# Idempotent and tropical mathematics. Complexity of algorithms and interval analysis

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#### Abstract

A very brief introduction to tropical and idempotent mathematics is presented. Tropical mathematics can be treated as a result of a dequantization of the traditional mathematics as the Planck constant tends to zero taking imaginary values. In the framework of idempotent mathematics usually constructions and algorithms are more simple with respect to their traditional analogs. We especially examine algorithms of tropical/idempotent mathematics generated by a collection of basic semiring (or semifield) operations and other "good" operations. Every algorithm of this type has an interval version. The complexity of this interval version coincides with the complexity of the initial algorithm. The interval version of an algorithm of this type gives exact interval estimates for the corresponding output data. Algorithms of linear algebra over idempotent and semirings are examined. In this case, basic algorithms are polynomial as well as their interval versions. This situation is very different from the traditional linear algebra, where basic algorithms are polynomial but the corresponding interval versions are NP-hard and interval estimates are not exact.

*Keywords:* Tropical mathematics, idempotent mathematics, complexity of algorithms, interval analysis.

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# 1. Introduction

Tropical mathematics can be treated as a result of a dequantization of the traditional mathematics as the Planck constant tends to zero taking imaginary values. This kind of dequantization is known as the Maslov dequantization and it leads to a mathematics over tropical algebras like the max-plus algebra. The

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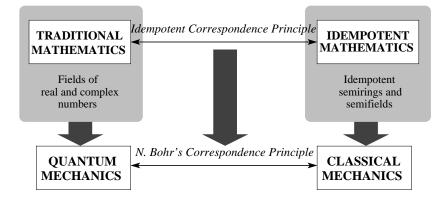


Figure 1: Relations between idempotent and traditional mathematics.

so-called idempotent dequantization is a generalization of the Maslov dequantization. The idempotent dequantization leads to mathematics over idempotent semirings (exact definitions see below in sections 2 and 3). For example, the field of real or complex numbers can be treated as a quantum object whereas idempotent semirings can be examined as "classical" or "semiclassical" objects (a semiring is called idempotent if the semiring addition is idempotent, i.e.  $x \oplus x = x$ ), see [25, 26, 27, 28, 29].

Tropical algebras are idempotent semirings (and semifields). Thus tropical mathematics is a part of idempotent mathematics. Tropical algebraic geometry can be treated as a result of the Maslov dequantization applied to the traditional algebraic geometry (O. Viro, G. Mikhalkin), see, e.g., [21, 52, 53, 66, 67, 68]. There are interesting relations and applications to the traditional convex geometry.

In the spirit of N.Bohr's correspondence principle there is a (heuristic) correspondence between important, useful, and interesting constructions and results over fields and similar results over idempotent semirings. A systematic application of this correspondence principle (which is a basic paradigm in idempotent/tropical mathematcs) leads to a variety of theoretical and applied results [25, 26, 27, 28, 29, 30, 38], see Fig.1.

The history of the subject is discussed, e.g., in [25]. There is a large list of references.

In the framework of idempotent mathematics usually constructions and algorithms are more simple with respect to their traditional analogs (however, there exist NP-hard problems in tropical linear algebra). We especially examine algorithms of tropical/idempotent mathematics generated by a collection of basic semiring (or semifield) operations and other "good" operations. Every algorithm of this type has an interval version. The complexity of this interval version coincides with the complexity of the initial algorithm. The interval version of an algorithm of this type gives exact interval estimates for the corresponding output data. Algorithms of linear algebra over idempotent and semirings are examined. In this case, basic algorithms are polynomial as well as their interval versions. This situation is very different from the traditional linear algebra, where basic algorithms are polynomial but the corresponding interval versions are NP-hard and interval estimates are not exact.

### 2. The Maslov dequantization

Let **R** and **C** be the fields of real and complex numbers. The so-called maxplus algebra  $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$  is defined by the operations  $x \oplus y = \max\{x, y\}$  and  $x \odot y = x + y$ .

The max-plus algebra can be treated as a result of the *Maslov dequantization* of the semifield  $\mathbf{R}_+$  of all nonnegative numbers with the usual arithmetics. The change of variables

$$x \mapsto u = h \log x_{i}$$

where h > 0, defines a map  $\Phi_h : \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$ . This logarithmic transform was used by many authors. Let the addition and multiplication operations be mapped from  $\mathbf{R}_+$  to  $\mathbf{R} \cup \{-\infty\}$  by  $\Phi_h$ , i.e. let

$$u \oplus_h v = h \log(\exp(u/h) + \exp(v/h)), \quad u \odot v = u + v,$$
$$\mathbf{0} = -\infty = \Phi_h(0), \quad \mathbf{1} = 0 = \Phi_h(1).$$

It can easily be checked that  $u \oplus_h v \to \max\{u, v\}$  as  $h \to 0$ . Thus we get the semifield  $\mathbf{R}_{\max}$  (i.e. the max-plus algebra) with zero  $\mathbf{0} = -\infty$  and unit  $\mathbf{1} = 0$  as a result of this deformation of the algebraic structure in  $\mathbf{R}_+$ .

The semifield  $\mathbf{R}_{\max}$  is a typical example of an *idempotent semiring*; this is a semiring with idempotent addition, i.e.,  $x \oplus x = x$  for arbitrary element x of this semiring.

The semifield  $\mathbf{R}_{\text{max}}$  is also called a *tropical algebra*. The semifield  $\mathbf{R}^{(h)} = \Phi_h(\mathbf{R}_+)$  with operations  $\oplus_h$  and  $\odot$  (i.e.+) is called a *subtropical algebra*.

The semifield  $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$  with operations  $\oplus = \min$  and  $\odot = +$  $(\mathbf{0} = +\infty, \mathbf{1} = 0)$  is isomorphic to  $\mathbf{R}_{\max}$ .

The analogy with quantization is obvious; the parameter h plays the role of the Planck constant. The map  $x \mapsto |x|$  and the Maslov dequantization for  $\mathbf{R}_+$ give us a natural transition from the field  $\mathbf{C}$  (or  $\mathbf{R}$ ) to the max-plus algebra  $\mathbf{R}_{\max}$ . We will also call this transition the Maslov dequantization. In fact the Maslov dequantization corresponds to the usual Schrödinger dequantization but for imaginary values of the Planck constant (see below). The transition from numerical fields to the max-plus algebra  $\mathbf{R}_{\max}$  (or similar semifields) in mathematical constructions and results generates the so called *tropical mathematics*. The so-called *idempotent dequantization* is a generalization of the Maslov dequantization; this is the transition from basic fields to idempotent semirings in mathematical constructions and results without any deformation. The idempotent dequantization generates the so-called *idempotent mathematics*, i.e. mathematics over idempotent semifields and semirings. Recently new versions of the Maslov dequantization appeared, see, e.g. [69]. **Remark.** The term 'tropical' appeared in [54] for a discrete version of the max-plus algebra (as a suggestion of Christian Choffrut). On the other hand V.P. Maslov used this term in 80s in his talks and works on economical applications of his idempotent analysis (related to colonial politics). For the most part of modern authors, 'tropical' means 'over  $\mathbf{R}_{\max}$  (or  $\mathbf{R}_{\min}$ )' and tropical algebras are  $\mathbf{R}_{\max}$  and  $\mathbf{R}_{\min}$ . The terms 'max-plus', 'max-algebra' and 'min-plus' are often used in the same sense.

# 3. Semirings and semifields

Consider a set S equipped with two algebraic operations:  $addition \oplus$  and  $multiplication \odot$ . It is a *semiring* if the following conditions are satisfied:

- the addition  $\oplus$  and the multiplication  $\odot$  are associative;
- the addition  $\oplus$  is commutative;
- the multiplication  $\odot$  is distributive with respect to the addition  $\oplus$ :

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

and

$$(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$$

for all  $x, y, z \in S$ .

A unity of a semiring S is an element  $\mathbf{1} \in S$  such that  $\mathbf{1} \odot x = x \odot \mathbf{1} = x$  for all  $x \in S$ . A zero of a semiring S is an element (if it exists)  $\mathbf{0} \in S$  such that  $\mathbf{0} \neq \mathbf{1}$  and  $\mathbf{0} \oplus x = x$ ,  $\mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}$  for all  $x \in S$ . A semiring S is called an *idempotent semiring* if  $x \oplus x = x$  for all  $x \in S$ . A semiring S with a neutral element  $\mathbf{1}$  is called a *semifield* if every nonzero element of S is invertible with respect to the multiplication. The theory of semirings and semifields is treated, e.g., in [14].

# 4. Idempotent analysis

Idempotent analysis deals with functions taking their values in an idempotent semiring and the corresponding function spaces. Idempotent analysis was initially constructed by V. P. Maslov and his collaborators and then developed by many authors. The subject is presented in the book of V. N. Kolokoltsov and V. P. Maslov [22] (a version of this book in Russian was published in 1994).

Let S be an arbitrary semiring with idempotent addition  $\oplus$  (which is always assumed to be commutative), multiplication  $\odot$ , and unit **1**. The set S is supplied with the *standard partial order*  $\preceq$ : by definition,  $a \preceq b$  if and only if  $a \oplus b = b$ . If the zero element exists, then all elements of S are nonnegative:  $\mathbf{0} \preceq a$  for all  $a \in S$ . Due to the existence of this order, idempotent analysis is closely related to the lattice theory, theory of vector lattices, and theory of ordered spaces. Moreover, this partial order allows to model a number of basic "topological" concepts and results of idempotent analysis at the purely algebraic level; this line of reasoning was examined systematically in [25]–[44] and [10].

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map  $X \to S$ , where X is an arbitrary set and S is an idempotent semiring. Functions with values in S can be added, multiplied by each other, and multiplied by elements of S pointwise.

The idempotent analog of a linear functional space is a set of S-valued functions that is closed under addition of functions and multiplication of functions by elements of S, or an S-semimodule. Consider, e.g., the S-semimodule B(X, S)of all functions  $X \to S$  that are bounded in the sense of the standard order on S.

If  $S = \mathbf{R}_{\max}$ , then the idempotent analog of integration is defined by the formula

$$I(\varphi) = \int_X^{\oplus} \varphi(x) \, dx = \sup_{x \in X} \varphi(x), \tag{1}$$

where  $\varphi \in B(X, S)$ . Indeed, a Riemann sum of the form  $\sum_{i} \varphi(x_i) \cdot \sigma_i$  corresponds to the expression  $\bigoplus_{i} \varphi(x_i) \odot \sigma_i = \max_{i} \{\varphi(x_i) + \sigma_i\}$ , which tends to the right-hand side of (1) as  $\sigma_i \to 0$ . Of course, this is a purely heuristic argument.

Formula (1) defines the *idempotent* (or *Maslov*) *integral* not only for functions taking values in  $\mathbf{R}_{\text{max}}$ , but also in the general case when any of bounded (from above) subsets of S has the least upper bound.

An *idempotent* (or *Maslov*) measure on X is defined by the formula  $m_{\psi}(Y) = \sup_{x \in Y} \psi(x)$ , where  $\psi \in B(X, S)$  is a fixed function. The integral with respect to this measure is defined by the formula

$$I_{\psi}(\varphi) = \int_{X}^{\oplus} \varphi(x) \, dm_{\psi} = \int_{X}^{\oplus} \varphi(x) \odot \psi(x) \, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)).$$
(2)

Obviously, if  $S = \mathbf{R}_{\min}$ , then the standard order is opposite to the conventional order  $\leq$ , so in this case equation (2) assumes the form

$$\int_X^{\oplus} \varphi(x) \, dm_{\psi} = \int_X^{\oplus} \varphi(x) \odot \psi(x) \, dx = \inf_{x \in X} (\varphi(x) \odot \psi(x)),$$

where inf is understood in the sense of the conventional order  $\leq$ .

#### 5. The superposition principle and linear problems

Basic equations of quantum theory are linear; this is the superposition principle in quantum mechanics. The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However, it is linear over the semirings  $\mathbf{R}_{\text{max}}$  and  $\mathbf{R}_{\text{min}}$ . Similarly, different versions of the Bellman equation, the basic equation of optimization theory, are linear over

suitable idempotent semirings; this is V. P. Maslov's idempotent superposition principle, see [48, 49, 50]. For instance, the finite-dimensional stationary Bellman equation can be written in the form  $X = H \odot X \oplus F$ , where X, H, F are matrices with coefficients in an idempotent semiring S and the unknown matrix X is determined by H and F [7, 8]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases  $S = \mathbf{R}_{\text{max}}$  and  $S = \mathbf{R}_{\text{min}}$ , respectively. It is known that principal optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman's shortest path algorithm corresponds to a version of Jacobi's algorithm, Ford's algorithm corresponds to the Gauss–Seidel iterative scheme, etc. [7, 8].

The linearity of the Hamilton–Jacobi equation over  $\mathbf{R}_{\min}$  and  $\mathbf{R}_{\max}$ , which is the result of the Maslov dequantization of the Schrödinger equation, is closely related to the (conventional) linearity of the Schrödinger equation and can be deduced from this linearity. Thus, it is possible to borrow standard ideas and methods of linear analysis and apply them to a new area.

Consider a classical dynamical system specified by the Hamiltonian

$$H = H(p, x) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x),$$

where  $x = (x_1, \ldots, x_N)$  are generalized coordinates,  $p = (p_1, \ldots, p_N)$  are generalized momenta,  $m_i$  are generalized masses, and V(x) is the potential. In this case the Lagrangian  $L(x, \dot{x}, t)$  has the form

$$L(x, \dot{x}, t) = \sum_{i=1}^{N} m_i \frac{\dot{x}_i^2}{2} - V(x),$$

where  $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_N)$ ,  $\dot{x}_i = dx_i/dt$ . The value function S(x, t) of the action functional has the form

$$S = \int_{t_0}^t L(x(t), \dot{x}(t), t) \, dt,$$

where the integration is performed along the factual trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle, or the least action principle).

For fixed values of t and  $t_0$  and arbitrary trajectories x(t), the action functional S = S(x(t)) can be considered as a function taking the set of curves (trajectories) to the set of real numbers which can be treated as elements of  $\mathbf{R}_{\min}$ . In this case the minimum of the action functional can be viewed as the Maslov integral of this function over the set of trajectories or an idempotent analog of the Euclidean version of the Feynman path integral. The minimum of the action functional corresponds to the maximum of  $e^{-S}$ , i.e. idempotent integral  $\int_{\{paths\}}^{\oplus} e^{-S(x(t))} D\{x(t)\}$  with respect to the max-plus algebra  $\mathbf{R}_{\max}$ . Thus the least action principle can be considered as an idempotent version of the well-known Feynman approach to quantum mechanics. The representation of a solution to the Schrödinger equation in terms of the Feynman integral corresponds to the Lax–Oleĭnik solution formula for the Hamilton–Jacobi equation.

Since  $\partial S/\partial x_i = p_i$ ,  $\partial S/\partial t = -H(p, x)$ , the following Hamilton–Jacobi equation holds:

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x_i}, x_i\right) = 0. \tag{3}$$

Quantization leads to the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \hat{H}\psi = H(\hat{p}_i, \hat{x}_i)\psi, \qquad (4)$$

where  $\psi = \psi(x,t)$  is the wave function, i.e., a time-dependent element of the Hilbert space  $L^2(\mathbf{R}^N)$ , and  $\hat{H}$  is the energy operator obtained by substitution of the momentum operators  $\hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial x_i}$  and the coordinate operators  $\hat{x}_i : \psi \mapsto x_i \psi$  for the variables  $p_i$  and  $x_i$  in the Hamiltonian function, respectively. This equation is linear in the conventional sense (the quantum superposition principle). The standard procedure of limit transition from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function:  $\psi(x,t) = a(x,t)e^{iS(x,t)/\hbar}$ , and to keep only the leading order as  $\hbar \to 0$  (the 'semiclassical' limit).

Instead of doing this, we switch to imaginary values of the Planck constant  $\hbar$  by the substitution  $h = i\hbar$ , assuming h > 0. Thus the Schrödinger equation (4) turns to an analog of the heat equation:

$$h\frac{\partial u}{\partial t} = H\left(-h\frac{\partial}{\partial x_i}, \hat{x}_i\right)u,\tag{5}$$

where the real-valued function u corresponds to the wave function  $\psi$ . A similar idea (the switch to imaginary time) is used in the Euclidean quantum field theory; let us remember that time and energy are dual quantities.

Linearity of equation (4) implies linearity of equation (5). Thus if  $u_1$  and  $u_2$  are solutions of (5), then so is their linear combination

$$u = \lambda_1 u_1 + \lambda_2 u_2. \tag{6}$$

Let  $S = h \ln u$  or  $u = e^{S/h}$  as in Section 2 above. It can easily be checked that equation (5) thus turns to

$$\frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^{N} \frac{1}{2m_i} \left(\frac{\partial S}{\partial x_i}\right)^2 + h \sum_{i=1}^{n} \frac{1}{2m_i} \frac{\partial^2 S}{\partial x_i^2}.$$
(7)

Thus we have a transition from (4) to (7) by means of the change of variables  $\psi = e^{S/h}$ . Note that  $|\psi| = e^{ReS/h}$ , where ReS is the real part of S. Now let us consider S as a real variable. The equation (7) is nonlinear in the conventional sense. However, if  $S_1$  and  $S_2$  are its solutions, then so is the function

$$S = \lambda_1 \odot S_1 \oplus_h \lambda_2 \odot S_2$$

obtained from (6) by means of our substitution  $S = h \ln u$ . Here the generalized multiplication  $\odot$  coincides with the ordinary addition and the generalized addition  $\oplus_h$  is the image of the conventional addition under the above change of variables. As  $h \to 0$ , we obtain the operations of the idempotent semiring  $\mathbf{R}_{\max}$ , i.e.,  $\oplus = \max$  and  $\odot = +$ , and equation (7) turns to the Hamilton–Jacobi equation (3), since the third term in the right-hand side of equation (7) vanishes.

Thus it is natural to consider the limit function  $S = \lambda_1 \odot S_1 \oplus \lambda_2 \odot S_2$  as a solution of the Hamilton–Jacobi equation and to expect that this equation can be treated as linear over  $\mathbf{R}_{\max}$ . This argument (clearly, a heuristic one) can be extended to equations of a more general form. For a rigorous treatment of (semiring) linearity for these equations see, e.g., [22, 30]. Notice that if h is changed to -h, then we have that the resulting Hamilton–Jacobi equation is linear over  $\mathbf{R}_{\min}$ .

The idempotent superposition principle indicates that there exist important nonlinear (in the traditional sense) problems that are linear over idempotent semirings. The idempotent linear functional analysis (see [22, 26, 30, 33, 34, 35, 36, 43, 44, 51, 10, 16]) is a natural tool for investigation of those nonlinear infinite-dimensional problems that possess this property. In practice infinitedimensional problems can be approximated by finite-dimensional problems. So algorithms of idempotent linear algebras are especially important (because of the superposition principle!). Below some examples are examined.

# 6. Applications

There are very many important applications of tropical/idempotent mathematics (and especially the correspondence and superposition principles) including optimization and control, algebraic geometry, dynamic programming, differential equations, mathematical biology, mathematical physics and chemistry, transport and energoenergetic netwoks, interval analysis, mathematical economics, game theory, computer technology etc., see, e.g. [7, 8, 19, 22, 25, 27, 28, 30, 31, 38, 41, 42, 48, 49, 51, 52, 53, 59, 66, 68, 69]. Applications of the idempotent correspondence principles to software and hardware design are examined, e.g. in [27, 28, 38]. Some applications are discucced in [32, 31].

### 7. Positive semirings and basic operations

# 7.1. Some definitions

Let the semiring S be partially ordered (see, e.g. [4, 14] and Subsection 9.2 below) by a relation  $\preceq$  such that **0** is the least element and the inequality  $x \preceq y$  implies that  $x \oplus z \preceq y \oplus z$ ,  $x \odot z \preceq y \odot z$ , and  $z \odot x \preceq z \odot y$  for all  $x, y, z \in S$ ; in this case the semiring S is called *positive* (see, e.g., [14]).

Recall that a semiring S is called *idempotent* if  $x \oplus x = x$  for all  $x \in S$ . In this case the addition  $\oplus$  defines a *canonical (or standard) partial order*  $\preceq$  on the semiring S by the rule:  $x \preceq y$  iff  $x \oplus y = y$ . It is easy to prove that any idempotent semiring is positive with respect to this order. Note also that  $x \oplus y = \sup\{x, y\}$  with respect to the canonical order. In the sequel, we shall assume that all idempotent semirings are ordered by the canonical partial order relation.

We shall say that a positive (e.g., idempotent) semiring S is complete if it is complete as an ordered set. This means that for every subset  $T \subset S$  there exist elements sup  $T \in S$  and  $\inf T \in S$ .

The most well-known and important examples of positive semirings are "numerical" semirings consisting of (a subset of) real numbers and ordered by the usual linear order  $\leq$  on **R**: the semiring **R**<sub>+</sub> with the usual operations  $\oplus = +$ ,  $\odot = \cdot$  and neutral elements  $\mathbf{0} = 0, \mathbf{1} = 1$ , the semiring  $\mathbf{R}_{\text{max}} = \mathbf{R} \cup \{-\infty\}$ with the operations  $\oplus = \max$ ,  $\odot = +$  and neutral elements  $\mathbf{0} = -\infty$ ,  $\mathbf{1} = 0$ , the semiring  $\mathbf{R}_{\max} = \mathbf{R}_{\max} \cup \{\infty\}$ , where  $x \preceq \infty$ ,  $x \oplus \infty = \infty$  for all x,  $x \odot \infty = \infty \odot x = \infty$  if  $x \neq 0$ , and  $0 \odot \infty = \infty \odot 0$ , the semirings  $\mathbf{Z}_{\max} = \mathbf{Z} \cup \{-\infty\}$ and  $\hat{\mathbf{Z}}_{\max} = \mathbf{Z}_{\max} \cup \{+\infty\}$  (subsemirings in  $\mathbf{R}_{\max}$  and  $\hat{\mathbf{R}}_{\max}$ ), and the semiring  $S_{\max,\min}^{[a,b]} = [a,b]$ , where  $-\infty \le a < b \le +\infty$ , with the operations  $\oplus = \max$ ,  $\odot$  = min and neutral elements **0** = *a*, **1** = *b*. The semirings **R**<sub>max</sub>,  $\hat{\mathbf{R}}_{max}$ ,  $\mathbf{Z}_{\max}$ ,  $\hat{\mathbf{Z}}_{\max}$  and  $S_{\max,\min}^{[a,b]} = [a,b]$  are idempotent. The semirings  $\hat{\mathbf{R}}_{\max}$ ,  $\hat{\mathbf{Z}}_{\max}$ ,  $S_{\max,\min}^{[a,b]}, \, \widehat{\mathbf{R}}_+ = \mathbf{R}_+ \bigcup \{\infty\}$  are complete. Remind that every partially ordered set can be imbedded to its completion (a minimal complete set containing the initial one). We shall say that all these semirings (as well as algebras isomorphic to them) are *basic numerical semiring*. These semirings are complete or their completions are complete semirings.

Denote by  $\operatorname{Mat}_{mn}(S)$  a set of all matrices  $A = (a_{ij})$  with m rows and n columns whose coefficients belong to a semiring S. The sum  $A \oplus B$  of matrices  $A, B \in \operatorname{Mat}_{mn}(S)$  and the product AB of matrices  $A \in \operatorname{Mat}_{lm}(S)$  and  $B \in \operatorname{Mat}_{mn}(S)$  are defined according to the usual rules of linear algebra:  $A \oplus B = (a_{ij} \oplus b_{ij}) \in \operatorname{Mat}_{mn}(S)$  and

$$AB = \left(\bigoplus_{k=1}^{m} a_{ij} \odot b_{kj}\right) \in \operatorname{Mat}_{ln}(S),$$

where  $A \in \operatorname{Mat}_{lm}(S)$  and  $B \in \operatorname{Mat}_{mn}(S)$ . Note that we write AB instead of  $A \odot B$ .

If the semiring S is positive, then the set  $\operatorname{Mat}_{mn}(S)$  is ordered by the relation  $A = (a_{ij}) \preceq B = (b_{ij})$  iff  $a_{ij} \preceq b_{ij}$  in S for all  $1 \leq i \leq m, 1 \leq j \leq n$ .

The matrix multiplication is consistent with the order  $\leq$  in the following sense: if  $A, A' \in \operatorname{Mat}_{lm}(S), B, B' \in \operatorname{Mat}_{mn}(S)$  and  $A \leq A', B \leq B'$ , then  $AB \leq A'B'$  in  $\operatorname{Mat}_{ln}(S)$ . The set  $\operatorname{Mat}_{nn}(S)$  of square  $(n \times n)$  matrices over a [positive, idempotent] semiring S forms a [positive, idempotent] semiring with a zero element  $O = (o_{ij})$ , where  $o_{ij} = \mathbf{0}, 1 \leq i, j \leq n$ , and a unit element  $I = (\delta_{ij})$ , where  $\delta_{ij} = \mathbf{1}$  if i = j and  $\delta_{ij} = \mathbf{0}$  otherwise.

The set  $Mat_{nn}$  is an example of a noncommutative semiring if n > 1.

#### 7.2. Closure operations

Let a positive semiring S be endowed with a partial unary closure operation or Kleene star operation \* such that  $a \leq b$  implies  $a^* \leq b^*$  and  $a^* = \mathbf{1} \oplus (a^* \odot a) =$  $\mathbf{1} \oplus (a \odot a^*)$  on its domain of definition. In particular,  $\mathbf{0}^* = \mathbf{1}$  by definition.

These axioms imply that  $a^* = \mathbf{1} \oplus a \oplus a^2 \oplus \cdots \oplus (a^* \odot a^n)$  if  $n \ge 1$ . Thus  $x^*$  can be considered as a 'regularized sum' of the series  $a^* = \mathbf{1} \oplus a \oplus a^2 \oplus \cdots$ . In a positive semiring, provided that it is closed under taking bounded ordered sup-operations and the operations  $\oplus$  and  $\odot$  distribute over such sup-operations we can define

$$a^* := \sup_{k \ge 0} \mathbf{1} \oplus a \oplus \ldots \oplus a^k, \tag{7.1}$$

if the sequence on the r.h.s. is bounded. In this case  $a^*$  is the **least solution** of the equations  $x = ax \oplus \mathbf{1}$  and  $x = xa \oplus \mathbf{1}$ , and more generally  $a^*b$  is the the least solution of the *Bellman equations*  $x = ax \oplus b$  and  $x = xa \oplus b$ . So if S is complete, then the closure operation is well-defined for every element  $x \in S$ .

In the case of idempotent addition (7.1) becomes particularly nice:

$$a^* = \bigoplus_{i \ge 0} a^i = \sup_{i \ge 0} a^i.$$

$$(7.2)$$

In numerical semirings the operation \* is usually very easy to implement:  $x^* = (1 - x)^{-1}$  if x < 1 in  $\mathbf{R}_+$ , or  $\hat{\mathbf{R}}_+$  and  $x^* = \infty$  if  $x \ge 1$  in  $\hat{\mathbf{R}}_+$ ;  $x^* = \mathbf{1}$ if  $x \preceq \mathbf{1}$  in  $\mathbf{R}_{\max}$  and  $\hat{\mathbf{R}}_{\max}$ ,  $x^* = \infty$  if  $x \succ \mathbf{1}$  in  $\hat{\mathbf{R}}_{\max}$ ,  $x^* = \mathbf{1}$  for all x in  $S_{\max,\min}^{[a,b]}$ . In all other cases  $x^*$  is undefined.

The closure operation in matrix semirings over a positive semiring S can be defined inductively:  $A^* = (a_{11})^* = (a_{11}^*)$  in  $Mat_{11}(S)$  and for any integer n > 1 and any matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where  $A_{11} \in \operatorname{Mat}_{kk}(S)$ ,  $A_{12} \in \operatorname{Mat}_{kn-k}(S)$ ,  $A_{21} \in \operatorname{Mat}_{n-kk}(S)$ ,  $A_{22} \in \operatorname{Mat}_{n-kn-k}(S)$ ,  $1 \le k \le n$ , by definition,

$$A^* = \begin{pmatrix} A_{11}^* \oplus A_{11}^* A_{12} D^* A_{21} A_{11}^* & A_{11}^* A_{12} D^* \\ D^* A_{21} A_{11}^* & D^* \end{pmatrix},$$
(7.3)

where  $D = A_{22} \oplus A_{21}A_{11}^*A_{12}$ . It can be proved that this definition of  $A^*$  implies that the equalities  $A^* = A^*A \oplus I = AA^* \oplus I$  are satisfied, and thus  $A^*$  is a 'regularized sum' of the series  $I \oplus A \oplus A^2 \oplus \ldots$ . Moreover, in the case when  $A^*$ is defined as the least solution of  $A^* = A^*A \oplus I$  and  $A^* = AA^* \oplus I$ , it can be shown that it satisfies (7.3).

Note that this recurrence relation coincides with the formulas of escalator method of matrix inversion in the traditional linear algebra over the field of real or complex numbers, up to the algebraic operations used. Hence this algorithm of matrix closure requires a polynomial number of operations in n, see [8, 16, 38, 41, 45, 46] for more details.

Let S be a positive semiring. The discrete stationary Bellman equation or matrix Bellman equation has the form

$$X = AX \oplus B,\tag{7.4}$$

where  $A \in \operatorname{Mat}_{nn}(S)$ ,  $X, B \in \operatorname{Mat}_{ns}(S)$ , and the matrix X is unknown. Let  $A^*$  be the closure of the matrix A. It follows from the identity  $A^* = A^*A \oplus I$  that the matrix  $A^*B$  satisfies this equation. As in the scalar case, it can be shown that for positive semirings under reasonable distributivity assumptions, if  $A^*$  is defined as in (7.1) then  $A^*B$  is the least in the set of solutions to equation (7.4) with respect to the partial order in  $\operatorname{Mat}_{ns}(S)$ . Recall that in the idempotent case

$$A^* = \bigoplus_{i \ge 0} A^i = \sup_{i \ge 0} A^i.$$

$$(7.5)$$

Consider also the case when  $A = (a_{ij})$  is  $n \times n$  strictly upper-triangular (such that  $a_{ij} = \mathbf{0}$  for  $i \ge j$ ), or  $n \times n$  strictly lower-triangular (such that  $a_{ij} = \mathbf{0}$  for  $i \le j$ ). In this case  $A^n = O$ , the all-zeros matrix, and it can be shown by iterating  $X = AX \oplus I$  that this equation has a unique solution, namely

$$A^* = I \oplus A \oplus \ldots \oplus A^{n-1}. \tag{7.6}$$

Curiously enough, formula (7.6) works more generally in the case of numerical idempotent semiring  $\mathbf{R}_{\text{max}}$  (and other idempotent semirings): in fact, the series (7.5) converges there if and only if it can be truncated to (7.6). This is closely related to the principal path interpretation of  $A^*$  explained in the next subsection.

# 7.3. Weighted directed graphs and matrices over semirings

Suppose that S is a semiring with zero **0** and unity **1**. It is well-known that any square matrix  $A = (a_{ij}) \in \operatorname{Mat}_{nn}(S)$  specifies a weighted directed graph. This geometrical construction includes three kinds of objects: the set X of n elements  $x_1, \ldots, x_n$  called nodes, the set  $\Gamma$  of all ordered pairs  $(x_i, x_j)$  such that  $a_{ij} \neq \mathbf{0}$  called arcs, and the mapping  $A \colon \Gamma \to S$  such that  $A(x_i, x_j) = a_{ij}$ . The elements  $a_{ij}$  of the semiring S are called weights of the arcs.

Conversely, any given weighted directed graph with n nodes specifies a unique matrix  $A \in Mat_{nn}(S)$ .

This definition allows for some pairs of nodes to be disconnected if the corresponding element of the matrix A is **0** and for some channels to be "loops" with coincident ends if the matrix A has nonzero diagonal elements. This concept is convenient for analysis of parallel and distributed computations and design of computing media and networks.

Recall that a sequence of nodes of the form

$$p = (y_0, y_1, \dots, y_k)$$

with  $k \ge 0$  and  $(y_i, y_{i+1}) \in \Gamma$ , i = 0, ..., k - 1, is called a *path* of length k connecting  $y_0$  with  $y_k$ . Denote the set of all such paths by  $P_k(y_0, y_k)$ . The

weight A(p) of a path  $p \in P_k(y_0, y_k)$  is defined to be the product of weights of arcs connecting consecutive nodes of the path:

$$A(p) = A(y_0, y_1) \odot \cdots \odot A(y_{k-1}, y_k).$$

By definition, for a 'path'  $p \in P_0(x_i, x_j)$  of length k = 0 the weight is **1** if i = j and **0** otherwise.

For each matrix  $A \in \operatorname{Mat}_{nn}(S)$  define  $A^0 = I = (\delta_{ij})$  (where  $\delta_{ij} = \mathbf{1}$  if i = jand  $\delta_{ij} = \mathbf{0}$  otherwise) and  $A^k = AA^{k-1}$ ,  $k \ge 1$ . Let  $a_{ij}^{[k]}$  be the (i, j)th element of the matrix  $A^k$ . It is easily checked that

$$a_{ij}^{[k]} = \bigoplus_{\substack{i_0=i, i_k=j\\1\leq i_1, \dots, i_{k-1}\leq n}} a_{i_0i_1} \odot \dots \odot a_{i_{k-1}i_k}$$

Thus  $a_{ij}^{[k]}$  is the supremum of the set of weights corresponding to all paths of length k connecting the node  $x_{i_0} = x_i$  with  $x_{i_k} = x_j$ .

Let  $A^*$  be defined as in (7.5). Denote the elements of the matrix  $A^*$  by  $a_{ij}^*$ ,  $i, j = 1, \ldots, n$ ; then

$$a_{ij}^* = \bigoplus_{0 \le k < \infty} \bigoplus_{p \in P_k(x_i, x_j)} A(p).$$

The closure matrix  $A^*$  solves the well-known algebraic path problem, which is formulated as follows: for each pair  $(x_i, x_j)$  calculate the supremum of weights of all paths (of arbitrary length) connecting node  $x_i$  with node  $x_j$ . The closure operation in matrix semirings has been studied extensively (see, e.g., [2, 8, 12, 14, 15, 16, 24, 22, 46] and references therein).

**Example 7.1** (The shortest path problem). Let  $S = \mathbf{R}_{\min}$ , so the weights are real numbers. In this case

$$A(p) = A(y_0, y_1) + A(y_1, y_2) + \dots + A(y_{k-1}, y_k).$$

If the element  $a_{ij}$  specifies the length of the arc  $(x_i, x_j)$  in some metric, then  $a_{ij}^*$  is the length of the shortest path connecting  $x_i$  with  $x_j$ .

**Example 7.2** (The maximal path width problem). Let  $S = \mathbf{R} \cup \{0, 1\}$  with  $\oplus = \max, \odot = \min$ . Then

$$a_{ij}^* = \max_{\substack{p \in \bigcup_{k \ge 1} P_k(x_i, x_j)}} A(p), \quad A(p) = \min(A(y_0, y_1), \dots, A(y_{k-1}, y_k)).$$

If the element  $a_{ij}$  specifies the "width" of the arc  $(x_i, x_j)$ , then the width of a path p is defined as the minimal width of its constituting arcs and the element  $a_{ij}^*$  gives the supremum of possible widths of all paths connecting  $x_i$  with  $x_j$ .

**Example 7.3** (A simple dynamic programming problem). Let  $S = \mathbf{R}_{\max}$  and suppose  $a_{ij}$  gives the *profit* corresponding to the transition from  $x_i$  to  $x_j$ . Define the vector  $B = (b_i) \in \operatorname{Mat}_{n1}(\mathbf{R}_{\max})$  whose element  $b_i$  gives the *terminal* 

profit corresponding to exiting from the graph through the node  $x_i$ . Of course, negative profits (or, rather, losses) are allowed. Let m be the total profit corresponding to a path  $p \in P_k(x_i, x_j)$ , i.e.

$$m = A(p) + b_j.$$

Then it is easy to check that the supremum of profits that can be achieved on paths of length k beginning at the node  $x_i$  is equal to  $(A^k B)_i$  and the supremum of profits achievable without a restriction on the length of a path equals  $(A^*B)_i$ .

**Example 7.4** (The matrix inversion problem). Note that in the formulas of this section we are using distributivity of the multiplication  $\odot$  with respect to the addition  $\oplus$  but do not use the idempotency axiom. Thus the algebraic path problem can be posed for a nonidempotent semiring S as well (this is well-known). For instance, if  $S = \mathbf{R}$ , then

$$A^* = I + A + A^2 + \dots = (I - A)^{-1}.$$

If ||A|| > 1 but the matrix I - A is invertible, then this expression defines a regularized sum of the divergent matrix power series  $\sum_{i>0} A^i$ .

We emphasize that this connection between the matrix closure operation and solution to the Bellman equation gives rise to a number of different algorithms for numerical calculation of the closure matrix. All these algorithms are adaptations of the well-known algorithms of the traditional computational linear algebra, such as the Gauss–Jordan elimination, various iterative and escalator schemes, etc. This is a special case of the idempotent superposition principle.

In fact, the theory of the discrete stationary Bellman equation can be developed using the identity  $A^* = AA^* \oplus I$  as an additional axiom without any substantive interpretation (the so-called *closed semirings*, see, e.g., [14, 24]. Such theory can be based on the following identities, true both for the case of idempotent semirings with path interpretation, and the real numbers with conventional arithmetic (assumed that A and B have appropriate sizes):

$$(A \oplus B)^* = (A^*B)^*A^*, (AB)^*A = A(BA)^*.$$
 (7.7)

### 7.4. Basic operations

Let S be an idempotent or positive semiring. Then S is a partial ordered set (or poset) with respect to the canonical order. Suppose that S is a lattice, i.e. for each pair of elements x, y, there exists the least lower bound  $x \vee y$  called supremum and the greatest lower bound  $x \wedge y$  called infimum. See details in Subsection 9.1 below. For basic numerical semirings (see Subsection 7.2 above) these operations are maximum and minimum.

We shall say that the semiring operations, supremum, infinum and the unary closure operation (Kleene star-operation) are *basic operations*.

We shall say that S is a completed semifield if S is a complete semiring and S without the element sup S is a semifield. Then the (unary) inversion  $x \mapsto x^{-1}$ 

is obviously well defined for every element. If S is a semifield or completed semifield, then we shall say that the inversion operation is also basic.

For basic numerical positive semirings all the basic operations are very simple and easy for computer implementations.

# 8. Algorithms of tropical/idempotent mathematics

#### 8.1. The correspondence principle for computations

Of course, the idempotent correspondence principle is valid for algorithms as well as for their software and hardware implementations [27, 28, 29, 37, 38, 41]. Thus:

If we have an important and interesting numerical algorithm, then there is a good chance that its semiring analogs are important and interesting as well.

In particular, according to the superposition principle, analogs of linear algebra algorithms are especially important. Note that numerical algorithms for standard infinite-dimensional linear problems over idempotent semirings (i.e., for problems related to idempotent integration, integral operators and transformations, the Hamilton-Jacobi and generalized Bellman equations) deal with the corresponding finite-dimensional (or finite) "linear approximations". Nonlinear algorithms often can be approximated by linear ones. Thus the idempotent linear algebra is a basis for the idempotent numerical analysis.

Moreover, it is well-known that linear algebra algorithms easily lend themselves to parallel computation; their idempotent analogs admit parallelization as well. Thus we obtain a systematic way of applying parallel computing to optimization problems. In this paper we do not deal with parallel algorithms and their implementations.

# 8.2. Universal algorithms

Computational algorithms are constructed on the basis of certain primitive operations. These operations manipulate data that describe "numbers." These "numbers" are elements of a "numerical domain," i.e., a mathematical object such as the field of real numbers, the ring of integers, numerical and idempotent semirings and semifields.

In practice, elements of the numerical domains are replaced by their computer representations, i.e., by elements of certain finite models of these domains. Examples of models that can be conveniently used for computer representation of real numbers are provided by various modifications of floating point arithmetics, approximate arithmetics of rational numbers [40], and interval arithmetics. The difference between mathematical objects ("ideal" numbers) and their finite models (computer representations) results in computational (e.g., rounding) errors.

An algorithm is called *universal* if it is independent of a particular numerical domain and/or its computer representation. A typical example of a universal algorithm is the computation of the scalar product (x, y) of two vectors  $x = (x_1, ..., x_n)$  and  $y = (y_1, ..., y_n)$  by the formula  $(x, y) = x_1y_1 + \dots + x_ny_n$ . This algorithm (formula) is independent of a particular domain and its computer implementation, since the formula is well-defined for any semiring. It is clear that one algorithm can be more universal than another. For example, the simplest Newton-Cotes formula, the rectangular rule, provides the most universal algorithm for numerical integration. In particular, this formula is valid also for idempotent integration (that is, over any idempotent semiring, see e.g. [22, 25, 48, 49, 50]. Other quadrature formulas (e.g., combined trapezoid rule or the Simpson formula) are independent of computer arithmetics and can be used (e.g., in the iterative form) for computations with arbitrary accuracy. In contrast, algorithms based on Gauss–Jacobi formulas are designed for fixed accuracy computations: they include constants (coefficients and nodes of these formulas) defined with fixed accuracy. (Certainly, algorithms of this type can be made more universal by including procedures for computing the constants; however, this results in an unjustified complication of the algorithms.)

Modern achievements in software development and mathematics make us consider numerical algorithms and their classification from a new point of view. Conventional numerical algorithms are oriented to software (or hardware) implementation based on floating point arithmetic and fixed accuracy. However, it is often desirable to perform computations with variable (and arbitrary) accuracy. For this purpose, algorithms are required that are independent of the accuracy of computation and of the specific computer representation of numbers. In fact, many algorithms are independent not only of the computer representation of numbers, but also of concrete mathematical (algebraic) operations on data. In this case, operations themselves may be considered as variables. Such algorithms are implemented in the form of *generic programs* based on abstract data types that are defined by the user in addition to the predefined types provided by the language. The corresponding program tools appeared as early as in Simula-67, but modern object-oriented languages (like  $C^{++}$ , see, e.g., [47, 61]) are more convenient for generic programming. Computer algebra algorithms used in such systems as Mathematica, Maple, REDUCE, and others are also highly universal.

A different form of universality is featured by iterative algorithms (beginning with the successive approximation method) for solving differential equations (e.g., methods of Euler, Euler–Cauchy, Runge–Kutta, Adams, a number of important versions of the difference approximation method, and the like), methods for calculating elementary and some special functions based on the expansion in Taylor's series and continuous fractions (Padé approximations). These algorithms are independent of the computer representation of numbers.

The concept of a generic program was introduced by many authors; for example, in [24] such programs were called 'program schemes.' In this paper, we discuss universal algorithms implemented in the form of generic programs.

#### 8.3. The correspondence principle for hardware design

A systematic application of the correspondence principle to computer calculations leads to a unifying approach to software and hardware design.

The most important and standard numerical algorithms have many hardware realizations in the form of technical devices or special processors. These devices often can be used as prototypes for new hardware units generated by substitution of the usual arithmetic operations for its semiring analogs and by addition tools for performing neutral elements **0** and **1** (the latter usually is not difficult). Of course, the case of numerical semirings consisting of real numbers (maybe except neutral elements) and semirings of numerical intervals is the most simple and natural . Note that for semifields (including  $\mathbf{R}_{max}$  and  $\mathbf{R}_{min}$ ) the operation of division is also defined.

Good and efficient technical ideas and decisions can be transposed from prototypes into new hardware units. Thus the correspondence principle generated a regular heuristic method for hardware design. Note that to get a patent it is necessary to present the so-called 'invention formula', that is to indicate a prototype for the suggested device and the difference between these devices [27, 28, 29, 37, 38, 41].

Consider (as a typical example) the most popular and important algorithm of computing the scalar product of two vectors:

$$(x,y) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$
(8.1)

The universal version of (12) for any semiring A is obvious:

$$(x,y) = (x_1 \odot y_1) \oplus (x_2 \odot y_2) \oplus \dots \oplus (x_n \odot y_n).$$
(8.2)

In the case  $A = \mathbf{R}_{\max}$  this formula turns into the following one:

$$(x,y) = \max\{x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n\}.$$
(8.3)

This calculation is standard for many optimization algorithms, so it is useful to construct a hardware unit for computing (14). There are many different devices (and patents) for computing (12) and every such device can be used as a prototype to construct a new device for computing (14) and even (13). Many processors for matrix multiplication and for other algorithms of linear algebra are based on computing scalar products and on the corresponding "elementary" devices respectively, etc.

# 8.4. The correspondence principle for software design

Software implementations for universal semiring algorithms are not as efficient as hardware ones (with respect to the computation speed) but they are much more flexible. Program modules can deal with abstract (and variable) operations and data types. Concrete values for these operations and data types can be defined by the corresponding input data. In this case concrete operations and data types are generated by means of additional program modules. For programs written in this manner it is convenient to use special techniques of the so-called object oriented (and functional) design, see, e.g., [47, 61]. Fortunately, powerful tools supporting the object-oriented software design have recently appeared including compilers for real and convenient programming languages (e.g.  $C^{++}$  and Java) and modern computer algebra systems.

Recently, this type of programming technique has been dubbed the so-called generic programming (see, e.g., [64]). To help automate the generic programming, the so-called Standard Template Library (STL) was developed in the framework of  $C^{++}$  [61, 64]. However, high-level tools, such as STL, possess both obvious advantages and some disadvantages and must be used with caution.

# 8.5. Complexity of algorithms in idempotent mathematics

We shall use the well known standard terminology of the complexity theory (time complexity, space complexity, asymptotic computational complexity, polynomial complexity, NP-hard problems etc.), see, e.g., the corresponding Wikipedia articles. The time complexity of an algorithm quantifies the amount of time taken by this algorithm to run as a function of the size of the input to the problem. The time complexity of an algorithm is commonly expressed using big O notation, which suppresses multiplicative constants and lower order terms. When expressed this way, the time complexity is said to be described asymptotically, i.e., as the input size goes to infinity. The time complexity is commonly estimated by counting the number of elementary operations performed by the algorithm. In idempotent mathematics (and mathematics over positive semirings) the elementary operations are basic operations described in Subsection 7.4 above.

For the space complexity, the situation is quite similar. The following definition is important for us. We shall say that two algorithms have the *same complexity* if they have the same asymptotic time and space complexity.

In principle idempotent mathematics and its algorithms are more simple with respect to traditional mathematics. That is why the most important algorithms of idempotent mathematics (and espcially idempotent linear algebra) are polynomial. For example, many algorithms of solving the stationary discrete (matrix) Bellman equations have the complexity  $O(n^3)$ , see [7, 8, 15, 16, 38, 39, 45, 46, 41]. Many other polynomial algorithms of linear algebra are examined in [2, 5, 6, 12, 17, 18, 20, 32, 65]. However, NP-hard problems exist (e.g., in tropical and idempotent linear algebra), see [17, 18, 32, 62]. Some frontiers of polynomial computations in tropical geometry are investigated in [65].

# 9. Interval analysis in idempotent mathematics and complexity of algorithms

#### 9.1. Interval extensions of algorithms

Interval analysis appears for treating input and output data with uncertainties (interval data). Traditional interval analysis is a nontrivial and popular mathematical area, see, e.g., [1, 13, 23, 55, 58]. An "idempotent" version of interval analysis (and moreover interval analysis over positive semirings) appeared in [45, 46, 63]. Rather many publications on the subject appeared later, see, e.g., [9, 13, 20, 56, 57]. Interval analysis over the positive semiring  $\mathbf{R}_+$  was discussed in [3]. In the framework of idempotent mathematics, interval analysis gives exact interval solutions without any conditions of smallness on uncertainty intervals.

Let a set S be partially ordered by a relation  $\preceq$ . Below (in Subsection 9.2 partially ordered sets (or posets for the sake of brevity) will be discussed in details. A *closed interval* in S is a subset of the form  $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{x \in S \mid \underline{\mathbf{x}} \leq \overline{\mathbf{x}} \leq \overline{\mathbf{x}}\}$ , where the elements  $\underline{\mathbf{x}} \leq \overline{\mathbf{x}}$  are called *lower* and *upper bounds* of the interval  $\mathbf{x}$ . The order  $\leq$  induces a partial ordering on the set of all closed intervals in S:  $\mathbf{x} \leq \mathbf{y}$  iff  $\underline{\mathbf{x}} \leq \mathbf{y}$  and  $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$ .

A weak interval extension I(S) of a positive semiring S is the set of all closed intervals in S endowed with operations  $\oplus$  and  $\odot$  defined as  $\mathbf{x} \oplus \mathbf{y} = [\mathbf{x} \oplus \mathbf{y}, \overline{\mathbf{x}} \oplus \overline{\mathbf{y}}], \mathbf{x} \odot \mathbf{y} = [\mathbf{x} \odot \mathbf{y}, \overline{\mathbf{x}} \odot \overline{\mathbf{y}}]$  and a partial order induced by the order in S. The closure operation in I(S) is defined by  $\mathbf{x}^* = [\mathbf{x}^*, \overline{\mathbf{x}}^*]$ . There are some other interval extensions (including the so-called strong interval extension [46]) but the weak extension is more convenient.

The extension I(S) is a positive semiring; I(S) is idempotent if S is an idempotent semiring. A universal algorithm over S can be applied to I(S) and we shall get an interval version of the initial algorithm. However, there are some conditions for interval extensions of algorithms to be sure that these extensions are good enough, see below. Usually both the versions have the same complexity. For the discrete stationary Bellman equation and the corresponding optimization problems on graphs, interval analysis was examined in [45, 46] in details. Other problems of idempotent linear algebra were examined in [9, 13, 20, 56, 57].

Idempotent mathematics appears to be remarkably simpler than its traditional analog. For example, in traditional interval arithmetic, multiplication of intervals is not distributive with respect to addition of intervals, whereas in idempotent interval arithmetic this distributivity is preserved. Moreover, in traditional interval analysis the set of all square interval matrices of a given order does not form even a semigroup with respect to matrix multiplication: this operation is not associative since distributivity is lost in the traditional interval arithmetic. On the contrary, in the idempotent (and positive) case associativity is preserved. Finally, in traditional interval analysis some problems of linear algebra, such as solution of a linear system of interval equations, can be very difficult (more precisely, they are NP-hard, see [11, 13, 23] and references therein). It was noticed in [45, 46] that in the idempotent case solving an interval linear system requires a polynomial number of operations (similarly to the usual Gauss elimination algorithm). Two properties that make the idempotent interval arithmetic so simple are monotonicity of arithmetic operations and positivity of all elements of an idempotent semiring.

Usually interval estimates in idempotent mathematics are exact. In the traditional theory such estimates tend to be overly pessimistic.

9.2. Intervals in partially ordered sets and interval regular mappings

Let us start with some basic notions of the theory of lattices and partially ordered sets. The reader is referred to [4] for more information.

**Definition 9.1.** Binary relation  $\leq$  on a set *S* is called a partial order if it satisfies the following axioms: 1)  $a \leq a$ , 2)  $a \leq b$  and  $b \leq a$  imply a = b, 3)  $a \leq b$  and  $b \leq c$  imply  $a \leq c$ . In this case *S* is called a partially ordered set or, briefly, a poset.

If S is a Cartesian product  $S_1 \times S_2$  where  $S_1$  and  $S_2$  are posets ordered with  $\leq_1$  and  $\leq_2$  respectively, one can naturally introduce relation  $\leq$  on S, by  $(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq_1 x_2$  and  $y_1 \leq_2 y_2$ .

For the dynamics, consider mappings of partially ordered sets. A mapping  $\phi: S \to T$  is a morphism of partially ordered sets S and T if  $x \leq y$  implies  $\phi(x) \leq \phi(y)$ . This mapping is an isomorphism if it is a bijection (i.e. one-to-one correspondence) between S and T. Also note that if  $\phi_1$  is a morphism between  $S_1$  and  $T_1$ , and  $\phi_2$  is a morphism (resp. isomorphism) between  $S_2$  and  $T_2$ , then mapping  $\phi_1 \times \phi_2: (x, y) \mapsto (\phi_1(x), \phi_2(y))$  is a morphism (resp. isomorphism) between  $S_1 \times T_1$  and  $S_2 \times T_2$ .

For a poset S and a subset  $X \subseteq S$ , an element  $t \in S$  is called an *upper bound* (resp. a *lower bound*) of X if  $t \ge x$  (resp.  $t \le x$ ) for every  $x \in X$ .

**Definition 9.2.** A poset S is called a lattice if for each pair of elements x, y, there exists the least lower bound  $x \lor y$  called supremum and the greatest lower bound  $x \land y$  called infimum.

A lattice S is complete if every subset  $H \subseteq S$  (not necessarily finite) has supremum and infimum in S, and it is conditionally complete if supremum exists for each subset bounded from above, and infimum exists for each subset bounded from below. Note that the last two statements are equivalent so that, formally, only one of them is needed.

**Example 9.3.** Consider the set of natural numbers N ordered in such a way that  $n_1 \leq n_2$  if and only if  $n_1$  divides  $n_2$ . Then it can be verified that  $m \vee n$  is the least upper bound of m and n while  $m \wedge n$  is their greatest common divisor. Other related examples are the lattice of subsets ordered by inclusion where  $\wedge = \cap$  and  $\vee = \cup$ , or the lattice of convex sets ordered by inclusion where  $\wedge = \cap$  but  $\vee$  is the convex hull of the arguments. However we emphasize that such examples are not important to us here.

**Example 9.4.** Any *linearly ordered* set S, i.e. such that for each  $x, y \in S$  there is  $x \leq y$  or  $y \leq x$ , is a lattice where both  $x \vee y \in \{x, y\}$  and  $x \wedge y \in \{x, y\}$ . For example we can take the real line  $\mathbf{R}$  or any subset of the real line, e.g., an interval [a, b] or a set of nonnegative numbers  $\mathbf{R}_+$ . However,  $\mathbf{R}$  and  $\mathbf{R}_+$ are only conditionally complete, and for their completion we can consider  $\hat{R} :=$  $\mathbf{R} \cup \{-\infty\} \cup \{\infty\}$  and  $\hat{\mathbf{R}}_+ := \mathbf{R}_+ \cup \{\infty\}$ .

From this one can construct slightly more complicated examples, which will not be linearly ordered, e.g., by means of Cartesian products. The following lattice-theoretic definition is of particular importance to us. **Definition 9.5.** Let S be a set partially ordered by a relation  $\preceq$ . A (closed) interval in S is a subset of the form  $\mathbf{x} := [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{t \in S : \underline{\mathbf{x}} \leq t \leq \overline{\mathbf{x}}\}$ , where  $\underline{\mathbf{x}}, \overline{\mathbf{x}} \in S$  and  $\underline{\mathbf{x}} \leq \overline{\mathbf{x}}$ . The elements  $\underline{\mathbf{x}}$  and  $\overline{\mathbf{x}}$  are called lower bound and upper bound of x respectively.

Intervals can be viewed as pairs of lower and upper bounds. The set of such pairs is denoted by I(S) and called *interval extension* of S. It is evident that  $I(S) \subseteq S \times S$  (but  $I(S) \neq S \times S$ ) and that I(S) is a poset. It can be shown that  $I(S \times T) = I(S) \times I(T)$  for posets S and T.

Consider an algorithm  $\mathcal{A}$  on posets. It takes input data  $(x_1, \ldots, x_n)$  and generates output  $(y_1, \ldots, y_m)$ . Here  $x_i \in S_i$  for  $i = 1, \ldots, n$  and  $y_j \in T_j$ for  $j = 1, \ldots, m$ , where  $S_i$  and  $T_j$  are posets. Hence, this algorithm induces a mapping  $\phi_{\mathcal{A}} \colon S \to T$  where  $S \coloneqq S_1 \times \cdots \times S_n$  and  $T \coloneqq T_1 \times \cdots \times T_m$ . We call an algorithm  $\mathcal{A}$  positive or nondecreasing if the corresponding mapping  $S_1 \times \cdots \times S_n \to T_1 \times \cdots \times T_m$  is nondecreasing.

**Proposition 9.6.** If an algorithm  $\mathcal{A}$  is positive, then applied to the lower and upper bounds of an interval of  $S_1 \times \cdots \times S_n$  where  $S_1, \ldots, S_n$  are posets, it yields exact interval bounds on the application of  $\mathcal{A}$  to the whole interval.

*Proof.* These properties follow immediately from the positivity of A.

It is clear that such algorithm and its interval extension have the same complexity. Using Proposition 9.6, the interval extension  $I(\mathcal{A})$  can be defined on the bounds of intervals only, and the result of  $I(\mathcal{A})$  is a mapping from  $I(S_1) \times \cdots \times I(S_n)$  to  $I(T_1) \times \cdots \times I(T_m)$ . The complexity of  $I(\mathcal{A})$  only doubles the complexity of  $\mathcal{A}$ .

We proceed with the following observations.

**Proposition 9.7.** Cartesian product of positive algorithms is a positive algorithmm.

**Proposition 9.8.** Let S be a complete or conditionally complete poset. Operations  $(x, y) \mapsto x \lor y$  and  $(x, y) \mapsto x \land y$  are positive mappings  $S \times S \to S$ .

**Definition 9.9.** A mapping  $f : S \to T$  will be called interval regular or, briefly, *i*-regular if the following condition hold: for any interval  $\mathbf{x} \subseteq S$  there exists  $\mathbf{y} \subseteq T$  such that  $f(\mathbf{x}) \subseteq \mathbf{y}, \mathbf{y} \in f(\mathbf{x})$  and  $\overline{\mathbf{y}} \in f(\mathbf{x})$ .

For example, consider  $\hat{\mathbf{R}}_+ := \mathbf{R}_+ \cup \{+\infty\}$  and unary operation  $x \mapsto x^-$  defined as  $x^- := x^{-1}$  on finite numbers,  $0^- = +\infty$  and  $\infty^- := 0$ . This mapping is not positive, but *i*-regular. Observe that an interval [a, b] is mapped to  $[b^{-1}, a^{-1}]$ .

Any algorithm  $\mathcal{A}$  induces a mapping  $f_{\mathcal{A}}$ , hence we can define *i*-regular algorithms.

**Definition 9.10.** An algorithm  $\mathcal{A}$  is called *i*-regular if  $f_{\mathcal{A}}$  is *i*-regular.

**Definition 9.11.** An algorithm  $\mathcal{A}$  is called *ci-regular* if  $f_{\mathcal{A}}$  is interval regular and this algorithm and its interval extension have the same (asymptotic) complexity.

For example, the inversion operations in semifields and completed semifields are ci-regular but not positive. The following simple and obvious proposition illustrates a typical application of the notion of ci-regular algorithms.

**Proposition 9.12.** Any composition or Cartesian product of ci-regular algorithms is a ci-regular algorithm.

**Corollary 9.13.** Any positive algorithm is ci-regular. The complexity of its interval version is the same.

# 9.3. Interval analysis over a fixed basic semiring

Fix a basic positive semiring K, e.g., a basic numerical semiring in the sense of Subsection 7.1. For the sake of simplicity, we suppose that K is complete.

So input output data run Cartesian products of several copies of K. Note that Cartesian product of positive semirings  $S_1 \times S_2$  is a positive semiring, with respect to the Cartesian product of orders in  $S_1$  and  $S_2$ .

**Definition 9.14.** An algorithm  $\mathcal{A}$  is called *elementary* if it can be realized as a composition or Cartesian product of a finite number of basic operations from K, see Subsection 7.4 above.

Algorithms which are not elementary might use if-else constructions, however, they may be still *ci*-regular.

From the results presented in Subsection 9.2 we can easily deduce the following result.

**Theorem 9.15.** The basic unary operation  $x \mapsto x^*$  induces a nondecreasing mapping  $S \to S$ , and the basic binary operations  $\oplus, \odot$ , as well as supremum and infimum, induce nondecreasing mappings  $S \times S \to S$ . If an elementary algorithm  $\mathcal{P}$  uses only these basic operations over a positive semiring, then it is positive. If K is a completed semifield, then the unary inversion operation is ciregular but not positive. Every composition of elementary (and other ci-regular algorithms) is a ci-regular algorithm.

**Example 9.16.** There exists an algorithm for computing  $A^*$  which can be represented as a composition of positive operations with respect to K, see Subsection 7.2 above. Hence this algorithm is positive. Moreover each standard algorithm solving the stationary discrete Bellman equation (see, e.g. [41]) is positive and *ci*-regular.

Some standard algorithms for solving the matrix equations Ax = b, Ax = Bx, Ax = By, and other problems of tropical/idempotent linear algebra use the so-called (binary) pseudodivision operation, see, e.g. [2, 20]. In principle, this operation is not elementary or *ci*-regular. However, using results presented in [20], it is possible to reduce this operation to *ci*-regular algorithms for all the basic numerical semirings. So the corresponding algorithms of linear algebra are *ci*-regular.

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