AN EFFICIENT NUMERICAL SCHEME FOR THE BIHARMONIC EQUATION BY WEAK GALERKIN FINITE ELEMENT METHODS ON POLYGONAL OR POLYHEDRAL MESHES

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Abstract. This paper presents a new and efficient numerical algorithm for the biharmonic equation by using weak Galerkin (WG) finite element methods. The WG finite element scheme is based on a variational form of the biharmonic equation that is equivalent to the usual H^2 -semi norm. Weak partial derivatives and their approximations, called discrete weak partial derivatives, are introduced for a class of discontinuous functions defined on a finite element partition of the domain consisting of general polygons or polyhedra. The discrete weak partial derivatives serve as building blocks for the WG finite element method. The resulting matrix from the WG method is symmetric, positive definite, and parameter free. An error estimate of optimal order is derived in an H^2 -equivalent norm for the WG finite element solutions. Error estimates in the usual L^2 norm are established, yielding optimal order of convergence for all the WG finite element algorithms except the one corresponding to the lowest order (i.e., piecewise quadratic elements). Some numerical experiments are presented to illustrate the efficiency and accuracy of the numerical scheme.

Key words. weak Galerkin, finite element methods, weak partial derivatives, biharmonic equation, polyhedral meshes.

AMS subject classifications. Primary, 65N30, 65N15, 65N12, 74N20; Secondary, 35B45, 35J50, 35J35

1. Introduction. This paper is concerned with new developments of numerical methods for the biharmonic equation with Dirichlet and Neumann boundary conditions. The model problem seeks an unknown function u = u(x) satisfying

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(1.1)
$$\begin{aligned} \Delta^{-}u &= f, & \text{in } \Omega, \\ u &= \xi, & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= \nu, & \text{on } \partial\Omega, \end{aligned}$$

where Ω is an open bounded domain in \mathbb{R}^d (d = 2, 3) with a Lipschitz continuous boundary $\partial \Omega$. The functions f, ξ , and ν are given on the domain or its boundary, as appropriate.

A variational formulation for the biharmonic problem (1.1) is given by seeking $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = \xi$, $\frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \nu$ and the following equation

(1.2)
$$\sum_{i,j=1}^{d} (\partial_{ij}^2 u, \partial_{ij}^2 v) = (f, v), \quad \forall v \in H_0^2(\Omega),$$

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where (\cdot, \cdot) stands for the usual inner product in $L^2(\Omega)$, ∂_{ij}^2 is the second order partial derivative in the direction x_i and x_j , and $H_0^2(\Omega)$ is the subspace of the Sobolev space $H^2(\Omega)$ consisting of functions with vanishing trace for the function itself and its gradient.

Based on the variational form (1.2), one may design various conforming finite element schemes for (1.1) by constructing finite element spaces as subspaces of $H^2(\Omega)$. Such H^2 -conforming methods essentially require C^1 -continuity for the underlying piecewise polynomials (known as finite element functions) on a prescribed finite element partition. The C^1 -continuity imposes an enormous difficulty in the construction of the corresponding finite element functions in practical computation. Due to the complexity in the construction of C^1 -continuous elements, H^2 -conforming finite element methods are rarely used in practice for solving the biharmonic equation.

As an alternative approach, nonconforming and discontinuous Galerkin finite element methods have been developed for solving the biharmonic equation over the last several decades. The Morley element [6] is a well-known example of nonconforming element for the biharmonic equation by using piecewise quadratic polynomials. Recently, a C^0 interior penalty method was studied in [2, 3]. In [8], a hp-version interior-penalty discontinuous Galerkin method was developed for the biharmonic equation. To avoid the use of C^1 -elements, mixed methods have been developed for the biharmonic equation by reducing the fourth order problem to a system of two second order equations [1, 4, 5, 7, 9].

Recently, weak Galerkin (WG) has emerged as a new finite element technique for solving partial differential equations. WG method refers to numerical techniques for partial differential equations where differential operators are interpreted and approximated as distributions over a set of generalized functions. The method/idea was first introduced in [10] for second order elliptic equations, and the concept was further developed in [11, 12, 14]. By design, WG uses generalized and/or discontinuous approximating functions on general meshes to overcome the barrier in the construction of "smooth" finite element functions. In [13], a WG finite element method was introduced and analyzed for the biharmonic equation by using polynomials of degree $k \geq 2$ on each element plus polynomials of degree k and k - 1 for u and $\frac{\partial u}{\partial \mathbf{n}}$ on the boundary of each element (i.e., elements of type $P_k/P_k/P_{k-1}$). The WG scheme of [13] is based on the variational form of $(\Delta u, \Delta v) = (f, v)$.

In this paper, we will develop a highly flexible and robust WG finite element method for the biharmonic equation by using an element of type $P_k/P_{k-2}/P_{k-2}$; i.e., polynomials of degree k on each element and polynomials of degree k-2 on the boundary of the element for u and ∇u . Our WG finite element scheme is based on the variational form (1.2), and has a smaller number of unknowns than that of [13] for the same order of element. Intuitively, our WG finite element scheme for (1.1) shall be derived by replacing the differential operator ∂_{ij}^2 in (1.2) by a discrete and weak version, denoted by $\partial_{ij,w}^2$. In general, such a straightforward replacement may not produce a working algorithm without including a mechanism that enforces a certain weak continuity of the underlying approximating functions. A weak continuity shall be realized by introducing an appropriately defined stabilizer, denoted as $s(\cdot, \cdot)$. Formally, our WG finite element method for (1.1) can be described by seeking a finite element function u_h satisfying

(1.3)
$$\sum_{i,j=1}^{d} (\partial_{ij,w}^2 u_h, \partial_{ij,w}^2 v)_h + s(u_h, v) = (f, v)$$

for all testing functions v. The main advantage of the present approach as compared to [13] lies in the fact that elements of type $P_k/P_{k-2}/P_{k-2}$ are employed, which greatly reduces the degrees of freedom and results in a smaller system to solve. The rest of the paper is to specify all the details for (1.3), and justifies the rigorousness of the method by establishing a mathematical convergence theory.

The paper is organized as follows. In Section 2, we introduce some standard notations for Sobolev spaces. Section 3 is devoted to a discussion of weak partial derivatives and their discretizations. In Section 4, we present a weak Galerkin algorithm for the biharmonic equation (1.1). In Section 5, we introduce some local L^2 projection operators and then derive some approximation properties which are useful in the convergence analysis. Section 6 will be devoted to the derivation of an error equation for the WG finite element solution. In Section 7, we establish an optimal order of error estimate for the WG finite element approximation in a H^2 -equivalent discrete norm. In Section 8, we shall derive an error estimate for the WG finite element method approximation in the usual L^2 -norm. Finally in Section 9, we present some numerical results to demonstrate the efficiency and accuracy of our WG method.

2. Preliminaries and Notations. Let D be any open bounded domain with Lipschitz continuous boundary in \mathbb{R}^d , d = 2, 3. We use the standard definition for the Sobolev space $H^s(D)$ and the associated inner product $(\cdot, \cdot)_{s,D}$, norm $\|\cdot\|_{s,D}$, and seminorm $|\cdot|_{s,D}$ for any $s \ge 0$. For example, for any integer $s \ge 0$, the seminorm $|\cdot|_{s,D}$ is given by

$$v|_{s,D} = \left(\sum_{|\alpha|=s} \int_{D} |\partial^{\alpha} v|^2 dD\right)^{\frac{1}{2}}$$

with the usual notation

$$\alpha = (\alpha_1, \cdots, \alpha_d), |\alpha| = \alpha_1 + \cdots + \alpha_d, \partial^{\alpha} = \prod_{j=1}^d \partial_{x_j}^{\alpha_j}.$$

The Sobolev norm $\|\cdot\|_{m,D}$ is given by

$$||v||_{m,D} = \left(\sum_{j=0}^{m} |v|_{j,D}^2\right)^{\frac{1}{2}}.$$

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot, \cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and inner product notation.

Throughout the paper, the letter C is used to denote a generic constant independent of the mesh size and functions involved.

3. Weak Partial Derivatives of Second Order. For the biharmonic problem (1.1) with variational form (1.2), the principle differential operator is ∂_{ij}^2 . Thus, we shall define weak partial derivatives, denoted by $\partial_{ij,w}^2$, for a class of discontinuous functions. For numerical purpose, we shall also introduce a discrete version for the weak partial derivative $\partial_{ij,w}^2$ in polynomial subspaces.

Let T be any polygonal or polyhedral domain with boundary ∂T . By a weak function on the region T, we mean a function $v = \{v_0, v_b, \mathbf{v}_g\}$ such that $v_0 \in L^2(T)$, $v_b \in L^2(\partial T)$ and $\mathbf{v}_g \in [L^2(\partial T)]^d$. The first and second components v_0 and v_b can be understood as the value of v in the interior and on the boundary of T. The third term, $\mathbf{v}_g \in \mathbb{R}^d$ with components $v_{gi}, i = 1, \dots, d$, intends to represent the gradient ∇v on the boundary of T. Note that v_b and \mathbf{v}_g may not necessarily be related to the trace of v_0 and ∇v_0 on ∂T , respectively.

Denote by W(T) the space of all weak functions on T; i.e.,

$$W(T) = \{ v = \{ v_0, v_b, \mathbf{v}_g \} : v_0 \in L^2(T), v_b \in L^2(\partial T), \mathbf{v}_g \in [L^2(\partial T)]^d \}.$$

Let $\langle \cdot, \cdot \rangle_{\partial T}$ be the inner product in $L^2(\partial T)$. Define G(T) by

$$G(T) = \{\varphi : \varphi \in H^2(T)\}.$$

DEFINITION 3.1. The dual of $L^2(T)$ can be identified with itself by using the standard L^2 inner product as the action of linear functionals. With a similar interpretation, for any $v \in W(T)$, the weak partial derivative ∂_{ij}^2 of $v = \{v_0, v_b, v_g\}$ is defined as a linear functional $\partial_{ij,w}^2 v$ in the dual space of G(T) whose action on each $\varphi \in G(T)$ is given by

(3.1)
$$(\partial_{ij,w}^2 v, \varphi)_T = (v_0, \partial_{ji}^2 \varphi)_T - \langle v_b n_i, \partial_j \varphi \rangle_{\partial T} + \langle v_{gi}, \varphi n_j \rangle_{\partial T}.$$

Here \mathbf{n} , with components n_i $(i = 1, \dots, d)$, is the outward normal direction of T on its boundary.

Unlike the classical second order derivatives, $\partial_{ij,w}^2 v$ is usually different from $\partial_{ji,w}^2 v$ when $i \neq j$.

The Sobolev space $H^2(T)$ can be embedded into the space W(T) by an inclusion map $i_W : H^2(T) \to W(T)$ defined as follows

$$i_W(\phi) = \{\phi|_T, \phi|_{\partial T}, \nabla \phi|_{\partial T}\}, \qquad \phi \in H^2(T).$$

With the help of the inclusion map i_W , the Sobolev space $H^2(T)$ can be viewed as a subspace of W(T) by identifying each $\phi \in H^2(T)$ with $i_W(\phi)$. Analogously, a weak function $v = \{v_0, v_b, \mathbf{v}_g\} \in W(T)$ is said to be in $H^2(T)$ if it can be identified with a function $\phi \in H^2(T)$ through the above inclusion map. It is not hard to see that $\partial_{ij,w}^2$ is identical with ∂_{ij}^2 in $H^2(T)$; i.e., $\partial_{ij,w}^2 v = \partial_{ij}^2 v$ for all functions $v \in H^2(T)$.

Next, for $i, j = 1, \dots, d$, we introduce a discrete version of $\partial_{ij,w}^2$ by approximating $\partial_{ij,w}^2$ in a polynomial subspace of the dual of G(T). To this end, for any non-negative integer $r \ge 0$, denote by $P_r(T)$ the set of polynomials on T with degree no more than r. A discrete $\partial_{ij,w}^2$ $(i, j = 1, \dots, d)$ operator, denoted by $\partial_{ij,w,r,T}^2$, is defined as the unique polynomial $\partial_{ij,w,r,T}^2 v \in P_r(T)$ satisfying the following equation

$$(3.2) \quad (\partial_{ij,w,r,T}^2 v,\varphi)_T = (v_0,\partial_{ji}^2 \varphi)_T - \langle v_b n_i, \partial_j \varphi \rangle_{\partial T} + \langle v_{gi},\varphi n_j \rangle_{\partial T}, \quad \forall \varphi \in P_r(T).$$

4. Numerical Algorithm by Weak Galerkin. Let \mathcal{T}_h be a partition of the domain Ω into polygons in 2D or polyhedra in 3D. Assume that \mathcal{T}_h is shape regular in the sense as defined in [11]. Denote by \mathcal{E}_h the set of all edges or flat faces in \mathcal{T}_h , and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$ be the set of all interior edges or flat faces.

For any given integer $k \geq 2$, denote by $W_k(T)$ the discrete weak function space given by

$$W_k(T) = \{\{v_0, v_b, \mathbf{v}_g\} : v_0 \in P_k(T), v_b \in P_{k-2}(e), \mathbf{v}_g \in [P_{k-2}(e)]^d, e \subset \partial T\}.$$

By patching $W_k(T)$ over all the elements $T \in \mathcal{T}_h$ through a common value on the interface \mathcal{E}_h^0 , we arrive at a weak finite element space V_h defined as follows

$$V_h = \{\{v_0, v_b, \mathbf{v}_g\} : \{v_0, v_b, \mathbf{v}_g\}|_T \in W_k(T), \forall T \in \mathcal{T}_h\}.$$

Denote by V_h^0 the subspace of V_h with vanishing trace; i.e.,

$$V_h^0 = \{\{v_0, v_b, \mathbf{v}_g\} \in V_h, v_b|_e = 0, \mathbf{v}_g|_e = \mathbf{0}, e \subset \partial T \cap \partial \Omega\}.$$

Intuitively, the finite element functions in V_h are piecewise polynomials of degree $k \ge 2$. The extra value on the boundary of each element is approximated by polynomials of degree k - 2 for the function itself and its gradient. For such functions, we may compute the weak second order derivative $\partial_{ij,w}^2 v$ by using the formula (3.1). For computational purpose, this weak partial derivative $\partial_{ij,w}^2 v$ has to be approximated by using polynomials, preferably one with degree k - 2. Denote by $\partial_{ij,w,k-2}^2$ the discrete weak partial derivative computed by using (3.2) on each element T for $k \ge 2$; i.e.,

$$(\partial_{ij,w,k-2}^2 v)|_T = \partial_{ij,w,k-2,T}^2 (v|_T), \qquad v \in V_h$$

For simplicity of notation and without confusion, we shall drop the subscript k-2 in the notation $\partial_{ii,w,k-2}^2$. We also introduce the following notation

$$(\partial_w^2 u, \partial_w^2 v)_h = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (\partial_{ij,w}^2 u, \partial_{ij,w}^2 v)_T, \quad \forall u, v \in V_h.$$

For each element T, denote by Q_0 the L^2 projection onto $P_k(T)$, $k \ge 2$. For each edge or face $e \subset \partial T$, denote by Q_b the L^2 projection onto $P_{k-2}(e)$ or $[P_{k-2}(e)]^d$, as appropriate. For any $w \in H^2(\Omega)$, we define a projection $Q_h w$ into the weak finite element space V_h such that on each element T,

$$Q_h u = \{Q_0 u, Q_b u, Q_b (\nabla u)\}$$

For any $w = \{w_0, w_b, \mathbf{w}_g\}$ and $v = \{v_0, v_b, \mathbf{v}_g\}$ in V_h , we introduce a bilinear form as follows

$$s(w,v) = \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b(\nabla w_0) - \mathbf{w}_g, Q_b(\nabla v_0) - \mathbf{v}_g \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_b w_0 - w_b, Q_b v_0 - v_b \rangle_{\partial T}.$$

The following is a precise statement of the WG finite element scheme for the biharmonic equation (1.1) based on the variational formulation (1.2).

WEAK GALERKIN ALGORITHM 1. Find $u_h = \{u_0, u_b, u_g\} \in V_h$ satisfying $u_b = Q_b\xi$, $u_g \cdot n = Q_b\nu$, $u_g \cdot \tau = Q_b(\nabla \xi \cdot \tau)$ on $\partial\Omega$ and the following equation:

(4.1)
$$(\partial_w^2 u_h, \partial_w^2 v)_h + s(u_h, v) = (f, v_0), \quad \forall v = \{v_0, v_b, v_g\} \in V_h^0,$$

where $\boldsymbol{\tau} \in \mathbb{R}^d$ is the tangential direction to the edges/faces on the boundary $\partial \Omega$.

The following is a useful observation concerning the finite element space V_h^0 .

LEMMA 4.1. For any $v \in V_h^0$, define |||v||| by

(4.2)
$$|||v|||^2 = (\partial_w^2 v, \partial_w^2 v)_h + s(v, v).$$

Then, $\|\cdot\|$ is a norm in the linear space V_h^0 .

Proof. We shall only verify the positivity property for $\| \cdot \|$. To this end, assume that $\| v \| = 0$ for some $v \in V_h^0$. It follows from (4.2) that $\partial_{ij,w}^2 v = 0$ on T, $Q_b(\nabla v_0) = \mathbf{v}_g$ and $Q_b v_0 = v_b$ on ∂T . We claim that $\partial_{ij}^2 v_0 = 0$ on each element T. To this end, for any $\varphi \in P_{k-2}(T)$, we use $\partial_{ij,w}^2 v = 0$ and the identity (10.4) to obtain

$$0 = (\partial_{ij,w}^2 v, \varphi)_T$$

= $(\partial_{ij}^2 v_0, \varphi)_T + \langle v_{gi} - Q_b(\partial_i v_0), \varphi \cdot n_j \rangle_{\partial T} + \langle Q_b v_0 - v_b, \partial_j \varphi \cdot n_i \rangle_{\partial T}$
= $(\varphi, \partial_{ij}^2 v_0)_T$,

which implies that $\partial_{ij}^2 v_0 = 0$ for $i, j = 1, \ldots, d$ on each element T. Thus, v_0 is a linear function on T and ∇v_0 is a constant on each element. The condition $Q_b(\nabla v_0) = \mathbf{v}_g$ on ∂T implies that $\nabla v_0 = \mathbf{v}_g$ on ∂T . Thus, ∇v_0 is continuous over the whole domain Ω . The fact that $\mathbf{v}_g = 0$ on $\partial \Omega$ leads to $\nabla v_0 = 0$ in Ω and $\mathbf{v}_g = 0$ on each edge/face. Thus, v_0 is a constant on each element T. This, together with the fact that $Q_b v_0 = v_b$ on ∂T , indicates that v_0 is continuous over the whole domain Ω . It follows from $v_b = 0$ on $\partial \Omega$ that $v_0 = 0$ everywhere in the domain Ω . Furthermore, $v_b = Q_b(v_0) = 0$ on each edge/face. This completes the proof of the lemma. \Box

LEMMA 4.2. The Weak Galerkin Algorithm (4.1) has a unique solution.

Proof. Let $u_h^{(1)}$ and $u_h^{(2)}$ be two different solutions of the Weak Galerkin Algorithm (4.1). It is clear that the difference $e_h = u_h^{(1)} - u_h^{(2)}$ is a finite element function in V_h^0 satisfying

(4.3)
$$(\partial_w^2 e_h, \partial_w^2 v)_h + s(e_h, v) = 0, \quad \forall v \in V_h^0.$$

By setting $v = e_h$ in (4.3), we obtain

$$(\partial_w^2 e_h, \partial_w^2 e_h)_h + s(e_h, e_h) = 0.$$

From Lemma 4.1, we get $e_h \equiv 0$, i.e., $u_h^{(1)} = u_h^{(2)}$.

The rest of the paper will provide a mathematical and computational justification for the WG finite element method (4.1).

5. L^2 Projections and Their Properties. The goal of this section is to establish some technical results for the L^2 projections. These results are valuable in the error analysis for the WG finite element method.

LEMMA 5.1. On each element $T \in \mathcal{T}_h$, let \mathcal{Q}_h be the local L^2 projection onto $P_{k-2}(T)$. Then, the L^2 projections Q_h and \mathcal{Q}_h satisfy the following commutative property:

(5.1)
$$\partial_{ij,w}^2(Q_hw) = \mathcal{Q}_h(\partial_{ij}^2w), \quad \forall i, j = 1, \dots, d,$$

for all $w \in H^2(T)$.

Proof. For $\varphi \in P_{k-2}(T)$ and $w \in H^2(T)$, from the definition of $\partial_{ij,w}^2$ and the usual integration by parts, we have

$$\begin{aligned} (\partial_{ij,w}^2(Q_hw),\varphi)_T &= (Q_0w,\partial_{ji}^2\varphi)_T - \langle Q_bw,\partial_j\varphi \cdot n_i \rangle_{\partial T} + \langle Q_b(\partial_iw) \cdot n_j,\varphi \rangle_{\partial T} \\ &= (w,\partial_{ji}^2\varphi)_T - \langle w,\partial_j\varphi \cdot n_i \rangle_{\partial T} + \langle \partial_iw \cdot n_j,\varphi \rangle_{\partial T} \\ &= (\partial_{ij}^2w,\varphi)_T \\ &= (\mathcal{Q}_h\partial_{ii}^2w,\varphi)_T, \quad \forall i,j = 1,\cdots,d, \end{aligned}$$

which completes the proof. \square

The commutative property (5.1) indicates that the discrete weak partial derivative of the L^2 projection of a smooth function is a good approximation of the classical partial derivative of the same function. This is a nice and useful property of the discrete weak partial differential operator $\partial_{ij,w}^2$ in application to algorithm design and analysis.

The following lemma provides some approximation properties for the projection operators Q_h and Q_h .

LEMMA 5.2. [11, 13] Let \mathcal{T}_h be a finite element partition of Ω satisfying the shape regularity assumption as defined in [11]. Then, for any $0 \leq s \leq 2$ and $1 \leq m \leq k$, we have

(5.2)
$$\sum_{T \in \mathcal{T}_h} h_T^{2s} \|u - Q_0 u\|_{s,T}^2 \le C h^{2(m+1)} \|u\|_{m+1}^2,$$

(5.3)
$$\sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d h_T^{2s} \|\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u\|_{s,T}^2 \le C h^{2(m-1)} \|u\|_{m+1}^2.$$

Using Lemma 5.2 we can prove the following result.

LEMMA 5.3. Let $1 \le m \le k$ and $u \in H^{\max\{m+1,4\}}(\Omega)$. There exists a constant C such that the following estimates hold true:

(5.4)
$$\left(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d h_T \|\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u\|_{\partial T}^2\right)^{\frac{1}{2}} \le Ch^{m-1} \|u\|_{m+1},$$

(5.5)
$$\left(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^d h_T^3 \|\partial_j(\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u)\|_{\partial T}^2\right)^{\frac{1}{2}} \le Ch^{m-1}(\|u\|_{m+1} + h\delta_{m,2}\|u\|_4),$$

(5.6)
$$\left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|Q_b(\nabla Q_0 u) - Q_b(\nabla u)\|_{\partial T}^2\right)^{\frac{1}{2}} \le Ch^{m-1} \|u\|_{m+1},$$

(5.7)
$$\left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|Q_b(Q_0 u) - Q_b u\|_{\partial T}^2\right)^{\frac{1}{2}} \le Ch^{m-1} \|u\|_{m+1}.$$

Here $\delta_{i,j}$ is the usual Kronecker's delta with value 1 when i = j and value 0 otherwise.

 $\mathit{Proof.}\,$ To prove (5.4), by the trace inequality (10.1) and the estimate (5.3), we get

$$\sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} h_{T} \|\partial_{ij}^{2} u - \mathcal{Q}_{h} \partial_{ij}^{2} u\|_{\partial T}^{2}$$

$$\leq C \sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} \left(\|\partial_{ij}^{2} u - \mathcal{Q}_{h} \partial_{ij}^{2} u\|_{T}^{2} + h_{T}^{2} |\partial_{ij}^{2} u - \mathcal{Q}_{h} \partial_{ij}^{2} u|_{1,T}^{2} \right)$$

$$\leq C h^{2m-2} \|u\|_{m+1}^{2}.$$

As to (5.5), by the trace inequality (10.1) and the estimate (5.3), we obtain

$$\sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} h_{T}^{3} \|\partial_{j}(\partial_{ij}u - \mathcal{Q}_{h}\partial_{ij}u)\|_{\partial T}^{2}$$

$$\leq C \sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} \left(h_{T}^{2} \|\partial_{j}(\partial_{ij}^{2}u - \mathcal{Q}_{h}\partial_{ij}^{2}u)\|_{T}^{2} + h_{T}^{4} |\partial_{j}(\partial_{ij}^{2}u - \mathcal{Q}_{h}\partial_{ij}^{2}u)|_{1,T}^{2}\right)$$

$$\leq Ch^{2m-2} \left(\|u\|_{m+1}^{2} + h^{2}\delta_{m,2}\|u\|_{4}^{2}\right).$$

As to (5.6), by the trace inequality (10.1) and the estimate (5.2), we have

$$\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \|Q_{b}(\nabla Q_{0}u) - Q_{b}(\nabla u)\|_{\partial T}^{2}$$

$$\leq \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \|\nabla Q_{0}u - \nabla u\|_{\partial T}^{2}$$

$$\leq C \sum_{T \in \mathcal{T}_{h}} \left(h_{T}^{-2} \|\nabla Q_{0}u - \nabla u\|_{T}^{2} + |\nabla Q_{0}u - \nabla u|_{1,T}^{2}\right)$$

$$\leq Ch^{2m-2} \|u\|_{m+1}^{2}.$$

Finally for (5.7), by the trace inequality (10.1) and the estimate (5.2), we have

$$\begin{split} &\sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_b(Q_0 u) - Q_b u\|_{\partial T}^2 \\ &\leq \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 u - u\|_{\partial T}^2 \\ &\leq C \sum_{T \in \mathcal{T}_h} \left(h_T^{-4} \|Q_0 u - u\|_T^2 + h_T^{-2} \|\nabla(Q_0 u - u)\|_T^2 \right) \\ &\leq C h^{2m-2} \|u\|_{m+1}^2. \end{split}$$

This completes the proof of the lemma. \Box

6. An Error Equation. Let u and $u_h = \{u_0, u_b, \mathbf{u}_g\} \in V_h$ be the solution (1.1) and (4.1) respectively. Denote by

$$(6.1) e_h = Q_h u - u_h$$

the error function between the L^2 projection of the exact solution u and its weak Galerkin finite element approximation u_h . An error equation refers to some identity that the error function e_h must satisfy. The goal of this section is to derive an error equation for e_h . The following is our main result.

LEMMA 6.1. The error function e_h as defined by (6.1) is a finite element function in V_h^0 and satisfies the following equation

(6.2)
$$(\partial_w^2 e_h, \partial_w^2 v)_h + s(e_h, v) = \phi_u(v), \qquad \forall v \in V_h^0$$

where

(6.3)

$$\phi_u(v) = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u - \mathcal{Q}_h(\partial_{ij}^2 u), (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T}$$

$$- \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j (\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T}$$

$$+ s(Q_h u, v).$$

Proof. From Lemma 5.1 we have $\partial_{ij,w}^2 Q_h u = \mathcal{Q}_h(\partial_{ij}^2 u)$. Now using (10.3) with $\varphi = \partial_{ij,w}^2 Q_h u$ we obtain

$$\begin{aligned} (\partial_{ij}^2 v, \partial_{ij,w}^2 Q_h u)_T = & (\partial_{ij}^2 v_0, \mathcal{Q}_h(\partial_{ij}^2 u))_T + \langle v_0 - v_b, \partial_j (\mathcal{Q}_h(\partial_{ij}^2 u)) \cdot n_i \rangle_{\partial T} \\ &- \langle (\partial_i v_0 - v_{gi}) \cdot n_j, \mathcal{Q}_h \partial_{ij}^2 u \rangle_{\partial T} \\ = & (\partial_{ij}^2 v_0, \partial_{ij}^2 u)_T + \langle v_0 - v_b, \partial_j (\mathcal{Q}_h(\partial_{ij}^2 u)) \cdot n_i \rangle_{\partial T} \\ &- \langle (\partial_i v_0 - v_{gi}) \cdot n_j, \mathcal{Q}_h \partial_{ij}^2 u \rangle_{\partial T}, \end{aligned}$$

which implies that

(6.4)
$$(\partial_{ij}^2 v_0, \partial_{ij}^2 u)_T = (\partial_{ij,w}^2 Q_h u, \partial_{ij,w}^2 v)_T - \langle v_0 - v_b, \partial_j (\mathcal{Q}_h (\partial_{ij}^2 u)) \cdot n_i \rangle_{\partial T} + \langle (\partial_i v_0 - v_{gi}) \cdot n_j, \mathcal{Q}_h \partial_{ij}^2 u \rangle_{\partial T}.$$

We emphasize that (6.4) is valid for any $v \in V_h^0$ and any smooth function $u \in H^r(\Omega)$, r > 3. Next, it follows from the integration by parts that

$$(\partial_{ij}^2 u, \partial_{ij}^2 v_0)_T = ((\partial_{ij}^2)^2 u, v_0)_T + \langle \partial_{ij}^2 u, \partial_i v_0 \cdot n_j \rangle_{\partial T} - \langle \partial_j (\partial_{ij}^2 u) \cdot n_i, v_0 \rangle_{\partial T}.$$

By summing over all T and using the identity that $(\triangle^2 u, v_0) = (f, v_0)$, we obtain

$$\sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (\partial_{ij}^2 u, \partial_{ij}^2 v_0)_T = (f, v_0) + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u, (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T}$$
$$- \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j (\partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T},$$

where we have used the fact that the sum for the terms associated with $v_{gi} \cdot n_j$ and $v_b n_i$ vanishes (note that both v_{gi} and v_b vanishes on $\partial \Omega$). Combining the above equation with (6.4) yields

$$(\partial_w^2 Q_h u, \partial_w^2 v)_h = (f, v_0) + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u - \mathcal{Q}_h(\partial_{ij}^2 u), (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T}$$
$$- \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j (\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T}.$$

Adding $s(Q_h u, v)$ to both side of the above equation gives

$$(6.5) \qquad \begin{aligned} (\partial_w^2 Q_h u, \partial_w^2 v)_h + s(Q_h u, v) \\ = (f, v_0) + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u - \mathcal{Q}_h(\partial_{ij}^2 u), (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T} \\ - \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j (\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T} + s(Q_h u, v). \end{aligned}$$

Subtracting (4.1) from (6.5) yields the following error equation

$$(\partial_w^2 e_h, \partial_w^2 v)_h + s(e_h, v) = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 u - \mathcal{Q}_h(\partial_{ij}^2 u), (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T}$$
$$- \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j(\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, v_0 - v_b \rangle_{\partial T} + s(Q_h u, v),$$

which completes the proof. \Box

7. Error Estimates in H^2 . The goal of this section is to derive some error estimate for the solution of Weak Galerkin Algorithm (4.1). From the error equation (6.2), it suffices to handle the term $\phi_u(v)$ defined by (6.3).

Let w be any smooth function in Ω . We rewrite $\phi_w(v)$ as follows:

(7.1)

$$\begin{aligned}
\phi_w(v) &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_{ij}^2 w - \mathcal{Q}_h(\partial_{ij}^2 w), (\partial_i v_0 - v_{gi}) \cdot n_j \rangle_{\partial T} \\
&- \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j(\partial_{ij}^2 w - \mathcal{Q}_h \partial_{ij}^2 w) \cdot n_i, v_0 - v_b \rangle_{\partial T} \\
&+ \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b(\nabla Q_0 w) - Q_b(\nabla w), Q_b(\nabla v_0) - \mathbf{v}_g \rangle_{\partial T} \\
&+ \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_b Q_0 w - Q_b w, Q_b v_0 - v_b \rangle_{\partial T} \\
&= I_1(w, v) + I_2(w, v) + I_3(w, v) + I_4(w, v),
\end{aligned}$$

where $I_j(w, v)$ are defined accordingly. Each $I_j(w, v)$ is to be handled as follows.

LEMMA 7.1. Assume that $w \in H^{r+1}(\Omega), v \in V_h^0$ with $r \in [2, k]$. Let $I_1(w, v)$ and

 $I_2(w,v)$ be given in (7.1). Then, we have

(7.2)
$$|I_1(w,v)| \le Ch^{r-1} ||w||_{r+1} ||v||, (7.3) \qquad |I_2(w,v)| \le Ch^{r-1} (||w||_{r+1} + \delta_{k,2} ||w||_4) ||v||.$$

Proof. For the term $I_1(w, v)$, we use Cauchy-Schwarz inequality, the estimate (5.4) with m = r and Lemma 10.7 to obtain

(7.4)

$$|I_{1}(w,v)| = \left| \sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} \langle \partial_{ij}^{2}w - \mathcal{Q}_{h}(\partial_{ij}^{2}w), (\partial_{i}v_{0} - v_{gi}) \cdot n_{j} \rangle_{\partial T} \right|$$

$$\leq \left(\sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} h_{T} \| \partial_{ij}^{2}w - \mathcal{Q}_{h}(\partial_{ij}^{2}w) \|_{\partial T}^{2} \right)^{\frac{1}{2}} \cdot \left(\sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} h_{T}^{-1} \| (\partial_{i}v_{0} - v_{gi}) \cdot n_{j} \|_{\partial T}^{2} \right)^{\frac{1}{2}}$$

$$\leq Ch^{r-1} \| w \|_{r+1} \| v \|,$$

which verifies (7.2).

As to the term $I_2(w, v)$, for the case of quadratic element k = 2, we use Lemma 10.6 to obtain

(7.5)
$$\left| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j (\partial_{ij}^2 w - \mathcal{Q}_h \partial_{ij}^2 w) \cdot n_i, v_0 - v_b \rangle_{\partial T} \right| \le Ch \|w\|_4 \|v\|.$$

For $k \geq 3$, we use Cauchy-Schwarz inequality, the estimate (5.5) with m = r, and Lemma 10.3 to obtain

(7.6)

$$\left| \sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} \langle \partial_{j} (\partial_{ij}^{2} w - \mathcal{Q}_{h} \partial_{ij}^{2} w) \cdot n_{i}, v_{0} - v_{b} \rangle_{\partial T} \right| \\
\leq \left(\sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} h_{T}^{3} \| \partial_{j} (\partial_{ij}^{2} w - \mathcal{Q}_{h} \partial_{ij}^{2} w) \|_{\partial T}^{2} \right)^{\frac{1}{2}} \cdot \left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3} \| v_{0} - v_{b} \|_{\partial T}^{2} \right)^{\frac{1}{2}} \\
\leq Ch^{r-1} \| w \|_{r+1} \| v \|.$$

Combining (7.5) with (7.6) yields

(7.7)
$$|I_2(w,v)| \le Ch^{r-1}(||w||_{r+1} + \delta_{k,2}||w||_4) |||v|||.$$

This completes the proof of the lemma. \square

LEMMA 7.2. Assume that $w \in H^{r+1}(\Omega), v \in V_h^0$ with $r \in [2, k]$. Let $I_3(w, v)$ and $I_4(w, v)$ be given in (7.1). Then, we have

(7.8)
$$|I_3(w,v)| + |I_4(w,v)| \le Ch^{r-1} ||w||_{r+1} |v|_h,$$

where

(7.9)
$$|v|_h = s(v,v)^{\frac{1}{2}}$$

Proof. To estimate the term $I_3(w, v)$, we use Cauchy-Schwarz inequality and the estimate (5.6) with m = r to obtain

(7.10)
$$|I_{3}(w,v)| = \left| \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \langle \nabla Q_{0}w - \nabla w, Q_{b}(\nabla v_{0}) - \mathbf{v}_{g} \rangle_{\partial T} \right|$$
$$\leq \left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \| \nabla Q_{0}w - \nabla w \|_{\partial T}^{2} \right)^{\frac{1}{2}} \cdot \left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \| Q_{b}(\nabla v_{0}) - \mathbf{v}_{g} \|_{\partial T}^{2} \right)^{\frac{1}{2}}$$
$$\leq Ch^{r-1} \| w \|_{r+1} \| v |_{h}.$$

As to the term $I_4(w, v)$, we use Cauchy-Schwarz inequality and the estimate (5.7) with m = r to obtain

(7.11)

$$|I_4(w,v)| = \left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 w - w, Q_b v_0 - v_b \rangle_{\partial T} \right|$$

$$\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 w - w\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_b v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}}$$

$$\leq C h^{r-1} \|w\|_{r+1} |v|_h.$$

This completes the proof. \Box

The following result is an estimate for the error function e_h in the trip-bar norm which is essentially an H^2 -equivalent norm in V_h^0 .

THEOREM 7.3. Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (4.1) with finite elements of order $k \geq 2$. Assume that the exact solution uof (1.1) is sufficiently regular such that $u \in H^{\max\{k+1,4\}}(\Omega)$. Then, there exists a constant C such that

(7.12)
$$||\!|u_h - Q_h u|\!|\!| \le Ch^{k-1} \Big(||u||_{k+1} + \delta_{k,2} ||u||_4 \Big).$$

In other words, we have an optimal order of convergence in the H^2 norm.

Proof. By letting $v = e_h$ in the error equation (6.2), we obtain the following identity

(7.13)
$$\|\|e_h\|\|^2 = \phi_u(e_h)$$
$$= I_1(u, e_h) + I_2(u, e_h) + I_3(u, e_h) + I_4(u, e_h)$$

Using the estimates (7.2), (7.3), and (7.8) with w = u and $v = e_h$ we arrive at

$$|||e_h|||^2 \le Ch^{k-1} (||u||_{k+1} + \delta_{k,2} ||u||_4) |||e_h|||,$$

which implies the desired error estimate (7.12).

8. Error Estimates in L^2 . This section shall establish an estimate for the first component of the error function e_h in the standard L^2 norm. To this end, we consider

the following dual problem:

(8.1)
$$\begin{aligned} \Delta^2 \psi &= e_0 & \text{ in } \Omega, \\ \psi &= 0 & \text{ on } \partial\Omega, \\ \frac{\partial \psi}{\partial \mathbf{n}} &= 0 & \text{ on } \partial\Omega. \end{aligned}$$

Assume the above dual problem has the following regularity estimate

$$\|\psi\|_4 \le C \|e_0\|.$$

THEOREM 8.1. Let $u_h \in V_h$ be the solution of the Weak Galerkin Algorithm (4.1) with finite elements of order $k \ge 2$. Let $t_0 = \min\{k, 3\}$. Assume that the exact solution of (1.1) is sufficiently regular so that $u \in H^4(\Omega)$ for k = 2 and $u \in H^{k+1}(\Omega)$ otherwise, and the dual problem (8.1) has the H^4 regularity. Then, there exists a constant C such that

(8.3)
$$\|Q_0u - u_0\| \le Ch^{k+t_0-2} \Big(\|u\|_{k+1} + \delta_{k,2} \|u\|_4 \Big).$$

In other words, we have a sub-optimal order of convergence for k = 2 and optimal order of convergence for $k \ge 3$.

Proof. By testing (8.1) against the error function e_0 on each element and using the integration by parts, we obtain

$$\begin{split} \|e_0\|^2 &= (\Delta^2 \psi, e_0) \\ &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \left\{ (\partial_{ij}^2 \psi, \partial_{ij}^2 e_0)_T - \langle \partial_{ij}^2 \psi, \partial_i e_0 \cdot n_j \rangle_{\partial T} + \langle \partial_j (\partial_{ij}^2 \psi) \cdot n_i, e_0 \rangle_{\partial T} \right\} \\ &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \left\{ (\partial_{ij}^2 \psi, \partial_{ij}^2 e_0)_T - \langle \partial_{ij}^2 \psi, (\partial_i e_0 - e_{gi}) \cdot n_j \rangle_{\partial T} \right. \\ &+ \left. \langle \partial_j (\partial_{ij}^2 \psi) \cdot n_i, e_0 - e_b \rangle_{\partial T} \right\}, \end{split}$$

where the added terms associated with e_b and e_{gi} vanish due to the cancelation for interior edges and the fact that e_b and e_{gi} have zero value on $\partial\Omega$. Using (6.4) with ψ and e_h in the place of u and v_0 respectively, we arrive at

$$(8.4) \qquad \|e_0\|^2 = (\partial_w^2 Q_h \psi, \partial_w^2 e_h)_h + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \left\{ \langle \partial_j (\partial_{ij}^2 \psi - \mathcal{Q}_h (\partial_{ij}^2 \psi)) \cdot n_i, e_0 - e_b \rangle_{\partial T} - \langle \partial_{ij}^2 \psi - \mathcal{Q}_h \partial_{ij}^2 \psi, (\partial_i e_0 - e_{gi}) \cdot n_j \rangle_{\partial T} \right\} \\ = (\partial_w^2 Q_h \psi, \partial_w^2 e_h)_h - \phi_\psi(e_h) + s(Q_h \psi, e_h).$$

Next, it follows from the error equation (6.2) that

(8.5)
$$(\partial_w^2 Q_h \psi, \partial_w^2 e_h)_h = \phi_u(Q_h \psi) - s(e_h, Q_h \psi).$$

Substituting (8.5) into (8.4) yields

(8.6)
$$||e_0||^2 = \phi_u(Q_h\psi) - \phi_\psi(e_h).$$

The term $\phi_{\psi}(e_h)$ can be handled by using Lemma 7.1 and Lemma 7.2 with $r = t_0 = \min\{k, 3\}$ as follows:

(8.7)
$$\begin{aligned} |\phi_{\psi}(e_{h})| &\leq Ch^{t_{0}-1}(\|\psi\|_{t_{0}+1} + h\|\psi\|_{4}) \|e_{h}\| \\ &\leq Ch^{t_{0}-1} \|\psi\|_{4} \|e_{h}\| \\ &\leq Ch^{t_{0}-1} \|e_{0}\| \|e_{h}\| , \end{aligned}$$

where we have used the regularity assumption (8.2) in the last inequality.

It remains to deal with the term $\phi_u(Q_h\psi)$ in (8.6). Note that from (7.1) we have

(8.8)
$$\phi_u(Q_h\psi) = \sum_{j=1}^4 I_j(u, Q_h\psi).$$

 $I_3(u,Q_h\psi)$ and $I_4(u,Q_h\psi)$ can be handled by using Lemma 7.2 with r=k as follows:

(8.9)
$$|I_3(u, Q_h \psi)| + |I_3(u, Q_h \psi)| \le Ch^{k-1} ||u||_{k+1} |Q_h \psi|_h.$$

From the definition (7.9) we have

$$\begin{aligned} |Q_h\psi|_h^2 &= \sum_{T \in \mathcal{T}_h} \left(h_T^{-3} \|Q_b(Q_0\psi) - Q_b\psi\|_{\partial T}^2 + h_T^{-1} \|Q_b(\nabla Q_0\psi) - Q_b\nabla\psi\|_{\partial T}^2 \right) \\ &\leq \sum_{T \in \mathcal{T}_h} \left(h_T^{-3} \|Q_0\psi - \psi\|_{\partial T}^2 + h_T^{-1} \|\nabla(Q_0\psi) - \nabla\psi\|_{\partial T}^2 \right) \end{aligned}$$

Thus, it follows from the trace inequality (10.1) and the error estimate for the projection operator Q_0 that

(8.10)
$$|Q_h\psi|_h \le Ch^{t_0-1} \|\psi\|_{t_0+1} \le Ch^{t_0-1} \|\psi\|_4 \le Ch^{t_0-1} \|e_0\|.$$

Substituting the above estimate into (8.9) yields

(8.11)
$$|I_3(u, Q_h\psi)| + |I_3(u, Q_h\psi)| \le Ch^{k+t_0-2} ||u||_{k+1} ||e_0||.$$

The estimate for $I_1(u, Q_h\psi)$ and $I_2(u, Q_h\psi)$ shall explore the special property of the "test" function $Q_h\psi$. To this end, using the orthogonality property of Q_b and the fact that $\psi = Q_b\psi = 0$ on $\partial\Omega$ we obtain

$$\sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j (\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, \psi - Q_b \psi \rangle_{\partial T}$$
$$= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j \partial_{ij}^2 u \cdot n_i, \psi - Q_b \psi \rangle_{\partial T} = 0.$$

Thus,

$$I_{2}(u,Q_{h}\psi) = -\sum_{T\in\mathcal{T}_{h}}\sum_{i,j=1}^{d} \langle \partial_{j}(\partial_{ij}^{2}u - \mathcal{Q}_{h}\partial_{ij}^{2}u) \cdot n_{i}, Q_{0}\psi - Q_{b}\psi \rangle_{\partial T}$$
$$= -\sum_{T\in\mathcal{T}_{h}}\sum_{i,j=1}^{d} \langle \partial_{j}(\partial_{ij}^{2}u - \mathcal{Q}_{h}\partial_{ij}^{2}u) \cdot n_{i}, Q_{0}\psi - \psi \rangle_{\partial T}.$$

Using the Cauchy-Schwarz inequality and the standard error estimate for L^2 projections we arrive at

(8.12)

$$|I_{2}(u,Q_{h}\psi)| \leq \sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} \|\partial_{j}(\partial_{ij}^{2}u - \mathcal{Q}_{h}\partial_{ij}^{2}u)\|_{\partial T} \|Q_{0}\psi - \psi\|_{\partial T}$$

$$\leq Ch^{k+t_{0}-2}(\|u\|_{k+1} + \delta_{k,2}\|u\|_{4}) \|\psi\|_{t_{0}+1}$$

$$\leq Ch^{k+t_{0}-2}(\|u\|_{k+1} + \delta_{k,2}\|u\|_{4}) \|\psi\|_{4}$$

$$\leq Ch^{k+t_{0}-2}(\|u\|_{k+1} + \delta_{k,2}\|u\|_{4}) \|e_{0}\|.$$

A similar argument can be employed to deal with the term $I_1(u, Q_h \psi)$, yielding

(8.13)
$$|I_1(u, Q_h \psi)| \le C h^{k+t_0-2} ||u||_{k+1} ||e_0||.$$

Substituting (8.11), (8.12), and (8.13) into (8.8) we arrive at

(8.14)
$$|\phi_u(Q_h\psi)| \le Ch^{k+t_0-2} (||u||_{k+1} + \delta_{k,2} ||u||_4) ||e_0||.$$

Finally, by inserting (8.7) and (8.14) into (8.6) we obtain

$$\|e_0\|^2 \le C(h^{t_0-1} \|\|e_h\|\| + h^{k+t_0-2}(\|u\|_{k+1} + \delta_{k,2} \|u\|_4)) \|e_0\|,$$

which, together with the estimate (7.12) in Theorem 7.3, gives rise to the desired L^2 error estimate (8.3). This completes the proof of the theorem.

The H^2 error estimate (7.12) and the L^2 error estimate (8.3) can be used to derive some error estimates for the WG solution u_b and \mathbf{u}_g . More precisely, observe that e_b and \mathbf{e}_g can be represented by e_0 and $\partial_{ij,w}^2 e_h$ by choosing special test functions v in the error equation (6.2). For example, e_b can be represented by e_0 and $\partial_{ij,w}^2 e_h$ by selecting $v = \{0, v_b, 0\}$. The representation is expressed through an equation defined locally on each edge $e \in \mathcal{E}_h^0$. The rest of the analysis should be straightforward. Details are omitted due to page limitation.

9. Numerical Experiments. In this section, we present some numerical results for the WG finite element method analyzed in previous sections. The goal is to demonstrate the efficiency and the convergence theory established for the method. For simplicity, we implement the lowest order scheme for the Weak Galerkin Algorithm (4.1). In other words, the implementation makes use of the following finite element space

$$\widetilde{V}_h = \{v = \{v_0, v_b, \mathbf{v}_g\}, v_0 \in P_2(T), v_b \in P_0(e), \mathbf{v}_g \in [P_0(e)]^2, T \in \mathcal{T}_h, e \in \mathcal{E}_h\}.$$

For any given $v = \{v_0, v_b, \mathbf{v}_g\} \in \widetilde{V}_h$, the discrete weak partial derivative $\partial_{ij,w,r,T}^2 v$ is computed as a constant locally on each element T by solving the following equation

$$(\partial_{ij,w,r,T}^2 v, \varphi)_T = (v_0, \partial_{ji}^2 \varphi)_T - \langle v_b, \partial_j \varphi \cdot n_i \rangle_{\partial T} + \langle v_{gi} \cdot n_j, \varphi \rangle_{\partial T},$$

for all $\varphi \in P_0(T)$. Since $\varphi \in P_0(T)$, the above equation can be simplified as

(9.1)
$$(\partial_{ij,w,r,T}^2 v, \varphi)_T = \langle v_{gi} \cdot n_j, \varphi \rangle_{\partial T}, \qquad \forall \varphi \in P_0(T), \ i, j = 1, 2.$$

The error for the solution of the Weak Galerkin Algorithm (4.1) is measured in four norms or semi-norms defined as follows:

(9.2)
$$|||v|||^{2} = \sum_{T \in \mathcal{T}_{h}} \left(\sum_{i,j=1}^{d} \int_{T} (\partial_{ij,w}^{2} v_{h})^{2} dx + h_{T}^{-1} \int_{\partial T} |Q_{b}(\nabla v_{0}) - \mathbf{v}_{g}|^{2} ds + h_{T}^{-3} \int_{\partial T} (Q_{b} v_{0} - v_{b})^{2} ds \right), \quad (A \text{ discrete } H^{2}\text{-norm}),$$

(9.3)
$$||v||^2 = \sum_{T \in \mathcal{T}_h} \int_T v_0^2 dx, \qquad (\text{ Element-based } L^2\text{-norm}),$$

(9.4)
$$||v_b||_{\infty} = \max_{e \in \mathcal{E}_h} ||v_b||_{\infty},$$
 (Edge-based L^{∞} -norm),

(9.5)
$$\|\mathbf{v}_g\|_{\infty} = \max_{e \in \mathcal{E}_h} \|\mathbf{v}_g\|_{\infty},$$
 (Edge-based L^{∞} -norm).

The numerical experiment is conducted for the biharmonic equation (1.1) on the unit square domain $\Omega = (0,1)^2$. The function f = f(x,y) and the two boundary conditions are computed to match the exact solution in each test case. The WG finite element scheme (4.1) was implemented on two type of partitions: (1) uniform triangular partition, and (2) uniform rectangular partition. The uniform rectangular partition was obtained by partitioning the domain into $n \times n$ sub-rectangles as tensor products of 1-d uniform partitions. The triangular meshes are constructed from the rectangular partition by dividing each square element into two triangles by the diagonal line with a negative slope. The mesh size is denoted by h = 1/n.

Table 9.1 demonstrates the performance of the code when the exact solution is given by $u = x^2 + y^2 + xy + x + y + 1$. In theory, the WG finite element method is exact for any quadratic polynomials. The computational results are in consistency with theory. This table indicates that the code should be working.

TABLE 9.1 Numerical error for the biharmonic equation with exact solution $u = x^2 + y^2 + xy + x + y + 1$ on triangular partitions.

h	$\ u_0 - Q_0 u\ $	$ u_h - Q_h u $	$\ u_b - Q_b u\ _{\infty}$	$\ \mathbf{u}_g - Q_b(\nabla u)\ _{\infty}$
1	1.73e-014	2.03e-014	1.60e-014	4.44e-016
5.0000e-01	4.35e-014	1.88e-013	5.82e-014	6.66e-015
2.5000e-01	1.64e-013	1.60e-012	2.86e-013	8.44e-014
1.2500e-01	7.68e-013	7.68e-013	1.48e-012	1.19e-012
6.2500e-02	3.65e-012	9.92e-011	7.71e-012	1.34e-011
3.1250e-02	1.43e-011	5.20e-010	5.19e-010	3.13e-011
1.5625e-02	4.85e-011	3.40e-009	1.15e-010	5.98e-010

Tables 9.2 and 9.3 show the numerical results when the exact solution is given by $u = x^2(1-x)^2y^2(1-y)^2$. This case has a homogeneous boundary condition for both Dirichlet and Neumann. It shows that the convergence rates for the solution of

the Weak Galerkin Algorithm in the H^2 and L^2 norms are of order O(h) and $O(h^2)$, respectively. The numerical results are in consistency with theory for the L^2 and H^2 norm of the error. For the approximation of u on the edge set \mathcal{E}_h , it appears that the L^{∞} error is of order $\mathcal{O}(h^2)$. But the order of convergence for the approximation of ∇u on the edge set \mathcal{E}_h is hard to extract from the data. It is interesting to see that the absolute error for both u_b and \mathbf{u}_g is quite small.

TABLE 9.2 Numerical error and convergence order for exact solution $u = x^2(1-x)^2y^2(1-y)^2$ on triangular partitions.

h	$\ u_0 - Q_0 u\ $	order	$\ u_h - Q_h u\ $	order
1	0.41325		0.52598	
5.0000e-01	0.07371	2.49	0.31309	0.75
2.5000e-01	0.019859	1.89	0.18972	0.72
1.2500e-01	0.005176	1.94	0.100557	0.92
6.2500e-02	0.0013833	1.90	0.05240	0.94
3.1250e-02	3.7499e-004	1.88	0.02729	0.94
1.5625e-02	9.977e-005	1.91	0.014058	0.96
7.8125e-03	2.583e-05	1.95	0.007145	0.98

TABLE 9.3 Numerical error and convergence order for exact solution $u = x^2(1-x)^2y^2(1-y)^2$ on triangular partitions.

h	$\ u_b - Q_b u\ _{\infty}$	order	$\ \mathbf{u}_g - Q_b(\nabla u)\ _{\infty}$	order
1	0.41494		8.6485e-018	
5.0000e-01	0.08806	2.24	0.00942	
2.5000e-01	0.037013	1.25	0.00491	0.94
1.2500e-01	0.01069	1.79	0.00354	0.47
6.2500e-02	0.00293	1.87	0.00222	0.67
3.1250e-02	7.935e-004	1.88	0.00102	1.12
1.5625e-02	2.096e-004	1.92	3.577e-004	1.51
7.8125e-03	5.401e-05	1.96	1.053e-04	1.76

Tables 9.4 and 9.5 present some numerical results when the exact solution is given by $u = \sin(x)\sin(y)$. We would like to invite the readers to draw conclusions from these data. TABLE 9.4

Numerical error and convergence order for exact solution $u = \sin(x)\sin(y)$ on triangular partitions.

h	$\ u_0 - Q_0 u\ $	order	$ u_h - Q_h u $	order
1	0.23000		0.37336	
5.0000e-01	0.03575	2.68	0.27641	0.43
2.5000e-01	0.00684	2.38	0.21911	0.34
1.2500e-01	0.00147	2.21	0.17661	0.31
6.2500e-02	4.427e-004	1.74	0.12349	0.52
3.1250e-02	1.549e-004	1.52	0.07290	0.76
1.5625e-02	4.658e-005	1.73	0.03916	0.90

TABLE 9.5

Numerical error and convergence order for exact solution $u = \sin(x)\sin(y)$ on triangular partitions.

h	$\ u_b - Q_b u\ _{\infty}$	order	$\ \mathbf{u}_g - Q_b(\nabla u)\ _{\infty}$	order
1	0.21688		0.06306	
5.0000e-01	0.05108	2.09	0.05601	0.17
2.5000e-01	0.01132	2.17	0.05062	0.15
1.2500e-01	0.002524	2.17	0.03606	0.49
6.2500e-02	8.032e-004	1.65	0.01772	1.03
3.1250e-02	3.226e-004	1.32	0.00590	1.59
1.5625e-02	1.038e-004	1.64	0.00163	1.85

Table 9.6 demonstrates the performance of the WG finite element method when the exact solution is a biquadratic polynomial. It shows that the L^2 convergence is of order $\mathcal{O}(h^2)$, and the H^2 convergence has a rate approximately $\mathcal{O}(h)$.

Γable	9.6
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Numerical error and convergence rates for the biharmonic equation with exact solution u = x(1-x)y(1-y) on triangular meshes.

h	$ u_0 - Q_0 u $	order	$ u_h - Q_h u $	order
1	2.05586		4.05772	
5.0000e-01	0.32234	2.67	1.59961	1.34
2.5000e-01	0.06654	2.28	0.70890	1.17
1.2500e-01	0.01588	2.07	0.34325	1.05
6.2500e-02	0.00394	2.01	0.17416	0.98
3.1250e-02	9.691e-4	2.02	0.09025	0.95
1.5625e-02	2.361e-4	2.04	0.046632	0.95

The rest of the section will present some numerical results on rectangular meshes. The lowest order WG element on rectangles consists of quadratic polynomials on each element enriched with constants on the edge of each element for both u and ∇u . Therefore, the total number of unknowns on each element is 18. Note that all the unknowns corresponding to u on each element can be eliminated locally, so that the actual number of unknowns on each element is 12. Table 9.7 shows the numerical

solution when the exact solution is a quadratic polynomial. It can be seen that the numerical solution is numerically the same as the exact solution, as predicted by the theory.

TABLE 9.7 Numerical error for the biharmonic equation with exact solution $u = x^2 + y^2 + xy + x + y + 1$ on rectangular partitions.

h	$\ u_0 - Q_0 u\ $	$\ u_h - Q_h u\ $	$\ u_b - Q_b u\ _{\infty}$	$\ \mathbf{u}_g - Q_b(\nabla u)\ _{\infty}$
1	2.09e-015	0	0	0
2.5000e-01	4.66e-015	2.94e-014	6.66e-015	3.55e-015
6.2500e-02	9.91e-014	2.30e-012	1.98e-013	2.74e-013
1.5625e-02	5.83e-012	2.06e-010	1.34e-011	4.10e-011

Tables 9.8 and 9.9 show the numerical results when the exact solution is given by $u = x^2(1-x)^2y^2(1-y)^2$. The result is in consistency with the theory.

TABLE 9.8 Numerical error and convergence order for exact solution $u = x^2(1-x)^2y^2(1-y)^2$ on rectangular partitions.

h	$\ u_0 - Q_0 u\ $	order	$\ u_h - Q_h u\ $	order
1	1.15052		0	
5.0000e-01	0.14880	2.95	0.35	
2.5000e-01	0.03786	1.97	0.24649	0.52
1.2500e-01	0.009724	1.96	0.13593	0.86
6.2500e-02	0.002494	1.96	0.070216	0.95
3.1250e-02	6.509e-004	1.94	0.035987	0.96
1.5625e-02	1.709e-004	1.93	0.018427	0.97
7.8125e-03	4.415e-005	1.95	0.009357	0.98

TABLE 9.9 Numerical error and convergence order for exact solution $u = x^2(1-x)^2y^2(1-y)^2$ on rectangular partitions.

h	$\ u_b - Q_b u\ _{\infty}$	order	$\ \mathbf{u}_g - Q_b(\nabla u)\ _{\infty}$	order
1	0		0	
5.0000e-01	0.15414		0.01343	
2.5000e-01	0.06724	1.20	0.008681	0.6297
1.2500e-01	0.01961	1.78	0.0034078	1.3490
6.2500e-02	0.00518	1.92	0.0014578	1.2251
3.1250e-02	0.001359	1.93	8.774e-004	0.7325
1.5625e-02	3.566e-004	1.93	3.788e-004	1.2116
7.8125e-03	9.195e-005	1.96	1.231e-004	1.6211

Tables 9.10 and 9.11 present some results for the exact solution $u = \sin(x) \sin(y)$. Readers are encouraged to compare the results here with those in Tables 9.4 and 9.5. TABLE 9.10

Numerical error and convergence order for exact solution $u = \sin(x)\sin(y)$ on triangular partitions.

h	$ u_0 - Q_0 u $	order	$ u_h - Q_h u $	order
1	0.60602		0	
5.0000e-01	0.08424	2.85	0.26684	
2.5000e-01	0.01549	2.44	0.22733	0.23
1.2500e-01	0.00360	2.10	0.18593	0.29
6.2500e-02	0.00101	1.83	0.13440	0.47
3.1250e-02	2.98e-004	1.77	0.081869	0.72
1.5625e-02	7.95e-005	1.91	0.044701	0.87

TABLE 9.11

Numerical error and convergence order for exact solution $u = \sin(x)\sin(y)$ on rectangular partitions.

h	$\ u_b - Q_b u\ _{\infty}$	order	$\ \mathbf{u}_g - Q_b(\nabla u)\ _{\infty}$	order
1	0		0	
5.0000e-01	0.10202		0.06063	
2.5000e-01	0.02488	2.04	0.051219	0.24
1.2500e-01	0.006110	2.03	0.039518	0.37
6.2500e-02	0.001981	1.62	0.021362	0.89
3.1250e-02	5.810e-004	1.77	0.007942	1.43
1.5625e-02	1.501e-004	1.95	0.002355	1.75

More numerical experiments should be conducted for the Weak Galerkin Algorithm (4.1), particularly for elements of order higher than k = 2. There is also a need of developing fast solution techniques for the matrix problem arising from the WG finite element scheme (4.1). Numerical experiments on finite element partitions with arbitrary polygonal element should be conducted for a further assessment of the WG method.

10. Appendix. The goal of this Appendix is to establish some fundamental estimates useful in the error estimate for general weak Galerkin finite element methods.

For any $T \in \mathcal{T}_h$, let φ be a regular function in $H^1(T)$. The following trace inequality holds true [11]:

(10.1)
$$\|\varphi\|_{e}^{2} \leq C(h_{T}^{-1}\|\varphi\|_{T}^{2} + h_{T}\|\nabla\varphi\|_{T}^{2}),$$

If φ is a polynomial on the element $T \in \mathcal{T}_h$, then we have from the inverse inequality (see also [11]) that

(10.2)
$$\|\varphi\|_{e}^{2} \leq Ch_{T}^{-1}\|\varphi\|_{T}^{2}.$$

Here e is an edge/face on the boundary of T.

LEMMA 10.1. For the discrete weak partial derivative $\partial_{ij,w}^2$, the following identity holds true on each element $T \in \mathcal{T}_h$:

(10.3)
$$(\partial_{ij,w}^2 v, \varphi)_T = (\partial_{ij}^2 v_0, \varphi)_T + \langle v_0 - v_b, \partial_j \varphi \cdot n_i \rangle_{\partial T} - \langle \partial_i v_0 - v_{gi}, \varphi n_j \rangle_{\partial T}$$

for all $\varphi \in P_{k-2}(T)$. Consequently, we have

(10.4)
$$(\partial_{ij,w}^2 v, \varphi)_T = (\partial_{ij}^2 v_0, \varphi)_T + \langle Q_b v_0 - v_b, \partial_j \varphi \cdot n_i \rangle_{\partial T} - \langle Q_b (\partial_i v_0) - v_{gi}, \varphi n_j \rangle_{\partial T}$$

Proof. From the definition (3.2) of the weak partial derivative, we have

$$\begin{aligned} (\partial_{ij,w}^2 v, \varphi)_T = &(v_0, \partial_{ji}^2 \varphi)_T + \langle v_{gi} \cdot n_j, \varphi \rangle_{\partial T} - \langle v_b, \partial_j \varphi \cdot n_i \rangle_{\partial T} \\ = &(\partial_{ij}^2 v_0, \varphi)_T - \langle \partial_i v_0, \varphi \cdot n_j \rangle_{\partial T} + \langle v_0, \partial_j \varphi \cdot n_i \rangle_{\partial T} \\ &+ \langle v_{gi} \cdot n_j, \varphi \rangle_{\partial T} - \langle v_b, \partial_j \varphi \cdot n_i \rangle_{\partial T} \\ = &(\partial_{ij}^2 v_0, \varphi)_T + \langle v_0 - v_b, \partial_j \varphi \cdot n_i \rangle_{\partial T} - \langle \partial_i v_0 - v_{gi}, \varphi n_j \rangle_{\partial T}. \end{aligned}$$

Here we have used the usual integration by parts in the second line. The result then follows. \square

LEMMA 10.2. Let $e_h \in V_h^0$ be any finite element function. Then, there holds

(10.5)
$$\sum_{T \in \mathcal{T}_h} |e_0|_{2,T} \le C ||\!|e_h|\!||,$$

where, by definition (4.2),

(10.6)
$$\|\|e_{h}\|\|^{2} = \sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} (\partial_{ij,w}^{2} e_{h}, \partial_{ij,w}^{2} e_{h})_{T} + \sum_{T \in \mathcal{T}_{h}} h_{T}^{-3} \langle Q_{b}e_{0} - e_{b}, Q_{b}e_{0} - e_{b} \rangle_{\partial T}$$
$$+ \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \langle Q_{b}(\nabla e_{0}) - e_{g}, Q_{b}(\nabla e_{0}) - e_{g} \rangle_{\partial T}.$$

Proof. Using (10.4) with $v = e_h$ and $\varphi = \partial_{ij}^2 e_0$ we obtain

$$\begin{aligned} (\partial_{ij,w}^2 e_h, \partial_{ij}^2 e_0)_T = & (\partial_{ij}^2 e_0, \partial_{ij}^2 e_0)_T - \langle Q_b(\partial_i e_0) - e_{gi}, \partial_{ij}^2 e_0 \cdot n_j \rangle_{\partial T} \\ &+ \langle Q_b e_0 - e_b, \partial_j(\partial_{ij}^2 e_0) \cdot n_i \rangle_{\partial T}. \end{aligned}$$

Thus,

(10.7)
$$(\partial_{ij}^2 e_0, \partial_{ij}^2 e_0)_T = (\partial_{ij,w}^2 e_h, \partial_{ij}^2 e_0)_T + \langle Q_b(\partial_i e_0) - e_{gi}, \partial_{ij}^2 e_0 \cdot n_j \rangle_{\partial T} - \langle Q_b e_0 - e_b, \partial_j(\partial_{ij}^2 e_0) \cdot n_i \rangle_{\partial T}.$$

It then follows from (10.7), Cauchy-Schwarz inequality, the inverse inequality and (10.2) that

$$\begin{aligned} (\partial_{ij}^{2}e_{0},\partial_{ij}^{2}e_{0})_{T} \leq & \|\partial_{ij,w}^{2}e_{h}\|_{T}\|\partial_{ij}^{2}e_{0}\|_{T} + \|Q_{b}(\partial_{i}e_{0}) - e_{gi}\|_{\partial T}\|\partial_{ij}^{2}e_{0}\|_{\partial T} \\ & + \|Q_{b}e_{0} - e_{b}\|_{\partial T}\|\partial_{j}(\partial_{ij}^{2}e_{0})\|_{\partial T} \\ \leq & \|\partial_{ij,w}^{2}e_{h}\|_{T}\|\partial_{ij}^{2}e_{0}\|_{T} + Ch_{T}^{-\frac{1}{2}}\|Q_{b}(\partial_{i}e_{0}) - e_{gi}\|_{\partial T}\|\partial_{ij}^{2}e_{0}\|_{T} \\ & + Ch_{T}^{-\frac{3}{2}}\|Q_{b}e_{0} - e_{b}\|_{\partial T}\|\partial_{ij}^{2}e_{0}\|_{T}, \end{aligned}$$

which implies

$$(10.8) \quad \|\partial_{ij}^2 e_0\|_T^2 \le \|\partial_{ij,w}^2 e_h\|_T^2 + Ch_T^{-1}\|Q_b(\partial_i e_0) - e_{gi}\|_{\partial T}^2 + Ch_T^{-3}\|Q_b e_0 - e_b\|_{\partial T}^2.$$

Summing over $T \in \mathcal{T}_h$ completes the proof of the lemma.

LEMMA 10.3. For any $e_h \in V_h^0$ and $k \ge 3$, there exists a constant C such that

(10.9)
$$\left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|e_0 - e_b\|_{\partial T}^2\right)^{\frac{1}{2}} \le C \|\|e_h\|\|.$$

Proof. By the triangle inequality and the error estimate for the projection Q_b , we have

$$h_T^{-3} \| e_0 - e_b \|_{\partial T}^2 \leq 2h_T^{-3} \Big(\| e_0 - Q_b e_0 \|_{\partial T}^2 + \| Q_b e_0 - e_b \|_{\partial T}^2 \Big)$$

$$\leq 2h_T^{-3} \Big(Ch_T^2 | e_0 |_{2,\partial T} \Big)^2 + 2h_T^{-3} \| Q_b e_0 - e_b \|_{\partial T}^2$$

$$\leq 2Ch_T | e_0 |_{2,\partial T}^2 + 2h_T^{-3} \| Q_b e_0 - e_b \|_{\partial T}^2$$

$$\leq 2C | e_0 |_{2,T}^2 + 2h_T^{-3} \| Q_b e_0 - e_b \|_{\partial T}^2.$$

Combining the above with (10.5) gives (10.9).

LEMMA 10.4. (Poincaré Inequality) There exists a constant C such that

(10.10)
$$\sum_{T \in \mathcal{T}_h} \|e_0\|_T^2 \le C \Big(\sum_{T \in \mathcal{T}_h} \|\nabla e_0\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|e_0 - e_b\|_{\partial T}^2 \Big),$$

where $e_h \in V_h$ is any finite element function with $e_b = 0$.

Proof. Consider the Laplace equation:

$$\begin{aligned} -\Delta \phi &= e_0 & \text{ in } \Omega, \\ \phi &= 0 & \text{ on } \partial \Omega. \end{aligned}$$

Assume that the solution ϕ is regular so that

(10.11)
$$\|\phi\|_2^2 \le C \|e_0\|^2.$$

The above assumption is always satisfied since otherwise we may extend the domain Ω to $\tilde{\Omega}$ in which the required regularity is satisfied, with e_0 being extended by zero outside of Ω .

By letting $\mathbf{w} = -\nabla \phi$, we have

$$\sum_{T \in \mathcal{T}_h} (e_0, e_0)_T = \sum_{T \in \mathcal{T}_h} (e_0, \nabla \cdot \mathbf{w})_T = \sum_{T \in \mathcal{T}_h} \langle e_0, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial T} - \sum_{T \in \mathcal{T}_h} (\mathbf{w}, \nabla e_0)_T$$
$$= \sum_{T \in \mathcal{T}_h} \langle (e_0 - e_b), \mathbf{w} \cdot \mathbf{n} \rangle_{\partial T} - \sum_{T \in \mathcal{T}_h} (\mathbf{w}, \nabla e_0)_T$$
$$\leq \sum_{T \in \mathcal{T}_h} \|\mathbf{w}\|_T \|\nabla e_0\|_T + \sum_{T \in \mathcal{T}_h} \|\mathbf{w}\|_{\partial T} \|e_0 - e_b\|_{\partial T}.$$

The trace inequality (10.1) implies

$$\|\mathbf{w}\|_{\partial T}^{2} \leq C(h_{T}^{-1}\|\mathbf{w}\|_{T} + h_{T}\|\nabla\mathbf{w}\|_{T}) \leq Ch_{T}^{-1}\|\mathbf{w}\|_{1,T}^{2}$$

Thus, from Cauchy-Schwarz and the regularity (10.11) we obtain

$$\begin{split} \sum_{T \in \mathcal{T}_{h}} (e_{0}, e_{0})_{T} &\leq \sum_{T \in \mathcal{T}_{h}} \|\mathbf{w}\|_{1,T} \|\nabla e_{0}\|_{T} + \sum_{T \in \mathcal{T}_{h}} Ch_{T}^{-\frac{1}{2}} \|\mathbf{w}\|_{1,T} \|e_{0} - e_{b}\|_{\partial T} \\ &\leq C \Big(\sum_{T \in \mathcal{T}_{h}} \|\nabla e_{0}\|_{T}^{2} + \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \|e_{0} - e_{b}\|_{\partial T}^{2} \Big)^{\frac{1}{2}} \|\phi\|_{2} \\ &\leq C \Big(\sum_{T \in \mathcal{T}_{h}} \|\nabla e_{0}\|_{T}^{2} + \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \|e_{0} - e_{b}\|_{\partial T}^{2} \Big)^{\frac{1}{2}} \|e_{0}\|, \end{split}$$

which verifies the estimate (10.10).

The following is another version of the Poincaré inequality for functions in V_h^0 . LEMMA 10.5. There exists a constant C such that

(10.12)
$$\left(\sum_{T\in\mathcal{T}_{h}} \|\nabla e_{0}\|_{T}^{2}\right)^{\frac{1}{2}} \leq C \|\|e_{h}\|$$

for all $e_h \in V_h^0$.

Proof. Since $e_h \in V_h^0$, then we have $\mathbf{e}_g = 0$. Thus, an application of (10.10) with e_0 replaced by ∇e_0 yields

(10.13)
$$\sum_{T \in \mathcal{T}_h} \|\nabla e_0\|_T^2 \le C \Big(\sum_{T \in \mathcal{T}_h} |e_0|_{2,T}^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\nabla e_0 - \mathbf{e}_g\|_{\partial T}^2 \Big).$$

For the second term on the right-hand side of (10.13), we have

(10.14)
$$\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \| \nabla e_{0} - \mathbf{e}_{g} \|_{\partial T}^{2} \\ \leq 2 \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \| \nabla e_{0} - Q_{b}(\nabla e_{0}) \|_{\partial T}^{2} + 2 \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1} \| Q_{b}(\nabla e_{0}) - \mathbf{e}_{g} \|_{\partial T}^{2}$$

Substituting (10.14) into (10.13) yields

$$\sum_{T \in \mathcal{T}_h} \|\nabla e_0\|_T^2 \le C \sum_{T \in \mathcal{T}_h} |e_0|_{2,T}^2 + C ||\!|e_h|\!||^2 \le C ||\!|e_h|\!||^2,$$

where we have used (10.5) in the last inequality. \Box

LEMMA 10.6. For quadratic element k = 2, we assume that the exact solution u of (1.1) is sufficiently regular such that $u \in H^4(\Omega)$. There exists a constant C such that the following inequality holds true:

(10.15)
$$\left| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle \partial_j (\partial_{ij}^2 u - \mathcal{Q}_h \partial_{ij}^2 u) \cdot n_i, e_0 - e_b \rangle_{\partial T} \right| \le Ch \|u\|_4 \|\|e_h\|.$$

Proof. Since \mathcal{Q}_h is the local L^2 projection onto $P_0(T)$, then we have

$$\langle \partial_j (\partial_{ij}^2 u - Q_h \partial_{ij}^2 u) \cdot n_i, e_0 - e_b \rangle_{\partial T} = \langle \partial_j \partial_{ij}^2 u \cdot n_i, e_0 - e_b \rangle_{\partial T} (10.16) = \langle \partial_j \partial_{ij}^2 u \cdot n_i, e_0 - Q_b e_0 \rangle_{\partial T} + \langle \partial_j \partial_{ij}^2 u \cdot n_i, Q_b e_0 - e_b \rangle_{\partial T} = \langle (I - Q_b) \partial_j \partial_{ij}^2 u \cdot n_i, e_0 - Q_b e_0 \rangle_{\partial T} + \langle \partial_j \partial_{ij}^2 u \cdot n_i, Q_b e_0 - e_b \rangle_{\partial T} = J_1 + J_2.$$

For the second term J_2 , by using Cauchy-Schwarz inequality, trace inequality (10.1) and (10.6), we have

(10.17)
$$\begin{aligned} \left| \sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} \langle \partial_{j} \partial_{ij}^{2} u \cdot n_{i}, Q_{b} e_{0} - e_{b} \rangle_{\partial T} \right| \\ \leq \left(\sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} h_{T}^{3} \| \partial_{j} \partial_{ij}^{2} u \|_{\partial T}^{2} \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3} \| Q_{0} e_{0} - e_{b} \|_{\partial T}^{2} \right)^{\frac{1}{2}} \\ \leq C \left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{3} (h_{T} | u |_{4,T}^{2} + h_{T}^{-1} | u |_{3,T}^{2}) \right)^{\frac{1}{2}} \| e_{h} \| \\ \leq C h \left(\| u \|_{3} + h \| u \|_{4} \right) \| e_{h} \|. \end{aligned}$$

As to the first term J_1 , by using Cauchy-Schwarz inequality, trace inequality (10.1), (10.2), and Lemma 10.5, we arrive at

(10.18)
$$\begin{aligned} \left| \sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} \langle (I - Q_{b}) \partial_{j} \partial_{ij}^{2} u \cdot n_{i}, e_{0} - Q_{b} e_{0} \rangle_{\partial T} \right| \\ \leq \left(\sum_{T \in \mathcal{T}_{h}} \sum_{i,j=1}^{d} \| (I - Q_{b}) \partial_{j} \partial_{ij}^{2} u \|_{\partial T}^{2} \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_{h}} \| e_{0} - Q_{b} e_{0} \|_{\partial T}^{2} \right)^{\frac{1}{2}} \\ \leq C \left(\sum_{T \in \mathcal{T}_{h}} h_{T} |u|_{4,T}^{2} \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_{h}} h_{T} |e_{0}|_{1,T}^{2} \right)^{\frac{1}{2}} \\ \leq Ch \| u \|_{4} \left(\sum_{T \in \mathcal{T}_{h}} |e_{0}|_{1,T}^{2} \right)^{\frac{1}{2}} \leq Ch \| u \|_{4} \| e_{h} \|. \end{aligned}$$

Combining all the above inequalities gives rise to the desired estimate (10.15).

LEMMA 10.7. There exists a constant C such that the following inequality holds true:

$$\Big(\sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d h_T^{-1} \| (\partial_i e_0 - e_{gi}) \cdot n_j \|_{\partial T}^2 \Big)^{\frac{1}{2}} \le C \| \| e_h \| \|$$

for any $e_h \in V_h^0$.

Proof. From the triangle inequality, we have

(10.19)
$$\begin{aligned} \|(\partial_{i}e_{0} - e_{gi}) \cdot n_{j}\|_{\partial T}^{2} &\leq \|\partial_{i}e_{0} - e_{gi}\|_{\partial T}^{2} \\ &\leq 2\Big(\|\partial_{i}e_{0} - Q_{b}(\partial_{i}e_{0})\|_{\partial T}^{2} + \|Q_{b}(\partial_{i}e_{0}) - e_{gi}\|_{\partial T}^{2}\Big) \\ &\leq Ch_{T}|e_{0}|_{2,T}^{2} + 2\|Q_{b}(\partial_{i}e_{0}) - e_{gi}\|_{\partial T}^{2}. \end{aligned}$$

Thus,

$$\sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^a h_T^{-1} \| (\partial_i e_0 - e_{gi}) \cdot n_j \|_{\partial T}^2 \le C \sum_{T \in \mathcal{T}_h} \left(|e_0|_{2,T}^2 + h_T^{-1} \| Q_b(\partial_i e_0) - e_{gi} \|_{\partial T}^2 \right) \\ \le C \| \|e_h\| \|^2.$$

This completes the proof of the lemma. \Box

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