

# Subdiffusion in classical and quantum nonlinear Schrödinger equations with disorder

A. Iomin\*

*Department of Physics, Technion, Haifa, 32000, Israel*

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## Abstract

The review is concerned with the nonlinear Schrödinger equation (NLSE) in the presence of disorder. Disorder leads to localization in the form of the localized Anderson modes (AM), while nonlinearity is responsible for the interaction between the AMs and transport. The dynamics of an initially localized wave packets are concerned in both classical and quantum cases. In both cases, there is a subdiffusive spreading, which is explained in the framework of a continuous time random walk (CTRW), and it is shown that subdiffusion is due to the transitions between those AMs, which are strongly overlapped. This overlapping being a common feature of both classical and quantum dynamics, leads to the clustering with an effective trapping of the wave packet inside each cluster. Therefore, the dynamics of the wave packet corresponds to the CTRW, where the basic mechanism of subdiffusion is an entrapping of the wave packet with delay, or waiting, times distributed by the power law  $w(t) \sim 1/t^{1+\alpha}$ , where  $\alpha$  is the transport exponent. It is obtained that  $\alpha = 1/3$  for the classical NLSE and  $\alpha = 1/2$  for the quantum NLSE.

*Keywords:* Nonlinear Schrödinger equation, Liouville equation, Continuous time random walk, Fractional Fokker-Planck equation, Subdiffusion, Quantum continuous time random walk

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## 1. Introduction

It is well known that wave propagation in random media can be described in the framework of the Fokker-Planck equation, under certain conditions [1]. In modern optical experiments with nonlinear media a suitable description can be developed in the framework of the fractional kinetics based on fractional integro-differentiation. This concept of differentiation of non-integer orders arises from works of Leibniz, Liouville, Riemann, Grunwald, and Letnikov, see *e.g.*, [2, 3, 4]. Its application is related to random processes with power law distributions. The latter corresponds to the absence of characteristic average values for processes exhibiting many scales [5, 6].

A typical example of fractional dynamics in optics is a realization of a competition between localization and nonlinearity that leads to anomalous transport [7, 8, 9, 10,

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\*Corresponding author

*Email address:* iomin@physics.technion.ac.il (A. Iomin)

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11, 12, 13]. This dynamics is described in the framework of the nonlinear Schrödinger equation (NLSE) in the presence of an external field  $V = V(x)$ ,  $x \in (-\infty, +\infty)$ . The wave spreading, described by the wave function, is governed by the NLSE in the presence of disorder

$$i\partial_t\psi = -\partial_x^2\psi + \beta|\psi|^2\psi + V\psi, \quad (1.1)$$

where  $\beta$  is a nonlinearity parameter. The variables are chosen in dimensionless units and the Planck constant is  $\hbar = 1$ . The random potential  $V = V(x)$ ,  $x \in (-\infty, +\infty)$  is such that for the linear case ( $\beta = 0$ ) Anderson localization takes place [14, 15], and the system is described by the exponentially localized Anderson modes (AM)s  $\Psi_{\omega_k} \equiv \Psi_k(x)$ , such that

$$[-\partial_x^2 + V(x)]\Psi_k(x) = \omega_k\Psi_k(x), \quad (1.2)$$

where  $\Psi_{\omega_k}(x)$  are real functions and the eigenspectrum  $\omega_k$  is discrete and dense [15]. The problem in question is an evolution of an initially localized wave function  $\psi(t=0) = \psi_0(x)$ . It can be also a stationary state of the NLSE, which is localized with the same Lyapunov exponent as in the AMs [16, 17].

This problem is relevant to experiments in nonlinear optics, for example disordered photonic lattices [18, 19], where Anderson localization was found in the presence of nonlinear effects, as well as to experiments on Bose-Einstein Condensates in disordered optical lattices [20, 21, 22, 23]. A discrete analog of Eq. (1.1) is extensively studied numerically [7, 8, 9, 10, 13], and a subdiffusive spreading of the initially localized wave packets has been observed with the mean squared displacement (MSD)  $\langle x^2(t) \rangle = \int |\psi(t)|^2 x^2 dx \sim t^\alpha$ , where a transport exponent  $\alpha$  was found to be  $2/5$  [10] and  $1/3$  [9]. This difference has been explained in [25], where it has been shown that the former result is for the Markovian subdiffusion due to the range-dependent diffusion coefficient, while the latter one corresponds to non-Markovian fractional diffusion of a percolation type.

Subdiffusion of wave packets was also obtained analytically [11, 24, 25] in the limit of the large times asymptotic. In that case the dynamics of the wave packet has been approximated by the fractional Fokker-Planck equation (FFPE) due to the arguments of a so-called continuous time random walk (CTRW).

The concept of the CTRW was originally developed for mean first passage time in a random walk on a lattice with further application to a semiconductor electronic motion [26]. The mathematical apparatus of the fractional CTRW is well established for many applications in physics, see *e.g.*, [5, 6, 27, 28, 29].

In the present work we concern with the physical mechanism of this subdiffusion obtained in Refs. [11, 24, 30, 31] and give a new insight of the subdiffusion transport exponent  $\alpha$ , related to both classical and quantum properties of the nonlinear interaction term in Eq. (1.1). To this end we consider the NLSE (1.1) and its quantum counterpart (quantum NLSE), when the wave functions  $\psi(x)$  are considered as operators  $\hat{\psi}(x)$  satisfying to the commutation rule  $[\hat{\psi}(x), \hat{\psi}^\dagger(x')] = \delta(x - x')$  [32, 33]. The paper consists of two parts. The first one is devoted to the classical analysis, which is based on mapping the nonlinear Eq. (1.1) onto the linear Liouville equation for the probability amplitude  $|\psi(x, t)|^2$ , where the transition elements in the Liouville operator are determined by the nonlinear term in Eq. (1.1). Therefore, we proceed by developing the CTRW approach for the corresponding Liouville equation [34]. The second part of the review is devoted to the quantum NLSE, which a quantum counterpart of Eq. (1.1). The quantum analysis

is based on mapping the quantum system on the basis of coherent states, and in the framework of the obtained equations we study four-modes decay processes and develop a quantum CTRW and construct a generalized master equation as a quantum counterpart of the Liouville equation.

We show that in both classical and quantum cases, there is subdiffusion which is explained in the framework the CTRW. We also concern with a mechanism of subdiffusion, which is due to the transitions between the strongly overlapped AMs. This overlapping being a common feature of both classical and quantum dynamics, leads to the clustering with an effective trapping of the wave packet inside each cluster. Therefore, the dynamics of the wave packet corresponds to the CTRW, where the basic mechanism of subdiffusion is an entrapping of the wave packet with delay, or waiting, times distributed by the power law  $w(t) \sim 1/t^{1+\alpha}$ , where  $\alpha$  is the transport exponent. We show that  $\alpha = 1/3$  for the classical NLSE and  $\alpha = 1/2$  for the quantum NLSE.

## 2. Phase space dynamics and the Liouville operator

First, we obtain the linear Liouville equation for  $|\psi(x, t)|^2$  [11, 24]. Projecting Eq. (1.1) on the basis of the AMs

$$\psi(x, t) = \sum_{\omega_k} C_{\omega_k}(t) \Psi_{\omega_k}(x) \equiv \sum_k C_k(t) \Psi_k(x), \quad (2.1)$$

we obtain a system of equations for coefficients of the expansion  $C_k$

$$i\partial_t C_k = \omega_k C_k + \beta \sum_{k_1, k_2, k_3} A_{k_2, k_3}^{k, k_1} C_{k_1}^* C_{k_2} C_{k_3}. \quad (2.2)$$

Here  $A(\mathbf{k}) \equiv A_{k_2, k_3}^{k, k_1}$  is an overlapping integral of the four AMs:

$$A_{k_2, k_3}^{k, k_1} = \int \Psi_k(x) \Psi_{k_1}(x) \Psi_{k_2}(x) \Psi_{k_3}(x) dx. \quad (2.3)$$

The initial conditions for the system of Eqs. (2.2) are such that  $\psi_0(x) = \sum_k a_k \Psi_k(x) = a_{l_0} \Psi_{l_0}$ . Equations (2.2) correspond to a system of interacting nonlinear oscillators with the Hamiltonian

$$H = \sum_k \omega_k C_k^* C_k + (\beta/2) \sum_{\mathbf{k}} A_{k_2, k_3}^{k_1, k_4} C_{k_1}^* C_{k_4}^* C_{k_2} C_{k_3}. \quad (2.4)$$

Therefore, Eqs. (2.2) are produced by the Poisson brackets  $\{H, \dots\}_{PB}$  by means of the Liouville operator

$$\hat{L} = \frac{1}{i} \{H, \dots\}_{PB} = \frac{1}{i} \sum_k \left( \frac{\partial H}{\partial C_k^*} \cdot \frac{\partial}{\partial C_k} - \frac{\partial H}{\partial C_k} \cdot \frac{\partial}{\partial C_k^*} \right). \quad (2.5)$$

Since  $\hat{L}H = 0$  and  $H(\{C, C^*\}) = H(\{a, a^*\})$ , we obtain that the Liouville operator is an operator function of the initial values:

$$\hat{L} = \frac{1}{i} \sum_k \left[ \frac{\partial H}{\partial a_k^*} \cdot \frac{\partial}{\partial a_k} - \frac{\partial H}{\partial a_k} \cdot \frac{\partial}{\partial a_k^*} \right] \quad (2.6)$$

and corresponds to an infinite system of linear equations  $\partial_t \mathbf{C} = \hat{L} \mathbf{C}$ , where  $\mathbf{C} = \mathbf{C}(\{a_k, a_k^*\}) = (\dots, C_{k-1}, C_k, C_{k+1}, \dots)$  is an infinite vector. Thus, the Liouville operator reads<sup>1</sup>

$$\hat{L} = -i \sum_k \omega_k \left( a_k \frac{\partial}{\partial a_k} - \text{c.c.} \right) - i \frac{\beta}{2} \sum_{\mathbf{k}} A_{k_2, k_3}^{k_1, k_4} \left[ a_{k_1}^* a_{k_2} a_{k_3} \frac{\partial}{\partial a_{k_4}} - \text{c.c.} \right], \quad (2.7)$$

where c.c. denotes the complex conjugation. Finally, we obtain that the system of *nonlinear* ordinary differential equations (2.2) is replaced by a system of *linear* partial differential equations:

$$\partial_t C_k(t) = \hat{L} C_k(t), \quad k = 1, 2, \dots, . \quad (2.8)$$

### 3. Initial time dynamics: Perturbation approach

Let us first understand how the Liouville operator (2.7) describes the dynamics of the physical variables. To this end, the Liouville operator is taken as a combination of linear and nonlinear parts [24]

$$\hat{L} = -i(\hat{L}_l + \beta \hat{L}_{nl}), \quad (3.1)$$

A formal solution of Eq. (2.8) is the expansion

$$\bar{C}_k(t) = \sum_{n=0}^{\infty} \left[ \frac{t^n}{n!} \hat{L}^n a_k \right]_{a_k = \delta_{k, l_0}}. \quad (3.2)$$

The nonzero contribution to the first power over  $t$  of the expansion (3.2) is due to the term

$$\hat{L}_{nl}^{(0)} = \frac{1}{2} \sum_k A(l_0, l_0, l_0, k) |a_{l_0}|^2 \left( a_{l_0} \frac{\partial}{\partial a_k} - \text{c.c.} \right), \quad (3.3)$$

while  $(\hat{L}_{nl} - \hat{L}_{nl}^{(0)}) a_k \equiv 0$  is due to the initial conditions  $a_k = a_{l_0} \delta_{k, l_0}$ . Moreover, the contribution of  $\hat{L}_{nl} - \hat{L}_{nl}^{(0)}$  without  $\hat{L}_{nl}^{(0)}$  is zero in all powers of the expansion (3.2). For example, the  $n$ th power term for  $k \neq l_0$  is

$$\left[ \sum_{l \neq l_0} A(l_0, l_0, l, l) |a_{l_0}|^2 \partial_{\phi_l} \right]^n a_k = i^n A^n(l_0, l_0, k, k) \delta_{k, l_0}.$$

It has a non zero contribution only in the power of the  $n+1$  order after the action of the  $\hat{L}_{nl}^{(0)}$  term. Therefore, keeping only the  $\hat{L}_{nl}^{(0)}$  term in Eq. (3.5) means neglecting  $O(\beta^2 t^2)$  terms in the expansion (3.2). This solution is valid up to a time scale  $t < t_\beta = 1/\beta$ .

To obtain a solution in the framework of this approximation, first we eliminate the linear term  $\hat{L}_l$  from Eq. (2.8) by substitution

$$\bar{C}_k(t) = \exp(-i \hat{L}_l t) C_k(t). \quad (3.4)$$

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<sup>1</sup>For the evolution of the dynamical variables like  $C_k$  and  $C_k^*$ , it is a so called Koopman operator [35].

After this substitution, Eq. (2.8) reads

$$\partial_t \bar{C}_k = -i\beta \hat{L}_{nl}(t) \bar{C}_k, \quad \hat{L}_{nl}(t) = e^{-i\hat{L}_l t} \hat{L}_{nl} e^{i\hat{L}_l t}. \quad (3.5)$$

Taking into account that

$$\exp[-i\hat{L}_l t] = \exp \left[ - \sum_k \omega_k t \frac{\partial}{\partial \phi_k} \right]$$

is the phase shift operator for the complex values  $a_k = |a_k| e^{i\phi_k}$ , we obtain

$$\hat{L}_{nl}(t) = \frac{1}{2} \sum_{\mathbf{k}} A(\mathbf{k}) \left[ \exp[i\Delta\omega t] a_{k_1}^* a_{k_2} a_{k_3} \frac{\partial}{\partial a_{k_4}} - \text{c.c.} \right], \quad (3.6)$$

where  $\Delta\omega = \omega_{k_1} + \omega_{k_4} - \omega_{k_2} - \omega_{k_3}$ .

Solutions of Eq. (3.5) for  $k \neq l_0$  are functions which are zero at  $t = 0$ . These are

$$\bar{C}_k(t) = a_k + \frac{\beta A_1 |a_{l_0}|^2 a_{l_0}}{\Delta\omega + \beta A_0 |a_{l_0}|^2} \cdot \left( e^{-i\beta A_0 |a_{l_0}|^2 t} - e^{i\Delta\omega t} \right). \quad (3.7)$$

Here  $A_0 = A(l_0, l_0, l_0, l_0)$  and  $A_1 \equiv A_1(k) = A(l_0, l_0, l_0, k)/2$ ,  $k \neq l_0$ , while  $\Delta\omega$  now is  $\Delta\omega = \omega_k - \omega_{l_0}$ . The complex conjugation of Eq. (3.7) can be a solution as well. A solution for  $k = l_0$  is a function of  $\phi_{l_0} - \beta A_0 |a_{l_0}|^2 t$ , which corresponds to the initial conditions. It reads

$$\bar{C}_{l_0}(t) = a_{l_0} \exp(-i\beta A_0 |a_{l_0}|^2 t). \quad (3.8)$$

Using these analytical form for the coefficients  $\bar{C}_k(t)$  and Eq. (3.4), one obtains the solution of the of NLSE (1.1) in the first order approximation over  $\beta$  as a sum

$$\begin{aligned} \psi(t) &= a_{l_0} \exp(-i\omega_{l_1} t) \Psi_{l_0}(x) - 4\beta |a_{l_0}|^2 a_{l_0} \\ &\times \sum_k ' A_1(k) \frac{\sin \left[ \frac{\omega_k - \omega_{l_2}}{2} t \right]}{\omega_k - \omega_{l_2}} \sin \left[ \frac{\omega_k + \omega_{l_2}}{2} t \right] \Psi_k(x), \end{aligned} \quad (3.9)$$

where  $\omega_{l_1} = \omega_{l_0} + \beta A_0 |a|^2$  and  $\omega_{l_2} = \omega_{l_0} - \beta A_0 |a|^2$ , while prime means that  $k \neq l_0$ . When  $\beta \rightarrow 0$ , we have at the asymptotically large times  $t_\beta \rightarrow \infty$  that  $\omega_{l_1} = \omega_{l_2} = \omega_{l_0}$ , and the sinc function is

$$\lim_{t \rightarrow \infty} \frac{2 \sin \left[ \frac{\omega_k - \omega_{l_2}}{2} t \right]}{\omega_k - \omega_{l_2}} = 2\pi \delta(\omega_k - \omega_{l_2}).$$

The sum in Eq. (3.9) equals zero. Therefore, for  $\beta = 0$ , one obtains  $\psi(t) = e^{-i\omega_{l_0} t} \Psi_{l_0}(x)$  that corresponds to a solution of the linear problem.

For nonzero values  $\beta$  and  $t < t_\beta$  the sinc function can be approximated by  $t_\beta$  for  $\omega_k \approx \omega_{l_2}$ . Then summation in Eq. (3.9) can be estimated by adding and subtracting the term with  $k = l_0$ . Using the definition of the overlapping integrals  $A_1(k)$  and  $\sum_k \Psi_k(x) \Psi_k(y) = \delta(y - x)$ , one obtains an approximation for Eq. (3.9)

$$\psi(t) \sim \Psi_{l_0}(x) e^{-i\omega_{l_1} t} - 4\beta t [\Psi_{l_0}^3(x) - A_0 \Psi_{l_0}(x)] \sin(\omega_{l_2} t). \quad (3.10)$$

It means that at  $t < t_\beta$  the wave function is localized and its evolution corresponds to the periodic oscillations with the frequencies  $\omega_{l_1}$  and  $\omega_{l_2}$ . It is worth mentioning that Eq. (3.9) is valid for both finite and infinite systems (either discrete or continuous).

Consideration of the dynamics beyond  $t > t_\beta$  relates to the calculation of nonzero contributions of operators  $\left[\hat{L}_{nl} - \hat{L}_{nl}^{(0)}\right]^q$  and  $\left[\hat{L}_{nl}^{(0)}\right]^p$ , acting on the initial conditions. This combinatorics leads to essential difficulties for analytical treatment. To overcome this obstacle the dynamics of the initially localized states can be considered qualitatively in the framework of a phenomenological probabilistic approach.

#### 4. Liouville Equation

The Liouville equation is valid for any functions of the initial conditions  $\{a_k, a_k^*\}$ . In particular, introducing the function  $F_{k,k'}(t) = C_k(t) \cdot C_{k'}^*(t)$ , one has the Liouville equation:

$$\partial_t F_{k,k'}(t) = \hat{L} F_{k,k'}(t), \quad F_{k,k'}(t=0) = F_{k,k'}^{(0)} = a_k a_{k'}^*.$$

Therefore, the probability amplitude

$$\mathcal{P}(x, t) = |\psi|^2 = \sum_{k,k'} F_{k,k'}(t) \Psi_k(x) \Psi_{k'}(x),$$

as a function of the initial conditions, satisfies the Liouville equation as well:

$$\partial_t \mathcal{P} = \hat{L} \mathcal{P}. \quad (4.1)$$

Here the initial condition is

$$\mathcal{P}(x, t=0) = \mathcal{P}_0(x) = \sum_{k,k'} F_{k,k'}^{(0)} \Psi_k(x) \Psi_{k'}(x). \quad (4.2)$$

Eventually, one considers the dynamics of the probability distribution function (PDF)  $\mathcal{P}$  in the framework of Liouville equation (4.1), which is the *linear* equation with a formal solution in the exponential form

$$\mathcal{P}(x, t) = e^{\hat{L}t} \mathcal{P}_0(x) = \sum_{k,k'} \Psi_k(x) \Psi_{k'}(x) \sum_{n=0}^{\infty} \left[ \frac{t^n}{n!} \hat{L}^n \right] F_{k,k'}^{(0)}. \quad (4.3)$$

It is worth noting that the linear property of the Liouville equation (4.1) and its formal solution (4.3) are important for the probabilistic approach.

According to the values of the overlapping integrals (2.3), we divide the transitions between the localized states into two main groups. The first one corresponds to the exponentially small overlapping integrals and the second one corresponds to the strong overlapping between four AMs when the overlapping integrals are of the order of 1. In the case of strong overlapping, the AMs form clusters, where the wave functions have the same averaged coordinates for each cluster. Consequently, all transitions inside one cluster do not lead to any appreciable differences in the coordinate space, and we regard these transitions as *trapping* of the wave packet, or a particle, inside this cluster. Contrary to that, transitions due to the exponentially small overlapping integrals between

the AMs belonging to different clusters lead to a change of the space coordinates that can be accounted for. We call these processes *jumps*. Contributions of trappings and jumps to the wave packet spreading described by Eqs. (4.1) and (3.2) are different, and correspond to different probabilistic interpretations.

Therefore, the Liouville operator can be considered as a sum of two operators  $\hat{L} = \hat{L}_0 + \hat{L}_1$ , where  $\hat{L}_0$  corresponds to the dynamics inside a cluster, determined by the overlapping integrals of the order of 1, while  $\hat{L}_1$  corresponds to jumps between these clusters due to the exponentially small overlapping integrals. Thus considering this dynamics in the “interaction picture”, where  $\hat{L}_1(t) = e^{-\hat{L}_0 t} \hat{L}_1 e^{\hat{L}_0 t}$ , one obtains the solution of the Liouville Eq. (6.10) in the iterative form

$$\begin{aligned} \mathcal{P}(x, t) &= e^{\hat{L}_0 t} \left[ 1 + \int_0^t dt_1 \hat{L}_1(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{L}_1(t_1) \hat{L}_1(t_2) \right. \\ &\quad \left. + \dots + \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \hat{L}_1(t_1) \dots \hat{L}_1(t_n) + \dots \right] \\ &= e^{\hat{L}_0 t} \mathcal{P}_0(x) + \int_0^t dt' e^{(t-t')\hat{L}_0} \hat{L}_1 \mathcal{P}_0(x, t'). \end{aligned} \quad (4.4)$$

The probabilistic interpretation of the last expression is as follows. The first term in Eq. (4.4) corresponds to a particle, which at the initial time is at the position  $x$  and there are no jumps to another clusters until time  $t$ , and therefore  $e^{\hat{L}_0 t} \mathcal{P}_0(x) \rightarrow W(t) \mathcal{P}_0(x)$ . Here  $W(t)$  denotes the probability of no jump until time  $t$ , since the particle is inside the same cluster. It is described by all possible transitions between AMs inside one cluster, which is characterized by the coordinate  $x$ . The last term in Eq. (4.4) corresponds to a particle to be at position  $x$  at time  $t$  due to all possible jumps from points  $(x', t')$  with the transition probability  $\mathcal{G}(x - x'; t - t')$  between different clusters, which is determined by operator  $\hat{L}_1$ . It is a composite operator, which consists of both inter-cluster dynamics due to the operator  $\hat{L}_1$  and trapping inside one cluster dynamics due to the operator  $\hat{L}_0$ . Note, also, that when  $\beta = 0$ , the transition probability is zero:  $\mathcal{G}(x - x'; t - t') = 0$ . Therefore, Eq. (4.4) now reads

$$\mathcal{P}(x, t) = W(t) \mathcal{P}_0(x) + \int_0^t dt' \int_{-\infty}^{\infty} dx' \mathcal{G}(x - x'; t - t') \mathcal{P}(x', t'). \quad (4.5)$$

This linear property can be used now for the continuous time random walks (CTRW) approach to obtain the Montroll-Weiss equation [5, 6, 26]. We also express here the dependence of the transition probability on  $\beta$  in the explicit form to stress that, for  $\beta \rightarrow 0$ , the dynamics is localized. This dependence on  $\beta$  will be reflected in a generalized transport coefficient. It should be admitted that CTRW processes are connected with a continuous time generalization of the Chapman-Kolmogorov equation [36, 37].

## 5. CTRW

In what follows we consider the dynamics of the initial wave packet  $\mathcal{P}_0(x)$  in the framework of the probabilistic approach, where the dynamics of the wave packet is considered as the CTRW. Since the dynamics of the probability distribution function (PDF)

$\mathcal{P}(x, t)$  is governed by the same Liouville operator in Eqs. (4.1) and (3.2), the overlapping integrals  $A(\mathbf{k}) = A_{\mathbf{k}}$ , defined in Eq. (2.3), play the dominant role in the wave packet spreading. As follows from Eq. (3.2), the overlapping integrals determine the spread of the initially localized wave packet  $\mathcal{P}_0(x)$  over all the AMs as transitions from one mode to another. Since all states are localized, these transitions between states determine the transitions (or jumps) in the coordinate space as well.

As discussed, transitions between strongly overlapped AMs contribute to the trapping of the wave packet inside one cluster, while transitions between AMs belonging to different clusters contribute to jumps. It is reasonable to assume that the transitions between these different states are independent of each other; therefore, the jumps are independent and obey the Markov property, and the PDF of a jump  $p(x)$  is determined by the overlapping integrals as  $p(x) = \xi \exp(-\xi|x|)/2$ , and  $\xi = 1/R$  is an inverse localization length of the eigenstates  $\Psi_k(x)$  in Eq. (2.1).

The trapping is associated with clusters with effective lengths  $\Delta$ . Due to the exponential localization, these values are distributed by the exponential law  $P_{\text{cl}}(\Delta) = \Delta_0^{-1} \exp(-\Delta/\Delta_0)$ , where  $\Delta_0$  is the effective (maximal) length of a cluster. The effective lengths are determined by overlapping integrals of four wave functions; therefore, the minimum length of the cluster is  $\Delta = R$ , while the effective length is  $\Delta_0 = 3R$ , which is a maximal distance between four strongly overlapping AMs. Now the probability that a particle exits this cluster and jumps to another one is of the order of  $\sim \exp(-\Delta/R)$ . This value is also proportional to the inverse waiting time,  $t \sim \exp(\Delta/R)$ . The probability to find the waiting time in the interval  $(t, t + dt)$  is equal to the probability to find the corresponding trapping length in the interval  $(\Delta, \Delta + d\Delta)$ , namely,  $w(t)dt = P_{\text{cl}}(\Delta)d\Delta$ . Therefore, after simple calculations one obtains that the PDF of the waiting times is

$$w(t) = P_{\text{cl}}(\Delta) \frac{d\Delta}{dt} \sim \frac{1}{(t/\tau)^{1+\alpha}}, \quad (5.1)$$

where  $\alpha = R/\Delta_0 = 1/3$  and  $\tau$  is a time scale related to the trapping<sup>2</sup>. It follows that the mean waiting time is infinite. Taking into account that the waiting time PDF is normalized, we have

$$w(t) = \frac{w_0}{1 + (t/\tau)^{4/3}}, \quad (5.2)$$

such that  $\int_0^\infty w(t)dt = 1$ , while  $\int_0^\infty tw(t)dt = \infty$ , where  $w_0$  is a normalization constant. We also admit that according to the definition

$$W(t) = \int_t^\infty w(t')dt', \quad (5.3)$$

where  $W(t) = 1 - \int_0^t w(t')dt'$  denotes the probability of no jump during the time interval  $(0, t)$ , introduced in Eq.(4.5). Performing the Fourier transform  $\bar{p}(k) = \hat{\mathcal{F}}[p(x)]$  and the Laplace transform  $\tilde{w}(u) = \hat{\mathcal{L}}w(t)$ , we obtain the Montroll-Weiss equation [26] from Eq. (4.5)

$$\bar{\mathcal{P}}(k, u) = \hat{\mathcal{F}}\hat{\mathcal{L}}\mathcal{P} = \frac{1 - \tilde{w}(u)}{u} \cdot \frac{\bar{\mathcal{P}}_0(k)}{1 - \bar{p}(k)\tilde{w}(u)}, \quad (5.4)$$

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<sup>2</sup>For example, taking  $\Delta = \Delta_0$ , one obtains  $t = \tau = e^{-1/3}$ .



where the transition probability  $\mathcal{G}(x; t)$  is considered in the multiplicative form  $\bar{\mathcal{G}}(k, u) = \bar{p}(k)\tilde{w}(u)$ . Equation (5.4) can be simplified for the long time  $u \ll 1$  and the large scale  $k \ll 1$  asymptotics that corresponds to the diffusion limit  $(k, u) \rightarrow (0, 0)$ . Taking into account the Fourier  $\bar{p}(k)$  and the Laplace  $\tilde{w}(u)$  images in Eq. (5.4):

$$\begin{aligned}\bar{p}(k) &= \frac{1}{1 + \beta R^2 k^2} \approx 1 - \beta R^2 k^2, \\ \tilde{w}(u) &= \frac{1}{1 + (u\tau)^\alpha} \approx 1 - (u\tau)^{1/3},\end{aligned}\quad (5.5)$$

we obtain for the PDF in the Fourier-Laplace domain (see also [5])

$$\bar{\mathcal{P}}(k, u) = \frac{\bar{\mathcal{P}}_0(k)/u}{1 + D_{\frac{1}{3}} u^{-1/3} k^2}, \quad (5.6)$$

where  $D_{\frac{1}{3}} = \beta R^2 / \tau^{1/3}$  is a generalized diffusion coefficient<sup>3</sup>. Using the Laplace transform of the fractional integration

$$\hat{\mathcal{L}}[\partial_t^{-\alpha} f(t)] = \hat{\mathcal{L}}\left[\frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}}\right] = u^{-\alpha} f(u), \quad \alpha > 0,$$

one obtains the fractional integral equation

$$\mathcal{P}(x, t) - \mathcal{P}_0(x) = \partial_t^{-\alpha} D_\alpha \partial_x^2 \mathcal{P}(x, t). \quad (5.7)$$

Differentiating this equation with respect to time, one obtains that the CTRW is described by the fractional Fokker-Planck equation<sup>4</sup> (FFPE)

$$\partial_t \mathcal{P}(x, t) - D_\alpha \partial_t^{1-\alpha} \partial_x^2 \mathcal{P}(x, t) = 0, \quad (5.8)$$

where  $\partial_t^\alpha$  is a designation of the Riemann-Liouville fractional derivative

$$\partial_t^\alpha f(t) = \frac{d}{dt} \partial_t^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^\alpha}$$

with  $\alpha = 1/3$ , while in general case  $0 < \alpha < 1$ . From Eq. (5.8) one obtains for the MSD  $\langle x^2(t) \rangle = \int_{-\infty}^{\infty} x^2 \mathcal{P}(x, t) dx$  the differential equation:

$$\frac{d}{dt} \langle x^2(t) \rangle = \frac{2D_\alpha t^{\alpha-1}}{\Gamma(\alpha)}.$$

Here  $\Gamma(z)$  is the gamma function,  $x(t=0) = 0$ , and we use the following property of the fractional derivative  $\partial_t^\alpha 1 = t^{-\alpha} / \Gamma(1-\alpha)$ . Therefore, Eq. (5.8) describes subdiffusion

$$\langle x^2(t) \rangle = \frac{2D_\alpha t^\alpha}{\Gamma(1+\alpha)} = \frac{2D_{\frac{1}{3}} t^{1/3}}{\Gamma(4/3)}, \quad (5.9)$$

with the transport exponent  $\alpha = 1/3$ . This result exactly corresponds to the numerical [9, 13, 38] and analytical [25, 30, 39] studies.

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<sup>3</sup>The nonlinear parameter is explicitly introduced in the generalized diffusion coefficient, such that for  $\beta = 0$ , there is no any transport and the initial wave packet remains localized.

<sup>4</sup>The solution of the FFPE is obtained in the form of the Fox  $H$  function, presented in Appendix B (see *e.g.*, [5]) and its asymptotic behavior corresponds to the stretched exponential function  $\mathcal{P}(y) \sim \frac{1}{\sqrt{D_\alpha t^\alpha}} y^{-(1-\alpha)/(2-\alpha)} e^{-y^{2/(2-\alpha)}}$ , where  $y \equiv \frac{|x|}{\sqrt{D_\alpha t^\alpha}} \gg 1$ . For  $\alpha = 1$  it corresponds to the normal Gaussian distribution.

## 6. Quantum NLSE

Now we concern with a quantum counterpart of the NLSE (1.1). In this case, an initial wave packet spreading is governed by the quantum nonlinear Schrödinger equation, which leads to the quantum counterpart of the Hamiltonian (2.4)

$$\hat{H} = \sum_k \tilde{h} \omega_k \hat{C}_k^\dagger \hat{C}_k + \frac{\tilde{h}^2 \beta}{2} \sum_{\mathbf{k}} A_{\mathbf{k}} \hat{C}_{k_1}^\dagger \hat{C}_{k_2}^\dagger \hat{C}_{k_3} \hat{C}_{k_4}, \quad (6.1)$$

where  $\tilde{h}$  is a dimensionless Planck constant and the operators are not commute

$$[\hat{C}_i, \hat{C}_j^\dagger] = \delta_{ij}. \quad (6.2)$$

The linear frequency is shifted by the nonlinear term due to the commutation  $\omega_k \rightarrow \omega_k + \tilde{h} \beta A_0$ , where  $A_0 = A_{kk}^{kk}$  is the diagonal overlapping integral in Eq. (2.3) and  $A_{\mathbf{k}} \equiv A_{k_3, k_4}^{k_1, k_2}$ . One should recognize that the classical NLSE (1.1) is independent of the Planck constant. Therefore, for the quantum analysis, an effective dimensionless Planck constant  $\tilde{h}$  is introduced explicitly.

For the quantum mechanical analysis we use a technique of mapping the Heisenberg equation of motion on a basis of the coherent states [40, 41, 33]. At the initial moment  $t = 0$ , one introduces the coherent states vector  $|\mathbf{a}\rangle = \prod_q |a_q\rangle$  as the product of eigenfunctions of annihilation operators  $\hat{a}_q = \hat{C}_q(t=0)$ , such that  $\hat{a}_q |a_q\rangle = a_q |a_q\rangle$  and correspondingly  $\hat{a}_q |\mathbf{a}\rangle = a_q |\mathbf{a}\rangle$ , where also the coherent state is constructed from a vacuum state  $|0\rangle$

$$|a_q\rangle = \exp[a_q \hat{a}_q^\dagger - a_q^* \hat{a}_q] |0\rangle, \quad \hat{a}_q |0\rangle = 0. \quad (6.3)$$

Introducing  $c$ -functions

$$\mathcal{C}_q(t) = \langle \mathbf{a} | \hat{C}_q(t) | \mathbf{a} \rangle = \mathcal{C}_q(t | \mathbf{a}^*, \mathbf{a}), \quad (6.4)$$

one maps the Heisenberg equation of motion on the basis  $|\mathbf{a}\rangle$  as follows

$$i\tilde{h} \dot{\mathcal{C}}_q(t) = \langle \mathbf{a} | [\hat{C}_q(t) \hat{H} - \hat{H} \hat{C}_q(t)] | \mathbf{a} \rangle. \quad (6.5)$$

Accounting Eqs. (6.3) and (6.4), one obtains the mapping rules

$$\begin{aligned} \langle \mathbf{a} | \hat{C}_q(t) \hat{a}_q^\dagger | \mathbf{a} \rangle &= e^{-|a_q|^2} \frac{\partial}{\partial a_q} e^{|a_q|^2} \mathcal{C}_q(t), \\ \langle \mathbf{a} | \hat{a}_q \hat{C}_q(t) | \mathbf{a} \rangle &= e^{-|a_q|^2} \frac{\partial}{\partial a_q^*} e^{|a_q|^2} \mathcal{C}_q(t). \end{aligned} \quad (6.6)$$

The Hamiltonian is the integral of motion  $\hat{H}(\{\hat{C}_q^\dagger, \hat{C}_q\}) = \hat{H}(\{\hat{a}_q^\dagger, \hat{a}_q\})$ , therefore the mapping rules (6.6) yield equation of motion (6.5) for  $\mathcal{C}_q(t)$  in the closed form

$$i\dot{\mathcal{C}}_q(t) = \hat{K} \mathcal{C}_q(t), \quad (6.7)$$

where

$$\begin{aligned} \hat{K} = & \frac{1}{\hbar} e^{-\sum_k |a_k|^2} \left[ \hat{H}(\{\frac{\partial}{\partial a_q}, a_q\}) - \hat{H}(\{a_q^*, \frac{\partial}{\partial a_q^*}, \}) \right] e^{\sum_k |a_k|^2} = \\ & \sum_q \left[ \omega_q a_q \frac{\partial}{\partial a_q} - c.c \right] + \frac{\tilde{\hbar}\beta}{2} \sum_{\mathbf{q}} A_{\mathbf{q}} \left[ 2a_{q_1} a_{q_2} a_{q_3}^* \frac{\partial}{\partial a_{q_4}} + a_{q_1} a_{q_2} \frac{\partial}{\partial a_{q_3}} \frac{\partial}{\partial a_{q_4}} - c.c \right]. \end{aligned} \quad (6.8)$$

In the limit  $\tilde{\hbar} \rightarrow 0$ , the second derivative terms vanish,  $\tilde{\hbar} a_{q_1} a_{q_2} \frac{\partial}{\partial a_{q_3}} \frac{\partial}{\partial a_{q_4}} \rightarrow 0$ , and the operator (6.8) reduces to the classical Liouville operator [11] in Eq. (2.7).

Following the strategy of the construction of the kinetic equation (5.8) for the classical NLSE, we consider the density operator  $\hat{\rho} = |\hat{\psi}|^2$  and map it on the basis of the coherent states  $|\mathbf{a}\rangle$  in Eq. (6.3), such that

$$\mathcal{P}_Q(t) \equiv \mathcal{P}_Q(\mathbf{a}^*, \mathbf{a}, t) = \langle \mathbf{a} | \hat{\rho}(t) | \mathbf{a} \rangle. \quad (6.9)$$

Therefore, from the mapping rules one obtains the Liouville equation for the mean probability density

$$\partial_t \mathcal{P}_Q(t) = \hat{L} \mathcal{P}_Q(t). \quad (6.10)$$

Here the quantum “Liouville” operator is determined in Eq. (6.8)

$$\hat{L} = -i\hat{K}. \quad (6.11)$$

In the complete analogy with Eq. (4.1), the Liouville operator in Eq. (6.11) is considered as a sum of two operators  $\hat{L} = \hat{L}_0 + \hat{L}_1$ , where  $\hat{L}_0$  corresponds to the dynamics inside clusters, determined by the overlapping integrals of the order of 1, while  $\hat{L}_1$  corresponds to jumps between these clusters due to the exponentially small overlapping integrals. Eventually, one arrives at the master equation (4.5) for the quantum mean probability density  $\mathcal{P}_Q(t)$ , which coincides exactly with Eq. (4.5)

$$\mathcal{P}_Q(x, t) = W_Q(t) \mathcal{P}_0(x) + \int_0^t dt' \int_{-\infty}^{\infty} dx' \mathcal{G}_Q(x - x'; t - t') \mathcal{P}_Q(x', t'). \quad (6.12)$$

In this construction of the quantum master equation, as a quantum CTRW, we follow a van-Kampen coarse-graining procedure (see details of the discussion in [42]) by lumping a set of quantum states making a trapping cluster, which is in complete analogy with the classical CTRW, constructed in [11]. The transition probabilities in Eq. (6.12) (Pauli-van-Kampen master equation) reduce to their classical analogs of the transition probabilities, since the quantum “Liouville” operator determined in Eqs. (6.8) and (6.11) contains the same overlapping integrals as its classical counterpart with  $\hbar = 0$ . As the result, the coarse-grained process of quantum transitions between the trapping clusters is also a Markov process. Transitions between quantum states inside the cluster do not contribute to the transport (to the spreading of the initial wave packet in the  $x$  coordinate), while transitions between any states of the different trapping clusters lead to the spreading of the wave packet in the  $x$  coordinate. The main contribution to this transport in the chain of the localized AMs (in a more general case, Wannier states) is due to the transitions between the nearest neighbor clusters and this is determined by

the jump length PDF  $p_Q(x)$ . The quantum transitions between coherent states inside the cluster do not contribute to the transport with trapping, or delay times, distributed by the waiting time PDF  $w_Q(t)$ . The latter is determined by quantum decay processes, therefore the probability to find a quantum particle inside the trapping cluster after time  $t$  (or no jump during time interval  $(0, t)$ ) is

$$W_Q(t) = \int_t^\infty w_Q(t') dt. \quad (6.13)$$

Following the standard CTRW approach of Sec. 5 [5, 6, 26], the composite transition probability operator  $\mathcal{G}_Q(x - x'; t - t')$  in Eq. (6.12) is considered in the multiplicative form by analogy with the classical CTRW. Performing the Fourier transform  $\bar{p}_Q(k) = \hat{\mathcal{F}}[p_Q(x)]$  and the Laplace transform  $\tilde{w}_Q(u) = \hat{\mathcal{L}}[w_Q(t)]$ , we obtain that the quantum distribution function is governed by the Montroll-Weiss equation in the Fourier-Laplace space  $\bar{\mathcal{P}}_Q(k, u) = \hat{\mathcal{L}}\hat{\mathcal{F}}[\mathcal{P}_Q(x, t)]$

$$\bar{\mathcal{P}}_Q(k, u) = \hat{\mathcal{F}}\hat{\mathcal{L}}\mathcal{P}_Q = \frac{1 - \tilde{w}_Q(u)}{u} \cdot \frac{\bar{\mathcal{P}}_0(k)}{1 - \bar{p}_Q(k)\tilde{w}_Q(u)}. \quad (6.14)$$

Here it was assumed that the transitions between different clusters are classical processes, which are independent of each other. Therefore, the jumps are independent and obey the Markov property, where the PDF of a jump  $p_Q(x)$  is determined by the overlapping integrals as  $p_Q(x) = p(x) = \xi \exp(-\xi|x|)/2$ , and  $\xi = 1/R$  is an inverse localization length of the AM  $\Psi_k(x)$  in Eq. (2.1).

The situation with waiting time PDF  $w_Q(t)$  changes drastically. The transitions inside a cluster are pure quantum transitions, which determine the survival probability. It corresponds to the inverse amplitude of the population of a quantum states  $|a_q\rangle$  with  $q \neq k$ , described by  $\mathcal{C}_q(t)$  with the initial condition  $\mathcal{C}_q(t = 0) = a_k$ .

Therefore, we study the dynamics of four modes, which is governed by the Hamiltonian (6.1), where we take into account only resonant terms in the interaction. These resonant processes are the fastest and as it is shown in Refs. [25, 30, 39], these terms have the strongest contribution to the wave packet spreading. Therefore, locally these modes are described by the same overlapping integrals  $\sim A_0 \delta_{k_1+k_2-k_3-k_4, 0}$ , where, however, the interaction  $A_0$  varies for different clusters.

## 7. Four-modes stability analysis

The local Hamiltonian of the quantum NLSE in a cluster reads

$$\hat{H}_l = \sum_k \tilde{\hbar}\omega_k \hat{C}_k^\dagger \hat{C}_k + \frac{\tilde{\hbar}^2 \beta A_0}{2} \sum_{\mathbf{k}} \hat{C}_{k_1}^\dagger \hat{C}_{k_2}^\dagger \hat{C}_{k_3} \hat{C}_{k_4} \delta_{k_1+k_2-k_3-k_4, 0}, \quad (7.1)$$

In this case the  $c$  number Heisenberg equation (6.7)

$$i\dot{\mathcal{C}}_q(t) = \hat{K}\mathcal{C}_q(t), \quad (7.2)$$

is controlled by the local Liouville operator, which describes the quantum dynamics inside one cluster.

$$\begin{aligned} \hat{K} &= \sum_q \left[ \omega_q a_q \frac{\partial}{\partial a_q} - c.c \right] + \frac{\tilde{h}\beta A_0}{2} \\ &\times \sum_{\mathbf{q}} \left[ 2a_{q_1} a_{q_2} a_{q_3}^* \frac{\partial}{\partial a_{q_4}} + a_{q_1} a_{q_2} \frac{\partial}{\partial a_{q_3}} \frac{\partial}{\partial a_{q_4}} - c.c \right] \delta_{q_1+q_2-q_3-q_4,0}. \end{aligned} \quad (7.3)$$

Taking the initial conditions in the form  $\mathcal{C}_q(t=0) = a_k$  for  $q = k$  and  $\mathcal{C}_q(t=0) = 0$  for  $q \neq k$ , one obtains by the straightforward substitution that the solution of Eq. (7.2) is a periodic wave with the finite amplitude [33, 41]

$$\begin{aligned} \mathcal{C}_k(t) &= a_k \exp(-i\omega_k t) \left[ (e^{-i\tilde{h}\beta A_0 t} - 1) |a_k|^2 \right], \\ \mathcal{C}_q(t) &= 0, \quad q \neq k. \end{aligned} \quad (7.4)$$

Now we follow the analysis of Ref. [33] to study the stability of this initially populated state. To this end we study the dynamics of the quantum states with  $q \neq k$ . We also study the dynamics of the survival probability to stay inside the cluster. Obviously, it corresponds to the inverse amplitude of the population of the quantum states  $|a_q\rangle$  with  $q \neq k$ , which are described by  $\mathcal{C}_q(t)$ . The stability of the solution (7.4) can be studied in the framework of the resonant four-wave decay processes  $2k \rightarrow k \pm p$ . We take into account that due to the non-resonant interactions all modes with  $q \neq k$  are instantly populated at the initial time  $t = 0$ , such that the conditions for the initial amplitudes are  $|a_q| \ll |a_k|$ . In this case, one looks for the solution of  $\mathcal{C}_{k+p}$  as an expansion over the powers of  $a_q$ . Taking into account only the first orders of the expansion for the “small” waves, we have

$$\mathcal{C}_{k+p}(t|\{a_q^*, a_q\}) = f_0(t|a_k^*, a_k) + \sum_{q \neq 0} [f_q(t|a_k^*, a_k) a_q + \tilde{f}_q(t|a_k^*, a_k) a_q^*] + o(a_q, a_q^*). \quad (7.5)$$

Due to the initial condition  $\mathcal{C}_{k+p}(0) \equiv \mathcal{C}_{k+p}(0|\{a_q^*, a_q\}) = a_{k+p}$ , one obtains the initial conditions for the amplitudes  $f_0(t) \equiv f_0(t|a_k^*, a_k)$ ,  $f_q(t) \equiv f_q(t|a_k^*, a_k)$ , and  $\tilde{f}_q(t) \equiv \tilde{f}_q(t|a_k^*, a_k)$  as follows

$$\begin{aligned} f_0(0) &= a_k, \quad f_q(0) = \tilde{f}_q(0) = 0 & \text{for } p = 0, \\ f_0(0) &= 0, \quad f_q(0) = \delta_{q,p}, \quad \tilde{f}_q(0) = 0 & \text{for } p \neq 0. \end{aligned} \quad (7.6)$$

The dynamics of the functions  $f_q$  and  $\tilde{f}_q$  determine the dynamics of small amplitudes  $\mathcal{C}_q(t)$ . Substituting expansion (7.5) in Eq. (7.2) and taking into account expression (7.3) for the quantum operator and collecting the same powers of the small initial amplitudes  $a_{k \pm p}$  and  $a_{k \mp p}^*$ , one obtains the system of equations [33]

$$\begin{aligned} i \frac{\partial f_0}{\partial t} &= \hat{K}_0 f_0 \\ i \frac{\partial f_q}{\partial t} &= \hat{K}_0 f_q + [\omega_{k+q} + 2\tilde{h}\beta A_0 |a_k|^2] f_q + 2\tilde{h}\beta A_0 a_k \frac{\partial}{\partial a_k} f_q - \tilde{h}\beta A_0 a_k^{*2} \tilde{f}_{-q}, \\ i \frac{\partial \tilde{f}_{-q}}{\partial t} &= \hat{K}_0 \tilde{f}_{-q} - [\omega_{k-q} + 2\tilde{h}\beta A_0 |a_k|^2] \tilde{f}_{-q} - 2\tilde{h}\beta A_0 a_k \frac{\partial}{\partial a_k} \tilde{f}_{-q} + \tilde{h}\beta A_0 a_k^2 f_q. \end{aligned} \quad (7.7)$$

Here

$$\hat{K}_0 = [\omega_k + \tilde{h}\beta A_0 |a_k|^2] a_k \frac{\partial}{\partial a_k} + \frac{\tilde{h}\beta A_0}{2} a_k^2 \frac{\partial^2}{\partial a_k^2} - c.c. \quad (7.8)$$

Solution of the first equation in system (7.7) coincides with the nonzero solution in Eq. (7.4), and in this first order approximation it does not describes the escape rate, and we disregard this solution. In the next two equations the nonzero solutions exist only for  $q = \pm p$ . Then performing the variable change

$$f_p = f e^{-i(2\omega_k - \omega_{k-p} + \tilde{h}\beta A_0)t}, \quad \tilde{f}_{-p} = \frac{\alpha_k}{\alpha_k^*} g \exp[-i(2\omega_k - \omega_{k-p} + \tilde{h}\beta A_0)t], \quad (7.9)$$

and introducing the angle variable  $I = \tilde{h}|a_k|^2$ , as it is shown in appendix C, one obtains equations for the amplitude  $f = f(I, t)$  and  $g = g(I, t)$

$$\begin{aligned} i\partial_t f &= -\Delta\omega f + 2\tilde{h}\beta A_0 I \partial_I f + 2\beta A_0 I f - \beta A_0 I g, \\ i\partial_t g &= \beta A_0 I f. \end{aligned} \quad (7.10)$$

Here we take into account that the resonant four-mode decay takes place at both the momentum conservation  $2k \rightarrow (k+p) + (k-p)$  and the energy detuning  $\Delta\omega = 2\omega_k - \omega_{k+p} - \omega_{k-p} - \tilde{h}\beta A_0$ . Introducing dimensionless variables  $\mathcal{T} = \Delta\omega t$ ,  $v = \beta A_0 I / \Delta\omega$ , and  $\varepsilon = \tilde{h}\beta A_0 / \Delta\omega$ , one obtains from Eq. (7.10)

$$i\partial_{\mathcal{T}} f = (2v - 1)f - vg + 2\varepsilon y \partial_v f, \quad i\partial_{\mathcal{T}} g = vf. \quad (7.11)$$

The initial conditions are  $f(\mathcal{T} = 0) = 1$  and  $g(\mathcal{T} = 0) = 0$ .

Equation (7.11) describes the decay instability in the quantum case. Following [33], they will be referred to as equations of quantum decay.

### 7.1. Solution for the quantum four-mode decays

A careful mathematica analysis of the system (7.11) has been performed in Ref. [33] in the form of the semiclassical expansion (see also recent results [43, 44]). The system (7.11) is of mixed type with hyperbolic degeneracy on the line  $v = 0$ . The general theory yields merely that it has a real analytic solution in the three-dimensional space  $(\tau, v, \varepsilon)$  in some neighborhood of the plane  $\mathcal{T} = 0$ . As it is shown in [33, 43, 44], the quantum decays run not faster than exponential  $\exp(\sigma\mathcal{T})$ , where  $\sigma$  does not depend on  $\mathcal{T}$ , such that this property enables one to apply the Laplace transform in the analysis of equations (7.11).

Applying the Laplace transform in time  $f_u(v) = \hat{\mathcal{L}}[f(\mathcal{T})]$  and excluding  $g_u(v)$ , one obtains an ordinary equation for  $f_u(v)$

$$2\varepsilon v \frac{df_u}{dv} + (+iv/u - iu - 2)f_u = -i, \quad f_u(v = 0) = \frac{i}{2 + iu}. \quad (7.12)$$

The solution of equation (7.11) is

$$f(v, \mathcal{T}) = \hat{\mathcal{L}}^{-1} \left[ -ie^{\frac{1}{2\varepsilon}(\frac{v^2}{2iu} - 2v)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{v}{4\varepsilon}\right)^n \sum_{m=0}^n \frac{C_m^n}{2\varepsilon(m+n) - (iu+2)} \left(-\frac{v}{4iu}\right)^m \right], \quad (7.13)$$

where  $C_m^n$  are binominal coefficients. For the inverse Laplace transform we expand the exponential in Eq. (7.11) in the series, which yields (see Appendix D)

$$f(v, \mathcal{T}) = e^{-\frac{v}{\varepsilon}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{v}{\varepsilon}\right)^n \sum_{m=0}^n C_m^n \left(\frac{v}{4i}\right)^m F_{m,n}(\mathcal{T}), \quad (7.14)$$

where the time dependent term  $F_{m,n}(\mathcal{T})$  is obtained in Appendix D in the form of the Bessel functions

$$F_{m,n}(\mathcal{T}) = (ic_{m,n})^{-m} \left[ e^{ic_{m,n}\mathcal{T}} e^{-\frac{v^2}{4\varepsilon c_{n,m}}} + \sum_{k=1}^{\infty} \frac{J_n(\Lambda)}{(-ic_{m,n})^k} + \sum_{k=0}^{m-1} \left(-\frac{4\varepsilon c_{n,m}}{v^2}\right)^k J_n(\Lambda) \right], \quad (7.15)$$

where  $\Lambda = \sqrt{iv^2\mathcal{T}/\varepsilon}$ . In the large time asymptotic with  $\varepsilon\mathcal{T} \gg 1$ , the main contribution to quantum amplitude is due to the term

$$f(v, \mathcal{T}) \simeq e^{-\frac{v}{\varepsilon}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{v}{\varepsilon}\right)^n \sum_{m=0}^n C_m^n \left(\frac{v}{4i}\right)^m \sum_{k=0}^{m-1} \left(-\frac{4\varepsilon c_{n,m}}{v^2}\right)^k J_n(\Lambda). \quad (7.16)$$

Then taking into account that the argument of the Bessel function  $\Lambda$  is a complex value, the Bessel functions grow exponentially with time at  $|\Lambda| \gg 1$  [45]:  $J_n(\Lambda) \sim (2\pi/\Lambda)^{1/2} \cos(\Lambda n + \pi/2)$ . Therefore the quantum amplitudes grow in time according the stretch exponential function

$$f \sim \exp(v\sqrt{\mathcal{T}/2\varepsilon}) \Phi(v, \mathcal{T}) = \exp(\sqrt{\beta\gamma t}) \tilde{\Phi}(\gamma, t), \quad (7.17)$$

where  $\Phi(v, \mathcal{T})$  is a slow varying and not increasing function of time<sup>5</sup>, while  $\gamma = A_0 I / \hbar$ .

## 8. Quantum CTRW

Summarizing results of the previous sections, we admit that for the construction of the quantum CTRW, one follows a coarse-graining procedure by lumping a set of quantum states making a trapping cell, or trapping potential, which is in complete analogy with the classical CTRW, constructed in Sec. 5. Mapping the density operator  $\hat{\rho}$  on the basis of the coherent states  $|\mathbf{a}\rangle$  in Eq. (6.3), such that  $\mathcal{P}_Q(t) \equiv \mathcal{P}_Q(\mathbf{a}^*, \mathbf{a}, t) = \langle \mathbf{a} | \hat{\rho}(t) | \mathbf{a} \rangle$ , one obtains for the quantum density of the probability the following quantum Liouville equation

$$\partial_t \mathcal{P}_Q(t) = \hat{K} \mathcal{P}_Q(t). \quad (8.1)$$

Here the quantum “Liouville” operator is determined in Eq. (6.8) and it contains the same overlapping integrals as its classical counterpart in Eqs. (4.1) and (4.5). As the result, the coarse-grained quantum process of the transitions between the trapping clusters is also a Markov process. Transitions between quantum states inside the potential

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<sup>5</sup>This result of the Laplace inverse transform can be also obtained by the stationary phase approximation for the long time asymptotics. In this case the stationary point of the exponential in integrand (7.13) is  $u_0 = v/2\sqrt{i\varepsilon\mathcal{T}}$ , which immediately yields the result of Eq. (7.17).

do not contribute to the transport (to the spreading of the initial wave packet in the  $x$  coordinate), while transitions between any states of the different trapping clusters lead to the spreading of the wave packet in the  $x$  coordinate. The main contribution to this transport in the chain of the localized AMs is due to the transition between the nearest neighbor clusters and are determined by the jump length PDF  $p_Q(x)$ . The quantum transitions between coherent states inside the clusters leads to the traps with trapping times, distributed by the waiting time PDF  $w_Q(t)$ . The transition probabilities in the Pauli- van-Kampen master equation reduces to the classical analogs of the transition probabilities. Therefore, following the classical consideration, presented in Sec. 5, the quantum distribution function is governed by the Montroll-Weiss equation (5.4) in the Fourier- Laplace space

$$\bar{\bar{\mathcal{P}}}_Q(k, u) = \hat{\mathcal{F}}\hat{\mathcal{L}}\mathcal{P}_Q = \frac{1 - \tilde{w}_Q(u)}{u} \cdot \frac{\bar{\mathcal{P}}_0(k)}{1 - \bar{p}_Q(k)\tilde{w}_Q(u)}. \quad (8.2)$$

As already admitted above  $\tilde{p}_Q(k)$  is determined by the transitions between the localized states  $\tilde{p}_Q(k) \approx 1 - \beta A_0 R^2 k^2$ .

### 8.1. Waiting time PDF $w_Q(t)$

To estimate the waiting time PDF  $w_Q(t)$  for the quantum trapping cluster, or “trapping potential”, we are interesting in the large time asymptotics. The waiting time PDF, has a quantum nature and is determined by the quantum four modes decay rate, which is the inverse value of the quantum amplitudes  $|f|^2$  in Eq. (7.17). The probability to find a quantum particle inside trapping cluster after time  $t$  (or no jump during time interval  $(0, t)$ ) is

$$W_Q(t) = \int_t^\infty w_Q(t') dt' = \langle |f|^{-2} \rangle.$$

Here  $\langle |f|^{-2} \rangle$  is the averaged value of the inverse quantum amplitudes. We take into account that parameter  $\gamma$  changes randomly for different clusters as a function of the random localized states. We take these value being exponentially distributed  $\frac{1}{\gamma_0} e^{-\gamma/\gamma_0}$ , which yields<sup>6</sup>  $\langle |f|^{-2} \rangle \propto 1/(1 + \sqrt{t/\bar{\tau}})$ , where  $\bar{\tau} = 1/2\beta\gamma_0^2$ . Therefore one obtains<sup>7</sup>

$$w_Q(t) \simeq \frac{1}{(t/\bar{\tau})^{3/2}}. \quad (8.3)$$

which yields  $\tilde{w}_Q(u) \approx 1 - (u\bar{\tau})^{1/2}$ . Again, this waiting time PDF leads to the FFPE (5.8) with the transport exponent  $\alpha = 1/2$ . The solution of the FFPE is obtained in the form of the Fox  $H$  function, presented in Appendix B. This reads

$$\mathcal{P}_Q(x, t) = \frac{1}{\sqrt{D_{\frac{1}{2}} t^{\frac{1}{2}}}} H_{1,1}^{1,0} \left[ \frac{x^2}{D_{\frac{1}{2}} t^{\frac{1}{2}}} \left| \begin{matrix} (\frac{3}{4}, \frac{1}{2}) \\ (0, 2) \end{matrix} \right. \right], \quad (8.4)$$

<sup>6</sup>This integration can be considered as the Laplace transform. Therefore the Tauberian theorem can be applied to the slow varying function  $\bar{\Phi}(\gamma, t)$ .

<sup>7</sup>The same asymptotic behavior one obtains by calculating first a local waiting time PDF  $w(t, \gamma)$  and then averaging it over  $\gamma$ .



and its asymptotic behavior corresponds to the stretched Gaussian exponential function  $\mathcal{P}_Q(y) \sim \frac{1}{\sqrt{D_\alpha t^\alpha}} y^{-(1-\alpha)/(2-\alpha)} e^{-y^{2/(2-\alpha)}}$ , where  $y \equiv \frac{|x|}{\sqrt{D_\alpha t^\alpha}} \gg 1$ . Note that for  $\alpha = 1$  it corresponds to the normal Gaussian distribution. This solution eventually leads to quantum subdiffusion of a wave packet spreading with the MSD

$$\langle x^2(t) \rangle \sim \sqrt{t}. \quad (8.5)$$

Contrary to classical subdiffusion, there is no numerical confirmation of this result. However, it corroborates to an experimental observation of the optically induced exciton transport in molecular crystals, which exhibits the intermediate asymptotic subdiffusion [46] with the experimental transport exponent of the order of  $\sim 0.57$ .

## 9. Conclusion

The review is concerned with the nonlinear Schrödinger equation in the presence of disorder. The dynamics of an initially localized wave packet is described in both classical and quantum cases. In both cases, we obtained a subdiffusive spreading, which is explained in the framework of a continuous time random walk (CTRW), and it is shown that subdiffusion is due to the transitions between those Anderson modes (AM)s, which are strongly overlapped. This overlapping is a common feature of both classical and quantum dynamics and leads to the clustering with an effective trapping of the wave packet inside each cluster by an effective potential. Therefore, the classical dynamics of the wave packet corresponds to the CTRW, where the basic mechanism of subdiffusion is an entrapping of the wave packet with delay, or waiting times distributed by the power law  $w(t) \sim 1/t^{1+\alpha}$  with  $\alpha = 1/3$ . The trapping mechanism determines the transport exponent  $\alpha$ , which is due to the number of AMs contributed to the overlapping integrals according to Eq. (2.3). Note, that the PDF  $\mathcal{P}(x, t)$  in Eq. (4.1) is the exact distribution, and it corresponds to the CTRW in the long time and the large scale asymptotics described by the FFPE (5.8).

This classical CTRW consideration can be extended on the wave packet spreading in the framework of the generalized nonlinear Schrödinger equation

$$i\partial_t \psi = \hat{\mathcal{H}}_0 \psi + \beta |\psi|^{2s} \psi, \quad (9.1)$$

where  $s \geq 1$  and the Hamiltonian  $\hat{\mathcal{H}}_0$  has a pure point spectrum with the localized eigenfunctions:  $\hat{\mathcal{H}}_0 \Psi_k = \mathcal{E}_k \Psi_k$ . For example, the Hamiltonian describes Wannier-Stark localization [47], and the discrete counterpart of Eq. (9.1) with  $s = 1$  corresponds to delocalization in a nonlinear Stark ladder [48, 49, 50]. Repeating probabilistic consideration of the CTRW based on the overlapping integrals  $A(\mathbf{k})$  of  $2s + 1$  eigenfunctions  $\Psi_k(x)$ , one obtains that Eq. (9.1) describes subdiffusion with the transport exponent

$$\alpha = \frac{1}{1 + 2s}. \quad (9.2)$$

For different values of  $s$ , this expression coincides with the numerical results of Refs. [38, 49]. This result also correspond to the topological approach considered in [30, 39].

The situation with the quantum NLSE differs essentially due the nonzero commutation rule (6.2). Therefore, performing  $c$ -number projection of the Heisenberg equations of

motion on the basis of the coherent states, one obtains a quantum master equation with the same structure of the Liouville operator as the classical one. In this construction of the quantum master equation, as a quantum continuous time random walk (QCTRW), in some extent, we follow a van-Kampen coarse-graining procedure (for example, see details of the discussion in [42]) by lumping a set of quantum states making a trapping cluster, which is in a complete analogy with the classical CTRW construction [11], presented in Sec. 5. The transition probabilities in Eq. (6.12) correspond to their classical analogs of the transition probabilities, since the quantum “Liouville” operator determined in Eqs. (6.8) and (6.11) contains the same overlapping integrals as in the classical counterpart with  $\hbar = 0$ . Transitions between quantum states inside the clusters do not contribute to the transport (to the spreading of the initial wave packet in the  $x$  coordinate), while transitions between any states belonging to different clusters lead to the spreading of the wave packet by changing the  $x$  coordinate. The main contribution to this transport in the chain of the localized AMs is due to the transitions between the nearest neighbor clusters, which are determined by the jump length PDF  $p_Q(x)$ .

Therefore, the dynamics of the wave packet corresponds to the CTRW, where the basic mechanism of subdiffusion is the entrapping of the wave packet with delay, or waiting, times distributed by the power law  $w_Q(t) \sim 1/t^{1+\alpha}$ . The trapping mechanism also determines the transport exponent  $\alpha = 1/2$ , which is due to the quantum four-mode decay, described by Eq (7.11). The four-mode decays determine the leakage probability from trapping clusters and correspondingly determine the waiting time PDF  $w_Q(t)$ .

Careful mathematical analysis of the system (7.11) has been performed in Ref. [33] in the form of the semiclassical expansion. These results were also verified in recent publications [43, 44] devoted to the semiclassical analysis of quantum singularities in the dynamics of a Bose-Einstein condensate trapped in a one-dimensional toroidal geometry and the semiclassical analysis of the four-wave decay in a quantum chain of oscillators. As it is shown in [33, 43, 44] by asymptotic expansion over  $\varepsilon \ll 1$ , a convergence of quantum solutions to the corresponding classical solutions exists only for the logarithmic time scale  $\sim \log(1/\varepsilon)$ , and beyond this time quantum processes cannot be violated. Therefore, the quantum decay processes determine the quantum kinetics, which is defined in the large time asymptotics of the quantum dynamics and correspondingly determine the kinetic coefficients of the generalized master equation of the quantum kinetics.

It should be noted that iteration equation (6.12) is exact in complete correspondence with its classical counterpart (4.5). However the fractional integral equation (5.8), which is valid for both classical and quantum NLSE, is obtained in the diffusion limit in the framework of the CTRW consideration. The latter consists of two steps of the Markov independent processes. These are the trapping with the power law waiting PDF and instant jumps. The waiting process is a quantum process, which is described by the quantum decay equations (7.11). The instant jumps are described classically. In this case, we neglect the quantum terms  $\sim O(\hbar)$ , which are the second derivatives in the Liouville operator  $\hat{L}_1$ . As discussed in Ref. [44] the dimensionless Planck constant (as a semiclassical parameter) is small  $\hbar \sim 1/N$  due to the large number of particles in mesoscopic Bose-Einstein condensate systems  $N \sim 10^3 \div 10^6$ , and violation of the quantum terms  $\sim O(1/N)$  is well justified for the long distance jumps described by  $p_Q(x) \rightarrow p(x)$ .

It is worth mentioning that contrary to classical NLSE (1.1), there is no a numerical confirmation of this quantum subdiffusion. However, an experimental observation of the

exciton transport in molecular crystals exhibits the intermediate asymptotic subdiffusion [46] with the experimental transport exponent  $\alpha \sim 0.57$ .

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### Appendix A. A brief survey on fractional integration

Extended reviews of fractional calculus can be found *e.g.*, in [2, 3, 4]. Fractional integration of the order of  $\alpha$  is defined by the operator

$${}_a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1} dy, \quad (\text{A.1})$$

where  $\alpha > 0$ ,  $x > a$  and  $\Gamma(z)$  is the Gamma function. Fractional derivation was developed as a generalization of integer order derivatives and is defined as the inverse operation to the fractional integral. Therefore, the fractional derivative is defined as the inverse operator to  ${}_a I_x^\alpha$ , namely  ${}_a D_x^\alpha f(x) = {}_a I_x^{-\alpha} f(x)$  and  ${}_a I_x^\alpha = {}_a D_x^{-\alpha}$ . Its explicit form is

$${}_a D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x f(y)(x-y)^{-1-\alpha} dy. \quad (\text{A.2})$$

For arbitrary  $\alpha > 0$  this integral diverges, and as a result of this a regularization procedure is introduced with two alternative definitions of  ${}_a D_x^\alpha$ . For an integer  $n$  defined as  $n-1 < \alpha < n$ , one obtains the Riemann-Liouville fractional derivative of the form

$${}^{RL}_a D_x^\alpha f(x) \equiv {}_a D_x^\alpha f(x) = \frac{d^n}{dx^n} {}_a I_x^{n-\alpha} f(x), \quad (\text{A.3})$$

and fractional derivative in the Caputo form

$${}_a^C D_x^\alpha f(x) = {}_a I_x^{n-\alpha} \frac{d^n}{dx^n} f(x). \quad (\text{A.4})$$

There is no constraint on the lower limit  $a$ . For example, when  $a = 0$ , one has  ${}_0^{RL} D_x^\alpha x^\beta = \frac{x^{\beta-\alpha} \Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}$ . This fractional derivation with the fixed low limit is also called the left fractional derivative. However, one can introduce the right fractional derivative, where the upper limit  $a$  is fixed and  $a > x$ . For example, the right fractional integral is

$${}_x I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^a (y-x)^{\alpha-1} f(y) dy. \quad (\text{A.5})$$

Another important property is  $D^\alpha I^\beta = I^{\beta-\alpha}$ , where other indexes are omitted for brevity. Note that the inverse combination is not valid, in general case,  $I^\beta D^\alpha \neq I^{\beta-\alpha}$ , since it depends on the lower limits of the integrations [3]. We also use here a convolution rule for the Laplace transform for  $0 < \alpha < 1$

$$\mathcal{L}[I_x^\alpha f(x)] = s^{-\alpha} \tilde{f}(s). \quad (\text{A.6})$$

Note that for arbitrary  $\alpha > 1$  the treatment of the Caputo fractional derivative by the Laplace transform is more convenient than the Riemann-Liouville one.

It should be admitted that solutions, considered here can be obtained by the Laplace inversion in the form of the Mittag-Leffler function [51, 52, 53]

$$\mathcal{E}_{(\nu, \beta)}(zr^\nu) = \frac{r^{1-\beta}}{2\pi i} \int_{\mathcal{C}} \frac{s^{\nu-\beta} e^{sr}}{s^\nu - z} ds, \quad (\text{A.7})$$

where  $\mathcal{C}$  is a suitable contour of integration, starting and finishing at  $-\infty$  and encompassing a circle  $|s| \leq |z|^{\frac{1}{\nu}}$  in the positive direction, and  $\nu, \beta > 0$ .

## Appendix B. Solution in the form of the Fox $H$ function

The Fox  $H$  function is defined in terms of the Milln-Barnes integral [52, 53, 54]

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\Omega} \Theta(s) z^{-s} ds \quad (\text{B.1})$$

where

$$\Theta(s) = \frac{\left\{ \prod_{j=1}^m \Gamma(b_j + sB_j) \right\} \left\{ \prod_{j=1}^n \Gamma(1 - a_j - sA_j) \right\}}{\left\{ \prod_{j=m+1}^q \Gamma(1 - b_j - sB_j) \right\} \left\{ \prod_{j=n+1}^p \Gamma(a_j + sA_j) \right\}} \quad (\text{B.2})$$

with  $0 \leq n \leq p$ ,  $1 \leq m \leq q$  and  $a_i, b_j \in C$ , while  $A_i, B_j \in R+$ , for  $i = 1, \dots, p$ , and  $j = 1, \dots, q$ . The contour  $\Omega$  starting at  $\sigma - i\infty$  and ending at  $\sigma + i\infty$ , separates the poles of the functions  $\Gamma(b_j + sB_j)$ ,  $j = 1, \dots, m$  from those of the function  $\Gamma(1 - a_i - sA_i)$ ,  $i = 1, \dots, n$ .

Now the Montroll-Weiss equation (5.6) can be solved in terms of the Fox  $H$  functions. Let us present the Montroll-Weiss equation (5.6) in the form

$$\bar{\mathcal{P}}(k, u) = \frac{u^{\alpha-1}}{u^\alpha + D_\alpha k^2}, \quad (\text{B.3})$$

where we take  $\bar{\mathcal{P}}_0(k, u) = 1$ . Then employing formula (A.7) for the Mittag-Leffler function [3, 51, 52, 53] one obtains

$$\bar{\mathcal{P}}(k, t) = \mathcal{E}_{(\alpha, 1)}(-D_\alpha k^2 t^\alpha). \quad (\text{B.4})$$

The two parameter Mittag-Leffler function (B.2) is a special case of the Fox  $H$ -function [52, 53], which can be represented by means of the Milln-Barnes integral (B.1)

$$\begin{aligned} \mathcal{E}_{(\alpha, \beta)}(-z) &= \frac{1}{2\pi i} \int_{\Omega} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)} z^{-s} ds = H_{1,2}^{1,1} \left[ z \left| \begin{matrix} (0, 1) \\ (0, 1), (1 - \beta, \alpha) \end{matrix} \right. \right] \\ &= \frac{1}{\delta} H_{1,2}^{1,1} \left[ z \left| \begin{matrix} (0, 1/\delta) \\ (0, 1), (1 - \beta, \alpha/\delta) \end{matrix} \right. \right] \end{aligned} \quad (\text{B.5})$$

Fourier-cosine transform of Eqs. (B.4) and (B.5) yields [53]

$$\begin{aligned} \mathcal{P}_\rho(x, t) &= \frac{1}{2\pi} \int_0^\infty dk k^{\rho-1} \cos(kx) H_{1,2}^{1,1} \left[ \sqrt{D_\alpha t^\alpha} |k| \left| \begin{matrix} (0, 1/2) \\ (0, 1), (0, \alpha/2) \end{matrix} \right. \right] \\ &= \frac{1}{|x|^\rho} H_{3,3}^{2,1} \left[ \frac{x^2}{D_\alpha t^\alpha} \left| \begin{matrix} (1, 1), (1, \alpha), (\frac{1+\rho}{2}, 1) \\ (1, 2), (1, 1), (\frac{1+\rho}{2}, 1) \end{matrix} \right. \right] \end{aligned} \quad (\text{B.6})$$

For  $\rho = 1$  one obtains the solution of Eq. (5.6). However, taking into account the properties of the Fox H function [54], one obtains

$$\frac{1}{|x|} H_{3,3}^{2,1} \left[ \frac{x^2}{D_\alpha t^\alpha} \left| \begin{matrix} (1,1), (1,\alpha), (1,1) \\ (1,2), (1,1), (1,1) \end{matrix} \right. \right] = \frac{1}{|x|} H_{2,2}^{2,0} \left[ \frac{x^2}{D_\alpha t^\alpha} \left| \begin{matrix} (1,\alpha), (1,1) \\ (1,2), (1,1) \end{matrix} \right. \right] \quad (\text{B.7})$$

Then using property  $x^\delta H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] = H_{p,q}^{m,n} \left[ x \left| \begin{matrix} (a_p + \delta A_p, A_p) \\ (b_q + \delta B_q, B_q) \end{matrix} \right. \right]$ , Eq. (B.6) reduces to

$$\mathcal{P}(x, t) = \frac{1}{\sqrt{D_\alpha t^\alpha}} H_{2,2}^{2,0} \left[ \frac{x^2}{D_\alpha t^\alpha} \left| \begin{matrix} (1 - \frac{\alpha}{2}, \alpha), (\frac{1}{2}, 1) \\ (0, 2), (\frac{1}{2}, 1) \end{matrix} \right. \right] \quad (\text{B.8})$$

Again using property of Eq. (B.7), one obtains

$$\mathcal{P}(x, t) = \frac{1}{\sqrt{D_\alpha t^\alpha}} H_{1,1}^{1,0} \left[ \frac{x^2}{D_\alpha t^\alpha} \left| \begin{matrix} (1 - \frac{\alpha}{2}, \alpha) \\ (0, 2) \end{matrix} \right. \right]. \quad (\text{B.9})$$

### Appendix C. Inferring Eq. (7.10) for functions $f_p$ and $\tilde{f}_{-p}$

Let us rewrite Eqs. (7.7) and (7.8) in the action-angle variables  $(I_k, \varphi_k)$ , where we use the variable change

$$\sqrt{\tilde{h}} a_k = \sqrt{I_k} \exp(i\varphi_k) \equiv \sqrt{I} \exp(i\varphi). \quad (\text{C.1})$$

Therefore, functions  $f_p$  and  $\tilde{f}_{-p}$  can be presented in the form

$$\begin{aligned} f_p &= f(I, \varphi, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f_n(I, t) e^{in\varphi}, \\ \tilde{f}_{-p} &= g(I, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} g_n(I, t) e^{in\varphi}. \end{aligned} \quad (\text{C.2})$$

Using relations of differentiations

$$\begin{aligned} a_k \frac{\partial}{\partial a_k} &= I \frac{\partial}{\partial I} - \frac{i}{2} \frac{\partial}{\partial \varphi}, \\ a_k^2 \frac{\partial^2}{\partial a_k^2} + 2|a_k|^2 a_k \frac{\partial}{\partial a_k} &= 2i \left( \frac{1}{2} \frac{\partial}{\partial \varphi} - I \frac{\partial}{\partial I} \frac{\partial}{\partial \varphi} - I \frac{\partial}{\partial \varphi} \right), \end{aligned} \quad (\text{C.3})$$

one obtains from Eqs. (7.7),(7.8) and (C.2) the following system of equations

$$i\partial_t f_n = (n\omega_k + \omega_{k+p}) f_n + \tilde{h}\beta A_0 [(2+n)I\partial_I f_n + (n/2 + nI/\tilde{h} + 2I/\tilde{h}) f_n - I g_{n-2}/\tilde{h}], \quad (\text{C.4})$$

$$i\partial_t g_n = (n\omega_k - \omega_{k-p}) g_n + \tilde{h}\beta A_0 [-I(2-n)\partial_I g_n + (n/2 - 2I/\tilde{h} + nI/\tilde{h}) g_n + I f_{n-2}/\tilde{h}]. \quad (\text{C.5})$$

Due to the initial conditions  $f_p(0) = 1$  and  $f_{-p}(0) = 0$ , the initial conditions for the system of Eqs. (C.4),(C.5) are  $f_n(0) = \delta_{n,0}$  and  $g_n(0) = 0$ . Therefore the solutions of Eqs. (C.4),(C.5) are

$$f_n(t) = 0 \quad \text{for } n \neq 0, \quad g_n(t) = 0 \quad \text{for } n \neq 2. \quad (\text{C.6})$$

For the functions  $f_0(t)$  and  $g_2(t)$ , one performs the following substitution

$$f_0(I, t) = \exp[-i(2\omega_k - \omega_{k-p} + \tilde{h}\beta A_0)t]f(I, t), \text{nonnumber} \quad (\text{C.7})$$

$$g_2(I, t) = \exp[-i(2\omega_k - \omega_{k-p} + \tilde{h}\beta A_0)t]g(I, t), \quad (\text{C.8})$$

and obtains Eqs. (C.4) and (C.5) in the form

$$\begin{aligned} i\partial_t f &= (\omega_{k+p} + \omega_{k-p} - 2\omega_k)f + \tilde{h}\beta A_0 I \partial_I f + 2\tilde{h}\beta A_0 I f - \beta A_0 I g, \\ i\partial_t g &= \beta V_0 I f. \end{aligned} \quad (\text{C.9})$$

#### Appendix D. Inverse Laplace transform of Eq.(7.13)

To perform the Laplace inversion in Eq. (7.13) let us present  $f_u(y)$  as follows

$$f_u(v) = -ie^{\frac{v}{\varepsilon}} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^n C_m^n G_{m,n}(u) \left(-\frac{v}{4i}\right)^m. \quad (\text{D.1})$$

Here

$$G_{m,n}(u) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{v^2}{4i\varepsilon}\right)^k \frac{1}{u - ic_{n,m}} \cdot \frac{1}{u^{m+k}}, \quad (\text{D.2})$$

where  $c_{n,m} = 2 - 2\varepsilon(m+n)$ . Thus, the Laplace inversion reduces to the integrals in  $G_{m,n}(u)$

$$\chi_{n,m,k} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{u\mathcal{T}} du}{(u - ic_{n,m})u^{m+k}}. \quad (\text{D.3})$$

The standard residue rules yield

$$\chi_{n,0,0} = e^{ic_{n,0}\mathcal{T}}, \quad \chi_{n,m,k} = \frac{e^{ic_{n,m}\mathcal{T}}}{(ic_{n,m})^{m+k}} - \frac{e^{ic_{n,m}\mathcal{T}} \Gamma(m+k, ic_{n,m})}{ic_{n,m} \Gamma(m+k)}, \quad (\text{D.4})$$

where  $\Gamma(l) = (l-1)!$  is a gamma function, while  $\Gamma(l, z)$  is an incomplete gamma function and  $1/\Gamma(0) = 0$ . Note that  $\varepsilon$  is such that  $c_{n,m} \neq 0$  for all  $n$  and  $m$ . Taking into account Eqs. (D.2),(D.3),(D.4), the inverse Laplace in Eq. (D.1) yields function  $f(v, \mathcal{T})$  as follows

$$f(v, \mathcal{T}) = e^{-\frac{v}{\varepsilon}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{v}{\varepsilon}\right)^n \sum_{m=0}^n C_m^n \left(\frac{v}{4i}\right)^m F_{m,n}(\mathcal{T}), \quad (\text{D.5})$$

where

$$\begin{aligned} F_{m,n}(\mathcal{T}) &= \frac{e^{ic_{n,m}\mathcal{T}}}{(ic_{n,m})^m} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{v^2}{4i\varepsilon}\right)^k \frac{\Gamma(m+k) - \Gamma(m+k, ic_{n,m}\mathcal{T})}{(ic_{n,m})^k \Gamma(m+k)} \\ &= \frac{1}{(ic_{n,m})^m} \left[ e^c e^b - \sum_{k=0}^{\infty} \frac{b^k}{k!} \sum_{l=0}^{m+k-1} \frac{c^l}{l!} \right], \end{aligned} \quad (\text{D.6})$$

where  $b = v^2/4\epsilon c_{n,m}$  and  $c = ic_{n,m}\mathcal{T}$ . Here we used an explicit expression for the gamma function  $\Gamma(k, z) = e^{-z}(k-1)! \sum_{l=0}^{k-1} \frac{z^l}{l!}$  and accounted that  $1/\Gamma(0) = 0$ . Then we change the index in  $k = n + l$  in the sums that yields

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{b^k}{k!} \sum_{l=0}^{m+k-1} \frac{c^l}{l!} &= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{b^{n+l}}{(n+l)!} \cdot \frac{c^l}{l!} + \sum_{k=0}^{\infty} \sum_{n=0}^{m-1} \frac{b^k}{k!} \cdot \frac{c^{k+n}}{(k+n)!} \\ &= \sum_{n=1}^{\infty} b^n \sum_{l=0}^{\infty} \frac{(-bc)^l}{(n+l)!l!} + \sum_{n=0}^{m-1} c^n \sum_{l=0}^{\infty} \frac{(-1)^l (-bc)^l}{(n+l)!l!} \\ &= \sum_{n=1}^{\infty} c^{-n} J_n(2\sqrt{-bc}) + \sum_{n=0}^{m-1} b^{-n} J_n(2\sqrt{-bc}). \end{aligned} \quad (\text{D.7})$$

Here we used the definition of the Bessel function [45]  $J_n(2z) = \sum_{l=0}^{\infty} \frac{(-1)^l}{(n+l)!l!} z^{2l}$

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