

A mathematical model for a class of frying processes[☆]

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Received 13 September 2005; accepted 27 February 2006

Abstract

We present a mathematical model for deep frying in absence of mechanical deformation and in one-dimensional geometry. In the generic stage of the process the inner zone is saturated with liquid water below the boiling point at atmospheric pressure. When the boiling point is reached partial vaporization occurs and a zone of water–vapor thermodynamical equilibrium is formed, followed by a region of pure vapor. The interface between mixed and vapor region can be either a zero saturation surface or a level set of pressure when the latter reaches an imposed constraint. The outmost layer is crust. In each region the governing equations are written for temperature and pressure and the conditions at the interfaces comes out as the corresponding Rankine–Hugoniot relations. Boundary conditions at the crust–oil interface are discussed. Rescaling leads to only moderate simplifications and we are left with a considerably difficult free boundary problem for a parabolic system.

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Keywords: Free boundary problems; Phase change; Heat and mass transfer

1. Introduction

Frying is a process of great importance in the food industry and there is a quite substantial literature on frying chips, tortillas or thicker specimens of tubers (see e.g. [1,2] for references). It is also the subject of intense studies for improving quality, reducing oil absorption and producing healthier products [3,4].

In this paper we present a mathematical model for the process of frying a sufficiently thick ($\simeq 1$ cm) potato slice, schematized as an infinite slab to formulate the model in one space dimension.

The three-dimensional formulation, which may look straightforward, presents nontrivial difficulties due to the presence of a constraint on pressure, as we shall see. Since the one-dimensional model is already remarkably complicated we confine our attention to this case.

Some of the basic ideas were anticipated in [5]. During frying the outer surface of the sample is exposed to an edible oil at a temperature well above the water boiling point at atmospheric pressure. Our “potato” is a porous material which before frying is saturated with water, so the definition applies to other vegetables of similar stiffness. The porous skeleton will be supposed undeformable, so with a constant porosity. In this way we avoid the difficulty connected with the mechanics of the skeleton, which on the contrary is of clear importance when considering the

[☆] Work partially supported by PRIN-MIUR *Free boundary Problems* (2004).

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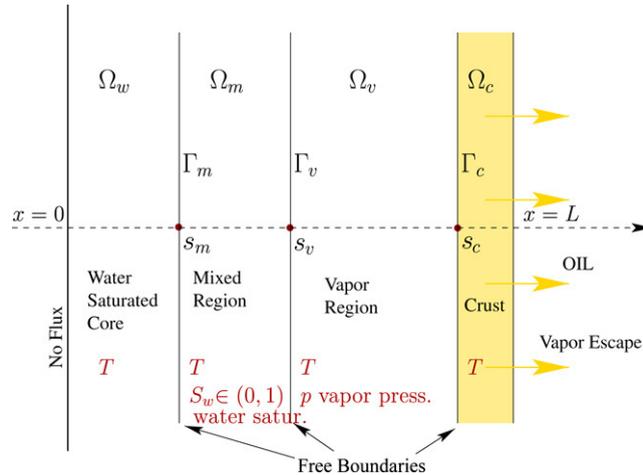


Fig. 1. Sketch of the generic stage.

frying of chips or tortillas [6–8]. It is precisely for this reason that we confine to the case of a thick sample. Therefore the problem we are going to deal with is in the spirit of [1,2], although we will pursue an entirely different approach.

Even if we are considering an idealized situation, we believe that our analysis can represent a progress in the mathematical modelling of the process, being adherent to its thermodynamical basis as much as possible since the relevant physics is included. Despite the many simplifications introduced, in the geometry (taken one-dimensional) and in the mechanical behaviour of the sample (considered undeformable), the model is remarkably complex. One of the reasons is that it contains several interfaces. A peculiar feature is the presence of a constraint on vapor pressure, not allowed to exceed some threshold.

Before we proceed to deriving the governing equations, let us sketch briefly the general structure we will be dealing with.

At the generic stage of the process, proceeding from the median section of the sample ($x = 0$) to the outer surface ($x = L$), we want to describe the evolution of the following regions (see Fig. 1):

- the water saturated region $0 < x < s_m(t)$, where temperature is below the boiling point T_0 ; here temperature is the only unknown,
- the mixed region $s_m(t) < x < s_v(t)$, in which water coexists with saturated vapor; besides temperature the water saturation S_w has to be found, while vapor pressure is a given function of temperature because of Clapeyron's law,
- the vapor region $s_v(t) < x < s_c(t)$, where no liquid water is present; pressure and temperature are independent variables
- the crust $s_c(t) < x < L$ which forms above some temperature T_c ; due to its structure we consider it completely permeable to vapor.

Enthalpy and mass balance must be imposed in each region (thus including the latent heat of vaporization) as well as on each of the unknown interfaces $x = s_m(t)$, $x = s_v(t)$, $x = s_c(t)$. In the region Ω_m a unilateral constraint on pressure will be imposed. Assumptions will be made for heat, liquid water (when needed) and vapor transport mechanisms.

We will carry out this programme in the next two sections and we will discuss the initial conditions in Section 4. In Section 5 we will select specific values of the physical parameter and on that basis we will try to obtain a nondimensional version of the model of somehow reduced complexity. We shall see however that no substantial simplification of the original model is possible unless some extreme conditions are verified. The corresponding mathematical theory looks exceedingly difficult and will not be considered in this paper, whose purpose is limited to the illustration of the basic ideas.

As we said, the model we are going to develop has several limitations concerning the geometry and the mechanics. There are other important phenomena that are difficult to describe in quantitative terms. One is the shielding action of the outgoing vapor, which reduces the contact of the sample with the oil. Another effect, which becomes important in the real industrial process, where the mass of the material to be fried is comparable to the mass of the oil, is the fact that

heat exchange makes the temperature of the oil variable in time and space. In our case the oil temperature is constant, but we will try to account for the vapor–oil interaction taking a variable heat transfer coefficient at the external surface. Needless to say, this is just an attempt to emphasize this phenomenon, more qualitatively than quantitatively, since we play with a coefficient which is hardly known with satisfactory accuracy even in standard cases.

We will completely ignore oil suction, because it occurs mainly after vapor has ceased to flow out. For more information on oil suction see [4,9,10].

2. The differential equations in the four regions $\Omega_w, \Omega_m, \Omega_v, \Omega_c$

2.1. Ω_w — Heat transport

In Ω_w we have $S_w = 1$ and $p = p_0$ (which is both the external atmospheric pressure and the initial pressure in the sample). The only unknown is temperature and the heat balance is expressed by

$$[\epsilon\rho_w^0 C_w + (1 - \epsilon)\rho_s C_s] \frac{\partial T}{\partial t} - [\epsilon k_w + (1 - \epsilon)k_s] \frac{\partial^2 T}{\partial x^2} = 0 \tag{2.1}$$

where ρ_w^0 is the liquid water density, ρ_s is the density of the solid components C_α, k_α denote the specific heat and the thermal conductivity of the species α ($\alpha = w, s$).

2.2. Ω_m — Mass transport

It is assumed that in Ω_m p and T are related by the Clapeyron equation

$$p = p_0 e^{\frac{\lambda}{R} \left(\frac{1}{T_0} - \frac{1}{T} \right)}, \tag{2.2}$$

when λ is the latent heat of vaporization, R the gas constant, T_0 the boiling temperature of water corresponding to p_0 .

Writing the mass balance for liquid water and vapor requires an assumption on the corresponding mass fluxes q_w, q_v . Following [1] we write

$$q_w = -D_w \epsilon \rho_w^0 \frac{\partial S_w}{\partial x} \tag{2.3}$$

suggesting that the water transport mechanism is diffusion, D_w representing diffusivity. This may be questionable but it could also be interpreted as a linearized description of the capillarity action.

As to q_v , we write Darcy’s law

$$q_v = -K_v(T) \rho_v \frac{\partial p}{\partial x} = -K_v(T) \rho_v p'(T) \frac{\partial T}{\partial x} \tag{2.4}$$

in which $K_v(T)$ is the hydraulic conductivity and

$$\rho_v = (1 - S_w) \hat{\rho}_v(T) \tag{2.5}$$

is the vapor mass actually present in the unit pore volume (irrespectively of the way it is filled) and $\hat{\rho}_v(T)$ is the vapor density in the pore volume fraction occupied by vapor.

Now the mass balance equations are

$$\epsilon \rho_w^0 \frac{\partial S_w}{\partial t} + \frac{\partial q_w}{\partial x} = -\omega_v, \tag{2.6}$$

$$\epsilon \frac{\partial \rho_v}{\partial t} + \frac{\partial q_v}{\partial x} = \omega_v, \tag{2.7}$$

where ω_v represents the rate of vapor production per unit volume.

Recalling (2.2), the final expression of q_v is

$$q_v = -K_v(T) \rho_v p(T) \frac{\lambda}{RT^2} \frac{\partial T}{\partial x}, \tag{2.8}$$

which, together with (2.5), allows the computation of ω_v from (2.7):

$$\begin{aligned} \omega_v = & -\epsilon \hat{\rho}_v \frac{\partial S_w}{\partial t} + \epsilon(1 - S_w) \frac{d\hat{\rho}_v}{dT} \frac{\partial T}{\partial t} - K_v \frac{\lambda p}{RT^2} \frac{\partial T}{\partial x} \left[-\hat{\rho}_v \frac{\partial S_w}{\partial x} + (1 - S_w) \frac{d\hat{\rho}_v}{dT} \frac{\partial T}{\partial x} \right] \\ & - K'_v \hat{\rho}_v (1 - S_w) \frac{\lambda p}{RT^2} \left(\frac{\partial T}{\partial x} \right)^2 + K_v \hat{\rho}_v (1 - S_w) \frac{\lambda p}{RT^3} \left(2 - \frac{\lambda}{RT} \right) \left(\frac{\partial T}{\partial x} \right)^2 \\ & - K_v \hat{\rho}_v (1 - S_w) \frac{\lambda p}{RT^2} \frac{\partial^2 T}{\partial x^2} \end{aligned} \quad (2.9)$$

inserting this expression in (2.6) we arrive at the equation governing water flux

$$\begin{aligned} \epsilon(\rho_w^0 - \hat{\rho}_v) \frac{\partial S_w}{\partial t} - D_w \epsilon \rho_w^0 \frac{\partial^2 S_w}{\partial x^2} + (1 - S_w) \left[\epsilon \frac{d\hat{\rho}_v}{dT} \frac{\partial T}{\partial t} - K_v \hat{\rho}_v \frac{\lambda p}{RT^2} \frac{\partial^2 T}{\partial x^2} \right] \\ + K_v \hat{\rho}_v \frac{\lambda p}{RT^2} \frac{\partial S_w}{\partial x} \frac{\partial T}{\partial x} - K_v \hat{\rho}_v (1 - S_w) \frac{\lambda p}{RT^2} \left\{ \frac{1}{\hat{\rho}_v} \frac{d\hat{\rho}_v}{dT} + \frac{K'_v}{K_v} + \frac{1}{T} \left(\frac{\lambda}{RT} - 2 \right) \right\} \left(\frac{\partial T}{\partial x} \right)^2 = 0. \end{aligned} \quad (2.10)$$

Typically $\frac{\hat{\rho}_v}{\rho_w^0} \ll 1$ and we can introduce some simplifications. This model relies on the assumption of water–vapor equilibrium, which may not be closely respected approaching the full vaporization interface. Therefore the model is expected to be more realistic if $q_v \ll q_w$ (meaning in practice $K_v \ll 10^{-3} \text{ g}^{-1} \text{ cm}^3 \text{ s}$). We also remark that, since q_v is driven by the temperature gradient, the vapor flux is directed towards the colder interface $x = s_m(t)$ and therefore it is anyway q_w which provides the mass supply to the opposite interface $x = s_v(t)$.

2.3. Ω_m — Heat transport

Let h_i be the enthalpy of the unit mass of the species i ($i = w, v, s$) and let C_i be the corresponding specific heat ($\frac{\partial h_i}{\partial t} = C_i \frac{\partial T}{\partial t}$ in each species). Of course

$$h_v = h_w + \lambda \quad (2.11)$$

at the vaporization temperature (λ latent heat $\simeq 9717 \text{ cal/mol}$).

The enthalpy per unit volume of the system in Ω_m is

$$H^{(m)} = \epsilon S_w \rho_w^0 h_w + \epsilon \rho_v h_v + (1 - \epsilon) \rho_s h_s. \quad (2.12)$$

The local balance equation is

$$\frac{\partial H^{(m)}}{\partial t} + \frac{\partial j^{(m)}}{\partial x} = 0, \quad (2.13)$$

where $j^{(m)}$ is the enthalpy flux

$$j^{(m)} = -(1 - \epsilon) k_s \frac{\partial T}{\partial x} - \epsilon S_w k_w \frac{\partial T}{\partial x} + q_w h_w + q_v h_v. \quad (2.14)$$

With the help of (2.6) and (2.7) Eq. (2.13) can be written as

$$\begin{aligned} \left[\epsilon S_w \rho_w^0 C_w + \epsilon \rho_v C_v + (1 - \epsilon) \rho_s C_s \right] \frac{\partial T}{\partial t} - \frac{\partial}{\partial x} \left\{ (\epsilon S_w k_w + (1 - \epsilon) k_s) \frac{\partial T}{\partial x} \right\} \\ + (C_v q_v + C_w q_w) \frac{\partial T}{\partial x} = -\lambda \omega_v. \end{aligned} \quad (2.15)$$

The thermal conductivity of the vapor has been neglected in (2.15).

Using once again the expression (2.9) of ω_v , we obtain the final form of the heat balance equation

$$\begin{aligned} \left[\epsilon S_w \rho_w^0 C_w + (1 - \epsilon) \rho_s C_s + \epsilon \rho_v C_v + \epsilon \lambda (1 - S_w) \frac{d\hat{\rho}_v}{dT} \right] \frac{\partial T}{\partial t} \\ - \left[\epsilon S_w k_w + (1 - \epsilon) k_s + \lambda (1 - S_w) K_v \hat{\rho}_v \frac{\lambda p}{RT^2} \right] \frac{\partial^2 T}{\partial x^2} \end{aligned}$$

$$\begin{aligned}
 & -\epsilon k_w \frac{\partial S_w}{\partial x} \frac{\partial T}{\partial x} - \left(C_v K_v \rho_v \frac{\lambda p}{RT^2} \frac{\partial T}{\partial x} + \epsilon C_w D_w \rho_w^0 \frac{\partial S_w}{\partial x} \right) \frac{\partial T}{\partial x} \\
 = & \epsilon \lambda \hat{\rho}_v \frac{\partial S_w}{\partial t} - K_v \frac{\lambda^2 p}{RT^2} \hat{\rho}_v \frac{\partial S_w}{\partial x} \frac{\partial T}{\partial x} \\
 & + K_v \frac{\lambda^2 p}{RT^2} \hat{\rho}_v (1 - S_w) \left[\frac{1}{\hat{\rho}_v} \frac{d\hat{\rho}_v}{dT} + \frac{K'_v}{K_v} + \frac{1}{T} \left(\frac{\lambda}{RT} - 2 \right) \right] \left(\frac{\partial T}{\partial x} \right)^2.
 \end{aligned} \tag{2.16}$$

Note that the contributions of latent heat to the coefficients of $\frac{\partial T}{\partial t}$ and of $\frac{\partial^2 T}{\partial x^2}$ are both in favor of the forward parabolicity of Eq. (2.16). However we point out that in Ω_m it is the system of two Eqs. (2.10) and (2.16) that has to be parabolic. We will return to this point later on (see Section 7).

2.4. Ω_v — Mass transport

In Ω_v we take a different conductivity \bar{K}_v for the vapor, which we expect to be larger than K_v to increase the efficiency of the transport mechanism. This change of K_v can be physically justified by the softening of the material making the porous skeleton.

What changes from Ω_m to Ω_v is that pressure becomes independent of T . Therefore the two unknowns in Ω_v are p and T .

We write the vapor flux as

$$q_v = -\bar{K}_v(T) \rho_v \frac{\partial p}{\partial x}, \tag{2.17}$$

where ρ_v is linked to p and T by the equation of state

$$\rho_v = \rho_v(p, T), \tag{2.18}$$

for instance

$$\rho_v = \frac{m_w p}{R T}, \tag{2.19}$$

if we consider the vapor as an ideal gas ($m_w = 18$ g).

Mass balance is expressed by

$$\epsilon \frac{\partial \rho_v}{\partial t} + \frac{\partial q_v}{\partial x} = 0, \tag{2.20}$$

which in combination with (2.17) and (2.18) yields

$$\frac{\partial \rho_v}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial \rho_v}{\partial T} \frac{\partial T}{\partial t} - \frac{\bar{K}_v}{\epsilon} \left\{ \left(\frac{\partial \rho_v}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial \rho_v}{\partial T} \frac{\partial T}{\partial x} \right) \frac{\partial p}{\partial x} + \rho_v(p, T) \frac{\partial^2 p}{\partial x^2} \right\} \frac{-\rho_v \bar{K}'_v}{\epsilon} \frac{\partial p}{\partial x} \frac{\partial T}{\partial x} = 0. \tag{2.21}$$

In the particular case (2.19) it reduces to

$$\frac{\partial p}{\partial t} - \frac{\bar{K}_v}{\epsilon} p \frac{\partial^2 p}{\partial x^2} - \frac{\bar{K}_v}{\epsilon} \left[\left(\frac{\partial p}{\partial x} \right)^2 - \frac{p}{T} \frac{\partial p}{\partial x} \frac{\partial T}{\partial x} \right] - \frac{\bar{K}'_v}{\epsilon} p \frac{\partial T}{\partial x} \frac{\partial p}{\partial x} = \frac{p}{T} \frac{\partial T}{\partial t}. \tag{2.22}$$

2.5. Ω_v — Heat transport

Neglecting again the vapor thermal conductivity we simply write

$$[\epsilon \rho_v C_v + (1 - \epsilon) \rho_s C_s] \frac{\partial T}{\partial t} - (1 - \epsilon) k_s \frac{\partial^2 T}{\partial x^2} + \epsilon C_v q_v \frac{\partial T}{\partial x} = 0. \tag{2.23}$$

2.6. Ω_c — Mass transfer

We write no equation, since we make the assumption that vapor finds no resistance to flow through the crust (which is in any case very thin). This corresponds to moving the boundary condition $p = p_0$ from the external surface to the interface $x = s_c(t)$.

2.7. Ω_c — Heat transfer

Since we are considering the case $[L - s_c(t)]/L \ll 1$ from the practical point of view the heat balance in the crust has little relevance. Anyway the corresponding equation is

$$[\epsilon\rho_v C_v + (1 - \epsilon)\rho_c C_c] \frac{\partial T}{\partial t} - (1 - \epsilon)k_c \frac{\partial^2 T}{\partial x^2} + \epsilon C_v q_v|_{x=s_v(t)} \frac{\partial T}{\partial x} = 0, \quad (2.24)$$

where ρ_c , C_c , k_c refer to the crust and q_v is computed at $x = s_v(t)$ according to the previous remark.

Remark 1. We have written (2.24) for the sake of completeness, but eventually we will treat the crust in a different way, exploiting the fact that it is really thin in this kind of process.

3. Boundary conditions

In this problem we have two given boundaries

$$\Gamma_0 : x = 0, \quad \text{median cross section}$$

$$\Gamma_{\text{ext}} : x = L, \quad \text{surface in contact with oil}$$

and three free boundaries

$$\Gamma_m : x = s_m(t), \quad \text{desaturation front}$$

$$\Gamma_v : x = s_v(t), \quad \text{total vaporization front}$$

$$\Gamma_c : x = s_c(t), \quad \text{crust front.}$$

3.1. Conditions on Γ_0

In Ω_w we have $p = p_0$ and we just impose the symmetry condition on the thermal field

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0. \quad (3.1)$$

3.2. Conditions on Γ_{ext}

Here we specify the pressure $p = p_0$ (but in practice this condition becomes active on Γ_c) and a condition on the heat flux.

$$(1 - \epsilon)k_c \left. \frac{\partial T}{\partial x} \right|_{x=L} = \chi(q_v)(T_{\text{oil}} - T), \quad (3.2)$$

where the heat transfer coefficient $\chi(q_v)$ is taken as a decreasing function of the vapor flux in order to mimic the shielding action of the vapor.

3.3. Conditions on Γ_m , $x = s_m(t)$

The desaturation front is the isotherm

$$T(s_m(t), t) = T_0. \quad (3.3)$$

Concerning mass conservation, we must say that on both sides of Γ_m the total mass balance has the form

$$\epsilon \frac{\partial}{\partial t} (\rho_w^0 S_w + \rho_v) + \frac{\partial}{\partial x} (q_w + q_v) = 0 \tag{3.4}$$

and we can impose the Rankine–Hugoniot condition

$$\epsilon [\rho_w^0 S_w + \rho_v]_w^m \dot{s}_m = [q_w + q_v]_w^m, \tag{3.5}$$

with

$$[f]_w^m = f(s_m(t)+, t) - f(s_m(t)-, t).$$

On the left hand side of Γ_m we have $S_w = 1, \rho_v = 0, q_w = q_v = 0$, so that $[S_w]_w^m = S_w - 1, [q_w + q_v]_w^m = q_w + q_v$, where on the right hand side S_w, q_w, q_v refer to the respective limits from Ω_m . Thus (3.5) can be written in the form

$$-\epsilon \rho_w^0 (1 - S_w) \dot{s}_m - q_w = -\epsilon \rho_v \dot{s}_m + q_v, \tag{3.6}$$

that we can interpret as follows:

- $-\epsilon \rho_w^0 (1 - \rho_w) \dot{s}_m =$ water loss rate due to the interface displacement,
- $q_w =$ water flux leaving the interface,
- the l.h.s. of (3.6) is the amount of water converted to vapor on the interface,
- $-\epsilon \rho_v \dot{s}_m =$ vapor gain accompanying the interface motion,
- $q_v =$ vapor flux leaving the interface,
- the r.h.s. of (3.6) is the total vapor production rate at the interface.

Recalling the expressions (2.3) and (2.4) of q_w, q_v and (2.5) of ρ_v , we can give (3.6) its explicit form

$$\epsilon \dot{s}_m (1 - S_w) \{ \rho_v(T_0) - \rho_w^0 \} = -D_w \epsilon \rho_w^0 \frac{\partial S_w}{\partial x} \Big|_{x=s_m(t)+} - K_v (1 - S_w) \lambda \hat{\rho}_v(T_0) \frac{p_0}{RT_0^2} \frac{\partial T}{\partial x} \Big|_{x=s_m(t)+} \tag{3.7}$$

The information just obtained about the vaporization rate allows us to write down the enthalpy balance

$$-[\epsilon \rho_w^0 (1 - S_w) \dot{s}_m + q_w] \lambda = -[\epsilon S_w + (1 - \epsilon) k_s] \frac{\partial T}{\partial x} \Big|_{x=s_m(t)-} + [\epsilon S_w k_w + (1 - \epsilon) k_s] \frac{\partial T}{\partial x} \Big|_{x=s_m(t)+} \tag{3.8}$$

that is

$$\begin{aligned} & - \left[\epsilon (1 - S_w) \dot{s}_m - D_w \epsilon \frac{\partial S_w}{\partial x} \Big|_{x=s_m(t)+} \right] \rho_w^0 \lambda \\ & = -[\epsilon k_w + (1 - \epsilon) k_s] \frac{\partial T}{\partial x} \Big|_{x=s_m(t)-} + [\epsilon S_w k_w + (1 - \epsilon) k_s] \frac{\partial T}{\partial x} \Big|_{x=s_m(t)+}. \end{aligned} \tag{3.9}$$

Remark 2. It is important to stress that the quantity S_w appearing both in (3.5) and (3.7) is not known a priori. However we can say that S_w cannot be 1, since it would imply $\rho_v = 0, q_v = 0$ and consequently $q_w = 0, \frac{\partial S_w}{\partial x} = 0$ and the latter equality is not consistent with the parabolic nature of Eq. (2.6) (Hopf’s boundary point principle) in which ω_v could not be negative in the vicinity of the interface if complete saturation is achieved there. Also we must exclude $S_w = 0$, which would mean the collapse of the region Ω_m .

3.4. Conditions on $\Gamma_v, x = s_v(t)$

This is the most delicate point of the model. The assumption that in Ω_m pressure is given by (2.2) produces a rapid increase of p with T . At $T = 407.2$ K (well below $T_{oil} \simeq 473$ K) p reaches the value $\simeq 3p_0$ (remember that $R = 1.987$ cal/mol, $\lambda = 9717$ cal/mol, $T_0 = 373$ K). For $T = 423$ K (the average between T_0 and T_{oil}) we find $p = 4.7p_0$, and if we let T reach 443 K than p becomes 7.9 times p_0 . It looks unreasonable that such a high pressure can build up within the sample. This suggest to put a constraint p^* on pressure in Ω_m (hence on Γ_v , since T is expected to increase from $x = 0$ to $x = L$). Therefore we distinguish two regimes: the unconstrained and the constrained regime.

3.4.1. The unconstrained regime

As long as

$$p(s_v(t), t) < p^* \tag{3.10}$$

the boundary $x = s_v(t)$ is characterized by total vaporization, i.e.

$$S_w(s_v(t), t) = 0. \tag{3.11}$$

We also impose

$$[p]_m^v = 0, \tag{3.12}$$

$$[\rho_v]_m^v = 0, \tag{3.13}$$

with $[\cdot]_m^v$ denoting jumps.

Note that as a consequence of (3.11) what (3.13) says is that $\rho_v = \hat{\rho}_v(T)$ on both sides of Γ_v . All the incoming water is vaporized. Thus the mass balance is

$$-\left[\rho_v K_v \frac{\partial p}{\partial x} \right]_m^v = -D_w \epsilon \rho_w^0 \frac{\partial S_w}{\partial x} \Big|_{x=s_v(t)-} \tag{3.14}$$

(the continuity of ρ_v has been used), and the enthalpy balance leads to

$$-\lambda D_w \epsilon \rho_w^0 \frac{\partial S_w}{\partial x} \Big|_{x=s_v(t)-} = (1 - \epsilon) k_s \left[\frac{\partial T}{\partial x} \right]_m^v \tag{3.15}$$

neglecting as usual vapor thermal conductivity.

3.4.2. The constrained regime

Now p hits the obstacle and on both sides of Γ_v

$$p(s_v(t), t) = p^* \tag{3.16}$$

before total vaporization is achieved. Therefore (3.11) is no longer valid and its place is taken by constraint induced by (3.16) and temperature:

$$T(s_v(t), t) = T^*, \tag{3.17}$$

where

$$T^* = T_0 \left(1 - \frac{RT_0}{\lambda} \log \frac{p^*}{p_0} \right)^{-1}. \tag{3.18}$$

The total mass balance reads

$$\epsilon [\rho_w^0 S_w + \rho_v]_m^v \dot{s}_v = [q_w + q_v]_m^v, \tag{3.19}$$

where we must remember that instead of (3.13) we now have

$$[\rho_v]_m^v = \hat{\rho}_v(T^*) [1 - (1 - S_w)|_{x=s_v(t)-}] = \hat{\rho}_v(T^*) S_w(s_v(t)-, t). \tag{3.20}$$

Hence (3.19) becomes

$$-\epsilon (\rho_w^0 - [\rho_v]_m^v) S_w(s_v(t)-, t) \dot{s}_v(t) = \epsilon D_w \rho_w^0 \frac{\partial S_w}{\partial x} \Big|_{x=s_v(t)-} + [q_v]_m^v, \tag{3.21}$$

where

$$[q_v]_m^v = \left\{ -\rho_v \bar{K}_v(T^*) \frac{\partial p}{\partial x} \Big|_{x=s_v(t)+} + \hat{\rho}_v K_v(T^*) [1 - S_w(s_v(t)-, t)] \frac{\lambda p^*}{R(T^*)^2} \frac{\partial T}{\partial x} \Big|_{x=s_v(t)-} \right\}. \tag{3.22}$$

The amount of water vaporized per unit time and unit surface of the front is given by the sum of the incoming water flux q_w and water loss rate associated to the displacement of the boundary, i.e. $(-\epsilon\rho_w^0 S_w \dot{s}_v + q_w)|_{x=s_v(t)-}$, yielding the enthalpy balance

$$\lambda(-\epsilon\rho_w^0 S_w \dot{s}_v + q_w)|_{x=s_v(t)-} = -[\epsilon S_w k_w + (1 - \epsilon)k_s] \frac{\partial T}{\partial x} \Big|_{x=s_v(t)-} + (1 - \epsilon)k_s \frac{\partial T}{\partial x} \Big|_{x=s_v(t)+}. \tag{3.23}$$

3.5. Conditions on Γ_c , $x = s_c(t)$

In the previous model [5] a crust formation temperature T_c was introduced (between T^* and T_{oil}). According to this picture the interface Γ_c supports boundary conditions of Stefan type, since it is a temperature level set:

$$T(s_c(t), t) = T_c \tag{3.24}$$

(on both sides of Γ_c), and the change induced in the material is accompanied by heat absorption needed to break chemical bonds in the solid matrix and to vaporize bonded water;

$$\lambda_c \rho_s \dot{s}_c(t) = \left[-(1 - \epsilon)k \frac{\partial T}{\partial x} \right]_v^c, \tag{3.25}$$

where λ_c is perceived as a “latent heat” and k jumps from k_s to k_c .

As we said we also imposed

$$p(s_c(t), t) = p_0. \tag{3.26}$$

Remark 3. The fact that $(L - s_c)/L \ll 1$ for a sufficiently long time requires either T_c sufficiently close to T_{oil} , or λ_c large enough. In the former case Ω_c could be simply neglected, while in the latter case Ω_c plays the role of a thin boundary layer in which temperature may be approximated linearly.

It may be objected that the above scheme is too simplified. Indeed, frying chips (which eventually consists of only crust) at low pressure and lower temperatures (see [3]) shows that the crust formation is not necessarily related to temperature only. It would be more appropriate to say that the formation of the crust is the result of two concurrent situations:

- temperature is above some threshold (apparently larger than 373 K, but not necessarily much larger),
- moisture content (in the form of vapor) is below some (small) threshold (i.e. the medium is almost dehydrated).

Since the latter condition always occurs when approaching the external surface, once $T(L, t)$ is greater than T_0 , the introduction of a sufficiently large crust formation temperature will produce the correct outcome of a thin crust. Since the role of the crust both in heat and mass transport seems marginal (except possibly for heat absorption if λ_c is large enough), determining its actual thickness does not seem a relevant issue.

4. Initial conditions

What happens at the moment in which the oil comes into contact with the sample may be difficult to describe, if e.g. the sample surface is covered by a film of water.

We simply suppose that $p = p_0$ everywhere and

$$T(x, 0) = T_i < T_0, \quad 0 < x < L, \tag{4.1}$$

$$S_w(x, 0) = 1, \quad 0 < x < L. \tag{4.2}$$

Thus during a very short time interval $(0, t_0)$ the whole medium will be occupied by the saturated domain Ω_w at a temperature below T_0 . During that time we just have to solve the heat transport equation (2.1) with conditions (4.1), (3.1) and (3.2) with $q_v = 0$. This may not be completely correct, but $(0, t_0)$ will be anyway so short that no significant error is expected.

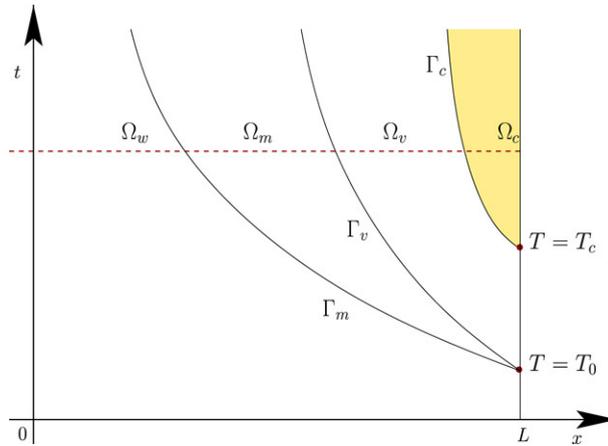


Fig. 2. (x, t) plane.

At time $t = t_0$ both regions Ω_m and Ω_v must appear because $p = p_0$ on $x = L$ prevents Ω_m to have points on the outer surface, hence we have the starting conditions

$$s_m(t_0) = s_v(t_0) = L \tag{4.3}$$

for the two boundaries Γ_m, Γ_v .

For some time the system will evolve in the pressure unconstrained regime, switching to the constrained regime at some later time $t^* > t_0$.

The appearance time of the crust t_c may or may not be larger than t^* , depending on the various parameters. In any case

$$s_c(t_c) = L. \tag{4.4}$$

The qualitative behavior of the onset of the free boundaries is sketched in Fig. 2.

5. Rescaling P.D.E.'s

We take the reference quantities $L = 1$ cm, $\lambda \simeq 9.7 \cdot 10^3$ cal/mol, $T_0 \simeq 373$ K, $\epsilon \simeq 0.3$, $k_w \simeq 1.27 \cdot 10^{-3}$ cal/(s cm K), $k_s \simeq 1.36 \cdot 10^{-3}$ cal/(s cm K), $C_w \simeq 1$ cal/(g K), $C_s \simeq 0.74$ cal/(g K), $\rho_w^0 \simeq 1$ g/cm³, $\rho_s \simeq 1.6$ g/cm³. We can define

$$k_{\text{sat}} = \epsilon k_w + (1 - \epsilon)k_s \quad (\simeq 1.33 \cdot 10^{-3} \text{ cal/(s cm K)})$$

$$C_{\text{sat}} = \epsilon \rho_w^0 C_w + (1 - \epsilon)\rho_s C_s \quad (\simeq 1.13 \text{ cal/(g K)}).$$

Some rescaling is obvious

$$\tilde{x} = \frac{x}{L}, \quad \tilde{T} = \frac{T}{T_0}, \quad \tilde{p} = \frac{p}{p_0}. \tag{5.1}$$

Choosing the reference time is more delicate. Since vaporization is the leading phenomenon, we define

$$t_\lambda = \epsilon \frac{L^2 \lambda \rho_w^0}{k_{\text{sat}} T_0} \quad (\simeq 3.2 \cdot 10^2 \text{ s}), \tag{5.2}$$

and we take

$$\tilde{t} = \frac{t}{t_\lambda}. \tag{5.3}$$

Each of the transport processes taking place in the sample has its own natural time scale which will have to be compared with t_λ .

Let us write down the nondimensional version of the equations.

We denote the rescaled domains by $\tilde{\Omega}_v, \tilde{\Omega}_w$, etc. with interfaces $\tilde{\Gamma}_w$ etc.

5.1. Saturated region $\tilde{\Omega}_w$ — Heat transfer

We define the diffusivity

$$\alpha_{\text{sat}} = \frac{\epsilon k_w + (1 - \epsilon)k_s}{\epsilon \rho_w^0 C_w + (1 - \epsilon)\rho_s C_s} \simeq 1.2 \cdot 10^{-3} \text{ cm}^2/\text{s} \tag{5.4}$$

and the corresponding natural scale for heat diffusion in Ω_w

$$t_{\text{sat}} = \frac{L^2}{\alpha_{\text{sat}}} \simeq 8.3 \cdot 10^2 \text{ s.} \tag{5.5}$$

Eq. (2.1) becomes

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} - \frac{t_\lambda}{t_{\text{sat}}} \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} = 0 \tag{5.6}$$

with

$$\frac{t_\lambda}{t_{\text{sat}}} \simeq 0.38. \tag{5.7}$$

5.2. Mixed region $\tilde{\Omega}_m$ — Mass transport

The nondimensional form of Clapeyron’s law (2.2) is

$$\tilde{p} = e^{\theta(1-\tilde{T}^{-1})}, \quad \theta = \frac{\lambda}{RT_0} \simeq 13.1. \tag{5.8}$$

In Ω_m a natural rescaling factor for $\hat{\rho}_v, \rho_v$ is

$$\rho_v^0 = \frac{m_w p_0}{R T_0} \simeq 5.8 \cdot 10^{-4} \text{ g/cm}^3 \tag{5.9}$$

i.e. the value of $\hat{\rho}$ for $p = p_0, T = T_0$ according to the equation of state of ideal gases. The corresponding nondimensional quantities will be denoted by $\bar{\rho}_v(\tilde{T}) = \hat{\rho}_v(\tilde{T}T_0)/\rho_v^0, \tilde{\rho}_v = \rho_v/\rho_v^0$.

Taking [2] $D_w = 4.10 \cdot 10^{-4} \text{ cm}^2/\text{s}$, we introduce

$$t_{D_w} = \frac{L^2}{D_w} = 2.5 \cdot 10^3 \text{ s} \tag{5.10}$$

such that

$$t_\lambda/t_{D_w} \simeq 0.13. \tag{5.11}$$

The mass balance equation (2.10) is transformed to

$$\begin{aligned} & \left[1 - \frac{\rho_v^0}{\rho_w^0} \tilde{\rho}_v(\tilde{T}) \right] \frac{\partial S_w}{\partial \tilde{t}} - \frac{t_\lambda}{t_{D_w}} \frac{\partial^2 S}{\partial \tilde{x}^2} + \frac{\rho_v^0}{\rho_w^0} (1 - S_w) \left[\frac{d\tilde{\rho}_v}{d\tilde{T}} \frac{\partial \tilde{T}}{\partial \tilde{t}} - \tilde{\rho}_v \frac{t_\lambda}{t_v} \frac{\tilde{p}}{\tilde{T}^2} \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} \right] + \frac{\rho_v^0}{\rho_w^0} \frac{t_\lambda}{t_v} \frac{\partial S_w}{\partial \tilde{x}} \frac{\partial \tilde{T}}{\partial \tilde{x}} \frac{\tilde{p}}{\tilde{T}^2} \\ & - \frac{\rho_v^0}{\rho_w^0} \frac{t_\lambda}{t_v} \tilde{\rho}_v (1 - S_w) \frac{\tilde{p}}{\tilde{T}^2} \left[\frac{1}{\tilde{\rho}_v} \frac{d\tilde{\rho}_v}{d\tilde{T}} + \frac{1}{\tilde{T}} \left(\frac{\theta}{\tilde{T}} - 2 \right) \right] \left(\frac{\partial \tilde{T}}{\partial \tilde{x}} \right)^2 = 0 \end{aligned} \tag{5.12}$$

where one more time scale has been introduced

$$t_v = \frac{\epsilon L^2}{K_v p_0 \theta} \simeq 0.23 \text{ s} \tag{5.13}$$

corresponding to $K_v \simeq 10^{-7} \text{ g}^{-1} \text{ cm}^2 \text{ s}$. Thus

$$\frac{t_\lambda}{t_v} \simeq 1.4 \cdot 10^3.$$

However $\frac{t_\lambda}{t_v}$ appears in (5.12) only through

$$\Theta = \frac{\rho_v^0}{\rho_w^0} \frac{t_\lambda}{t_v} \simeq 1.1 \tag{5.14}$$

and the only negligible terms in (5.12) are the ones containing the ratio $\frac{\rho_v^0}{\rho_w^0}$ not multiplying $\frac{t_\lambda}{t_v}$. Thus the reduced form of (5.12) is

$$\begin{aligned} \frac{\partial S_w}{\partial \tilde{t}} - \frac{t_\lambda}{t_{D_w}} \frac{\partial^2 S_w}{\partial \tilde{x}^2} - \Theta(1 - S_w) \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} \\ + \Theta \frac{\tilde{p}}{\tilde{T}^2} \left\{ \frac{\partial S_w}{\partial \tilde{x}} - \bar{\rho}_v(1 - S_w) \left[\frac{1}{\bar{\rho}_v} \frac{d\bar{\rho}_v}{d\tilde{T}} + \frac{1}{\tilde{T}} \left(\frac{\theta}{\tilde{T}} - 2 \right) \right] \frac{\partial \tilde{T}}{\partial \tilde{x}} \right\} \frac{\partial \tilde{T}}{\partial \tilde{x}} = 0. \end{aligned} \tag{5.15}$$

Remark 4. Eq. (5.15) simply reduces to

$$\frac{\partial S_w}{\partial \tilde{t}} - \frac{t_\lambda}{t_{D_w}} \frac{\partial^2 S_w}{\partial \tilde{x}^2} = 0 \tag{5.16}$$

when $\Theta \ll 1$, decoupling the equation for S_w . However this requires a much smaller value of K_v . Reducing K_v by an order of magnitude would greatly slow down vapor recirculation in Ω_m . Physical information about K_v and \bar{K}_v is scarce, but for sure \bar{K}_v has to be large enough to allow vapor to escape at a sufficiently high rate.

5.3. $\tilde{\Omega}_m$ — Heat transfer

In (2.16) we neglect $\epsilon \rho_v C_v$ in the heat capacity and again we take $K_v = \text{constant}$. We define

$$\gamma_C^w = \frac{\rho_w^0 C_w}{C_{\text{sat}}}, \quad \gamma_C^s = \frac{\rho_s C_s}{C_{\text{sat}}}, \quad \gamma_k^w = \frac{k_w}{k_{\text{sat}}}, \quad \gamma_k^s = \frac{k_s}{k_{\text{sat}}} \tag{5.17}$$

and

$$\sigma = \epsilon \frac{\lambda \rho_v^0}{C_{\text{sat}} T_0} \simeq 2.2 \cdot 10^{-4} \tag{5.18}$$

$$t_w = \frac{L^2 \rho_v^0 C_w}{k_w} \simeq 7.8 \cdot 10^2 \text{ s}. \tag{5.19}$$

We conclude that

$$\frac{t_\lambda}{t_w} \simeq 0.41, \quad \frac{t_\lambda}{t_v} \sigma \simeq 0.31$$

and the nondimensional form of the heat balance equation is

$$\begin{aligned} \left[\epsilon S_w \gamma_C^w + (1 - \epsilon) \gamma_C^s \right] \frac{\partial \tilde{T}}{\partial \tilde{t}} - \left\{ \frac{t_\lambda}{t_{\text{sat}}} [\epsilon S_w \gamma_k^w + (1 - \epsilon) \gamma_k^s] + (1 - S_w) \frac{t_\lambda}{t_v} \sigma \right\} \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} \\ - \left\{ \gamma_C^w \frac{t_\lambda}{t_w} + \gamma_C^s \frac{t_\lambda}{t_{D_w}} + \frac{t_\lambda}{t_v} \sigma \bar{\rho}_v \frac{\tilde{p}}{\tilde{T}^2} \right\} \frac{\partial S_w}{\partial \tilde{x}} \frac{\partial \tilde{T}}{\partial \tilde{x}} \\ = \frac{t_\lambda}{t_v} \sigma \bar{\rho}_v \frac{\tilde{p}}{\tilde{T}^2} (1 - S_w) \left\{ \frac{1}{\bar{\rho}_v} \frac{d\bar{\rho}_v}{d\tilde{T}} + \frac{1}{\tilde{T}} \left(\frac{\theta}{\tilde{T}} - 2 \right) \right\} \left(\frac{\partial \tilde{T}}{\partial \tilde{x}} \right)^2 \end{aligned} \tag{5.20}$$

where we have omitted the terms $\sigma \frac{d\bar{\rho}_v}{d\tilde{T}} \frac{\partial \tilde{T}}{\partial \tilde{t}}$ and $\sigma \bar{\rho}_v \frac{\partial \tilde{S}_v}{\partial \tilde{t}}$ due to the smallness of σ .

Also in this case reducing K_v (i.e. $\frac{t_\lambda}{t_v} \sigma$) by one order of magnitude would imply some (through non-radical) simplification.

5.4. Vapor region $\tilde{\Omega}_v$ — Mass transfer

Eq. (2.22) takes the form

$$\frac{\partial \tilde{p}}{\partial \tilde{t}} - \frac{t_\lambda}{\tilde{t}_v} \left[\tilde{p} \frac{\partial^2 \tilde{p}}{\partial \tilde{x}^2} + \left(\frac{\partial \tilde{p}}{\partial \tilde{x}} \right)^2 - \frac{\tilde{p}}{\tilde{T}} \frac{\partial \tilde{p}}{\partial \tilde{x}} \frac{\partial \tilde{T}}{\partial \tilde{x}} \right] = \frac{\tilde{p}}{\tilde{T}} \frac{\partial \tilde{T}}{\partial \tilde{t}} \tag{5.21}$$

when \bar{K}_v is taken constant and

$$\tilde{t}_v = \frac{\epsilon L^2}{\bar{K}_v p_0} \simeq 0.3 \text{ s.} \tag{5.22}$$

If $\frac{\partial \tilde{T}}{\partial \tilde{t}}$ is not too large, since $\frac{t_\lambda}{t_v} \simeq 10^3$ we can write

$$\tilde{p} \frac{\partial^2 \tilde{p}}{\partial \tilde{x}^2} + \left(\frac{\partial \tilde{p}}{\partial \tilde{x}} \right)^2 - \frac{\tilde{p}}{\tilde{T}} \frac{\partial \tilde{p}}{\partial \tilde{x}} \frac{\partial \tilde{T}}{\partial \tilde{x}} = 0 \tag{5.23}$$

to be solved with the data

$$\tilde{p}|_{\tilde{x}=\tilde{s}_c} = 1, \quad \tilde{p}|_{\tilde{x}=\tilde{s}_v} = e^{\theta(1-\tilde{T}^{-1})}|_{\tilde{x}=\tilde{s}_v} = f \tag{5.24}$$

in the unconstrained case, while the data change to

$$\tilde{p}|_{\tilde{x}=\tilde{s}_c} = 1, \quad \tilde{p}|_{\tilde{x}=\tilde{s}_v} = \tilde{p}^* = e^{\theta(1-(\tilde{T}^*)^{-1})} = f^* \tag{5.25}$$

with obvious meaning of the symbols.¹

Eq. (5.18) can be rewritten as

$$\frac{\partial^2 P}{\partial \tilde{x}^2} - \frac{1}{\tilde{T}} \frac{\partial P}{\partial \tilde{x}} \frac{\partial \tilde{T}}{\partial \tilde{x}} = 0, \tag{5.26}$$

by setting $\frac{1}{2} \tilde{p}^2 = P$ and integrated, yielding

$$\frac{1}{2} \tilde{p}^2(\tilde{x}, \tilde{t}) = \frac{1}{2} f^2 - \frac{1}{2} (f^2 - 1) \int_{\tilde{s}_v(\tilde{t})}^{\tilde{x}} \tilde{T}(\xi, \tilde{t}) d\xi \bigg/ \int_{\tilde{s}_v(\tilde{t})}^{\tilde{s}_c(\tilde{t})} \tilde{T}(\xi, \tilde{t}) d\xi, \tag{5.27}$$

in the case (5.24). In the case (5.25) f has to be replaced with f^* .

Thus pressure becomes a known functional of temperature.

5.5. $\tilde{\Omega}_v$ — Heat transfer

Neglecting the contribution of vapor to heat capacity, we define

$$t_s = \frac{\rho_s C_s L^2}{k_s} \simeq 8.7 \cdot 10^2 \text{ s}, \quad \frac{t_\lambda}{t_s} \simeq 0.36 \tag{5.28}$$

and (2.23) becomes

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} - \frac{t_\lambda}{t_s} \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} - \frac{\epsilon^2 C_v \rho_v^0}{(1-\epsilon)\rho_s C_s} \tilde{\rho}_v \frac{t_\lambda}{\tilde{t}_v} \frac{\partial \tilde{p}}{\partial \tilde{x}} \frac{\partial \tilde{T}}{\partial \tilde{x}} = 0. \tag{5.29}$$

¹ As we will see in the following, the approximation $s_c = 1$ can be meaningful.

Since $\frac{\epsilon^2 C_v \rho_v^0}{(1-\epsilon)\rho_s C_s} \frac{t_\lambda}{t_v} \simeq 10^{-2}$ it is reasonable to use the approximation

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} - \frac{t_\lambda}{t_s} \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} = 0. \tag{5.30}$$

5.6. Crust region $\tilde{\Omega}_c$

We think that in the present problem it is convenient to ignore mass and heat transfer through the crust, concentrating its influence on the boundary conditions.

6. Rescaling boundary conditions

Let us start with the condition on the interfaces.

6.1. Conditions on $\tilde{\Gamma}_m, \tilde{x} = \tilde{s}_c(t)$

Conditions (3.3), (3.7) and (3.9) take the form

$$\tilde{T}(\tilde{s}_m(\tilde{t}), \tilde{t}) = 1, \tag{6.1}$$

$$\frac{d\tilde{s}_m}{d\tilde{t}}(1 - S_w) = \left[\frac{t_\lambda}{t_{D_w}} \frac{\partial S_w}{\partial \tilde{x}} + (1 - S_w) \Theta \frac{\partial \tilde{T}}{\partial \tilde{x}} \right]_{\tilde{x}=\tilde{s}_m(\tilde{t})_+}, \tag{6.2}$$

$$-(1 - S_w) \frac{d\tilde{s}_m}{d\tilde{t}} + \frac{t_\lambda}{t_{D_w}} \frac{\partial S_w}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_m(\tilde{t})_+} = \left[(\epsilon S_w \gamma_k^w + (1 - \epsilon) \gamma_k^s) \frac{\partial \tilde{T}}{\partial \tilde{x}} \right]_{\tilde{x}=\tilde{s}_m(\tilde{t})_+}^m. \tag{6.3}$$

We remark that in deducing (6.1) from (3.3) we have neglected a term containing the ratio $\frac{\rho_v^0}{\rho_w^0} \simeq 5.18 \cdot 10^{-4}$. Recall that such a ratio appears also in the definition of $\Theta = \frac{\rho_v^0 t_\lambda}{\rho_w^0 t_v}$ which is however $O(1)$.

6.2. Conditions on $\tilde{\Gamma}_v, \tilde{x} = \tilde{s}_v(t)$

6.2.1. (A) Unconstrained ($\tilde{p} < \tilde{p}^*$)

Here we have

$$S_w(\tilde{s}_v(\tilde{t}), \tilde{t}) = 0, \tag{6.4}$$

$$[\tilde{\rho}_v]_m^v = [\tilde{\rho}_v]_m^v = 0 \tag{6.5}$$

as a consequence of pressure continuity $[\tilde{p}]_m^v = 0$, while (3.14) becomes

$$\frac{t_{D_w}}{t_v} \frac{\rho_v^0}{\rho_w^0} \tilde{\rho}_v \left[\frac{1}{\theta} \frac{\tilde{K}_v}{K_v} \frac{\partial \tilde{p}}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_v(\tilde{t})_+} - \frac{\tilde{p}}{\tilde{T}^2} \frac{\partial \tilde{T}}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_v(\tilde{t})_-} \right] = \frac{\partial S_w}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_v(\tilde{t})_-}. \tag{6.6}$$

Note that $\frac{t_{D_w}}{t_v} \frac{\rho_v^0}{\rho_w^0} \simeq 5.6$ and that $\frac{1}{\theta} \frac{\tilde{K}_v}{K_v} = O(1)$.

Here we used that in $\tilde{\Omega}_m$

$$\frac{\partial \tilde{p}}{\partial \tilde{x}} = \theta \frac{\tilde{p}}{\tilde{T}^2} \frac{\partial \tilde{T}}{\partial \tilde{x}}. \tag{6.7}$$

Eq. (3.15) takes the form

$$-\frac{t_\lambda}{t_{D_w}} \frac{\partial S_w}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_v^-} = (1 - \epsilon) \tilde{k}_s \left[\frac{\partial \tilde{T}}{\partial \tilde{x}} \right]_m^v. \tag{6.8}$$

6.2.2. (B) Constrained ($\tilde{p} = \tilde{p}^*$)

The whole set of conditions (6.4)–(6.7) must be changed, keeping only (6.8).

The new conditions are

$$\tilde{T}(\tilde{s}_v(\tilde{t}), \tilde{t}) = T^*/T_0 = \tilde{T}^*, \quad [\tilde{T}]_m^v = 0 \tag{6.9}$$

$$\tilde{p}(\tilde{s}_v(\tilde{t}), \tilde{t}) = \tilde{p}^*, \quad [\tilde{p}]_m^v = 0. \tag{6.10}$$

6.3. Conditions on $\tilde{T}_c, \tilde{x} = \tilde{s}_c(t)$

In view of the fact that $1 - \tilde{s}_c$ is small (unless the sample is “overfried”), we have little interest in describing the phenomena within the crust. Instead we extend $\tilde{\Omega}_v$ up to $\tilde{x} = 1$. The crust influence goes into the boundary condition at $\tilde{x} = 1$, as we shall see below. We identify \tilde{T}_c as the isotherm

$$\tilde{T}(\tilde{s}_c(\tilde{t}), \tilde{t}) = \tilde{T}_c = T_c/T_0. \tag{6.11}$$

6.4. Fixed boundaries

For $\tilde{x} = 0$ we have

$$\left. \frac{\partial \tilde{T}}{\partial \tilde{x}} \right|_{\tilde{x}=0} = 0. \tag{6.12}$$

If the progression of the crust front is accompanied by heat absorption, for $\tilde{x} = 1$ we write the following balance: the heat flux coming from Ω_v (extended up to $\tilde{x} = 1$) minus the heat flux leaving the system equals the absorption rate $\lambda_c \rho_s \dot{\tilde{s}}_c$. Since, neglecting the crust thickness, the outgoing heat flux is $\chi(q_v)(T_{oil} - T)|_{\tilde{x}=1}$, passing to nondimensional variables we write

$$-\frac{t_\lambda^c}{t_\lambda} \frac{d\tilde{s}_c}{d\tilde{t}} = -(1 - \epsilon) \frac{\partial \tilde{T}}{\partial \tilde{x}} + \Xi(\tilde{q}_v)(\tilde{T}_{oil} - \tilde{T}), \quad \tilde{x} = 1, \tag{6.13}$$

where

$$t_\lambda^c = \frac{(1 - \epsilon)\lambda_c L^2 \rho_s}{k_s T_0}, \tag{6.14}$$

$$\Xi(\tilde{q}_v) = \frac{T_0 L}{k_s} \chi(q_v), \tag{6.15}$$

and

$$\tilde{q}_v = \frac{q_v}{Q}, \quad Q = \bar{K}_v \rho_v^0 \frac{P_0}{L} \tag{6.16}$$

i.e.

$$\tilde{q}_v = -\frac{\partial \tilde{p}}{\partial \tilde{x}} \tilde{\rho}_v(1, \tilde{T}). \tag{6.17}$$

A possible choice for $\Xi(\xi)$ can be

$$\Xi(\xi) = \Xi_0 e^{-\xi^2} \tag{6.18}$$

(slowly decreasing near $\xi = 0$). Since we want $|\frac{\partial \tilde{s}_c}{\partial \tilde{t}}| \ll 1$ it seems that its coefficient in (6.13) should be large. Even in this situation (6.13) and (6.15) are a peculiar and nontrivial pair of free boundary conditions. However, if on the contrary $\frac{t_\lambda^c}{t_\lambda} \ll 1$ we may replace (6.13) with

$$-(1 - \epsilon) \frac{\partial \tilde{T}}{\partial \tilde{x}} + \Xi(\tilde{q}_v)(\tilde{T}_{oil} - \tilde{T}) = 0, \quad \text{for } \tilde{x} = 1. \tag{6.19}$$

In that case the penetration of the isotherm $T = T_c$ will be slow, as we suppose, if $\Xi(\tilde{q}_v)$ becomes small, due to the effect of the outgoing vapor flux.

Remark 5. Since (6.13) (or (6.19)) is the result of the approximation $\tilde{s}_c \simeq 1$, condition (6.11) strictly speaking is no longer applicable and it can be used to monitor how good the approximation $\tilde{s}_c \simeq 1$ is, computing \tilde{s}_c by means of (6.13) and evaluating \tilde{T} on it.

7. Summary of the rescaled problem. The mathematical structure

With the nondimensional quantities introduced in Sections 5 and 6 the problem to be solved consists in the equations:

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} - \frac{t_\lambda}{t_{\text{sat}}} \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} = 0 \quad \text{in } \tilde{\Omega}_w, \tag{7.1}$$

$$\begin{aligned} \frac{\partial S_w}{\partial \tilde{t}} - \frac{t_\lambda}{t_{D_w}} \frac{\partial^2 S_w}{\partial \tilde{x}^2} - \Theta(1 - S_w) \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} \\ + \Theta \frac{\tilde{p}}{\tilde{T}^2} \left\{ \frac{\partial S_w}{\partial \tilde{x}} - \bar{\rho}_v(1 - S_w) \left[\frac{1}{\bar{\rho}_v} \frac{d\bar{\rho}_v}{d\tilde{T}} + \frac{1}{\tilde{T}} \left(\frac{\theta}{\tilde{T}} - 2 \right) \right] \right\} \frac{\partial \tilde{T}}{\partial \tilde{x}} \frac{\partial \tilde{T}}{\partial \tilde{x}} = 0 \quad \text{in } \tilde{\Omega}_m, \end{aligned} \tag{7.2}$$

$$\begin{aligned} [\epsilon S_w \gamma_C^w + (1 - \epsilon) \gamma_C^s] \frac{\partial \tilde{T}}{\partial \tilde{t}} - \left\{ \frac{t_\lambda}{t_{\text{sat}}} [\epsilon S_w \gamma_k^w + (1 - \epsilon) \gamma_k^s] + (1 - S_w) \frac{t_\lambda}{t_v} \sigma \right\} \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} \\ - \left\{ \gamma_C^w \frac{t_\lambda}{t_w} + \gamma_C^w \frac{t_\lambda}{t_{D_w}} + \frac{t_\lambda}{t_v} \sigma \bar{\rho}_v \frac{\tilde{p}}{\tilde{T}^2} \right\} \frac{\partial S_w}{\partial \tilde{x}} \frac{\partial \tilde{T}}{\partial \tilde{x}} \\ = \frac{t_\lambda}{t_v} \sigma \bar{\rho}_v \frac{\tilde{p}}{\tilde{T}^2} (1 - S_w) \left\{ \frac{1}{\bar{\rho}_v} \frac{d\bar{\rho}_v}{d\tilde{T}} + \frac{1}{\tilde{T}} \left(\frac{\theta}{\tilde{T}} - 2 \right) \right\} \left(\frac{\partial \tilde{T}}{\partial \tilde{x}} \right)^2 \quad \text{in } \tilde{\Omega}_m, \end{aligned} \tag{7.3}$$

$$\frac{1}{2} \tilde{p}^2(\tilde{x}, \tilde{t}) = \frac{1}{2} f^2 - \frac{1}{2} (f^2 - 1) \int_{\tilde{s}_v(\tilde{t})}^{\tilde{x}} \tilde{T}(\xi, \tilde{t}) d\xi \Big/ \int_{\tilde{s}_v(\tilde{t})}^{\tilde{s}_c(\tilde{t})} \tilde{T}(\xi, \tilde{t}) d\xi \quad \text{in } \tilde{\Omega}_v, \tag{7.4}$$

with f given by (5.24) or by (5.25),

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} - \frac{t_\lambda}{t_s} \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} = 0 \quad \text{in } \tilde{\Omega}_v. \tag{7.5}$$

On the free boundaries $\tilde{\Gamma}_m, \tilde{\Gamma}_v$ both temperature and pressure are continuous, and

$$\tilde{T}(\tilde{s}_m(\tilde{t}), \tilde{t}) = 1 \quad \text{on } \tilde{\Gamma}_m, \tag{7.6}$$

$$\frac{d\tilde{s}_m}{d\tilde{t}} (1 - S_w) = \left[\frac{t_\lambda}{t_{D_w}} \frac{\partial S_w}{\partial \tilde{x}} + (1 - S_w) \Theta \frac{\partial \tilde{T}}{\partial \tilde{x}} \right]_{\tilde{x}=\tilde{s}_m(\tilde{t})_+} \quad \text{on } \tilde{\Gamma}_m, \tag{7.7}$$

$$\begin{aligned} -(1 - S_w) \frac{d\tilde{s}_m}{d\tilde{t}} + \frac{t_\lambda}{t_{D_w}} \frac{\partial S_w}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_m(\tilde{t})_+} &= \left[(\epsilon S_w \gamma_k^w + (1 - \epsilon) \gamma_k^s) \frac{\partial \tilde{T}}{\partial \tilde{x}} \right]_{\tilde{x}=\tilde{s}_m(\tilde{t})_+} \\ &- \left[(\epsilon S_w \gamma_k^w + (1 - \epsilon) \gamma_k^s) \frac{\partial \tilde{T}}{\partial \tilde{x}} \right]_{\tilde{x}=\tilde{s}_m(\tilde{t})_-} \quad \text{on } \tilde{\Gamma}_m, \\ - \frac{t_\lambda}{t_{D_w}} \frac{\partial S_w}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_v^-} &= (1 - \epsilon) \tilde{k}_s \left[\frac{\partial \tilde{T}}{\partial \tilde{x}} \right]_m^v \quad \text{on } \tilde{\Gamma}_v \end{aligned} \tag{7.8}$$

and, again on \tilde{T}_v , either

$$S_w(\tilde{s}_v(\tilde{t}), \tilde{t}) = 0, \tag{7.9}$$

$$[\tilde{\rho}_v]_m^v = [\tilde{\rho}_v]_m^v = 0 \tag{7.10}$$

$$\frac{t_{D_w}}{t_v} \frac{\rho_w^0}{\rho_w^0} \tilde{\rho}_v \left[\frac{1}{\theta} \frac{\tilde{K}_v}{K_v} \frac{\partial \tilde{p}}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_v(\tilde{t})+} - \frac{\tilde{p}}{\tilde{T}^2} \frac{\partial \tilde{T}}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_v(\tilde{t})-} \right] = \frac{\partial S_w}{\partial \tilde{x}} \Big|_{\tilde{x}=\tilde{s}_v(\tilde{t})-} \tag{7.11}$$

in the unconstrained case ($\tilde{p} < \tilde{p}^*$), or

$$\tilde{T}(\tilde{s}_v(\tilde{t}), \tilde{t}) = T^*/T_0 = \tilde{T}^*, \tag{7.12}$$

$$\tilde{p}(\tilde{s}_v(\tilde{t}), \tilde{t}) = \tilde{p}^* \tag{7.13}$$

in the constrained case,

$$\frac{\partial \tilde{T}}{\partial \tilde{x}} \Big|_{\tilde{x}=0} = 0 \quad \text{on } \tilde{x} = 0, \tag{7.14}$$

and finally, on \tilde{T}_c , approximated as $\tilde{x} = 1$,

$$-\frac{t_\lambda^c}{t_\lambda} \frac{d\tilde{s}_c}{d\tilde{t}} = -(1 - \epsilon) \frac{\partial \tilde{T}}{\partial \tilde{x}} + \Xi(\tilde{q}_v)(\tilde{T}_{oil} - \tilde{T}), \tag{7.15}$$

(with (6.18) as a possible choice of Ξ), possibly simplified to

$$-(1 - \epsilon) \frac{\partial \tilde{T}}{\partial \tilde{x}} + \Xi(\tilde{q}_v)(\tilde{T}_{oil} - \tilde{T}) = 0, \quad \text{for } \tilde{x} = 1. \tag{7.16}$$

We remind the values of the ratios of all time scales to t_λ :

$$\frac{t_\lambda}{t_{sat}} \simeq 0.38, \quad \frac{t_\lambda}{t_{D_w}} \simeq 0.13, \quad \frac{t_\lambda}{t_v} \simeq 1.4 \cdot 10^3 \left(\frac{t_\lambda}{t_v} \sigma = 0.31 \right), \tag{7.17}$$

$$\frac{t_\lambda}{t_v} \simeq 10^3, \quad \frac{t_\lambda}{t_w} \simeq 0.41, \quad \frac{t_\lambda}{t_s} \simeq 0.36. \tag{7.18}$$

In the scheme above the only region in which we have to solve a system of partial differential equations is $\tilde{\Omega}_m$. Introducing the vector

$$\vec{U} = \begin{pmatrix} S_w \\ \tilde{T} \end{pmatrix} \tag{7.19}$$

the system (7.2), (7.3) can be written in the form

$$\frac{\partial \vec{U}}{\partial t} = \mathcal{A}(\vec{U}) \frac{\partial^2 \vec{U}}{\partial x^2} + \vec{B} \left(\vec{U}, \frac{\partial \vec{U}}{\partial x} \right) \tag{7.20}$$

where \mathcal{A} is the 2×2 matrix

$$\begin{pmatrix} \frac{t_\lambda}{t_{D_w}} & \Theta(1 - S_w) \\ 0 & \frac{\bar{k}}{\bar{C}} \end{pmatrix}, \tag{7.21}$$

\bar{k} , \bar{C} denoting the coefficients of $\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2}$ and of $\frac{\partial \tilde{T}}{\partial \tilde{t}}$ in (7.3), respectively.

It is immediate to recognize that (7.20) is uniformly parabolic in the sense of Petrowski, following the definition of [11].

Thus problem (7.1)–(7.16) has the structure of a parabolic free boundary problem. We remark that if we can set $\Theta \simeq 0$ the results of [12], in which \mathcal{A} is diagonal, are directly applicable. If not it seems crucial to assume $\frac{\tilde{t}_v}{\tilde{D}_w} \neq \frac{\tilde{k}}{\tilde{C}}$ so that the matrix is not of Jordan type. Both the theoretical and numerical investigation of (7.1)–(7.16) look extremely difficult. Very delicate questions are for instance the simultaneous appearance of the free boundaries $\tilde{\Gamma}_m$, $\tilde{\Gamma}_v$ and the implicit nature of the boundary $\tilde{\Gamma}_m$ in the pressure unconstrained regime. We leave these questions open.

8. Conclusions

We have formulated a mathematical model for deep frying processes, assuming that the sample undergoing frying is sufficiently thick so as to neglect deformation. Thus frying is described as vaporization of water in an initially saturated undeformable porous material. The chosen geometrical setting is one dimensional. In the generic stage of the process, proceeding from the median section to the outer surface (in contact with the oil) one finds a saturated region below the boiling point, a mixed region in which liquid water and vapor coexist in thermodynamical equilibrium, a pure vapor zone, the crust. The latter is assumed to be thin and transparent to vapor, but its formation possibly requiring some latent heat. The temperature and the pressure of vapor (when present) are the field variables, the interfaces separating the regions listed above are free boundaries. A double regime is proposed on the total vaporization interface: unconstrained pressure or constrained pressure. The latter has the role to prevent excessive pressure build-up due to temperature rise.

Using data available in the literature the model has been reduced to a nondimensional version, emphasizing that its intrinsic complication survives rescaling. Only when the permeability to the vapor on the mixed region is small enough is some more effective simplification possible. It is shown that the governing equations are parabolic. The questions of well-posedness and numerical computations are left open. Nevertheless, we believe that the material presented here represents some progress in the direction of providing a description of deep frying processes with a solid physical basis.

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