{9}{16}$	$\frac{-1}{16}$	$\frac{6467093}{599155630080}$	$\frac{3350219}{6241204480}$	$\frac{240113233}{42796830720}$	$\frac{117720139}{10699207680}$	$\frac{127667567}{21398415360}$	$\frac{232353}{1783201280}$	$\frac{10247443}{42796830720}$	$\frac{-5073967}{74894453760}$	$\frac{5073967}{599155630080}$
5	$\frac{1}{16}$	$\frac{-5}{16}$	$\frac{15}{16}$	$\frac{5}{16}$	$\frac{-22206487}{449366722560}$	$\frac{-7851623}{33702504192}$	$\frac{-170484485}{25678098432}$	$\frac{-24839819}{2139841536}$	$\frac{-1452326597}{77034295296}$	$\frac{-1312179}{1783201280}$	$\frac{-32565011}{25678098432}$	$\frac{50091379}{134810016768}$	$\frac{-8464657}{179746689024}$
7	$\frac{-5}{16}$	$\frac{21}{16}$	$\frac{-35}{16}$	$\frac{35}{16}$	$\frac{18840599}{22009798656}$	$\frac{-2545829}{687806208}$	$\frac{307001783}{6113832960}$	$\frac{72905329}{2751224832}$	$\frac{54387101}{343903104}$	$\frac{666435}{50948608}$	$\frac{378701281}{11004899328}$	$\frac{-95397103}{13756124160}$	$\frac{705171}{815177728}$
8	-1	4	-6	4	$\frac{11813507}{3510677520}$	$\frac{-11102912}{658252035}$	$\frac{22361461}{125381340}$	$\frac{178464}{3482815}$	$\frac{427169213}{752288040}$	$\frac{178464}{3482815}$	$\frac{22361461}{125381340}$	$\frac{-11102912}{658252035}$	$\frac{11813507}{3510677520}$

Problem (25) has the exact solution $y(x) = \ln(x + 1)$. Table 6 shows the numerical results for Problem (25). The maximum absolute errors obtained by the Galerkin method in [8] and the $(m + 1)$th-step block BM method in [29] are 4.917×10^{-6} and 3.84959×10^{-7} respectively for $N = 10$ subintervals on the domain [0, 1], while the maximum error obtained with the BHM is 2.06366×10^{-7}, for $N = 4$ subintervals on the domain [0, 1]. For comparison purposes, we set $N = 20$ to obtain the global errors at the grid points 0.1(0.1)0.9.

5. Conclusion

A block hybrid method (BHM) based on continuous linear multistep formulas has been developed and applied to solve fourth order linear and non linear BVPs in ordinary differential equations. It was shown that the method is very flexible, easy to derive and can be applied to solve diverse kinds of fourth order BVPs, with either Neumann or Dirichlet boundary conditions as seen in the examples presented. The method shows a very high accuracy when compared to the exact solution and hence it is competitive with existing methods in the literature cited.

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Appendix. Coefficients of main and additional formulas

See Tables 7–10.

Table 8
Coefficients of additional formulas for hy'_{n+j}.

j	α^1_{0j}	α^1_{1j}	α^1_{2j}	α^1_{3j}	β^1_{0j}	β^1_{1j}	β^1_{2j}	β^1_{3j}	β^1_{4j}	β^1_{5j}	β^1_{6j}	β^1_{7j}	β^1_{8j}
0	$\frac{-11}{6}$	3	$\frac{-3}{2}$	$\frac{1}{3}$	$\frac{-50448799}{47286676800}$	$\frac{-43207204}{1034396055}$	$\frac{-544432879}{5516778960}$	$\frac{-11165306}{147770865}$	$\frac{-1027916543}{33100673760}$	$\frac{-2352}{2487725}$	$\frac{-584479}{472866768}$	$\frac{377738}{1034396055}$	$\frac{-1040471}{22067115840}$
1	$\frac{-23}{24}$	$\frac{7}{8}$	$\frac{1}{8}$	$\frac{-1}{24}$	$\frac{518995537}{177949222133760}$	$\frac{239222998933}{27804565958400}$	$\frac{7063594199}{353073853440}$	$\frac{6507924329}{794416170240}$	$\frac{62481285047}{12710658723840}$	$\frac{-86813}{445800320}$	$\frac{79303727}{496510106400}$	$\frac{-52075097}{2780456595840}$	$\frac{131071}{6590711930880}$
2	$\frac{-1}{3}$	$\frac{-1}{2}$	1	$\frac{-1}{6}$	$\frac{17465663}{463409432640}$	$\frac{66502421}{21722317155}$	$\frac{330790549}{11821669200}$	$\frac{2126281}{59108346}$	$\frac{1524953173}{99302021280}$	$\frac{1486}{3482815}$	$\frac{3699029}{5516778960}$	$\frac{-43380361}{217223171550}$	$\frac{11893159}{463409432640}$
3	$\frac{1}{24}$	$\frac{-9}{8}$	$\frac{9}{8}$	$\frac{-1}{24}$	$\frac{-496767163}{59316407377920}$	$\frac{-123564389}{370727546112}$	$\frac{-228641783}{58845642240}$	$\frac{-521578507}{1324026950400}$	$\frac{3842422291}{847377248256}$	$\frac{-11007}{445800320}$	$\frac{28507505}{211844312064}$	$\frac{-2958553}{115852358160}$	$\frac{73538077}{32953559654400}$
4	$\frac{1}{6}$	-1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{-285896147}{6951141489600}$	$\frac{-146849558}{108611585775}$	$\frac{-416954521}{27583894800}$	$\frac{-76547177}{2216562975}$	$\frac{-3046673843}{99302021280}$	$\frac{-10916}{17414075}$	$\frac{-322316441}{248255053200}$	$\frac{39132763}{108611585775}$	$\frac{-34242907}{772349054400}$
5	$\frac{1}{24}$	$\frac{-1}{8}$	$\frac{-7}{8}$	$\frac{23}{24}$	$\frac{-40710074833}{296582036889600}$	$\frac{3819938693}{5560913191680}$	$\frac{-7860240547}{1059221560320}$	$\frac{-635601419}{264805390080}$	$\frac{-353757522817}{12710658723840}$	$\frac{-4527137}{2229001600}$	$\frac{-25014937}{7355705280}$	$\frac{86428031}{86889268620}$	$\frac{-1502575853}{11863281475584}$
6	$\frac{-1}{3}$	$\frac{3}{2}$	-3	$\frac{11}{6}$	$\frac{276963649}{463409432640}$	$\frac{-51987389}{36203861925}$	$\frac{247289899}{5516778960}$	$\frac{92187563}{2068792110}$	$\frac{4602143233}{33100673760}$	$\frac{31398}{3482815}$	$\frac{1449300077}{82751684400}$	$\frac{-68021927}{14481544770}$	$\frac{18092743}{30893962176}$
7	$\frac{-23}{24}$	$\frac{31}{8}$	$\frac{-47}{8}$	$\frac{71}{24}$	$\frac{15797883989}{5084263489536}$	$\frac{-12079166569}{794416170240}$	$\frac{49272768719}{294228211200}$	$\frac{6342337547}{113488024320}$	$\frac{6749625424349}{12710658723840}$	$\frac{3024991}{63685760}$	$\frac{67011585461}{453952097280}$	$\frac{-47322469987}{1986040425600}$	$\frac{1753684073}{564918165504}$
8	$\frac{-11}{6}$	7	$\frac{-19}{2}$	$\frac{13}{3}$	$\frac{673093717}{92681886528}$	$\frac{-861439924}{21722317155}$	$\frac{5916680327}{16550336880}$	$\frac{187789298}{5171980275}$	$\frac{22745989835}{19860404256}$	$\frac{77264}{696563}$	$\frac{2509840733}{5516778960}$	$\frac{53843858}{21722317155}$	$\frac{19190084621}{2317047163200}$

Table 9
Coefficients of additional formulas for $h^2y''_{n+j}$.

j	α^2_{0j}	α^2_{1j}	α^2_{2j}	α^2_{3j}	β^2_{0j}	β^2_{1j}	β^2_{2j}	β^2_{3j}	β^2_{4j}	β^2_{5j}	β^2_{6j}	β^2_{7j}	β^2_{8j}
0	2	-5	4	-1	$\frac{125156183}{7801505600}$	$\frac{818786078}{3291260175}$	$\frac{2250585863}{7522880400}$	$\frac{42631697}{156726675}$	$\frac{201297517}{3009152160}$	$\frac{214724}{17414075}$	$\frac{769933}{278625200}$	$\frac{-1749317}{1097086725}$	$\frac{5800455}{210640651200}$
1	$\frac{3}{2}$	$\frac{-7}{2}$	$\frac{5}{2}$	$\frac{-1}{2}$	$\frac{-309230083}{561708403200}$	$\frac{79321033}{15603011200}$	$\frac{31188338921}{240732172800}$	$\frac{12705279997}{120366086400}$	$\frac{162276931}{3209762304}$	$\frac{-96419}{557250400}$	$\frac{71366329}{34390310400}$	$\frac{-45799489}{93618067200}$	$\frac{43601021}{842562604800}$
2	1	-2	1	0	$\frac{5943493}{70213550400}$	$\frac{-37318171}{3291260175}$	$\frac{-153270907}{2507626800}$	$\frac{-9855983}{940360050}$	$\frac{-274373}{334350240}$	$\frac{7286}{17414075}$	$\frac{-479177}{7522880400}$	$\frac{-104101}{6582520350}$	$\frac{41221}{7801505600}$
3	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{-612147041}{6740500838400}$	$\frac{-593261461}{140427100800}$	$\frac{-5514026299}{120366086400}$	$\frac{-184248483}{17195155200}$	$\frac{-4679875943}{96292869120}$	$\frac{-288697}{278625200}$	$\frac{-16201043}{8597577600}$	$\frac{447758173}{842562604800}$	$\frac{-447758173}{6740500838400}$
4	0	1	-2	1	$\frac{-5049397}{30091521600}$	$\frac{618736}{470180025}$	$\frac{-33156581}{7522880400}$	$\frac{-6568}{2487725}$	$\frac{-215258051}{3009152160}$	$\frac{-6568}{2487725}$	$\frac{-33156581}{7522880400}$	$\frac{618736}{470180025}$	$\frac{-5049397}{30091521600}$
5	$\frac{-1}{2}$	$\frac{5}{2}$	$\frac{-7}{2}$	$\frac{3}{2}$	$\frac{61808423}{481464345600}$	$\frac{80232979}{20061014400}$	$\frac{3661827577}{80244057600}$	$\frac{12987031441}{120366086400}$	$\frac{1578882883}{12036608640}$	$\frac{141769}{79607200}$	$\frac{137019779}{80244057600}$	$\frac{-91077403}{120366086400}$	$\frac{5562929}{53496038400}$
6	-1	4	-5	2	$\frac{677735887}{210640651200}$	$\frac{-51543173}{3291260175}$	$\frac{1309020983}{7522880400}$	$\frac{56625599}{940360050}$	$\frac{111601333}{200610144}$	$\frac{858818}{17414075}$	$\frac{849687439}{7522880400}$	$\frac{-177618587}{6582520350}$	$\frac{694453399}{210640651200}$
7	$\frac{-3}{2}$	$\frac{11}{2}$	$\frac{-13}{2}$	$\frac{5}{2}$	$\frac{44363036453}{6740500838400}$	$\frac{-15860557283}{421281302400}$	$\frac{18722975027}{60183043200}$	$\frac{-760580869}{120366086400}$	$\frac{96876124951}{96292869120}$	$\frac{3462247}{34828150}$	$\frac{17596779121}{40122028800}$	$\frac{-27025583383}{842562604800}$	$\frac{40303467289}{6740500838400}$
8	-2	7	-8	3	$\frac{2242500397}{210640651200}$	$\frac{-212136446}{3291260175}$	$\frac{1157012837}{2507626800}$	$\frac{-13958849}{156726675}$	$\frac{4385526221}{3009152160}$	$\frac{2971148}{17414075}$	$\frac{814405169}{1074697200}$	$\frac{203965861}{1097086725}$	$\frac{618190309}{23404516800}$

Table 10
Coefficients of additional formulas for $h^3y'''_{n+j}$.

j	α^3_{0j}	α^3_{1j}	α^3_{2j}	α^3_{3j}	β^3_{0j}	β^3_{1j}	β^3_{2j}	β^3_{3j}	β^3_{4j}	β^3_{5j}	β^3_{6j}	β^3_{7j}	β^3_{8j}
0	-1	3	-3	1	$\frac{-1467997511}{9873780525}$	$\frac{-1467115409}{1974756105}$	$\frac{-99355337}{1253813400}$	$\frac{-1635773617}{2821080150}$	$\frac{138688129}{1128432060}$	$\frac{-1389102}{17414075}$	$\frac{1125583}{322409160}$	$\frac{105036829}{19747561050}$	$\frac{-5491103}{4388346900}$
1	-1	3	-3	1	$\frac{97781931881}{20221502515200}$	$\frac{-111251338171}{631921953600}$	$\frac{-43613710037}{80244057600}$	$\frac{-769837307}{5158546560}$	$\frac{-41928306083}{288878607360}$	$\frac{7463313}{557250400}$	$\frac{-3062279881}{722196518400}$	$\frac{-245443991}{1263843907200}$	$\frac{66649141}{449366722560}$
2	-1	3	-3	1	$\frac{-2055743}{1579804884}$	$\frac{248368817}{9873780525}$	$\frac{-12293131}{89558100}$	$\frac{-871809643}{2821080150}$	$\frac{-29671325}{451372824}$	$\frac{-31218}{3482815}$	$\frac{-25562519}{5642160300}$	$\frac{39095743}{19747561050}$	$\frac{-2675059}{8776693800}$
3	-1	3	-3	1	$\frac{9559098569}{20221502515200}$	$\frac{3335799077}{631921953600}$	$\frac{1765193807}{16048811520}$	$\frac{1906410671}{180549129600}$	$\frac{-36609495107}{288878607360}$	$\frac{1569777}{557250400}$	$\frac{-1870306153}{722196518400}$	$\frac{46653269}{252768781440}$	$\frac{4111549}{2246833612800}$
4	-1	3	-3	1	$\frac{-2243329}{2821080150}$	$\frac{23800157}{1410540075}$	$\frac{71713639}{1253813400}$	$\frac{902095151}{2821080150}$	$\frac{137896441}{1128432060}$	$\frac{24354}{2487725}$	$\frac{92735011}{11284320600}$	$\frac{9090331}{2821080150}$	$\frac{288443}{626906700}$
5	-1	3	-3	1	$\frac{65480917097}{20221502515200}$	$\frac{-2045343551}{126384390720}$	$\frac{14275275307}{80244057600}$	$\frac{9840387887}{180549129600}$	$\frac{28349661307}{41268372480}$	$\frac{26057361}{557250400}$	$\frac{9434690059}{144439303680}$	$\frac{-26891767319}{1263843907200}$	$\frac{6254650889}{2246833612800}$
6	-1	3	-3	1	$\frac{311092063}{39495122100}$	$\frac{-518396239}{9873780525}$	$\frac{188493071}{626906700}$	$\frac{-98953391}{564216030}$	$\frac{2094473639}{2256864120}$	$\frac{2171334}{17414075}$	$\frac{2445342373}{5642160300}$	$\frac{-1494434369}{19747561050}$	$\frac{15575461}{1755338760}$
7	-1	3	-3	1	$\frac{4446069809}{808860100608}$	$\frac{-22062362587}{631921953600}$	$\frac{19774383499}{80244057600}$	$\frac{-16219117777}{180549129600}$	$\frac{50413288025}{57775721472}$	$\frac{8113149}{111450080}$	$\frac{81061508849}{103170931200}$	$\frac{178132507177}{1263843907200}$	$\frac{1818780521}{2246833612800}$
8	-1	3	-3	1	$\frac{114804103}{9873780525}$	$\frac{-772851157}{9873780525}$	$\frac{93073163}{250762680}$	$\frac{-105426823}{403011450}$	$\frac{1055854753}{1128432060}$	$\frac{4152786}{17414075}$	$\frac{5121885773}{11284320600}$	$\frac{2646097721}{3949512210}$	$\frac{697976281}{4388346900}$

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