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Numerical analysis of a dynamic bilateral thermoviscoelastic contact problem with nonmonotone friction law

Krzysztof Bartosz [a,*], David Danan [b], Paweł Szafraniec [a]

[a] Jagiellonian University, Faculty of Mathematics and Computer Science, ul. Lojasiewicza 6, 30348 Krakow, Poland

[b] Laboratoire de Mathématiques et Physique Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France

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ABSTRACT

We study a fully dynamic thermoviscoelastic contact problem. The contact is assumed to be bilateral and frictional, where the friction law is described by a nonmonotone relation between the tangential stress and the tangential velocity. Weak formulation of the problem leads to a system of two evolutionary, possibly nonmonotone subdifferential inclusions of parabolic and hyperbolic type, respectively. We study both semidiscrete and fully discrete approximation schemes, and bound the errors of the approximate solutions. Under regularity assumptions imposed on the exact solution, optimal order error estimates are derived for the linear element solution. This theoretical result is illustrated numerically.

1. Introduction

Problems involving thermoviscoelastic effects arise in many fields of industry and everyday life. A physical body may change its properties and shape when heated. In particular a contact between two physical bodies usually leads to a heat exchange and a frictional contact is directly related to heat generation. For this reason many experts in contact mechanics pay a special attention to this kind of problems.

The first existence and uniqueness results for contact problems with friction in elastodynamics were obtained by Duvaut and Lions [1]. Later, Martins and Oden [2] studied the normal compliance model of contact with friction and showed existence and uniqueness results for a viscoelastic material. These results were extended by Figueiredo and Trabucho [3] to thermoelastic and thermoviscoelastic models. Recently dynamic viscoelastic frictional contact problems with or without thermal effects have been investigated in a large number of papers, see e.g. Adly et al. [4] Amassad et al. [5], Andrews et al. [6,7], Chau et al. [8], Han and Sofonea [9], Jarušek [10], Kuttler and Shillor [11], Migórski [12], Migórski and Ochal [13], Migórski et al. [14,15], Rochdi and Shillor [16] and the references therein. Recently, a dynamic thermoviscoelastic contact problem involving nonmonotone and nonsmooth friction law has been studied by Migórski and Szafraniec [17]. A mathematical formulation of the above problem has been reduced to a system of parabolic and hyperbolic hemivariational inequalities. Under some reasonable assumptions, an existence and uniqueness of a solution has been proved.

Existence and uniqueness results for contact problems in mechanics are definitely important from a theoretical point of view. Nevertheless, it is also useful and interesting to approximate the solution numerically, study the error estimate for the numerical method and carry out a computer simulation. There is a huge number of publications that deal with this task. We refer the reader to [18–24] to learn more about numerical aspects of contact problems in mechanics. Our goal is to provide

* Corresponding author.
E-mail address: krzysztof.bartosz@ii.uj.edu.pl (K. Bartosz).

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a numerical analysis of problem studied by Migórski and Szafraniec in [17]. Namely, we deal with a system of two dynamic (first and second order, respectively) inclusions describing the displacement and the temperature of a thermoviscoelastic body, which is in a contact with a foundation. However, in contrast to [17], we neglect a history dependent term in the constitutive law and we deal with a bilateral contact instead of considering a normal damped condition. Nevertheless, similarly to [17], we involve a nonmonotone friction law and possibly nonmonotone boundary heat flux law in our model. Both relations are described by means of the Clarke subdifferential inclusion. In order to approximate the solution of our problem, we introduce two numerical schemes: semidiscrete and fully discrete ones. In both cases we estimate the error between the exact solution and the numerical one. In case of the first order finite element spatial approximation, we guarantee a linear error estimate with respect to the size of spatial mesh and the time step length provided the exact solution is sufficiently regular. Our theoretical result concerning the linear bound of the error is confirmed by computer simulations, which illustrate the behaviour of the concrete physical body.

Our paper can be also seen as a continuation of [25], where the analogous result has been obtained for a dynamic viscoelastic contact problem without a thermal effect.

The rest of the paper is organized as follows. In Section 2 we introduce some preliminary material to be used in the rest of the paper. In Section 3 we formulate the mathematical model of the dynamic thermoviscoelastic contact problem and present its variational formulation. The main results concerning the error estimates for semidiscrete and fully discrete numerical schemes are presented in Sections 4 and 5, respectively. Finally, in Section 6, we present numerical results for simulations of a two dimensional contact problem, and provide numerical evidence of optimal order convergence for the linear elements.

2. Preliminaries

In this section we introduce notation and recall some definitions and results needed in the sequel, cf. [26,9,15,27].

We denote by $\mathbb{S}^d$ the space $\mathbb{R}^{d\times d}_s$ of symmetric matrices of order d, where $d = 2, 3$. We recall that the canonical inner product and the corresponding norm on $\mathbb{S}^d$ is given by

$$\sigma : \tau = \sigma_{ij}\,\tau_{ij}, \qquad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{1/2} \quad \text{for all } \sigma = (\sigma_{ij}),\ \tau = (\tau_{ij}) \in \mathbb{S}^d.$$

Here and below, the indices i and j run from 1 to d, and the summation convention over repeated indices is adopted.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ. The unit outward normal vector ν is defined a.e. on Γ. We use notation $L^2(\Omega) = L^2(\Omega;\mathbb{R})$ and $H^1(\Omega) = W^{1,2}(\Omega;\mathbb{R})$. Moreover, we introduce the following function spaces:

$$\begin{aligned} &H = L^2(\Omega;\mathbb{R}^d) = \{u = (u_i) \mid u_i \in L^2(\Omega)\}, && Q = \{\sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},\\ &H_1 = \{u \in H \mid \varepsilon(u) \in Q\}, && Q_1 = \{\sigma \in Q \mid \mathrm{Div}\sigma \in H\}. \end{aligned}$$

Here $\varepsilon\colon H_1 \to Q$ and $\mathrm{Div}\colon Q_1 \to H$ are the *deformation* and *divergence* operators, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \mathrm{Div}\,\sigma = (\sigma_{ij,j}),$$

respectively, where the index following a comma indicates the partial derivative with respect to the corresponding component of the independent variable.

For every elements $u \in H^1(\Omega)$ and $v \in H_1$ we use the same symbols u and v to denote the traces of u and v on Γ. For every $v \in H_1$, we denote by v_ν and v_τ the *normal* and *tangential* components of v on the boundary Γ given by

$$v_\nu = v \cdot \nu, \qquad v_\tau = v - v_\nu \nu.$$

Similarly, for a regular tensor field $\sigma : \Omega \to \mathbb{S}^d$ we denote by σ_ν and σ_τ the *normal* and *tangential* component of σ,

$$\sigma_\nu = (\sigma\nu)\cdot\nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu\nu.$$

We recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, where X is a Banach space (see [28]). The generalized directional derivative of φ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y\to x,\ t\downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$

The generalized gradient of φ at x, denoted by $\partial\varphi(x)$, is a subset of a dual space X^* given by $\partial\varphi(x) = \{\zeta \in X^* \mid \varphi^0(x; v) \geq \langle\zeta, v\rangle_{X^*\times X}$ for all $v \in X\}$.

We denote by $\mathcal{L}(X, Y)$ the space of linear continuous mappings from X to Y. Given a reflexive Banach space Y, we denote by $\langle\cdot,\cdot\rangle_Y$ the duality pairing between the dual space Y^* and Y. We complete this section with the following version of the Gronwall lemma which we use in next sections.

Lemma 1. *Let $T > 0$ be given. For a positive integer N we define $k = T/N$. Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying*

$$e_n \le c g_n + c \sum_{j=1}^{n} k e_j \quad \text{for } n = 1, \dots, N,$$

with a positive constant c independent of N and k. Then there exists a positive constant $\hat{c}$, independent of N and k, such that

$$\max_{1 \le n \le N} e_n \le \hat{c} \max_{1 \le n \le N} g_n.$$

3. Mechanical problem and variational formulation

Let Ω be an open bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary Γ. The boundary Γ is composed of three sets $\overline{\Gamma}_D$, $\overline{\Gamma}_N$ and $\overline{\Gamma}_C$, with mutually disjoint relatively open sets Γ_D, Γ_N and Γ_C such that meas $(\Gamma_D) > 0$. We consider a viscoelastic body, which in the reference configuration, occupies volume Ω and which is supposed to be stress free and at a constant temperature, conveniently set as zero. We assume that the temperature changes accompanying the deformations are small and they do not produce any changes in the material parameters which are regarded temperature independent. We are interested in a mathematical model that describes the evolution of the mechanical state of the body and its temperature during the time interval $[0, T]$, where $0 < T < \infty$. To this end, we denote by $\sigma = \sigma(x, t) = (\sigma_{ij}(x, t))$ the stress field, by $u = u(x, t) = (u_i(x, t))$ the displacement field, and by $\theta = \theta(x, t)$ the temperature, where $x \in \Omega$ and $t \in [0, T]$ denote the spatial and the time variables, respectively. The functions $u\colon \Omega \times [0, T] \to \mathbb{R}^d$, $\sigma\colon \Omega \times [0, T] \to \mathbb{S}^d$ and $\theta\colon \Omega \times [0, T] \to \mathbb{R}$ will play the role of the unknowns of the frictional contact problem. From time to time, we suppress the explicit dependence of the quantities on the spatial variable x, or both x and t.

We suppose that the body is clamped on Γ_D, the volume forces of density $f_0 = f_0(x, t)$ act in Ω and the surface tractions of density $f_2 = f_2(x, t)$ are applied on Γ_N. Moreover, the body is subjected to a heat source term per unit volume $g = g(x, t)$, its temperature is fixed on $\Gamma_D \cup \Gamma_N$ and it comes in contact with an obstacle, the so-called foundation, over the contact surface Γ_C.

The classical formulation of the mechanical problem is the following.

Problem 2. Find a displacement $u\colon \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma\colon \Omega \times [0, T] \to \mathbb{S}^d$ and temperature $\theta\colon \Omega \times [0, T] \to \mathbb{R}$ such that

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}\varepsilon(u) + C\theta \qquad \text{in } \Omega \times (0, T), \tag{1}$$

$$\rho \ddot{u} = \operatorname{Div} \sigma + f_0 \qquad \text{in } \Omega \times (0, T), \tag{2}$$

$$\rho c_p \dot{\theta} - \operatorname{div}(K \nabla \theta) = c_{ij} \frac{\partial \dot{u}_i}{\partial x_j} + g \qquad \text{in } \Omega \times (0, T), \tag{3}$$

$$\sigma \nu = f_2 \qquad \text{on } \Gamma_N \times (0, T), \tag{4}$$

$$u = 0 \qquad \text{on } \Gamma_D \times (0, T), \tag{5}$$

$$\theta = 0 \qquad \text{on } \Gamma_D \cup \Gamma_N \times (0, T), \tag{6}$$

$$u_\nu = 0 \qquad \text{on } \Gamma_C \times (0, T), \tag{7}$$

$$-\sigma_\tau \in \partial j_\tau(\dot{u}_\tau) \qquad \text{on } \Gamma_C \times (0, T), \tag{8}$$

$$-K(x, t, \nabla \theta(t)) \cdot \nu \in \partial j(\theta(t)) - h_\tau(x, t, \|\dot{u}_\tau(x, t)\|_{\mathbb{R}^d}) \qquad \text{on } \Gamma_C \times (0, T), \tag{9}$$

$$u(0) = u_0, \qquad \dot{u}(0) = u_1, \qquad \theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{10}$$

Here, Eq. (1) represents the thermoviscoelastic constitutive law of the material, where $\mathcal{A}$, $\mathcal{B}$ and $C = (c_{ij})$ are respectively the viscosity, elasticity and heat expansion tensors. Eq. (2) is the equation of motion, with ρ being the density of the material. Then, Eq. (3) is the energy equation expressed in terms of the temperature θ with c_p, the heat capacity. Eq. (4) represents the traction boundary condition on the part of the boundary Γ_N. Eqs. (5) and (6) are the displacement and temperature boundary conditions, respectively. Conditions (7) and (8) represent bilateral contact coupled with a law of friction. Eq. (9) describes the heat transfer between the body and the foundation and also takes into account the velocity heat generation phenomena. Finally, (10) is the initial conditions in which u_0, u_1, and θ_0 denote the given initial displacement, velocity and temperature, respectively. We assume the density ρ and a specific heat capacity c_p are greater than zero and constant. In Sections 3–5, without loss of generality, we assume $\rho = c_p = 1$. In Section 6 we put specific values for numerical simulations.

In the study of the contact problem we need the following assumptions on its data.

$\underline{H(\mathcal{A})}$: The viscosity operator $\mathcal{A} : \Omega \times [0, T] \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

(a) $\mathcal{A}(\cdot, \cdot, \varepsilon)$ is measurable on $\Omega \times [0, T]$ for all $\varepsilon \in \mathbb{S}^d$;

(b) $\|\mathcal{A}(x, t, \varepsilon)\|_{\mathbb{S}^d} \le a_0(x, t) + a_1 \|\varepsilon\|_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x, t) \in \Omega \times (0, T)$ with $a_0 \in L^2(\Omega \times (0, T))$, $a_0 \ge 0$ and $a_1 > 0$;

(c) $(\mathcal{A}(x,t,\varepsilon_1)-\mathcal{A}(x,t,\varepsilon_2)):(\varepsilon_1-\varepsilon_2)\ge m_{\mathcal{A}}\|\varepsilon_1-\varepsilon_2\|^2_{\mathbb{S}^d}$ for all $\varepsilon_1,\varepsilon_2\in\mathbb{S}^d$, a.e. $(x,t)\in\Omega\times(0,T)$ with $m_{\mathcal{A}}>0$;
(d) $\mathcal{A}(x,t,\varepsilon):\varepsilon\ge\alpha_{\mathcal{A}}\|\varepsilon\|^2_{\mathbb{S}^d}$ for all $\varepsilon\in\mathbb{S}^d$, a.e. $(x,t)\in\Omega\times(0,T)$ with $\alpha_A>0$;
(e) $\|\mathcal{A}(x,t,\varepsilon_1)-\mathcal{A}(x,t,\varepsilon_2)\|_{\mathbb{S}^d}\le L_{\mathcal{A}}\|\varepsilon_1-\varepsilon_2\|_{\mathbb{S}^d}$ for all $\varepsilon_1,\varepsilon_2\in\mathbb{S}^d$, a.e. $(x,t)\in\Omega\times(0,T)$ with $L_{\mathcal{A}}>0$.

$\underline{H(\mathcal{B})}$: The elasticity operator $\mathcal{B}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$ is bounded, symmetric, positive fourth order tensor, i.e.
(a) $\mathcal{B}_{ijkl}\in L^\infty(\Omega)$, $1\le i,j,k,l\le d$;
(b) $\mathcal{B}\sigma:\tau=\sigma:\mathcal{B}\tau$ for all $\sigma,\tau\in\mathbb{S}^d$, a.e. in Ω;
(c) $\mathcal{B}\tau:\tau\ge 0$ for all $\tau\in\mathbb{S}^d$, a.e. in Ω.

$\underline{H(C)}$: The thermal expansion tensor $C=(c_{ij})$ satisfies
(a) $c_{ij}\in L^\infty(\Omega)$, $1\le i,j\le d$;
(b) $c_{ij}=c_{ji}$, $1\le i,j\le d$.

$\underline{H(K)}$: The thermal conductivity operator $K:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies
(a) $K(\cdot,\cdot,\xi)$ is measurable on $\Omega\times[0,T]$ for all $\xi\in\mathbb{R}^d$;
(b) $\|K(x,t,\xi)\|_{\mathbb{R}^d}\le k_0(x,t)+k_1\|\xi\|_{\mathbb{R}^d}$ for all $\xi\in\mathbb{R}^d$, a.e. $(x,t)\in\Omega\times(0,T)$ with $k_0\in L^2(\Omega\times(0,T))$, $k_0\ge 0$, $k_1>0$;
(c) $(K(x,t,\xi_1)-K(x,t,\xi_2))\cdot(\xi_1-\xi_2)\ge m_K\|\xi_1-\xi_2\|^2_{\mathbb{R}^d}$ for all $\xi_1,\xi_2\in\mathbb{R}^d$, a.e. $(x,t)\in\Omega\times(0,T)$ with $m_K>0$;
(d) $K(x,t,\xi)\cdot\xi\ge\alpha_K\|\xi\|^2_{\mathbb{R}^d}$ for all $\xi\in\mathbb{R}^d$, a.e. $(x,t)\in\Omega\times(0,T)$ with $\alpha_K>0$;
(e) $\|K(x,t,\xi_1)-K(x,t,\xi_2)\|_{\mathbb{R}^d}\le L_K\|\xi_1-\xi_2\|_{\mathbb{R}^d}$ for all $\xi_1,\xi_2\in\mathbb{R}^d$, a.e. $(x,t)\in\Omega\times(0,T)$ with $L_K>0$.
The potentials j_τ and j satisfy the following hypotheses.

$\underline{H(j_\tau)}$: $j_\tau:\Gamma_C\times(0,T)\times\mathbb{R}^d\to\mathbb{R}$ is such that
(a) $j_\tau(\cdot,\cdot,\xi)$ is measurable on $\Gamma_C\times(0,T)$ for all $\xi\in\mathbb{R}^d$ and there exists $e_1\in L^2(\Gamma_C;\mathbb{R}^d)$ such that $j_\tau(\cdot,\cdot,e_1(\cdot))\in L^1(\Gamma_C\times(0,T))$;
(b) $j_\tau(x,t,\cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $(x,t)\in\Gamma_C\times(0,T)$;
(c) $\|\partial j_\tau(x,t,\xi)\|_{\mathbb{R}^d}\le c_\tau$ for all $\xi\in\mathbb{R}^d$, a.e. $(x,t)\in\Gamma_C\times(0,T)$ with $c_\tau\ge 0$;
(d) $(\zeta_1-\zeta_2)\cdot(\xi_1-\xi_2)\ge -m_\tau\|\xi_1-\xi_2\|^2_{\mathbb{R}^d}$ for all $\zeta_i\in\partial j_\tau(x,t,\xi_i)$, $\xi_i\in\mathbb{R}^d$, $i=1,2$, a.e. $(x,t)\in\Gamma_C\times(0,T)$ with $m_\tau\ge 0$;
(e) $j_\tau(x,t,\cdot)$ is regular in the sense of Clarke for a.e. $(x,t)\in\Gamma_C\times(0,T)$.

$\underline{H(j)}$: $j:\Gamma_C\times(0,T)\times\mathbb{R}\to\mathbb{R}$ is such that
(a) $j(\cdot,\cdot,r)$ is measurable on $\Gamma_C\times(0,T)$ for all $r\in\mathbb{R}$ and there exists $e_2\in L^2(\Gamma_C)$ such that $j(\cdot,\cdot,e_2(\cdot))\in L^1(\Gamma_C\times(0,T))$;
(b) $j(x,t,\cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $(x,t)\in\Gamma_C\times(0,T)$;
(c) $|\partial j(x,t,r)|\le c_\theta(1+|r|)$ for all $r\in\mathbb{R}$, a.e. $(x,t)\in\Gamma_C\times(0,T)$ with $c_\theta\ge 0$;
(d) $|\zeta_1-\zeta_2|\le m_\theta|r_1-r_2|$ for all $\zeta_i\in\partial j(x,t,r_i)$, $r_i\in\mathbb{R}$, $i=1,2$, a.e. $(x,t)\in\Gamma_C\times(0,T)$ with $m_\theta\ge 0$;
(e) $j(x,t,\cdot)$ is regular in the sense of Clarke for a.e. $(x,t)\in\Gamma_C\times(0,T)$.

$\underline{H(h_\tau)}$: $h_\tau:\Gamma_C\times\mathbb{R}_+\to\mathbb{R}_+$ is such that
(a) $h_\tau(\cdot,r)\in L^2(\Gamma_C)$ for all $r\in\mathbb{R}_+$.
(b) $|h_\tau(x,r_1)-h_\tau(x,r_2)|\le L_\tau|r_1-r_2|$ for all $r_1,r_2\in\mathbb{R}_+$, a.e. $x\in\Gamma_C$ with $L_\tau>0$.

$\underline{H(f)}$: The force and traction densities satisfy

$$f_0\in L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\qquad f_2\in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d)).$$

Remark 3. The assumption $H(j)$(d) implies that the Clarke subdifferential ∂j is single-valued, and, in a consequence, the heat transfer inclusion (9) reduces to an equation. Nevertheless, in order to underline the connection of our result with one obtained in [17], we decided to keep the same convention as used in [17] and to present the heat transfer law in a form of inclusion involving Clarke subdifferential of the potential j. The results obtained hereafter are still open when the assumption $H(j)$(d) is omitted.

In order to provide the variational formulation of Problem 2, we introduce the following spaces

$$E=\left\{v\in H^1(\Omega;\mathbb{R}^d)\mid v=0 \text{ on } \Gamma_D,\ v_\nu=0 \text{ on } \Gamma_C\right\},$$
$$V=\left\{\eta\in H^1(\Omega)\mid \eta=0 \text{ on } \Gamma_D\cup\Gamma_N\right\}.$$

On the spaces E and V we consider the inner products and the corresponding norms given by

$$(u,v)_E=\langle\varepsilon(u),\varepsilon(v)\rangle_{L^2(\Omega;\mathbb{S}^d)},\qquad \|v\|_E=\|\varepsilon(v)\|_{L^2(\Omega;\mathbb{S}^d)} \text{ for } u,v\in E,$$
$$(u,v)_V=\langle\nabla u,\nabla v\rangle_{L^2(\Omega;\mathbb{R}^d)},\qquad \|v\|_V=\|\nabla v\|_{L^2(\Omega;\mathbb{R}^d)} \text{ for } u,v\in V.$$

From the Korn inequality $\|v\|_{H^1(\Omega;\mathbb{R}^d)}\le c\|\varepsilon(v)\|_{L^2(\Omega;\mathbb{S}^d)}$ for $v\in E$ with $c>0$, it follows that $\|\cdot\|_{H^1(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|_E$ are the equivalent norms on E. Analogously, by the Poincare inequality, we know, that the norm $\|\cdot\|_V$ is equivalent with $\|\cdot\|_{H^1(\Omega)}$. Let $Z=H^\delta(\Omega;\mathbb{R}^d)$ with a fixed $\delta\in(1/2,1)$. We denote by $i_Z:E\to Z$ the embedding injection and by c_Z its norm $\|i_Z\|_{\mathcal{L}(E,Z)}$. Let $\gamma_Z:Z\to L^2(\Gamma_C;\mathbb{R}^d)$ be the trace operator. For simplicity we omit the notation of the embedding and trace and write

$v = \gamma_Z(i_Z v)$ for $v \in E$. We also introduce the following spaces of vector valued functions $\mathcal{E} = L^2(0, T; E)$, $\mathcal{Z} = L^2(0, T; Z)$, $\widehat{\mathcal{H}} = L^2(0, T; H)$ and $\mathbb{E} = \{v \in \mathcal{E} \mid \dot{v} \in \mathcal{E}^*\}$.

Similarly, we introduce the space $Y = H^\delta(\Omega)$ with the same $\delta \in (1/2, 1)$. We denote by $i_Y : V \to Y$ the embedding injection and by c_Y its norm $\|i_Y\|_{\mathcal{L}(V,Y)}$. Let $\gamma_Y : Y \to L^2(\Gamma_C)$ be the trace operator. For simplicity, we omit the notation of the embedding and trace and write $v = \gamma_Y(i_Y v)$ for $v \in V$. We introduce the spaces $\mathcal{V} = L^2(0, T; V)$, $\mathcal{Y} = L^2(0, T; Y)$ and $\mathcal{W} = \{\eta \in \mathcal{V} \mid \dot{\eta} \in \mathcal{V}^*\}$.

Next, we define operators $A\colon (0, T) \times E \to E^*$, $B\colon E \to E^*$, $C_1 : V \to E^*$, $C_2 : V \to V^*$, $C_3 : E \to V^*$ by

$$\begin{aligned}
&\langle A(t, u), v\rangle_E = (\mathcal{A}(t, \varepsilon(u)), \varepsilon(v))_Q \quad \text{for } u, v \in E,\ t \in (0, T),\\
&\langle Bu, v\rangle_E = (\mathcal{B}\varepsilon(u), \varepsilon(v))_Q \quad \text{for } u, v \in E,\\
&\langle C_1\theta, v\rangle_E = \int_\Omega c_{ij}\frac{\partial v_i}{\partial x_j}\theta\, dx \quad \text{for } v \in E, \theta \in V,\\
&\langle C_2(t, \theta), \eta\rangle_V = \langle K(t, \nabla\theta), \nabla\eta\rangle_{L^2(\Omega)} \quad \text{for } \theta, \eta \in V,\\
&\langle C_3 w, \eta\rangle_V = -\int_\Omega c_{ij}\frac{\partial w_i}{\partial x_j}\eta\, dx \quad \text{for } w \in E\ \eta \in V.
\end{aligned}$$

We define the function $f\colon (0, T) \to E^*$ by

$$\langle f(t), v\rangle_{E^*\times E} = \langle f_0(t), v\rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle f_2(t), v\rangle_{L^2(\Gamma_N;\mathbb{R}^d)}$$

for all $v \in E$, a.e. $t \in (0, T)$ and functionals $J_\tau : L^2(\Gamma_C; \mathbb{R}^d)$, $J\colon L^2(\Gamma_C) \to \mathbb{R}$ by

$$\begin{aligned}
&J_\tau(v) = \int_{\Gamma_C} j_\tau(v)\, d\Gamma \quad \text{for } v \in L^2(\Gamma_C; \mathbb{R}^d)\\
&J(\eta) = \int_{\Gamma_C} j(\eta)\, d\Gamma \quad \text{for } \eta \in L^2(\Gamma_C).
\end{aligned}$$

We need additional hypothesis on the data.

$\underline{H_0}$: $g \in \mathcal{V}^*$, $u_0 \in E$, $u_1 \in H$, $\theta_0 \in V$.

Now, we formulate the following lemmas concerning the properties of the above operators and functionals.

Lemma 4. *Under hypothesis $H(\mathcal{A})$, operator $A\colon (0, T) \times E \to E^*$ satisfies*

(a) *$A(\cdot, v)$ is measurable on $(0, T)$ for all $v \in E$;*
(b) *$A(t, \cdot)$ is strongly monotone for a.e. $t \in (0, T)$, i.e. $\langle A(t, v) - A(t, u), v - u\rangle_E \ge m_{\mathcal{A}}\|v - u\|_E^2$ for all $u, v \in E$, a.e. $t \in (0, T)$;*
(c) *$\|A(t, v)\|_{E^*} \le \widetilde{a}_0(t) + \widetilde{a}_1\|v\|_E$ for all $v \in V$, a.e. $t \in (0, T)$ with $\widetilde{a}_0 \in L^2(0, T)$, $\widetilde{a}_0 \ge 0$ and $\widetilde{a}_1 > 0$, where $\widetilde{a}_0(t) = \sqrt{2}\,\|a_0(t)\|_{L^2(\Omega)}$ and $\widetilde{a}_1 = \sqrt{2}\,a_1$;*
(d) *$\langle A(t, v), v\rangle_E \ge \alpha_{\mathcal{A}}\|v\|_E^2$ for all $v \in E$, a.e. $t \in (0, T)$;*
(e) *$A(t, \cdot)$ is pseudomonotone for a.e. $t \in (0, T)$;*
(f) *$A(t, \cdot)$ is Lipschitz continuous for a.e. $t \in (0, T)$, i.e., $\|A(t, v_1) - A(t, v_2)\|_{E^*} \le L_{\mathcal{A}}\|v_1 - v_2\|_E$ for all $v_1, v_2 \in E$, a.e. $t \in (0, T)$.*

Lemma 5. *Under hypotheses $H(\mathcal{B})$ and $H(C)$, operators $B\colon E \to E^*$, $C_1 : V \to E^*$ and $C_3 : E \to V^*$ satisfy*

(a) *$B \in \mathcal{L}(E, E^*)$;*
(b) *$C_1 \in \mathcal{L}(V, E^*)$;*
(c) *$C_3 \in \mathcal{L}(E, V^*)$.*

Lemma 6. *Under hypothesis $H(K)$, the operator $C_2 : (0, T) \times V \to V^*$ satisfies*

(a) *$C_2(\cdot, \theta)$ is measurable on $(0, T)$ for all $\theta \in V$;*
(b) *$C_2(t, \cdot)$ is strongly monotone for a.e. $t \in (0, T)$, i.e.,*
$\langle C_2(t, \theta_1) - C_2(t, \theta_2), \theta_1 - \theta_2\rangle_V \ge m_K\|\theta_1 - \theta_2\|_V^2$ for all $\theta_1, \theta_2 \in V$;
(c) *$\|C_2(t, \theta)\|_{V^*} \le \widetilde{k}_0(t) + \widetilde{k}_1\|\theta\|_V$ for all $\theta \in V$, a.e. $t \in (0, T)$ with $\widetilde{k}_0 \in L^2(0, T)$, $\widetilde{k}_0 \ge 0$ and $\widetilde{k}_1 > 0$;*
(d) *$\langle C_2(t, \theta), \theta\rangle_V \ge \alpha_K\, \|\theta\|_V^2$ for all $\theta \in V$, a.e. $t \in (0, T)$;*
(e) *$C_2(t, \cdot)$ is pseudomonotone for a.e. $t \in (0, T)$;*
(f) *$C(t, \cdot)$ is Lipschitz continuous for a.e. $t \in (0, T)$, i.e. $\|C(t, \theta_1) - C(t, \theta_2)\|_{V^*} \le L_K\|\theta_1 - \theta_2\|_E$ for all $\theta_1, \theta_2 \in V$, a.e. $t \in (0, T)$.*

Proceeding in a standard way, we obtain the following variational formulation of Problem 2.

Problem 7. Find the displacement field $u \in \mathcal{E}$ with $\dot{u} \in \mathbb{E}$, the temperature $\theta \in \mathcal{W}$ and $\xi \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$, $\zeta \in L^2(0,T;L^2(\Gamma_C))$ such that

$$\langle \ddot{u}(t) + A(t,\dot{u}(t)) + Bu(t) + C_1\theta(t) - f(t), v\rangle_E = \int_{\Gamma_C} \xi(t)\cdot v_\tau \, d\Gamma$$
$$\text{for all } v \in E, \text{ a.e. } t \in (0,T), \tag{11}$$
$$\langle \dot{\theta}(t) + C_2(t,\theta(t)) + C_3\dot{u}(t) - g(t), \eta\rangle_V = \int_{\Gamma_C} \zeta(t)\eta \, d\Gamma$$
$$+ \int_{\Gamma_C} h_\tau(t, \|\dot{u}_\tau(t)\|_{\mathbb{R}^d})\eta \, d\Gamma \quad \text{for all } \eta \in V, \text{ a.e. } t \in (0,T), \tag{12}$$
$$-\xi \in \partial j_\tau(\dot{u}_\tau), \qquad -\zeta \in \partial j(\theta) \quad \text{on } \Gamma_C \times (0,T), \tag{13}$$
$$u(0) = u_0, \qquad \dot{u}(0) = u_1, \qquad \theta(0) = \theta_0.$$

Next we consider an auxiliary problem.

Problem 8. Find displacement field $u \in \mathcal{E}$ with $\dot{u} \in \mathbb{E}$ and the temperature $\theta \in \mathcal{W}$ such that

$$\ddot{u}(t) + A(t,\dot{u}(t)) + Bu(t) + C_1\theta(t) + \gamma_Z^*\partial J_\tau(\gamma_Z\dot{u}_\tau(t)) \ni f(t) \quad \text{for a.e. } t \in (0,T),$$
$$\dot{\theta}(t) + C_2(t,\theta(t)) + C_3\dot{u}(t) + \gamma_Y^*\partial J(\gamma_Y\theta(t)) \ni g(t) \quad \text{for a.e. } t \in (0,T),$$
$$u(0) = u_0, \qquad \dot{u}(0) = u_1, \qquad \theta(0) = \theta_0.$$

We complete this section with a result on existence and uniqueness of solution to Problem 8.

Theorem 9. *Assume $H(\mathcal{A})$, $H(\mathcal{B})$, $H(C)$, $H(f)$, $H(K)$, $H(j_\tau)$, $H(j)$, H_0 and the following conditions*

$$\text{either } j_\tau(x,t,\cdot) \quad \text{or} \quad -j_\tau(x,t,\cdot) \text{ is regular}, \tag{14}$$
$$\text{either } j(x,t,\cdot) \quad \text{or} \quad -j(x,t,\cdot) \text{ is regular}, \tag{15}$$

$$m_{\mathcal{A}} \geq m_\tau \, c_Z^2 \, \|\gamma_Z\|^2_{\mathcal{L}(E,Z)}, \tag{16}$$
$$m_K \geq m_\theta \, c_Y^2 \, \|\gamma_Y\|^2_{\mathcal{L}(V,Y)}, \tag{17}$$
$$\alpha_K > c_\theta \, c_Y^2 \, \|\gamma_Y\|^2_{\mathcal{L}(V,Y)}. \tag{18}$$

Then there exists a unique solution to Problem 8.

Note, that regularity of functions imposed by (14) and (15) is understood in the sense of Clarke (see Definition 3.25 of [15]).

Remark 10. Using the basic properties of Clarke subdifferential of integral functionals J_τ and J (see Theorem 3.47 (v) of [15]) it is easy to see that every solution of Problem 8 is also a solution of Problem 7, provided $H(j_\tau)$ and $H(j)$ hold. Moreover, under regularity conditions (14) and (15), both problems are equivalent. In that case, they are also equivalent to a system of two hemivariational inequalities corresponding to Problem P_V of [17] without history dependent term.

In view of Remark 10, the proof of Theorem 9 follows the lines of the proof of a more general result, Theorem 9 in [17], that deals with the unique solvability of Problem P_V of [17].

4. Spatially semidiscrete error estimates

In this section we consider a spatially semidiscrete approximation of Problem 7 and examine the error between its solution and the solution of the approximate problem.

Let V^h and E^h be finite dimensional subspaces of V and E, respectively, where $h > 0$ denotes a spatial discretization parameter. Let $u_0^h, u_1^h \in E^h$ and $\theta_0^h \in V^h$ be suitable approximations of u_0, u_1, θ_0 characterized by

$$\langle u_0^h - u_0, v^h\rangle_E = 0, \qquad \langle u_1^h - u_1, v^h\rangle_H = 0, \qquad \langle \theta_0^h - \theta_0, \eta^h\rangle_{L^2(\Omega)} = 0 \tag{19}$$

for all $v^h \in E^h$, $\eta^h \in V^h$. It is easy to observe, that

$$\|u_0^h\|_E \leq \|u_0\|_E, \qquad \|u_1^h\|_H \leq \|u_1\|_H, \qquad \|\theta_0^h\|_{L^2(\Omega)} \leq \|\theta_0\|_{L^2(\Omega)}.$$

Then we have the following semidiscrete approximation of Problem 7.

Problem 11. Find displacement field $u^h \in L^2(0,T;E^h)$, $\dot u^h, \ddot u^h \in L^2(0,T;E^h)$, the temperature $\theta^h \in L^2(0,T;V^h)$ and $\xi^h \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$, $\zeta^h \in L^2(0,T;L^2(\Gamma_C))$ such that

$$\langle \ddot u^h(t) + A(t,\dot u^h(t)) + Bu^h(t) + C_1\theta^h(t) - f(t), v^h\rangle_E = \int_{\Gamma_C} \xi^h(t)\cdot v^h_\tau \, d\Gamma$$
$$\text{for all } v^h \in E^h,\ \text{a.e. } t\in(0,T), \tag{20}$$
$$\langle \dot\theta^h(t) + C_2(t,\theta^h(t)) + C_3\dot u^h(t) - g(t), \eta^h\rangle_V = \int_{\Gamma_C} \zeta^h(t)\eta^h\, d\Gamma$$
$$+ \int_{\Gamma_C} h_\tau(t, \|\dot u^h_\tau(t)\|_{\mathbb{R}^d})\eta^h\, d\Gamma \quad \text{for all } \eta^h\in V^h,\ \text{a.e. } t\in(0,T), \tag{21}$$
$$-\xi^h \in \partial j_\tau(\dot u^h_\tau), \qquad -\zeta^h \in \partial j(\theta^h) \quad \text{on } \Gamma_C\times(0,T), \tag{22}$$
$$u^h(0) = u^h_0, \qquad \dot u^h(0) = u^h_1, \qquad \theta^h(0) = \theta^h_0.$$

Under the assumptions of Theorem 9 we have the existence and uniqueness of a solution to Problem 11.
We provide a result on the error estimate between the solutions to Problem 7 and Problem 11.

Theorem 12. *Assume $H(\mathcal{A})$, $H(\mathcal{B})$, $H(C)$, $H(K)$, $H(f)$, H_0 and (18) hold. Suppose, moreover, that*

$$c_1 := m_{\mathcal{A}} - \left(m_\tau + \frac{1}{2}L_\tau\right) c_Z^2 \,\|\gamma_Z\|^2_{\mathcal{L}(E,Z)} \ge 0, \tag{23}$$
$$c_2 := m_K - \left(m_\theta + \frac{1}{2}L_\tau\right) c_Y^2 \,\|\gamma_Y\|^2_{\mathcal{L}(V,Y)} \ge 0. \tag{24}$$

Let u,θ and u^h,θ^h be solutions to Problem 7 and Problem 11, respectively. Then there exists a positive constant c depending only on the data of the problem, such that for any $v^h\in L^2(0,T;E^h)\cap\mathbb{E}$, $\eta^h\in L^2(0,T;V^h)\cap\mathcal{W}$, we have

$$\begin{aligned}
&\|u-u^h\|^2_{C(0,T;E)} + \|\dot u-\dot u^h\|^2_{C(0,T;H)} + \|\dot u-\dot u^h\|^2_{\mathcal{E}} + \|\theta-\theta^h\|^2_{C(0,T;L^2(\Omega))} + \|\theta-\theta^h\|^2_{\mathcal{V}}\\
&\quad \le c\Big(\|u_0-u_0^h\|^2_E + \|u_1-u_1^h\|_H\|u_1-v^h(0)\|_H + \|\dot u-v^h\|^2_{\mathcal{E}} + \|\ddot u-\dot v^h\|_{\mathcal{E}^*}\\
&\qquad + \|\dot u-v^h\|^2_{C(0,T;H)} + \|\dot u_\tau - v^h_\tau\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))} + \|\theta-\eta^h\|^2_{\mathcal{V}} + \|\theta-\eta^h\|^2_{C(0,T;L^2(\Omega))}\\
&\qquad + \|\theta_0-\theta_0^h\|_{L^2(\Omega)}\|\theta_0-\eta^h(0)\|_{L^2(\Omega)} + \|\theta-\eta^h\|_{L^2(0,T;L^2(\Gamma_C))} + \|\dot\theta-\dot\eta^h\|^2_{\mathcal{V}^*}\Big).
\end{aligned}\tag{25}$$

Proof. Define $w(t) = \dot u(t)$ and $w^h(t) = \dot u^h(t)$. Then, we have

$$u(t) = (Iw)(t) := u_0 + \int_0^t w(s)\,ds,$$
$$u^h(t) = (Iw^h)(t) := u^h_0 + \int_0^t w^h(s)\,ds$$

for $t\in(0,T)$. Using the above notation in (11), (12), (20) and (21), we obtain

$$\begin{aligned}
&\langle \dot w(t) - \dot w^h(t), v^h\rangle_E + \langle A(t,w(t)) - A(t,w^h(t)), v^h\rangle_E\\
&\quad + \langle B(Iw)(t) - B(I^hw^h)(t), v^h\rangle_E + \langle C_1\theta(t) - C_1\theta^h(t), v^h\rangle_E\\
&\quad + \langle \dot\theta(t) - \dot\theta^h(t), \eta^h\rangle_V + \langle C_2(t,\theta(t)) - C_2(t,\theta^h(t)), \eta^h\rangle_V\\
&\quad + \langle C_3 w(t) - C_3 w^h(t), \eta^h\rangle_V + \int_{\Gamma_C} (\xi^h(t) - \xi(t))\cdot v^h_\tau\, d\Gamma\\
&\quad + \int_{\Gamma_C} (\zeta^h(t) - \zeta(t))\eta^h\, d\Gamma + \int_{\Gamma_C} (h_\tau(\|w^h(t)\|_{\mathbb{R}^d}) - h_\tau(\|w(t)\|_{\mathbb{R}^d}))\eta^h\, d\Gamma = 0
\end{aligned}\tag{26}$$

for $t\in(0,T)$ and all $v^h\in E^h$, $\eta^h\in V^h$. Note that

$$\langle \dot w(t) - \dot w^h(t), w(t) - w^h(t)\rangle_E = \frac{1}{2}\frac{d}{dt}\|w(t) - w^h(t)\|^2_H, \tag{27}$$
$$\langle \dot\theta(t) - \dot\theta^h(t), \theta(t) - \theta^h(t)\rangle_V = \frac{1}{2}\frac{d}{dt}\|\theta - \theta^h(t)\|^2_{L^2(\Omega)} \tag{28}$$

for $t \in (0, T)$. Moreover, by Lemma 4(b), Lemma 6(b) and the assumptions $H(j_\tau)$(d) and $H(j)$(d) and $H(h_\tau)$(b), we have

$$\langle A(t, w(t)) - A(t, w^h(t)), w(t) - w^h(t)\rangle_E \ge m_A \|w(t) - w^h(t)\|_E^2, \tag{29}$$

$$\langle C_2(t, \theta(t)) - C_2(t, \theta^h(t)), \theta(t) - \theta^h(t)\rangle_E \ge m_K \|\theta(t) - \theta^h(t)\|_V^2, \tag{30}$$

$$\langle B(Iw)(t) - B(I^h w^h)(t), w(t) - w^h(t)\rangle_E = \frac{1}{2}\frac{d}{dt}\langle Bu(t) - Bu^h(t), u(t) - u^h(t)\rangle_E, \tag{31}$$

$$\int_{\Gamma_C} (\xi^h(t) - \xi(t)) \cdot (w_\tau(t) - w_\tau^h(t))\, d\Gamma \ge -m_\tau c_Z^2 \|\gamma_Z\|^2_{\mathcal{L}(E,Z)} \|w(t) - w^h(t)\|_E^2, \tag{32}$$

$$\int_{\Gamma_C} (\zeta^h(t) - \zeta(t))(\theta(t) - \theta^h(t))\, d\Gamma \ge -m_\theta c_Y^2 \|\gamma_Y\|^2_{\mathcal{L}(V,Y)} \|\theta(t) - \theta^h(t)\|_V^2, \tag{33}$$

$$\begin{aligned}
&\int_{\Gamma_C} (h_\tau(\|w^h(t)\|_{\mathbb{R}^d}) - h_\tau(\|w(t)\|_{\mathbb{R}^d}))(\theta(t) - \theta^h(t))\, d\Gamma \\
&\quad \ge -L_\tau \|w^h(t) - w(t)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \|\theta^h(t) - \theta(t)\|_{L^2(\Gamma_C)} \\
&\quad \ge -\frac{1}{2} L_\tau c_Z^2 \|\gamma_Z\|^2_{\mathcal{L}(E,Z)} \|w^h(t) - w(t)\|_E^2 - \frac{1}{2} L_\tau c_Y^2 \|\gamma_Y\|^2_{\mathcal{L}(V,Y)} \|\theta^h(t) - \theta(t)\|_V^2.
\end{aligned} \tag{34}$$

Let the constants c_1 and c_2 be defined by (23) and (24). Taking into account (26)–(33), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t) - w^h(t)\|_H^2 + c_1\|w(t) - w^h(t)\|_E^2 + \frac{1}{2}\frac{d}{dt}\|\theta(t) - \theta^h(t)\|^2_{L^2(\Omega)} \\
&\qquad + c_2\|\theta(t) - \theta^h(t)\|_V^2 + \frac{1}{2}\frac{d}{dt}\langle Bu(t) - Bu^h(t), u(t) - u^h(t)\rangle_E \\
&\quad \le \langle \dot w(t) - \dot w^h(t), w(t) - w^h(t)\rangle_E + \langle A(t, w(t)) - A(t, w^h(t)), w(t) - w^h(t)\rangle_E \\
&\qquad + \langle B(Iw)(t) - B(I^h w^h)(t), w(t) - w^h(t)\rangle_E + \langle \dot\theta(t) - \dot\theta^h(t), \theta(t) - \theta^h(t)\rangle_V \\
&\qquad + \langle C_1\theta(t) - C_1\theta^h(t), w(t) - w^h(t)\rangle_V + \langle C_2(t, \theta(t)) - C_2(t, \theta^h(t)), \theta(t) - \theta^h(t)\rangle_V \\
&\qquad + \langle C_3 w(t) - w^h(t), \theta(t) - \theta^h(t)\rangle_V + \int_{\Gamma_C} (\zeta^h(t) - \zeta(t))(\theta(t) - \theta^h(t))\, d\Gamma \\
&\qquad + \int_{\Gamma_C} (\xi^h(t) - \xi(t)) \cdot (w_\tau(t) - w_\tau^h(t))\, d\Gamma + \int_{\Gamma_C} (h_\tau(\|w^h(t)\|_{\mathbb{R}^d}) - h_\tau(\|w(t)\|_{\mathbb{R}^d}))(\theta(t) - \theta^h(t))\, d\Gamma \\
&\quad = \langle \dot w(t) - \dot w^h(t), w(t) - v^h(t)\rangle_E + \langle A(t, w(t)) - A(t, w^h(t)), w(t) - v^h(t)\rangle_E \\
&\qquad + \langle B(Iw)(t) - B(I^h w^h)(t), w(t) - v^h(t)\rangle_E + \langle \dot\theta(t) - \dot\theta^h(t), \theta(t) - \eta^h(t)\rangle_V \\
&\qquad + \langle C_1\theta(t) - C_1\theta^h(t), w(t) - v^h(t)\rangle_E + \langle C_2(t, \theta(t)) - C_2(t, \theta^h(t)), \theta(t) - \eta^h(t)\rangle_V \\
&\qquad + \langle C_3 w(t) - w^h(t), \theta(t) - \eta^h(t)\rangle_V + \int_{\Gamma_C} (\zeta^h(t) - \zeta(t))(\theta(t) - \eta^h(t))\, d\Gamma \\
&\qquad + \int_{\Gamma_C} (\xi^h(t) - \xi(t)) \cdot (w_\tau(t) - v_\tau^h(t))\, d\Gamma + \int_{\Gamma_C} (h_\tau(\|w^h(t)\|_{\mathbb{R}^d}) - h_\tau(\|w(t)\|_{\mathbb{R}^d}))(\theta(t) - \eta^h(t))\, d\Gamma,
\end{aligned} \tag{35}$$

where $v^h \in E^h$, $\eta^h \in V^h$. For $t \in (0, T)$, assuming $v^h \in H^1(0, T; E)$, we integrate by parts

$$\begin{aligned}
&\int_0^t \langle \dot w(s) - \dot w^h(s), w(s) - v^h(s)\rangle_E\, ds = \langle w(t) - w^h(t), w(t) - v^h(t)\rangle_H \\
&\quad - \langle w(0) - w^h(0), w(0) - v^h(0)\rangle_H - \int_0^t \langle w(s) - w^h(s), \dot w(s) - \dot v^h(s)\rangle_E\, ds.
\end{aligned} \tag{36}$$

Thus, for $\varepsilon > 0$, we obtain

$$\begin{aligned}
&\int_0^t \langle \dot w(s) - \dot w^h(s), w(s) - v^h(s)\rangle_E\, ds \le \frac{1}{4}\|w(t) - w^h(t)\|_H^2 + \|w(t) - v^h(t)\|_H^2 \\
&\quad + \|u_1 - u_1^h\|_H \|u_1 - v^h(0)\|_H + \varepsilon \int_0^t \|w(s) - w^h(s)\|_E^2\, ds + \frac{1}{4\varepsilon}\|\dot w - \dot v^h\|^2_{\mathcal{E}^*}.
\end{aligned} \tag{37}$$

Similarly, for $\varepsilon > 0$, $\eta^h \in H^1(0,T;V)$ and $t \in (0,T)$, we have

$$\int_0^t \langle \dot\theta(s) - \dot\theta^h(s), \theta(s) - \eta^h(s)\rangle_V\, ds \le \frac{1}{4}\|\theta(t) - \theta^h(t)\|^2_{L^2(\Omega)} + \|\theta(t) - \eta^h(t)\|^2_{L^2(\Omega)}$$
$$+\ \|\theta_0 - \theta_0^h\|_{L^2(\Omega)}\|\theta_0 - \eta^h(0)\|_{L^2(\Omega)} + \varepsilon\int_0^t \|\theta(s) - \theta^h(s)\|_V^2\, ds + \frac{1}{4\varepsilon}\|\dot\theta - \dot\eta^h\|^2_{\mathcal{V}^*}. \tag{38}$$

Using the Lipschitz properties of A and C_2, we find that

$$\int_0^t \langle A(s,w(s)) - A(s,w^h(s)), w(s) - v^h(s)\rangle_E\, ds \le \varepsilon\int_0^t \|w(s) - w^h(s)\|_E^2\, ds + \frac{L_{\mathcal{A}}^2}{4\varepsilon}\|w - v^h\|^2_{\mathcal{E}}, \tag{39}$$

$$\int_0^t \langle C_2(s,\theta(s)) - C_2(s,\theta^h(s)), \theta(s) - \eta^h(s)\rangle_V\, ds \le \varepsilon\int_0^t \|\theta(s) - \theta^h(s)\|_V^2\, ds + \frac{L_K^2}{4\varepsilon}\|\theta - \eta^h\|^2_{\mathcal{V}}. \tag{40}$$

Using the properties of B, and the classical inequality $\|u(t) - u^h(t)\|_E \le \|u_0 - u_0^h\|_E + \int_0^t \|w(s) - w^h(s)\|_E\, ds$, we deduce

$$\int_0^t \langle B(Iw)(s) - B(I^h w^h)(s), w(s) - v^h(s)\rangle_E\, ds \le \varepsilon\int_0^t \|u(s) - u^h(s)\|_E^2\, ds + \frac{\|B\|^2_{\mathcal{L}(E,E^*)}}{4\varepsilon}\|w - v^h\|^2_{\mathcal{E}}$$
$$\le 2\varepsilon T\|u_0 - u_0^h\|_E^2 + 2\varepsilon T^2\int_0^t \|w(s) - w^h(s)\|_E^2\, ds + \frac{\|B\|^2_{\mathcal{L}(E,E^*)}}{4\varepsilon}\|w - v^h\|^2_{\mathcal{E}}. \tag{41}$$

Since C_1 and C_3 are linear, we have

$$\int_0^t \langle C_1\theta(s) - C_1\theta^h(s), w(s) - v^h(s)\rangle_E\, ds \le \varepsilon\int_0^t \|\theta(s) - \theta^h(s)\|_V^2 + \frac{\|C_1\|^2_{\mathcal{L}(V,E^*)}}{4\varepsilon}\|w - v^h\|^2_{\mathcal{E}}, \tag{42}$$

$$\int_0^t \langle C_3 w(s) - w^h(s), \theta(s) - \eta^h(s)\rangle_V\, ds \le \varepsilon\int_0^t \|w(s) - w^h(s)\|_E^2\, ds + \frac{\|C_3\|^2_{\mathcal{L}(E,V^*)}}{4\varepsilon}\|\theta - \eta^h\|^2_{\mathcal{V}}. \tag{43}$$

Finally, from $H(j_\tau)$(c), $H(j)$(d), $H(h_\tau)$(b), (13) and (22), we have

$$\int_0^t\int_{\Gamma_C} (\xi^h(s) - \xi(s))\cdot(w_\tau(s) - v_\tau^h(s))\, d\Gamma\, ds \le 2c_\tau\sqrt{T\mathrm{meas}(\Gamma_C)}\|w_\tau - v_\tau^h\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))} \tag{44}$$

and

$$\int_0^t\int_{\Gamma_C} (\zeta^h(s) - \zeta(s))(\theta(s) - \eta^h(s))\, d\Gamma\, ds \le m_\theta\int_0^t \|\theta^h(s) - \theta(s)\|_{L^2(\Gamma_C)}\|\theta(s) - \eta^h(s)\|_{L^2(\Gamma_C)}\, ds$$
$$\le \varepsilon\int_0^t \|\theta^h(s) - \theta(s)\|_V^2\, ds + \frac{m_\theta^2 c_Y^4\|\gamma_Y\|^4_{\mathcal{L}(V,Y)}}{4\varepsilon}\|\theta - \eta^h\|^2_{\mathcal{V}}. \tag{45}$$

for $t \in (0,T)$. Moreover, we get

$$\int_0^t\int_{\Gamma_C} (h_\tau(\|w^h(s)\|_{\mathbb{R}^d}) - h_\tau(\|w(s)\|_{\mathbb{R}^d}))(\theta(s) - \eta^h(s))\, d\Gamma\, ds$$
$$\le L_\tau\int_0^t \|w^h(s) - w(s)\|_{L^2(\Gamma_C;\mathbb{R}^d)}\|\theta(s) - \eta^h(s)\|_{L^2(\Gamma_C)}\, ds$$
$$\le \varepsilon\int_0^t \|w^h(s) - w(s)\|_E^2 + \frac{L_\tau^2 c_Y^2\|\gamma_Y\|^2_{\mathcal{L}(V,Y)} c_Z^2\|\gamma_Z\|^2_{\mathcal{L}(E,Z)}}{4\varepsilon}\|\theta - \eta^h\|^2_{\mathcal{V}} \tag{46}$$

for $t \in (0,T)$. Let us denote by r the following quantity

$$r = \|u_0 - u_0^h\|_E^2 + \|u_1 - u_1^h\|_H\|u_1 - v^h(0)\|_H + \|w - v^h\|^2_{\mathcal{E}} + \|\dot w - \dot v^h\|_{\mathcal{E}^*}$$
$$+\ \|w - v^h\|^2_{C(0,T;H)} + \|w_\tau - v_\tau^h\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))} + \|\theta - \eta^h\|^2_{\mathcal{V}}$$
$$+\ \|\theta - \eta^h\|^2_{C(0.T;L^2(\Omega))} + \|\theta_0 - \theta_0^h\|_{L^2(\Omega)}\|\theta_0 - \eta^h(0)\|_{L^2(\Omega)} + \|\dot\theta - \dot\eta^h\|^2_{\mathcal{V}^*}.$$

We integrate (35) over $(0, t)$, use $H(\mathcal{B})$(c) and apply (36)–(46) to get

$$\frac{1}{4}\|w(t)-w^h(t)\|_H^2+\frac{1}{4}\|\theta(t)-\theta^h(t)\|_{L^2(\Omega)}^2+\left(c_1-(4+2T^2)\varepsilon\right)\int_0^t \|w(s)-w^h(s)\|_E^2\,ds$$
$$+\ (c_2-4\varepsilon)\int_0^t \|\theta(s)-\theta^h(s)\|_V^2\,ds \le c_3 r \tag{47}$$

for all $t \in (0, T)$ with a constant c_3 depending only on the data of the problem. Since $t \in (0, T)$ is arbitrary, we obtain that

$$\|w-w^h\|_{C(0,T;H)}+\|w-w^h\|_{\mathcal{E}}^2+\|\theta-\theta^h\|_{C(0,T;L^2(\Omega))}+\|\theta-\theta^h\|_{\mathcal{V}}^2 \le c_4 r \tag{48}$$

with $c_4 > 0$. Since

$$\|u-u^h\|_{C(0,T;E)}^2 \le 2\|u_0-u_0^h\|^2+2T\|w-w^h\|_{\mathcal{V}}^2 \le c_4 r \tag{49}$$

with $c_4 > 0$, we obtain the result. $\square$

Corollary 13. *Let the hypotheses of Theorem 12 be satisfied. Assume Ω is a polygon/polyhedral domain, and let $\{V^h\}$ and $\{E^h\}$ be families of linear element spaces, corresponding to a regular family of finite element triangulations of $\overline{\Omega}$ into triangles or tetrahedrons. Let u, θ and u^h, θ^h be solutions of Problems 7 and 11, respectively. Assume $u_0 \in H^2(\Omega;\mathbb{R}^d)$, $u_1 \in H^1(\Omega;\mathbb{R}^d)$, $\theta_0 \in H^2(\Omega)$ and take u_0^h, u_1^h, θ_0^h to be projections of u_0, u_1, and θ_0 characterized by (19). Under regularity conditions*

$$\dot u \in C(0,T;H^2(\Omega;\mathbb{R}^d)), \qquad \ddot u \in L^2(0,T;H^2(\Omega;\mathbb{R}^d)),$$
$$\dot u_\tau \in L^2(0,T;H^2(\Gamma_C;\mathbb{R}^d)), \qquad \dot u_\nu \in L^2(0,T;H^2(\Gamma_C)),$$
$$\theta \in C(0,T;H^2(\Omega))\cap L^2(0,T;H^2(\Gamma_C)), \qquad \dot\theta \in L^2(0,T;H^2(\Omega)),$$

we have the optimal order error estimate

$$\|u-u^h\|_{C(0,T;E)}+\|\dot u-\dot u^h\|_{C(0,T;H)}+\|\dot u-\dot u^h\|_{\mathcal{E}}+\|\theta-\theta^h\|_{C(0,T;L^2(\Omega))}+\|\theta-\theta\|_{\mathcal{V}} \le ch$$

for a constant c independent of h.

Proof. Note that under the stated regularity assumptions, for a.e. $t \in [0, T]$, $\dot u(t)$, $\ddot u(t)$ are continuous on $\overline{\Omega}$, and $\dot u_\tau(t)$ is continuous on Γ_C. Moreover, θ is continuous on $\overline{\Omega}$ and its trace, denoted still by θ is continuous on Γ_C. Let $v^h(t) = \Pi^h \dot u(t) \in E^h$ and $\eta^h(t) = \Pi^h\theta(t) \in V^h$ be the finite element interpolants of $\dot u(t)$, and $\theta(t)$, respectively a.e. $t \in [0, T]$ (we refer to (2.3.29) of [29] for the definition of interpolation operator Π^h). Note that $v_\tau^h(t) = (\Pi^h\dot u(t))_\tau$ is the continuous piecewise linear interpolant of $\dot u_\tau(t)$ on Γ_C. Moreover, $\dot v^h(t)$ is the continuous piecewise linear interpolant of $\ddot u(t)$.

By standard finite element interpolation error estimates, cf. [29], we have the following approximation properties for $t \in (0, T)$

$$\|\dot u(t)-v^h(t)\|_E \le ch\|\dot u(t)\|_{H^2(\Omega;\mathbb{R}^d)}, \qquad \|\theta(t)-\eta^h(t)\|_V \le ch\|\theta(t)\|_{H^2(\Omega)},$$
$$\|\ddot u(t)-\dot v^h(t)\|_{E^*} \le ch\|\ddot u(t)\|_{H^2(\Omega;\mathbb{R}^d)}, \qquad \|\dot\theta(t)-\dot\eta^h(t)\|_{V^*} \le ch\|\dot\theta(t)\|_{H^2(\Omega)},$$
$$\|\dot u(t)-v^h(t)\|_H \le ch^2\|\dot u(t)\|_{H^2(\Omega;\mathbb{R}^d)}, \qquad \|\theta(t)-\eta^h(t)\|_{L^2(\Omega)} \le ch^2\|\theta(t)\|_{H^2(\Omega)},$$
$$\|\dot u_\tau(t)-v_\tau^h(t)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le ch^2\|\dot u_\tau(t)\|_{H^2(\Gamma_C;\mathbb{R}^d)} \qquad \|\theta(t)-\eta^h(t)\|_{L^2(\Gamma_C)} \le ch^2\|\theta(t)\|_{H^2(\Gamma_C)},$$

and

$$\|u_0-u_0^h\|_E \le ch\|u_0\|_{H^2(\Omega;\mathbb{R}^d)}, \qquad \|u_1-u_1^h\|_H \le ch\|u_1\|_{H^1(\Omega;\mathbb{R}^d)},$$
$$\|\theta_0-\theta_0^h\|_V \le ch\|\theta_0\|_{H^2(\Omega)}.$$

It follows that

$$\|\dot u-v^h\|_{\mathcal{V}} \le ch\|\dot u\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))},$$
$$\|\ddot u-\dot v^h\|_{\mathcal{E}^*} \le ch\|\ddot u\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^d))},$$
$$\|\dot u-v^h\|_{C(0,T;H)} \le ch^2\|\dot u\|_{C(0,T;H^2(\Omega;\mathbb{R}^d))}$$
$$\|\dot u_\tau-v_\tau^h\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))} \le ch^2\|\dot u_\tau\|_{L^2(0,T;H^2(\Gamma_C;\mathbb{R}^d))}.$$
$$\|\theta-\eta^h\|_{\mathcal{V}} \le ch\|\theta\|_{L^2(0,T;H^2(\Omega))},$$
$$\|\dot\theta-\dot\eta^h\|_{\mathcal{V}^*} \le ch\|\dot\theta\|_{L^2(0,T;H^2(\Omega))},$$
$$\|\theta-\eta^h\|_{C(0,T;L^2(\Omega))} \le ch^2\|\theta\|_{C(0,T;H^2(\Omega))},$$
$$\|\theta-\eta^h\|_{L^2(0,T;L^2(\Gamma_C))} \le ch^2\|\theta\|_{L^2(0,T;H^2(\Gamma_C))}.$$

Then the error bound follows from Theorem 12. $\square$

5. Fully discrete error estimates

In this section we introduce a fully discrete approximation of Problem 7. We need to impose the following regularity assumptions

$$A(\cdot, v) \in C(0, T; E^*) \quad \text{for all } v \in E, \tag{50}$$

$$C_2(\cdot, \eta) \in C(0, T; V^*) \quad \text{for all } \eta \in V, \tag{51}$$

$$f \in C(0, T; E^*), \quad g \in C(0, T; V^*). \tag{52}$$

In addition to spatial discretization, we need temporal discretization. We define a uniform partition of $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. Let $k = T/N$ be a time step size and for a continuous function h we denote $h_n = h(t_n)$. For a sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ for $n = 1, \ldots, N$ the backward divided difference. The fully discrete approximation of the problem is the following.

Problem 14. Find a velocity field $\{w_n^{hk}\}_{n=0}^N \subset E^h$, a temperature $\{\theta_n^{hk}\}_{n=0}^N \subset V^h$ and $\{\xi_n^{hk}\}_{n=0}^N \subset L^2(\Gamma_C; \mathbb{R}^d)$, $\{\zeta_n^{hk}\}_{n=0}^N \subset L^2(\Gamma_C)$ such that for $n = 0, \ldots, N$, we have

$$\langle \delta w_n^{hk} + A_n(w_n^{hk}) + Bu_n^{hk} + C_1\theta_n^{hk} - f_n, v^h\rangle_E = \int_{\Gamma_C} \xi_n^{hk} \cdot v_\tau^h \, d\Gamma \quad \text{for all } v^h \in E^h, \tag{53}$$

$$\langle \delta\theta_n^{hk} + C_{2n}(\theta_n^{hk}) + C_3 w_n^{hk} - g_n, \eta^h\rangle_V = \int_{\Gamma_C} \zeta_n^{hk}\, \eta^h \, d\Gamma + \int_{\Gamma_C} h_{\tau n}(\|w_n^{hk}\|_{\mathbb{R}^d})\eta^h \, d\Gamma \quad \text{for all } \eta^h \in V^h, \tag{54}$$

$$\xi_n^{hk} \in \partial j_\tau(w_{n\tau}^{hk}), \qquad -\zeta_n^{hk} \in \partial j(\theta_n^{hk}), \tag{55}$$

$$w_0^{hk} = u_1^h, \qquad \theta_0^{hk} = \theta_0^h, \tag{56}$$

where $u_0^{hk} = u_0^h$ and $\{u_n^{hk}\}_{n=1}^N \subset E^h$ is given by

$$u_n^{hk} = u_0^h + \sum_{j=1}^n k w_j^{hk} \quad \text{for } n = 1, \ldots, N. \tag{57}$$

Under assumptions stated in Theorem 9, from [17], it follows that there exists a solution to Problem 14.

Now we pass to a fully discrete error estimate result. In what follows, we assume, that u and θ solve Problem 7 and define $w(t) = \dot{u}(t)$ for $t \in [0, T]$. Let $\{w_n^{hk}\}_{n=0}^N \subset E^h$, $\{\theta_n^{hk}\}_{n=0}^N \subset V^h$ be a solution of Problem 14. We formulate the following theorem.

Theorem 15. *Let the assumptions $H(\mathcal{A})$, $H(\mathcal{B})$, $H(C)$, $H(K)$, $H(j_\tau)$, $H(j)$, $H(f)$, H_0, (16)–(18) and (50)–(52) hold. Moreover, we impose the following additional regularity conditions $u \in C^2(0, T; H) \cap C^1(0, T; E)$, $\dot{u}_\tau \in C(0, T; L^2(\Gamma_C; \mathbb{R}^d))$, $\theta \in C(0, T; L^2(\Gamma_C))$. Then, there exist $c > 0$ such that for all $\{v_n^h\}_{=1}^N \subset E^h$ and $\{\eta_n^h\}_{n=1}^N \subset V^h$ we have*

$$\begin{aligned}
&\max_{1\le n\le N}\{\|w_n - w_n^{hk}\|_H^2 + \|\theta_n - \theta_n^{hk}\|_H^2\} + \sum_{j=1}^N k\|w_j - w_j^{hk}\|_E^2 + \sum_{j=1}^N k\|\theta_j - \theta_j^{hk}\|_V^2 \\
&\le c\Bigg[k\sum_{j=1}^N(\|\dot{w}_j - \delta w_j\|_H^2 + \|w_j - v_j^h\|_E^2) + \max_{1\le n\le N}\|w_{n\tau} - v_{n\tau}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)} \\
&\quad + \frac{1}{k}\sum_{j=1}^{N-1}\|(w_j - v_j^h) - (w_{j+1} - v_{j+1}^h)\|_H^2 + \max_{1\le n\le N}\|w_n - v_n^h\|_H^2 \\
&\quad + \|u_0 - u_0^h\|_E^2 + k^2\|u\|_{H^2(0,T;E)} + \|w_0 - u_1^h\|_H^2 + \|\theta_0 - \theta_0^h\|_{L^2(\Omega)}^2 \\
&\quad + k\sum_{j=1}^N(\|\dot{\theta}_j - \delta\theta_j\|_{L^2(\Omega)} + \|\theta_j - \eta_j^h\|_V^2) + \max_{1\le n\le N}\|\theta_n - \eta_n^h\|_{L^2(\Gamma_C)} \\
&\quad + \frac{1}{k}\sum_{j=1}^{N-1}\|(\theta_j - \eta_j^h) - (\theta_{j+1} - \eta_{j+1}^h)\|_{L^2(\Omega)}^2 + \max_{1\le n\le N}\|\theta_n - \eta_n^h\|_{L^2(\Omega)}\Bigg].
\end{aligned}$$

Proof. Taking $v^h \in E^h$ and $\eta^h \in V^h$ in both Problems 7 and 14, we obtain for $n = 1, \ldots, N$

$$\begin{aligned}
&\langle \dot{w}_n - \delta w_n^{hk}, v^h\rangle_H + \langle A_n(w_n) - A_n(w_n^{hk}), v^h\rangle_E + \langle B(u_n - u_n^{hk}), v^h\rangle_E \\
&\quad + \langle C_1(\theta_n - \theta_n^{hk}), v^h\rangle_E + \int_{\Gamma_C}(\xi_n^{hk} - \xi_n)\cdot v_\tau^h \, d\Gamma
\end{aligned}$$

$$
\begin{aligned}
&+\ \langle \dot{\theta}_n - \delta\theta_n^{hk}, \eta^h\rangle_{L^2(\Omega)} + \langle C_{2n}\theta_n - C_{2n}\theta_n^{hk}, \eta^h\rangle_V + \langle C_3(w_n - w_n^{hk}), \eta^h\rangle_V \\
&+ \int_{\Gamma_C} (\zeta_n^{hk} - \zeta_n)\, \eta^h \, d\Gamma + \int_{\Gamma_C} \big(h_{\tau n}(\|w_n^{hk}\|_{\mathbb{R}^d}) - h_{\tau n}(\|w_n\|_{\mathbb{R}^d})\big)\, \eta^h \, d\Gamma = 0.
\end{aligned}
$$

Thus, for given sequences $\{v_n^h\}_{n=1}^N \subset E^h$ and $\{\eta_n^h\}_{n=1}^N \subset V^h$, we get for $n = 1, \ldots, N$

$$
\begin{aligned}
&\langle \dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\rangle_H + \langle A_n(w_n) - A_n(w_n^{hk}), w_n - w_n^{hk}\rangle_E \\
&\quad + \langle B(u_n - u_n^{hk}), w_n - w_n^{hk}\rangle_E + \langle C_1(\theta_n - \theta_n^{hk}), w_n - w_n^{hk}\rangle_E + \langle \dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\rangle_V \\
&\quad + \langle C_{2n}\theta_n - C_{2n}\theta_n^{hk}, \theta_n - \theta_n^{hk}\rangle_V + \langle C_3(w_n - w_n^{hk}), \theta_n - \theta_n^{hk}\rangle_V \\
&\quad + \int_{\Gamma_C} (\xi_n^{hk} - \xi_n)\cdot(w_{n\tau} - w_{n\tau}^{hk})\, d\Gamma + \int_{\Gamma_C} (\zeta_n^{hk} - \zeta_n)\,(\theta_n - \theta_n^{hk})\, d\Gamma \\
&\quad + \int_{\Gamma_C} \big(h_{\tau n}(\|w_n^{hk}\|_{\mathbb{R}^d}) - h_{\tau n}(\|w_n\|_{\mathbb{R}^d})\big)\,(\theta_n - \theta_n^{hk})\, d\Gamma \\
&= \langle \dot{w}_n - \delta w_n^{hk}, w_n - v_n^h\rangle_H + \langle A_n(w_n) - A_n(w_n^{hk}), w_n - v_n^h\rangle_E \\
&\quad + \langle B(u_n - u_n^{hk}), w_n - v_n^h\rangle_E + \langle C_1(\theta_n - \theta_n^{hk}), w_n - v_n^h\rangle_E + \langle \dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \eta_n^h\rangle_V \\
&\quad + \langle C_{2n}\theta_n - C_{2n}\theta_n^{hk}, \theta_n - \eta_n^h\rangle + \langle C_3(w_n - w_n^{hk}), \theta_n - \eta_n^h\rangle \\
&\quad + \int_{\Gamma_C} (\xi_n^{hk} - \xi_n)\cdot(w_{n\tau} - v_{n\tau}^h)\, d\Gamma + \int_{\Gamma_C} (\zeta_n^{hk} - \zeta_n)\,(\theta_n - \eta_n^h)\, d\Gamma \\
&\quad + \int_{\Gamma_C} \big(h_{\tau n}(\|w_n^{hk}\|_{\mathbb{R}^d}) - h_{\tau n}(\|w_n\|_{\mathbb{R}^d})\big)\,(\theta_n - \eta_n^h)\, d\Gamma.
\end{aligned}
$$

After some reformulation, taking into account Lemma 5(d), we obtain

$$
\begin{aligned}
&\langle \delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\rangle_H + \langle A_n w_n - A_n w_n^{hk}, w_n - w_n^{hk}\rangle_E + \langle \delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\rangle_{L^2(\Omega)} \\
&\quad + \langle C_{2n}\theta_n - C_{2n}\theta_n^{hk}, \theta_n - \theta_n^{hk}\rangle_V + \int_{\Gamma_C} (\xi_n^{hk} - \xi_n)\cdot(w_{n\tau} - w_{n\tau}^{hk})\, d\Gamma \\
&\quad + \int_{\Gamma_C} (\zeta_n^{hk} - \zeta_n)(\theta_n - \theta_n^{hk})\, d\Gamma + \int_{\Gamma_C} \big(h_{\tau n}(\|w_n^{hk}\|_{\mathbb{R}^d}) - h_{\tau n}(\|w_n\|_{\mathbb{R}^d})\big)\,(\theta_n - \theta_n^{hk})\, d\Gamma \\
&= \langle \delta w_n - \delta w_n^{hk}, w_n - v_n^h\rangle_H + \langle \dot{w}_n - \delta w_n, (w_n - v_n^h) + (w_n^{hk} - w_n)\rangle_H \\
&\quad + \langle A_n(w_n) - A_n(w_n^{hk}), w_n - v_n^h\rangle_E + \langle B(u_n - u_n^{hk}), (w_n - v_n^h) + (w_n^{hk} - w_n)\rangle_E \\
&\quad + \langle C_1(\theta_n - \theta_n^{hk}), w_n - v_n^h\rangle_E + \langle \delta\theta_n - \delta\theta_n^{hk}, \theta_n - \eta_n^h\rangle_{L^2(\Omega)} \\
&\quad + \langle \dot{\theta}_n - \delta\theta_n, (\theta_n - \eta_n^h) + (\theta_n^{hk} - \theta_n)\rangle_{L^2(\Omega)} + \langle C_{2n}\theta_n - C_2\theta_n^{hk}, \theta_n - \eta_n^h\rangle_V \\
&\quad + \langle C_3(w_n - w_n^{hk}), \theta_n - \eta_n^h\rangle_V + \int_{\Gamma_C} (\xi_n^{hk} - \xi_n)\cdot(w_{n\tau} - v_{n\tau}^h)\, d\Gamma \\
&\quad + \int_{\Gamma_C} (\zeta_n^{hk} - \zeta_n)(\theta_n - \eta_n^h)\, d\Gamma + \int_{\Gamma_C} \big(h_{\tau n}(\|w_n^{hk}\|_{\mathbb{R}^d}) - h_{\tau n}(\|w_n\|_{\mathbb{R}^d})\big)\,(\theta_n - \eta_n^h)\, d\Gamma.
\end{aligned}
\tag{58}
$$

Using formula $2\langle a - b, a\rangle_H = \|a - b\|_H^2 + \|a\|_H^2 - \|b\|_H^2$ for $a = w_n - w_n^{hk}$ and $b = w_{n-1} - w_{n-1}^{hk}$, we get

$$
\frac{1}{2k}(\|w_n - w_n^{hk}\|_H^2 - \|w_{n-1} - w_{n-1}^{hk}\|_H^2) \le \langle \delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\rangle_H. \tag{59}
$$

Similarly,

$$
\frac{1}{2k}(\|\theta_n - \theta_n^{hk}\|_{|L^2(\Omega)}^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_{L^2(\Omega)}^2) \le \langle \delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\rangle_{L^2(\Omega)}. \tag{60}
$$

Let the constants c_1 and c_2 be defined by (23) and (24). Using Lemma 4(b), Lemma 6(b), $H(j_\tau)$(d), $H(j)$(d), $H(h_\tau)$(b), (22) and (55) and using inequality analogous to (34), we get

$$\begin{aligned}&\langle A_n w_n - A_n w_n^{hk}, w_n - w_n^{hk}\rangle_E + \int_{\Gamma_C} (\xi_n^{hk} - \xi_n)\cdot(w_{n\tau} - w_{n\tau}^{hk})\, d\Gamma\\ &\quad + \langle C_{2n}\theta_n - C_{2n}\theta_n^{hk}, \theta_n - \theta_n^{hk}\rangle_V + \int_{\Gamma_C} (\zeta_n^{hk} - \zeta_n)(\theta_n - \theta_n^{hk})\, d\Gamma\\ &\quad + \int_{\Gamma_C} \left(h_{\tau n}(\|w_n^{hk}\|_{\mathbb{R}^d}) - h_{\tau n}(\|w_n\|_{\mathbb{R}^d})\right)(\theta_n - \theta_n^{hk})\, d\Gamma\\ &\quad \geq c_1\|w_n - w_n^{hk}\|_E^2 + c_2\|\theta_n - \theta_n^{hk}\|_V^2. \end{aligned} \tag{61}$$

Next, from the Lipschitz continuity of A and C_2, we have

$$\langle A_n w_n - A_n w_n^{hk}, w_n - v^h\rangle_E \leq \varepsilon\|w_n - w_n^{hk}\|_E^2 + \frac{L_{\mathcal{A}}^2}{4\varepsilon}\|w_n - v^h\|_E^2, \tag{62}$$

$$\langle C_{2n}\theta_n - C_{2n}\theta_n^{hk}, \theta_n - \eta^h\rangle_V \leq \varepsilon\|\theta_n - \theta_n^{hk}\|_V^2 + \frac{L_K^2}{4\varepsilon}\|\theta_n - \eta^h\|_V^2. \tag{63}$$

From linearity of B, C_1 and C_3, we obtain

$$\begin{aligned}&\langle B(u_n - u_n^{hk}), (w_n - v_n^h) + (w_n^{hk} - w_n)\rangle_E \leq \varepsilon\|w_n - w_n^{hk}\|_E^2\\ &\quad + \frac{\|B\|_{\mathcal{L}(E,E^*)}^2}{4\varepsilon}\|u_n - u_n^{hk}\|_E^2 + \frac{1}{2}\|B\|_{\mathcal{L}(E,E^*)}\left(\|u_n - u_n^{hk}\|_E^2 + \|w_n - v_n^h\|_E^2\right), \end{aligned} \tag{64}$$

$$\langle C_1(\theta_n - \theta_n^{hk}), w_n - v_n^h\rangle_E \leq \varepsilon\|\theta_n - \theta_n^{hk}\|_V^2 + \frac{\|C_1\|_{\mathcal{L}(V,E^*)}^2}{4\varepsilon}\|w_n - v_n^h\|_E^2, \tag{65}$$

$$\langle C_3(w_n - w_n^{hk}), \theta_n - \eta_n^h\rangle_V \leq \varepsilon\|w_n - w_n^{hk}\|_E^2 + \frac{\|C_3\|_{\mathcal{L}(E,V^*)}^2}{4\varepsilon}\|\theta_n - \eta_n^h\|_V^2. \tag{66}$$

Finally, by $H(j_\tau)$(c), $H(j)$(d) and $H(h_\tau)$(b), we get

$$\int_{\Gamma_C} (\xi_n^{hk} - \xi_n)\cdot(w_{n\tau} - v_{n\tau}^h)\, d\Gamma \leq 2c_\tau\sqrt{\mathrm{meas}(\Gamma_C)}\|w_{n\tau} - v_{n\tau}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)}, \tag{67}$$

$$\begin{aligned}&\int_{\Gamma_C} (\zeta_n^{hk} - \zeta_n)(\theta_n - \eta_n^h)\, d\Gamma \leq m_\theta\|\theta_n^{hk} - \theta_n\|_{L^2(\Gamma_C)}\|\theta_n - \eta_n^h\|_{L^2(\Gamma_C)}\\ &\quad \leq \varepsilon\|\theta_n^{hk} - \theta_n\|_V^2 + \frac{m_\theta^2 c_Y^4\|\gamma_Y\|_{\mathcal{L}(V,Y)}^4}{4\varepsilon}\|\theta_n - \eta_n^h\|_V^2, \end{aligned} \tag{68}$$

$$\begin{aligned}&\int_{\Gamma_C} \left(h_{\tau n}(\|w_n^{hk}\|_{\mathbb{R}^d}) - h_{\tau n}(\|w_n\|_{\mathbb{R}^d})\right)(\theta_n - \eta_n^h)\, d\Gamma\\ &\quad \leq \varepsilon\|w_n^{hk} - w_n\|_E^2 + \frac{L_\tau^2 c_Y^2\|\gamma_Y\|_{\mathcal{L}(V,Y)}^2 c_Z^2\|\gamma_Z\|_{\mathcal{L}(E,Z)}^2}{4\varepsilon}\|\theta_n - \eta_n^h\|_V^2. \end{aligned} \tag{69}$$

Applying (59)–(69) in (58), we deduce

$$\begin{aligned}&\frac{1}{2k}(\|w_n - w_n^{hk}\|_H^2 - \|w_{n-1} - w_{n-1}^{hk}\|_H^2) + (c_1 - 4\varepsilon)\|w_n - w_n^{hk}\|_E^2\\ &\quad + \frac{1}{2k}(\|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_{L^2(\Omega)}^2) + (c_2 - 3\varepsilon)\|\theta_n - \theta_n^{hk}\|_V^2\\ &\quad \leq C\Big(\|\dot{w}_n - \delta w_n\|_H^2 + \|w_n - v_n^h\|_E^2 + \|u_n - u_n^{hk}\|_E^2 + \|\dot{\theta}_n - \delta\theta_n\|_{L^2(\Omega)}^2\\ &\quad + \|\theta_n - \eta_n^h\|_V^2 + \|w_{n\tau} - v_{n\tau}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)} + \|\theta_n - \eta_n^h\|_{L^2(\Gamma_C)}^2\Big)\\ &\quad + \langle\delta w_n - \delta w_n^{hk}, w_n - v_n^h\rangle_H + \langle\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \eta_n^h\rangle_{L^2(\Omega)}. \end{aligned}$$

Replacing n by j in the above relation, and summing up from 1 to n, we obtain

$$\begin{aligned}&\|w_n - w_n^{hk}\|_H^2 + 2k(c_1 - 4\varepsilon)\sum_{j=1}^{n}\|w_j - w_j^{hk}\|_E^2 + \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2\\&\quad + 2k(c_2 - 3\varepsilon)\sum_{j=1}^{n}\|\theta_j - \theta_j^{hk}\|_V^2 \le \|w_0 - w_0^{hk}\|_E^2 + \|\theta_0 - \theta_0^{hk}\|_V^2\\&\quad + Ck\sum_{j=1}^{n}\left(\|\dot{w}_j - \delta w_j\|_H^2 + \|w_j - v_j^h\|_E^2 + \|u_j - u_j^{hk}\|_E^2 + \|\dot{\theta}_j - \delta\theta_j\|_{L^2(\Omega)}^2\right)\\&\quad + Ck\sum_{j=1}^{n}\left(\|\theta_j - \eta_j^h\|_V^2 + \|w_{j\tau} - v_{j\tau}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)} + \|\theta_j - \eta_j^h\|_{L^2(\Gamma_C)}^2\right)\\&\quad + 2k\sum_{j=1}^{n}\langle\delta w_j - \delta w_j^{hk}, w_j - v_j^h\rangle_H + 2k\sum_{j=1}^{n}\langle\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h\rangle_{L^2(\Omega)}\end{aligned}$$

for all $\{v_j^h\}_{j=1}^n \subset E^h$ and $\{\eta_j^h\}_{j=1}^n \subset V^h$. We have

$$\begin{aligned}&\sum_{j=1}^{n} k\langle\delta w_j - \delta w_j^{hk}, w_j - v_j^h\rangle_H \le \varepsilon\|w_n - w_n^{hk}\|_H^2 + C\|w_n - v_n^h\|_H^2 + C\|w_0 - w_0^{hk}\|_H^2\\&\quad + C\|w_1 - v_1^h\|_H^2 + \sum_{j=1}^{n-1}\frac{k}{4}\|w_j - w_j^{hk}\|_H^2 + \frac{1}{k}\sum_{j=1}^{n-1}\|(w_j - v_j^h) - (w_{j+1} - v_{j+1}^h)\|_H^2\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{n} k\langle\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h\rangle_{L^2(\Omega)} \le \varepsilon\|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 + C\|\theta_n - \eta_n^h\|_{L^2(\Omega)}^2\\&\quad + C\|\theta_0 - \theta_0^{hk}\|_{L^2(\Omega)}^2 + C\|\theta_1 - \eta_1^h\|_{L^2(\Omega)}^2 + \sum_{j=1}^{n-1}\frac{k}{4}\|\theta_j - \theta_j^{hk}\|_{L^2(\Omega)}^2 + \frac{1}{k}\sum_{j=1}^{n-1}\|(\theta_j - \eta_j^h) - (\theta_{j+1} - \theta_{j+1}^h)\|_{L^2(\Omega)}^2.\end{aligned}$$

Since

$$\|u_j - u_j^{hk}\|_E \le \|u_0 - u_0^h\|_E + \sum_{l=1}^{j} k\|w_l - w_l^{hk}\|_E + I_j, \tag{70}$$

where I_j is the integration error

$$I_j = \left\|\int_0^{t_j} w(s)\,ds - \sum_{l=1}^{j} kw_l\right\|_E.$$

We know that $I_j \le k\|u\|_{H^2(0,T;E)}$. Hence, we get

$$\|u_j - u_j^{hk}\|_E^2 \le C\left(\|u_0 - u_0^{hk}\|_E^2 + j\sum_{l=1}^{j} k^2\|w_l - w_l^{hk}\|_E^2 + k^2\|u\|_{H^2(0,T;E)}^2\right) \tag{71}$$

and using the fact that $Nk = T$, we have the estimate

$$\sum_{j=1}^{n} k\|u_j - u_j^{hk}\|_E^2 \le CT\left(\|u_0 - u_0^h\|_E^2 + k^2\|u\|_{H^2(0,T;E)}^2\right) + T\sum_{j=1}^{n} k\sum_{l=1}^{j} k\|w_l - w_l^{hk}\|_E^2.$$

We denote

$$\begin{aligned}e_n &= \|w_n - w_n^{hk}\|_H^2 + \sum_{j=1}^{n} k\|w_j - w_j^{hk}\|_E^2 + \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 + \sum_{j=1}^{n} k\|\theta_j - \theta_j^{hk}\|_V^2,\\g_n &= k\sum_{j=1}^{n}\left(\|\dot{w}_j - \delta w_j\|_H^2 + \|w_j - v_j^h\|_E^2 + \|\dot{\theta}_j - \delta\theta_j\|_{L^2(\Omega)}^2 + \|\theta_j - \eta_j^h\|_V^2\right)\\&\quad + \frac{1}{k}\sum_{j=1}^{n-1}\left(\|w_j - v_j^h - (w_{j+1} - v_{j+1}^h)\|_H^2 + \|\theta_j - \eta_j^h - (\theta_{j+1} - \theta_{j+1}^{hk})\|_{L^2(\Omega)}^2\right)\end{aligned}$$

$$
\begin{aligned}
&+k\sum_{j=1}^{n}\left(\|w_{j\tau}-v_{j\tau}^{h}\|_{L^2(\Omega;\mathbb{R}^d)}+\|\theta_j-\eta_j^h\|_{L^2(\Gamma_C)}\right)+k^2\|u\|^2_{H^2(0,T;E)}+\|w_0-w_0^{hk}\|^2_E\\
&+\|w_0-w_0^{hk}\|^2_H+\|u_0-u_0^h\|^2_E+\|\theta_0-\theta_0^{hk}\|^2_{L^2(\Omega)}+\|\theta_0-\theta_0^{hk}\|^2_V\\
&+\|\theta_1-\eta_1^h\|^2_{L^2(\Omega)}+\|w_1-v_1^h\|^2_H+\|w_n-v_n^h\|^2_H+\|\theta_n-\eta_n^h\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Then, we find that

$$
e_n\le Cg_n+C\sum_{j=1}^{n}ke_j\quad\text{for } n=1,\ldots,N \tag{72}
$$

with $C>0$. Using Lemma 1, we obtain the thesis. □

Corollary 16. *Assume the hypotheses of Theorem 15. Assume Ω is a polygon/polyhedral domain, and let $\{V^h\}$, $\{E^h\}$ be families of linear element spaces, corresponding to a regular family of finite element triangulations of $\overline{\Omega}$ into triangles or tetrahedrons. Let u,θ be solution to Problem 7 and the corresponding function w be defined as $w=\dot{u}$. Let $\{w_n^{hk}\}_{n=0}^N$, $\{\theta_n^{hk}\}_{n=0}^N$ be solution to Problem 14, and the corresponding sequence $\{u_n^{hk}\}_{n=1}^N$ be defined by (57). Assume $u_0\in H^2(\Omega;\mathbb{R}^d)$, $u_1\in H^1(\Omega;\mathbb{R}^d)$, $\theta_0\in H^1(\Omega)$ and let u_0^h, u_1^h, θ_0^h be defined by (19). Under the regularity conditions*

$$
\begin{aligned}
&u\in C^1(0,T;H^2(\Omega;\mathbb{R}^d))\cap H^3(0,T;H),\qquad \dot{u}_\tau\in C(0,T;H^2(\Gamma_C;\mathbb{R}^d)),\\
&\dot{u}_\nu\in C(0,T;H^2(\Gamma_C)),\qquad \theta\in C^1(0,T;H^2(\Omega))\cap H^3(0,T;L^2(\Omega))
\end{aligned}
$$

we have the optimal order error estimate

$$
\max_{1\le n\le N}\left\{\|u_n-u_n^{hk}\|_E+\|w_n-w_n^{hk}\|_H+\|\theta_n-\theta_n^{hk}\|_{L^2(\Omega)}+k\sum_{j=1}^{n}\|\theta_j-\theta_j^{hk}\|^2_V\right\}\le c(h+k). \tag{73}
$$

Proof. Let $u_j^h\in E^h$, $\theta^h\in V^h$ be the finite element interpolations of u_j, θ_j. Note that

$$
\begin{aligned}
&k\sum_{j=1}^{N}\|\dot{w}_j-\delta w_j\|^2_H\le ck^2\|u\|^2_{H^2(0,T;H)},\\
&k\sum_{j=1}^{N}\|\dot{\theta}_j-\delta\theta_j\|^2_{L^2(\Omega)}\le ck^2\|\theta\|^2_{H^2(0,T;L^2(\Omega))},\\
&\frac{1}{k}\sum_{j=1}^{N-1}\|(w_j-v_j^h)-(w_{j+1}-v_{j+1}^h)\|^2_H\le ch^2\|u\|^2_{H^2(0,T;E)},\\
&\frac{1}{k}\sum_{j=1}^{N-1}\|(\theta_j-\eta_j^h)-(\theta_{j+1}-\eta_{j+1}^h)\|^2_{L^2(\Omega)}\le ch^2\|\theta\|^2_{H^2(0,T;V)}.
\end{aligned}
$$

Then by using similar arguments as in the proof of Corollary 13, we obtain (73). □

6. Numerical simulations

The aim of this Section is to present the numerical strategy used to solve the frictional contact Problem 7, to provide numerical simulations and also to get a numerical evidence of the convergence of the discrete scheme established in Section 5. Note that the different numerical methods have been implemented in a code which is based on Finite Element Library in C++ under the GNU Public license: GEneric Tools for Finite Elements Methods (GETFEM++) developed by Julien Pommier and Yves Renard. For more details, we refer to http://download.gna.org/getfem/html/homepage/.

The numerical solution is based on a iterative procedure which leads to a sequence of convex programming problems already used in [25,30]. For each "convexification" iteration, the value of the friction coefficient $\mu(\|w_\tau||_{\mathbb{R}^d})$ is fixed to a given value depending on the tangential velocity solution w_τ found in the previous iteration. Then, the resulting nonsmooth convex iterative problems are solved by classical numerical methods. Furthermore, the frictional contact conditions are treated by using a numerical approach based on the augmented Lagrangian method. To this end, we consider additional fictitious nodes for the Lagrange multiplier in the initial mesh. The construction of these nodes depends on the contact element used for the geometrical discretization of the interface Γ_C. In the case presented below, the discretization is based on "node-to-rigid" contact element, which is composed by one node of Γ_C and one Lagrange multiplier node. For more details on the discretization step and Computational Contact Mechanics, we refer to [31–34]. The numerical solution of the nonsmooth nonconvex variational Problem 7 is based on the iterative scheme given below.

Let $\epsilon_{conv} > 0$, $w_0^{hk,(0)}$ be given.
For $n = 1, 2 \ldots N$, (time stepping)

For $m = 0, 1 \ldots,$ (convexification loop: sequence of convex problems)

PROBLEM 17_n^m. *Find a velocity field* $\{w_n^{hk,(m+1)}\}_{n=0}^N \subset E^h$, *a temperature* $\{\theta_n^{hk,(m+1)}\}_{n=0}^N \subset V^h$, *a friction stress field* $\{\xi_n^{hk,(m+1)}\}_{n=0}^N \subset L^2(\Gamma_C; \mathbb{R}^d)$ *and a field* $\{\zeta_n^{hk,(m+1)}\}_{n=0}^N \subset L^2(\Gamma_C)$ *such that*

$$\langle \delta w_n^{hk,(m+1)} + A_n(w_n^{hk,(m+1)}) + Bu_n^{hk,(m+1)} + C_1\theta_n^{hk,(m+1)} - f_n, v^h\rangle_E \tag{74}$$

$$= \int_{\Gamma_C} \xi_n^{hk,(m+1)} \cdot v_\tau^h \, d\Gamma$$

for all $v^h \in E^h$,

$$\langle \delta\theta_n^{hk,(m+1)} + C_{2n}(\theta_n^{hk,(m+1)}) + C_3 w_n^{hk,(m+1)} - g_n, \eta^h\rangle_V \tag{75}$$

$$= \int_{\Gamma_C} \zeta_n^{hk,(m+1)} \eta^h \, d\Gamma + \int_{\Gamma_C} h_{\tau n}(\|w_n^{hk,(m+1)}\|_{\mathbb{R}^d})\eta^h \, d\Gamma,$$

for all $\eta^h \in V^h$,

$$\textit{with} \quad -\zeta_n^{hk,(m+1)} \in \partial j(\theta_n^{hk,(m+1)}) \tag{76}$$

$$\textit{and} \quad -\xi_n^{hk,(m+1)} \in \mu(\|\Pi_h w_{n\tau}^{hk,(m)}\|_{\mathbb{R}^d}) S \partial \|\Pi_h w_{n\tau}^{hk,(m+1)}\|_{\mathbb{R}^d} \textit{ on } \Gamma_C,$$

$$\textit{until } \|u_n^{hk,(m+1)} - u_n^{hk,(m)}\|_E \le \epsilon_{conv} \|u_n^{hk,(m)}\|_E, \tag{77}$$

$$\|\theta_n^{hk,(m+1)} - \theta_n^{hk,(m)}\|_V \le \epsilon_{conv} \|\theta_n^{hk,(m)}\|_V, \tag{78}$$

$$\|\xi_n^{hk,(m+1)} - \xi_n^{hk,(m)}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le \epsilon_{conv} \|\xi_n^{hk,(m)}\|_{L^2(\Gamma_C;\mathbb{R}^d)}, \tag{79}$$

$$\textit{and } \|\zeta_n^{hk,(m+1)} - \zeta_n^{hk,(m)}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le \epsilon_{conv} \|\zeta_n^{hk,(m)}\|_{L^2(\Gamma_C)}. \tag{80}$$

Note that $w_n^{hk,(p)}$, $u_n^{hk,(p)}$, $\theta_n^{hk,(p)}$, $\xi_n^{hk,(p)}$ and $\zeta_n^{hk,(p)}$ used in the algorithm above are in fact analogous to the fully discrete unknowns introduced at the beginning of Section 5 with p, the index of the convexification iteration. In particular $u_n^{hk,(p)}$ is defined by relation analogous to (57). We also remind that the symbol δ stands for the backward divided difference defined at the beginning of Section 5. For a given time step n, when the difference between the solution of two consecutive convex problem fulfils the stop-criteria (77)–(80), we obtain the solution corresponding to this time step.

Numerical example. We consider the physical setting depicted in Fig. 1.

There, $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ with $L_1, L_2 > 0$ and

$$\Gamma_D = \{0\} \times [0, L_2], \qquad \Gamma_N = ([0, L_1] \times \{L_2\}) \cup (\{L_1\} \times [0, L_2]), \qquad \Gamma_C = [0, L_1] \times \{0\}.$$

The domain Ω represents the cross section of a three-dimensional linearly viscoelastic body subjected to the action of tractions in such a way that a plane stress hypothesis is valid. On $\Gamma_D = \{0\} \times [0, L_2]$ the body is clamped, i.e. the displacement field vanishes there while on $\Gamma_D \cup \Gamma_N = (\{0\} \times [0, L_2]) \cup ([0, L_1] \times \{L_2\}) \cup (\{L_1\} \times [0, L_2])$, the temperature is fixed. Vertical compressions act on the part $[0, L_1] \times \{L_2\}$ of the boundary and the part $\{L_1\} \times [0, L_2]$ is traction free. The body is in frictional bilateral contact with an obstacle on the part $\Gamma_C = [0, L_1] \times \{0\}$ of the boundary.

The friction follows a nonmonotone law in which the friction coefficient depends on the tangential velocity $\|w_\tau\|_{\mathbb{R}^d}$. For the coefficient of friction, we choose the function $\mu\colon \mathbb{R} \to \mathbb{R}$ of the form

$$\mu(r) = (a - b)\, e^{-\alpha r} + b \quad \text{for } r \in \mathbb{R} \tag{81}$$

with $a, b, \alpha > 0$, $a \ge b$. Note that such a function was used to describe the slip weakening phenomenon which appears in the study of geophysical problems; see [35] for details. Defining $j_\tau : \mathbb{R}^d \to \mathbb{R}$ by

$$j_\tau(\xi) = S \int_0^{\|\xi\|_{\mathbb{R}^d}} \mu(s)\, ds \quad \text{for all } \xi \in \mathbb{R}^d, \tag{82}$$

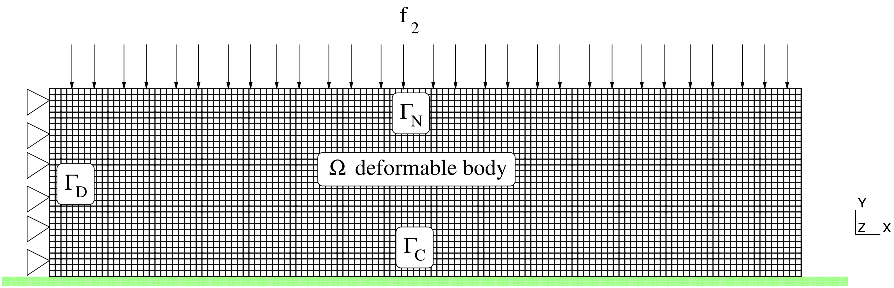

Fig. 1. Reference configuration of the two dimensional example.

we observe that the contact condition (8) reduces to the following one

$$\|\sigma_\tau\|_{\mathbb{R}^d} \le \mu(0)S \quad \text{if } \dot{u}_\tau = 0, \qquad -\sigma_\tau = \mu(\|\dot{u}_\tau\|_{\mathbb{R}^d})S\frac{\dot{u}_\tau}{\|\dot{u}_\tau\|_{\mathbb{R}^d}} \quad \text{if } \dot{u}_\tau \neq 0. \tag{83}$$

We also define $j(x,t,r) = \frac{1}{2}k_e(r-\theta_R)^2$ for $r \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$, k_e being the heat exchange coefficient between the body and the foundation and θ_R being the temperature of the foundation. Then the condition (9) reduces to the equation

$$-\frac{\partial\theta}{\partial\nu_K} = k_e(\theta-\theta_R) - h_\tau(x,t,\|\dot{u}_\tau\|_{\mathbb{R}^d}) \quad \text{on } \Gamma_C \times (0,T)$$

which was studied in [4,8]. As a simple tangential function h_τ in (9), we take

$$h_\tau(x,t,r) = \lambda(x,t)\,r \quad \text{for all } r \in \mathbb{R}_+, \text{ a.e. } (x,t) \in \Sigma_C,$$

where $\lambda \in L^\infty(\Sigma_C)$ represents a time-dependent rate coefficient for the gradient of the temperature.

The compressible material response is governed by a linearly viscoelastic constitutive law in which the viscosity tensor $\mathcal{A}$, the elasticity tensor $\mathcal{B}$ and the thermal expansion tensor C are given by

$$(\mathcal{A}\varepsilon(\dot{u}))_{\alpha\beta} = \mu_1(\varepsilon(\dot{u})_{11} + \varepsilon(\dot{u})_{22})\delta_{\alpha\beta} + \mu_2\varepsilon(\dot{u})_{\alpha\beta}, \quad 1 \le \alpha,\beta \le 2,$$
$$(\mathcal{B}\varepsilon(u))_{\alpha\beta} = \frac{E\kappa}{(1+\kappa)(1-2\kappa)}(\varepsilon(u)_{11} + \varepsilon(u)_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\varepsilon(u)_{\alpha\beta}, \quad 1 \le \alpha,\beta \le 2,$$
$$C = -\alpha_{exp}\frac{E}{1-2\kappa}I,$$

where μ_1 and μ_2 are viscosity constants, E and κ are Young's modulus and Poisson's ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol.

For computation we use the following data:

$$\begin{aligned}
&L_1 = 40\text{ m}, \quad L_2 = 10\text{ m}, \quad \rho = 100\text{ kg/m}^3, \quad T = 1\text{ s},\\
&u_0 = 0\text{ m}, \quad u_1 = 0\text{ m/s}, \quad \theta_0 = 25\,^\circ\text{C},\\
&\mu_1 = 50\text{ N s/m}^2, \quad \mu_2 = 200\text{ N s/m}^2, \quad E = 10\,000\text{ N/m}^2, \quad \kappa = 0.3,\\
&k = 0.5\text{ W/(m }^\circ\text{C)}, \quad c_p = 0.1\text{ J/(kg }^\circ\text{C)}, \quad k_e = 2\text{ W/(m}^2\,^\circ\text{C)},\\
&g = 0\text{ W/m}^3, \quad \theta_R = 0\,^\circ\text{C}, \quad \lambda = 10t\text{ J/m}^3, \quad \alpha_{exp} = 3.10^{-4}\,^\circ\text{C}^{-1},\\
&S = 150\text{ N/m}, \quad f_0 = (0,-10)\text{ N/m}^2, \quad f_2 = \begin{cases} (0,0)\text{ N/m} & \text{on } \{L\}\times[0,L],\\ (0,-800t)\text{ N/m} & \text{on } [0,L]\times\{L\},\end{cases}\\
&a = 1, \quad b = 0.2, \quad \alpha = 100, \quad \epsilon_{conv} = 10^{-6}.
\end{aligned}$$

Our results are presented below.

Mechanical behaviour of the solution. In Fig. 2 we plot the deformed configuration as well as the temperature distribution in the body and the interface forces on Γ_C at the moments $t = 0.25$ s, $t = 0.5$ s, $t = 0.75$ s and $t = 1$ s. At the beginning of the process, most of the nodes are in status of stick (4 of them are in the slip status on the right side of the contact boundary Γ_C at $t = 0.25$ s). During the dynamic process, as the deformation increases, a large proportion of nodes switch to the slip status since the compression of the domain is stronger. Therefore, the friction bound $\mu(\|w_\tau\|_{\mathbb{R}^d})S$, which is, recall a decreasing function with respect to $\|w_\tau\|_{\mathbb{R}^d}$, is reached for these nodes. At $t = 1$ s, we have 30 nodes in the slip status.

Thermal expansion. In order to highlight the influence of the temperature on the model, we plot the deformed meshes and the interface forces on Γ_C for two different values of α_{exp} (see Fig. 3).

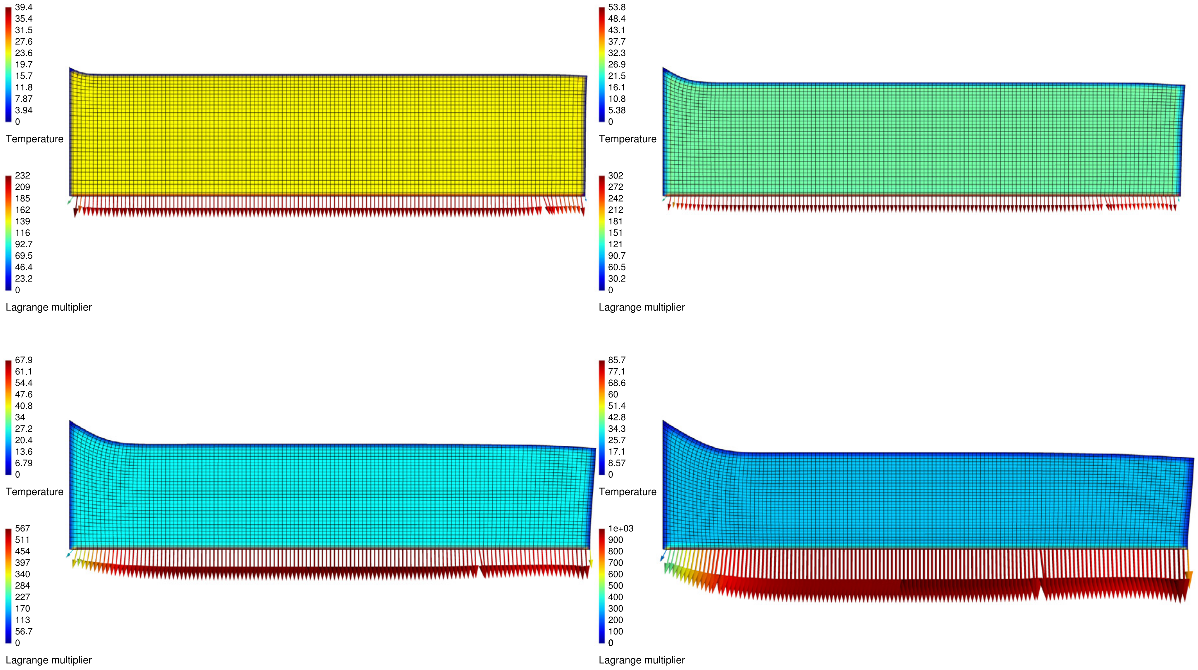

Fig. 2. Evolution of deformed meshes and frictional contact forces during the dynamic process.

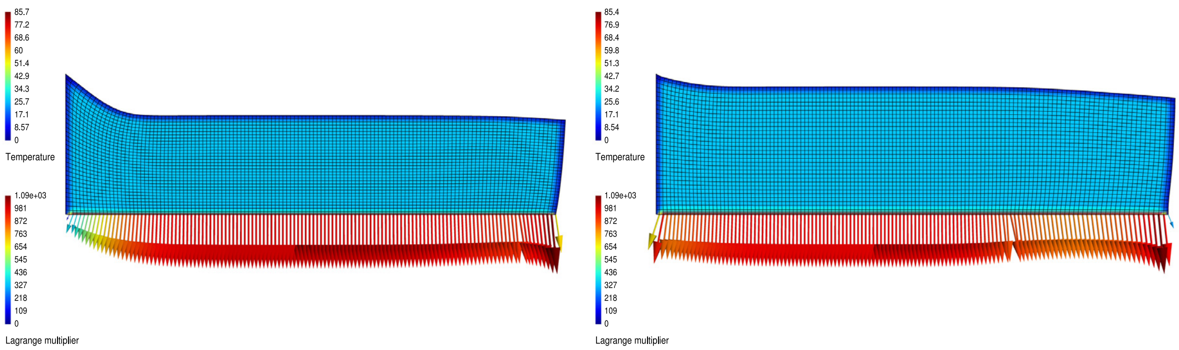

Fig. 3. Deformed meshes and the interface forces for $\alpha_{exp} = 1.10^{-4}$ and $\alpha_{exp} = 1.10^{-3}$ at $t = 1$ s.

Note that in the case $\alpha_{exp} = 1.10^{-4}$, the contribution of the temperature to the deformation of the body is small; therefore the body is compressed by the actions of tractions. However, in the second case, the shape of the body changes greatly because of the difference in temperature. Such a phenomena is particularly visible on the contact boundary: while there are only 5 nodes in slip status in the first case, there are 33 in the second case. It seems consistent since it shows that the slip rate is an increasing function of the thermal expansion on the boundary.

Error estimates. In order to check the convergence of the discrete scheme and to illustrate the optimal error estimate obtained in Section 5, we computed a sequence of numerical solutions by using uniform triangulations of the body according the spatial discretization parameter h and time step k. The numerical estimated error values $E^{hk} = \|u - u^{hk}\|_E$ are computed for several discretization parameters of h and k. Here, the boundary Γ_C of Ω is divided into $4/h$ equal parts; the rest of the boundary is divided accordingly in order to obtain uniform triangulations of the body. We start with $h = 1/2$ and $k = 1/2$ which are successively halved. The numerical solution corresponding to $h = 1/64$ and $k = 1/64$ was taken as the "exact" solution, which was used to compute the errors of the numerical solutions with higher values of h and k. This fine discretization corresponds to a problem with 51 143 degrees of freedom and 32 768 elements at each time level and was computed in 2898 CPU time (expressed in seconds) on a computer equipped with Intel Quad core processors (2.00 GHz). The numerical results are presented in Fig. 4 where the dependence of the error estimate E^{hk} with respect to h and k is plotted.

The curve of the numerical error estimate is asymptotically linear, which is consistent with the theoretically predicted optimal linear convergence of the numerical solution established in Theorem 15.

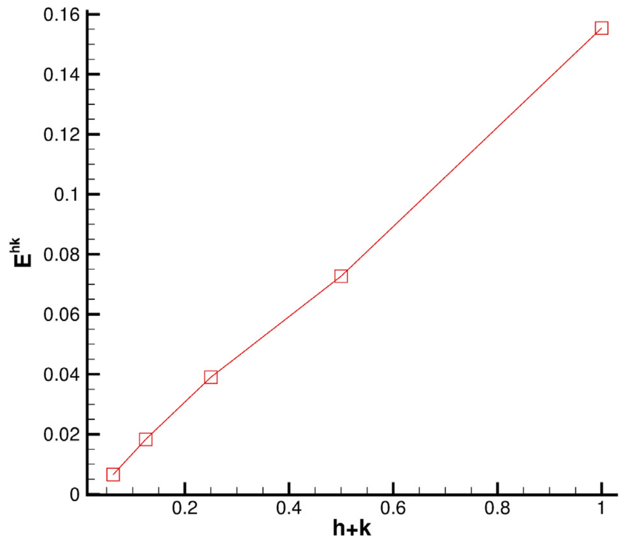

Fig. 4. Numerical errors.

Acknowledgements

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