# Numerical Methods and Analysis via Random Field Based Malliavin Calculus for Backward Stochastic PDEs ${ }^{1}$ 

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#### Abstract

We study the adapted solution, numerical methods, and related convergence analysis for a unified backward stochastic partial differential equation (B-SPDE). The equation is vector-valued, whose drift and diffusion coefficients may involve nonlinear and highorder partial differential operators. Under certain generalized Lipschitz and linear growth conditions, the existence and uniqueness of adapted solution to the B-SPDE are justified. The methods are based on completely discrete schemes in terms of both time and space. The analysis concerning error estimation or rate of convergence of the methods is conducted. The key of the analysis is to develop new theory for random field based Malliavin calculus to prove the existence and uniqueness of adapted solutions to the firstorder and second-order Malliavin derivative based B-SPDEs under random environments.


Key words and phrases: Numerical Method, Error Estimation, Convergence, Random Field Based Malliavin Calculus, Backward SPDE, High-Order Partial Differential Operator, Nonlinear, Random Environment
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## 1 Introduction

In this paper, we study the adapted solution, numerical schemes, and related convergence analysis for a unified backward stochastic partial differential equation (B-SPDE) given by

$$
\begin{equation*}
V(t, x)=H(x)+\int_{t}^{T} \mathcal{L}(s, x, V) d s+\int_{t}^{T}(\mathcal{J}(s, x, V)-\bar{V}(s, x)) d W(s) . \tag{1.1}
\end{equation*}
$$

The equation in (1.1) is vector valued. The nonlinear partial differential operators $\mathcal{L}$ and $\mathcal{J}$ depend not only on $V$ and/or $\bar{V}$ but also on their associated high-order partial derivatives, e.g., up to the $k$ th, $m$ th, and $n$th orders for $k, m, n \in\{0,1,2, \ldots\}$,

$$
\begin{aligned}
\mathcal{L}(s, x, V) & \equiv \mathcal{L}\left(s, x, V(s, x), \ldots, V^{(k)}(s, x), \bar{V}(s, x), \ldots, \bar{V}^{(m)}(s, x)\right), \\
\mathcal{J}(s, x, V) & \equiv \mathcal{J}\left(s, x, V(s, x), \ldots, V^{(n)}(s, x)\right) .
\end{aligned}
$$

[^0]The unified B-SPDE in (1.1) covers many existing systems as special cases. For examples, when $\mathcal{J}=0$ and $\mathcal{L}$ depends only on $x, V, \bar{V}$, but not on their associated partial derivatives, it reduces to a conventional backward (ordinary) stochastic differential equation (BSDE) (see, e.g., Pardoux and Peng [25]); Furthermore, when $\mathcal{J}=0$ and $\mathcal{L}$ depends on both the derivatives of $V$ and $\bar{V}$, it reduces to a well-known example of strongly nonlinear B-SPDE derived in Musiela and Zariphopoulou [19] for the purpose of optimal-utility based portfolio choice. We here note that the strongly nonlinearity concerning the operator $\mathcal{L}$ is in the sense addressed in Lions and Souganidis [17], Pardoux [23]. Besides these existing examples, our motivations to study the B-SPDE in (1.1) are also from optimal portfolio management in finance (see, e.g., Becherer [2], Dai [10, 11], Musiela and Zariphopoulou [19]), and multichannel (or multi-valued) image regularization such as color images in computer vision and network application (see, e.g., Caselles et al. [8], Tschumperlé and Deriche [31, 32, 33]).

Under certain generalized Lipschitz and linear growth conditions, we adopt a method to prove the unified B-SPDE in (1.1) to be well-posed in a suitable functional space. Although the approach is partially embedded in the discussion of unique existence of solution to a more general system with jumps in the preprint of Dai [12]. We refine it here and make it consistent with the system in (1.1) to develop theoretical foundation of random field based Malliavin calculus to conduct convergence analysis and error estimation for our newly designed numerical schemes.

Currently, there are numerous discussions concerning the numerical schemes for resolving SDEs (see, e.g., Kloeden and Platen [16]), SPDEs (see, e.g, Barth and Lang [3], Juan et al. [18]), and BSDEs (see, e.g., Bender and Denk [4], Bouchard and Touzi [7], Bouchard and ELIE [6], Gobet et al. [13], Hu et al. [14], Zhang [36]). Furthermore, there are also numerical techniques available in computing the stationary distributions of reflecting SDEs (see, e.g., Dai [9], Shen et al. [29]). However, to the best of our knowledge, there is no numerical technique available in the literature for B-SPDEs. Thus, in this paper, we make such an attempt to develop some numerical methods for the unified B-SPDE in (1.1).

More precisely, we design two algorithms to compute the adapted solution of the B-SPDE in (1.1). Comparing with most of the existing schemes for BSDEs and considering computer implementation, both of the algorithms are handled with completely discrete schemes in terms of time and space. The first one (named Algorithm 3.1) is an iterative one while the second one (named Algorithm 3.2) is not a purely iterative one since it needs to solve linear or nonlinear equations at each time point. Hence, Algorithm 3.2 is expensive when $x$ is in a higher-dimensional domain. Nevertheless, for the purpose of comparison and for the case that $x$ is in a lower-dimensional domain, Algorithm 3.2 is useful. Owing to the similarity of discussions, the convergence analysis for the algorithms is focused on Algorithm 3.1.

The analysis concerning error estimation or rate of convergence of Algorithm 3.1 is conducted with respect to a completely discrete criterion. Comparing with existing discussions for BSDEs, we need to develop new theory for random field based Malliavin calculus to prove the existence and uniqueness of adapted solutions to related first-order and second-order Malliavin derivative based B-SPDEs under random environments.

The remainder of the paper is organized as follows. In Section 2, we state conditions to guarantee the existence and uniqueness of adapted solution to the B-SPDE in (1.1). In Section 3, we design our numerical schemes and state our main convergence theorem. Related notations of random field based calculus are also introduced. In Section 4, we prove our BSPDE in (1.1) to be well-posed. In Section 5, we develop new theory for random field based Malliavin calculus to provide theoretical foundation in proving our main convergence theorem.

## 2 The Adapted Solution to the B-SPDE: Existence and Uniqueness

### 2.1 Preliminary Notations

First, we use $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}, P\right)$ to denote a complete filtered probability space on which are defined a standard $d$-dimensional Brownian motion $W \equiv\{W(t), t \in[0, T]\}$ with $W(t)=$ $\left(W_{1}(t), \ldots, W_{d}(t)\right)^{\prime}$ and a filtration $\left\{\mathcal{F}_{t}, t \in[0, T]\right\}$ with $\mathcal{F}_{t}=\sigma(W(s), s \leq t)$, where $T \in[0, \infty)$ and the prime denotes the corresponding transpose of a matrix or a vector.

Second, we consider the $p$-dimensional rectangle $D=\left[0, b_{1}\right] \times \cdots \times\left[0, b_{p}\right]$ with a given $p \in \mathcal{N}=\{1,2, \ldots\}$. Let $C^{k}\left(D, R^{q}\right)$ for each $k, q \in \mathcal{N}$ denote the Banach space of all functions $f$ having continuous derivatives up to the order $k$ with the uniform norm,

$$
\begin{equation*}
\|f\|_{C^{k}(D, q)}=\max _{c \in\{0,1, \ldots, k\}} \max _{j \in\{1, \ldots, r(c)\}} \sup _{x \in D}\left|f_{j}^{(c)}(x)\right| \tag{2.1}
\end{equation*}
$$

for each $f \in C^{k}\left(D, R^{q}\right)$. The $r(c)$ in (2.1) for each $c \in\{0,1, \ldots, k\}$ is the total number of the partial derivatives of the order $c$

$$
\begin{equation*}
f_{r,\left(i_{1} \ldots i_{p}\right)}^{(c)}(x)=\frac{\partial^{c} f_{r}(x)}{\partial x_{1}^{i_{1}} \ldots \partial x_{p}^{i_{p}}} \tag{2.2}
\end{equation*}
$$

with $i_{l} \in\{0,1, \ldots, c\}, l \in\{1, \ldots, p\}, r \in\{1, \ldots, q\}$, and $i_{1}+\ldots+i_{p}=c$. Furthermore, let

$$
\begin{align*}
f_{\left(i_{1}, \ldots, i_{p}\right)}^{(c)} & \equiv\left(f_{1,\left(i_{1}, \ldots, i_{p}\right)}^{(c)}, \ldots, f_{q,\left(i_{1}, \ldots, i_{p}\right)}^{(c)}\right)  \tag{2.3}\\
f^{(c)}(x) & \equiv\left(f_{1}^{(c)}(x), \ldots, f_{r(c)}^{(c)}(x)\right), \tag{2.4}
\end{align*}
$$

where each $j \in\{1, \ldots, r(c)\}$ corresponds to a $p$-tuple $\left(i_{1}, \ldots, i_{p}\right)$ and a $r \in\{1, \ldots, q\}$.
Third, we use $C^{\infty}\left(D, R^{q}\right)$ to denote the Banach space

$$
\begin{equation*}
C^{\infty}\left(D, R^{q}\right) \equiv\left\{f \in \bigcap_{c=0}^{\infty} C^{c}\left(D, R^{q}\right),\|f\|_{C^{\infty}(D, q)}<\infty\right\} \tag{2.5}
\end{equation*}
$$

where

$$
\begin{equation*}
\|f\|_{C^{\infty}(D, q)}^{2}=\sum_{c=0}^{\infty} \xi(c)\|f\|_{C^{c}(D, q)}^{2} \tag{2.6}
\end{equation*}
$$

for some discrete function $\xi(c)$ in terms of $c \in\{0,1,2, \ldots\}$, which is fast decaying in $c$. For convenience, we take $\xi(c)=\frac{1}{\left(\left(c^{10}\right)!\right)\left(\eta(c)!!e^{c}\right.}$ with

$$
\eta(c)=\left[\max \left\{\left|x_{1}\right|+\ldots+\left|x_{p}\right|, x \in D\right\}\right]^{c},
$$

where the notation [] denotes the summation of the unity and the integer part of a real number.

Fourth, we define some measurable spaces to support random fields considered in this paper. Let $L_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D ; R^{q}\right)\right)$ denote the set of all $R^{q}$-valued (or called $C^{\infty}\left(D ; R^{q}\right)$ valued) measurable stochastic processes $Z(t, x)$ adapted to $\left\{\mathcal{F}_{t}, t \in[0, T]\right\}$ for each $x \in D$, which are in $C^{\infty}\left(D, R^{q}\right)$ for each fixed $\left.t \in[0, T]\right)$, such that

$$
\begin{equation*}
E\left[\int_{0}^{T}\|Z(t)\|_{C^{\infty}(D, q)}^{2} d t\right]<\infty \tag{2.7}
\end{equation*}
$$

Let $L_{\mathcal{F}, p}^{2}\left([0, T], C^{\infty}\left(D, R^{q}\right)\right)$ denote the corresponding set of predictable processes (see, e.g., Definition 5.2 and Definition 1.1 respectively in pages 21 and 45 of Ikeda and Watanabe [15]). Furthermore, let $L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(D ; R^{q}\right)\right)$ denote the set of all $R^{q}$-valued, $\mathcal{F}_{T}$-measurable random variables $\zeta(x)$ for each $x \in D$, where $\zeta(x) \in C^{\infty}\left(D, R^{q}\right)$ satisfies

$$
\begin{equation*}
\|\zeta\|_{L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(D, R^{q}\right)\right)}^{2} \equiv E\left[\|\zeta\|_{C^{\infty}(D, q)}^{2}\right]<\infty . \tag{2.8}
\end{equation*}
$$

In addition, we define

$$
\begin{equation*}
\mathcal{Q}_{\mathcal{F}}^{2}([0, T] \times D) \equiv L_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q}\right)\right) \times L_{\mathcal{F}, p}^{2}\left([0, T], C^{\infty}\left(D, R^{q \times d}\right)\right) \tag{2.9}
\end{equation*}
$$

### 2.2 The Conditions

In this subsection, we impose some conditions to guarantee the unique existence of adapted solution to (1.1) and to be used in the convergence analysis of our designed algorithms. First, let "a.s." denote "almost surely". Then, suppose that, for each $s \in[0, T]$,

$$
\begin{equation*}
\bar{V}(s, \cdot)=\left(\bar{V}_{1}(s, \cdot), \ldots, \bar{V}_{d}(s, \cdot)\right) \in C^{\infty}\left(D, R^{q \times d}\right) \quad \text { a.s. } \tag{2.10}
\end{equation*}
$$

and in (1.1), $\mathcal{L}$ is a $q$-dimensional partial differential operator satisfying the generalized Lipschitz condition a.s.

$$
\begin{equation*}
\left\|\Delta \mathcal{L}^{(c+l+o)}(s, x, u, v)\right\| \leq K_{D, c}\left(\|u-v\|_{C^{k+c}(D, q)}+\|\bar{u}-\bar{v}\|_{C^{m+c}(D, q d)}\right) \tag{2.11}
\end{equation*}
$$

for any $(u, \bar{u}),(v, \bar{v}) \in C^{\infty}\left(D, R^{q}\right) \times C^{\infty}\left(D, R^{q \times d}\right)$, where $K_{D, c}$ with each $c \in\{0,1,2, \ldots\}$ is a nonnegative constant. Note that $K_{D, c}$ depends on the domain $D$ and the differential order $c$ with respect to each $x \in D$ and may be unbounded as $c \rightarrow \infty$ and $D \rightarrow R^{p} . l \in\{0,1,2\}$ denotes the $l$ th order of partial derivative of $\Delta \mathcal{L}^{(c)}(s, x, u, v)$ in time variable $t . o \in\{0,1,2\}$ denotes the $o$ th order of partial derivative of $\Delta \mathcal{L}^{(c+l)}(s, x, u, v)$ in terms of a component of $u$ or $v .\|A\|$ is the largest absolute value of entries (or components) of the given matrix (or vector) $A$, and

$$
\Delta \mathcal{L}^{(c+l+o)}(s, x, u, v) \equiv \mathcal{L}^{(c+l+o)}(s, x, u, \bar{u})-\mathcal{L}^{(c+l+o)}(s, x, v, \bar{v})
$$

Similarly, $\mathcal{J}=\left(\mathcal{J}_{1}, \ldots, \mathcal{J}_{d}\right)$ is a $q \times d$-dimensional partial differential operator satisfying, a.s.,

$$
\begin{equation*}
\left\|\Delta \mathcal{J}^{(c+l+o)}(s, x, u, v)\right\| \leq K_{D, c}\left(\|u-v\|_{C^{m+c}(D, q)}\right) . \tag{2.12}
\end{equation*}
$$

In addition, we assume that the generalized linear growth conditions hold,

$$
\begin{align*}
\left\|\mathcal{L}^{(c+l+o)}(s, x, u, \bar{u})\right\| & \leq K_{D, c}\left(\delta_{0 c}+\|u\|_{C^{k+c}(D, q)}+\|\bar{v}\|_{C^{m+c}(D, q d)}\right)  \tag{2.13}\\
\left\|\mathcal{J}^{(c+l+o)}(s, x, u)\right\| & \leq K_{D, c}\left(\delta_{0 c}+\|u\|_{C^{m+c}(D, q)}\right) \tag{2.14}
\end{align*}
$$

where $\delta_{0 c}=1$ if $c=0$ and $\delta_{0 c}=0$ if $c>0$.

### 2.3 The Adapted Solution

Theorem 2.1 Assume that $H(x) \in L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(D ; R^{q}\right)\right)$ for each $x \in D$. Then, under conditions of (2.11)-(2.14), if $\mathcal{L}(t, x, v, \cdot)$ and $\mathcal{J}(t, x, v, \cdot)$ are $\left\{\mathcal{F}_{t}\right\}$-adapted for each fixed $x \in D$ and any given $(v, \bar{v}) \in C^{\infty}\left(D, R^{q}\right) \times C^{\infty}\left(D, R^{q \times d}\right)$ with

$$
\begin{equation*}
\mathcal{L}(\cdot, x, 0, \cdot) \in L_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q}\right)\right), \mathcal{J}(\cdot, x, 0, \cdot) \in L_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q \times d}\right)\right), \tag{2.15}
\end{equation*}
$$

the $B-S P D E$ (1.1) has a unique adapted solution,

$$
\begin{equation*}
(V(\cdot, \cdot), \bar{V}(\cdot, \cdot)) \in \mathcal{Q}_{\mathcal{F}}^{2}([0, T] \times D) \tag{2.16}
\end{equation*}
$$

The proof of Theorem 2.1 is provided in Section 4, which is partially embedded in the discussion of unique existence of solution to a more general system with jumps in the preprint of Dai [12]. Since the techniques adopted in the proof are frequently used in the rest of this paper, we refine them here and make them consistent with the system in (1.1) for convenience.

## 3 Numerical Schemes and Their Convergence

### 3.1 The Schemes

Consider a partition $\pi$ for the product of the time interval $[0, T]$ and the $p$-dimensional rectangle $D=\left[0, b_{1}\right] \times \cdots \times\left[0, b_{p}\right]$ with a given $p \in \mathcal{N}=\{1,2, \ldots\}$ as follows,

$$
\begin{array}{ll}
\pi: & 0=t_{0}<t_{1}<\cdots<t_{n_{0}}=T \quad \text { with } n_{0} \in\{0,1, \ldots\},  \tag{3.1}\\
& 0=x_{l}^{0}<x_{l}^{1}<\cdots<x_{l}^{n_{l}}=b_{l} \text { with } l \in\{1, \ldots, p\}, n_{l} \in\{0,1, \ldots\} .
\end{array}
$$

In the sequel, for all $l \in\{0,1, \ldots, p\}$ and $j_{l} \in\left\{1, \ldots, n_{l}\right\}$, we take

$$
\begin{align*}
\Delta_{j_{0}}^{\pi} & =t_{j_{0}}-t_{j_{0}-1},  \tag{3.2}\\
\Delta_{l}^{\pi} & =x_{l}^{j_{l}}-x_{l}^{j_{l}-1}=\frac{b_{l}}{n_{l}},  \tag{3.3}\\
\Delta^{\pi} W_{j_{0}} & =W\left(t_{j_{0}}\right)-W\left(t_{j_{0}-1}\right), \tag{3.4}
\end{align*}
$$

and let

$$
\begin{align*}
|\pi| & \equiv \max _{j_{0} \in\left\{1, \ldots, n_{0}\right\}, l \in\{1, \ldots, p\}}\left\{\Delta_{j_{0}}^{\pi}, \Delta_{l}^{\pi}\right\},  \tag{3.5}\\
D^{j_{1} \ldots j_{p}} & \equiv\left[x_{1}^{j_{1}-1}, x_{1}^{j_{1}}\right) \times \cdots \times\left[x_{p}^{j_{p}-1}, x_{p}^{j_{p}}\right),  \tag{3.6}\\
\mathcal{X} & \equiv\left\{x: x=\left(x_{1}^{j_{1}}, \ldots, x_{p}^{j_{p}}\right), j_{l} \in\left\{0,1, \ldots, n_{l}\right\}, l \in\{1, \ldots, p\}\right\} . \tag{3.7}
\end{align*}
$$

To suitably describe the approximations of partial derivatives appeared in (1.1) and (2.2), we assume that the orders $k$ and $m$ are less than $2 \max \left\{n_{1}, \ldots, n_{p}\right\}$. Then we can use the forward and the backward difference techniques to approximate the partial derivatives in (1.1) and (2.2) as follows. For each $f \in\{V, \bar{V}\}, x \in \mathcal{X}, l \in\{1, \ldots, p\}$, and each integer $c$ satisfying $1 \leq c \leq k$ or $m$ or $n$, define

$$
f_{i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}, \pi}^{(c)}(t, x) \equiv \begin{cases}\frac{f_{i_{1} \ldots i_{p}, \pi}^{(c-1)}\left(t, x+\Delta_{l}^{\pi} e_{l}\right)-f_{i_{1} \ldots i_{p}, \pi}^{(c-1)}(t, x)}{\Delta_{l}^{\pi}} & \text { if } j_{l}<n_{l},  \tag{3.8}\\ \frac{f_{i_{1} \ldots i_{p}, \pi}^{(c-1)}\left(t, x-\Delta_{l}^{\pi} e_{l}\right)-f_{i_{1} \ldots i_{p}, \pi}^{(c-1)}(t, x)}{\Delta_{l}^{\pi}} & \text { if } j_{l}=n_{l},\end{cases}
$$

where $e_{l}$ is the unit vector whose $l$ th component is the unity and others are zero, $f^{(0)}=f$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c-1}$ with

$$
\begin{equation*}
\mathcal{I}^{c} \equiv\left\{\left(i_{1}, \ldots, i_{p}\right): i_{1}, \ldots, i_{p} \text { are nonnegative integers satisfying } i_{1}+\ldots+i_{p}=c\right\} \tag{3.9}
\end{equation*}
$$

Furthermore, we define the following vector for all given $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$

$$
\begin{equation*}
f_{\pi}^{(c)}(t, x)=\left(f_{i_{1} \ldots i_{p}, \pi}^{(c)}(t, x)\right) \tag{3.10}
\end{equation*}
$$

according to an increasing order indexed by $i_{p} c^{p}+i_{p-1} c^{p-1}+\ldots+i_{2} c+i_{1}$. Next, to be simple for notations, we define

$$
\begin{align*}
\mathcal{L}\left(t, x, V_{\pi}(t, x)\right) & \equiv \mathcal{L}\left(t, x, V_{\pi}(t, x), \ldots, V_{\pi}^{(k)}(t, x), \bar{V}_{\pi}(t, x), \ldots, \bar{V}_{\pi}^{(m)}(t, x)\right)  \tag{3.11}\\
\mathcal{J}\left(t, x, V_{\pi}(t, x)\right) & \equiv \mathcal{J}\left(t, x, V_{\pi}(t, x), \ldots, V_{\pi}^{(n)}(t, x)\right) \tag{3.12}
\end{align*}
$$

for each $x \in \mathcal{X}$. Thus, based on spacial discretization, we can design the following direct discrete approximations of a solution to the B-SPDE displayed in (1.1).
Algorithm 3.1 This algorithm is an iterative one in terms of $\left\{V^{(c)}\left(t_{j_{0}}, x\right), \bar{V}^{(c)}\left(t_{j_{0}}, x\right)\right.$ for all $x \in \mathcal{X}\}$ with $j_{0}$ decreasing from $n_{0}$ to 1 in a backward manner and $c=0,1, \ldots, M$ with $M=\max \{m, n, k\}$,

$$
\begin{align*}
V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{n_{0}}, x\right)= & H_{i_{1} \ldots i_{p}, \pi}^{(c)}(x), \bar{V}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{n_{0}}, x\right)=0,  \tag{3.13}\\
V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x\right)= & E\left[V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}}, x\right)+\mathcal{L}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}}, x, V_{\pi}\left(t_{j_{0}}, x\right)\right) \Delta_{j_{0}}^{\pi} \mid \mathcal{F}_{t_{j_{0}-1}}\right],  \tag{3.14}\\
\bar{V}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x\right)= & \frac{1}{\Delta_{j_{0}}^{\pi}} E\left[V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}}, x\right) \Delta^{\pi} W_{j_{0}} \mid \mathcal{F}_{t_{j_{0}-1}}\right]  \tag{3.15}\\
& +E\left[\mathcal{L}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}}, x^{\pi}, V_{\pi}\left(t_{j_{0}}, x\right)\right) \Delta^{\pi} W_{j_{0}} \mid \mathcal{F}_{t_{j_{0}-1}}\right] \\
& +\mathcal{J}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x, V_{\pi}\left(t_{j_{0}-1}, x\right)\right) .
\end{align*}
$$

Algorithm 3.2 For all $x \in \mathcal{X}$ and $c=0,1, \ldots, M$, we have the following algorithm with respect to $j_{0}$ decreasing from $n_{0}$ to 1 ,

$$
\begin{align*}
V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{n_{0}}, x\right)= & H_{i_{1} \ldots i_{p}, \pi}^{(c)}(x)  \tag{3.16}\\
V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x\right)= & E\left[V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{t_{j_{0}-1}}\right]  \tag{3.17}\\
& +\mathcal{L}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x, V_{\pi}\left(t_{j_{0}-1}, x\right)\right) \Delta_{j_{0}}^{\pi}, \\
\bar{V}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x\right)= & \frac{1}{\Delta_{j_{0}}^{\pi}} E\left[V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}}, x\right) \Delta^{\pi} W_{j_{0}} \mid \mathcal{F}_{t_{j_{0}-1}}\right]  \tag{3.18}\\
& +\mathcal{J}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x, V_{\pi}\left(t_{j_{0}-1}, x\right)\right) .
\end{align*}
$$

In nature, Algorithm 3.2 is a sort of generalization of the scheme considered in Bouchard and Touzi [7] from BSDEs to the B-SPDEs. Comparing with Algorithm 3.1, it is not a purely iterative one since it needs to solve linear or nonlinear equations at each time $t_{j_{0}-1}$ to obtain $V_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x\right)$ and $\bar{V}_{i_{1} \ldots i_{p}, \pi}^{(c)}\left(t_{j_{0}-1}, x\right)$, which is expensive when $x$ is in a higher-dimensional domain (e.g., $p \geq 3$ ). Furthermore, since the convergence analysis is similar, we will focus our discussion on Algorithm 3.1 in the rest of this paper.

### 3.2 Additional Notations for Random Field Based Malliavin Calculus

Let $H=L^{2}\left([0, T], R^{d}\right)$ denote the separable Hilbert space of all square integrable real-valued $d$-dimensional functions over the time interval $[0, T]$ with inner product

$$
<\cdot, \cdot>_{H}=\int_{0}^{T}\left\langle h^{1}(t), h^{2}(t)\right\rangle d t \text { for any } h^{1}, h^{2} \in H,
$$

and

$$
\begin{equation*}
\left\langle h^{1}(t), h^{2}(t)\right\rangle=\sum_{i=1}^{d} h_{i}^{1}(t) h_{i}^{2}(t) . \tag{3.19}
\end{equation*}
$$

For each $h \in H$, we define $W(h)=\int_{0}^{T}\langle h(t), d W(t)\rangle$. Furthermore, let $\mathcal{S}$ denote the set of all the random variables $F(x, \omega)$ of the following form with $x \in D$ and $\omega \in \Omega$,

$$
\begin{equation*}
F(x)=\phi\left(W\left(h^{1}\right), \ldots, W\left(h^{g}\right), x\right) \text { with } \phi \in C_{b}^{\infty}\left(R^{g+p}, R^{q}\right), h^{1}, \ldots, h^{g} \in H \tag{3.20}
\end{equation*}
$$

for some nonnegative integer $g$, where the lower index $b$ appeared in $C_{b}^{\infty}$ means bounded. For each $F \in \mathcal{S}$, we define

$$
\begin{equation*}
\|F\|_{\alpha, 2}^{\infty, 2}=\sum_{v=0}^{\infty} \xi(v)\|F\|_{\alpha, 2}^{v, 2}, \tag{3.21}
\end{equation*}
$$

where, the norm $\|\cdot\|_{\alpha, 2}^{v}$ with $\alpha \in\{1,2\}$ and $v \in\{0,1,2, \ldots\}$ is defined in the following way.

First, we define the first-order Malliavin derivative of the $c$ th order partial derivative $F_{r, i_{1} \ldots i_{p}}^{(c)}(x)$ in terms of $x \in D$ for each $r \in\{1, \ldots, q\}, c \in\{0,1, \ldots\}$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$ to be the $H$-valued random variable,

$$
\begin{equation*}
\mathcal{D}_{\theta_{1}} F_{r, i_{1} \ldots i_{p}}^{(c)}(x)=\sum_{l=1}^{g} \frac{\partial \phi_{r, i_{1} \ldots i_{p}}^{(c)}}{\partial y_{l}}\left(W\left(h^{1}\right), \ldots, W\left(h^{g}\right), x\right) h^{l}\left(\theta_{1}\right), 0 \leq \theta_{1} \leq T \tag{3.22}
\end{equation*}
$$

Second, for each $j \in\{1, \ldots, d\}$, we define the associated second-order Malliavin derivative

$$
\begin{equation*}
\mathcal{D}_{\theta_{2}} \mathcal{D}_{\theta_{1}}^{j} F_{r, i_{1} \ldots i_{p}}^{(c)}(x)=\sum_{l=1}^{g} \frac{\partial \mathcal{D}_{\theta_{1}}^{j} F_{r, i_{1} \ldots i_{p}}^{(c)}(x)}{\partial y_{l}} h^{l}\left(\theta_{2}\right), 0 \leq \theta_{2} \leq T . \tag{3.23}
\end{equation*}
$$

Third, we define

$$
\begin{align*}
& \|F\|_{1,2}^{v, 2}=E\left[\Lambda_{v}\left\|F_{r, i_{1} \ldots i_{p}}^{(c)}\right\|_{C^{v}(D, 1)}^{2}+\int_{0}^{T} \Lambda_{v}\left\|\mathcal{D}_{\theta_{1}} F_{r, i_{1} \ldots i_{p}}^{(c)}\right\|_{C^{v}(D, d)}^{2} d \theta_{1}\right],  \tag{3.24}\\
& \|F\|_{2,2}^{v, 2}=\|F\|_{1,2}^{v, 2}+E\left[\int_{0}^{T} \int_{0}^{T} \Lambda_{v}\left\|\mathcal{D}_{\theta_{2}} \mathcal{D}_{\theta_{1}} F_{r, i_{1} \ldots i_{p}}^{(c)}\right\|_{C^{v}(D, d \times d)}^{2} d \theta_{1} d \theta_{2}\right], \tag{3.25}
\end{align*}
$$

where the notation $\Lambda_{v}$ is defined by

$$
\begin{equation*}
\Lambda_{v}=\max _{c \in\{0,1, \ldots, v\}} \max _{r \in\{1, \ldots, q\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}} \tag{3.26}
\end{equation*}
$$

Note that if $F_{r, i_{1} \ldots i_{p}}^{(c)}(x)$ for each $x \in D$ is $\mathcal{F}_{t}$-measurable, then $\mathcal{D}_{\theta} F_{r, i_{1} \ldots i_{p}}^{(c)}(x)=0$ for $\theta \in(t, T]$ and we use $\mathcal{D}_{\theta}^{j} F_{r, i_{1} \ldots i_{p}}^{(c)}(x)$ for each $j \in\{1, \ldots, d\}$ to denote the $j$ th component of $\mathcal{D}_{\theta} F_{r, i_{1} \ldots i_{p}}^{(c)}(x)$.

Next, let $L^{2}\left(\Omega, C^{\infty}\left(D, R^{q}\right)\right)$ be the space corresponding to (2.8) with no measurable property imposed, and let $L_{\alpha, 2}^{2}\left(\Omega,\left(C^{\infty}(D, H)\right)^{q}\right)$ be the space of $H^{q}(q$ product space $H \times \ldots \times H)$ valued processes, which is endowed with the norm (3.21). Then, we can use $\mathcal{D}_{\infty}^{\alpha, 2}$ to denote the domain of the following unbounded operator,

$$
\mathcal{D}: L^{2}\left(\Omega, C^{\infty}\left(D, R^{q}\right)\right) \rightarrow L_{\alpha, 2}^{2}\left(\Omega,\left(C^{\infty}(D, H)\right)^{q}\right) .
$$

Owing to Lemma 5.1 proved in Section 5, this domain is the closure of the class of smooth random variables $\mathcal{S}$ with the norm (3.21). In the sequel, we use $D_{\theta} F(x, \omega)$ with each $F \in \mathcal{D}_{\infty}^{1,2}$ and $\theta \in[0, T]$ to denote the following infinite-dimensional vector

$$
\begin{equation*}
\left\{\left(D_{\theta}^{j} F_{r, i_{1} \ldots i_{p}}^{(c)}(t, x): r \in\{1, \ldots, q\}, j \in\{1, \ldots, d\}, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}\right\} .\right. \tag{3.27}
\end{equation*}
$$

Similarly, we use $D_{\theta_{2}} D_{\theta_{1}} F(x, \omega)$ with each $F \in \mathcal{D}_{\infty}^{2,2}$ and $\theta_{1}, \theta_{2} \in[0, T]$ to denote the following infinite-dimensional vector

$$
\begin{equation*}
\left\{\left(D_{\theta_{2}}^{j_{2}} D_{\theta_{1}}^{j_{1}} F_{r, i_{1} \ldots i_{p}}^{(c)}(t, x): r \in\{1, \ldots, q\}, j_{1}, j_{2} \in\{1, \ldots, d\}, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}\right\} .\right. \tag{3.28}
\end{equation*}
$$

Finally, based on the above notations, we can impose the following terminal conditions for the $q$-dimensional B-SPDEs in (1.1),

$$
\begin{array}{rlr}
H(x, \omega) & \in \mathcal{D}_{\infty}^{2,2} \bigcap L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(D, R^{q}\right)\right), \\
D_{\theta_{1}} H(x, \omega) & \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; C^{\infty}\left(D, R^{q \times d}\right)\right), & \theta_{1} \in[0, T], \\
\mathcal{D}_{\theta_{2}} \mathcal{D}_{\theta_{1}} H(x, \omega) & \in L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(D, R^{q \times d \times d}\right)\right), & \theta_{2} \in[0, T] . \tag{3.31}
\end{array}
$$

### 3.3 Convergence Theorem for Algorithm 3.1

Theorem 3.1 Consider Algorithm 3.1 under conditions required by Theorem 2.1 and with additional terminal conditions (3.29)-(3.31). Then, there exists some nonnegative constant $C$ depending only on the terminal time $T$, the region $D$, and the constant $\kappa$ such that

$$
\begin{equation*}
\sum_{c=0}^{M} \max _{x \in \mathcal{X}}\left(\sup _{t \in[0, T]} E\left[\left\|\Delta V^{(c)}(t, x)\right\|^{2}\right]+\sup _{t \in[0, T]} E\left[\left\|\Delta \bar{V}^{(c)}(t, x)\right\|^{2}\right]\right) \leq C|\pi|, \tag{3.32}
\end{equation*}
$$

for all sufficiently small $|\pi|$, where

$$
\begin{aligned}
\Delta V^{(c)}(t, x) & =V^{(c)}(t, x)-V_{\pi}^{(c)}(t, x), \\
\Delta \bar{V}^{(c)}(t, x) & =\bar{V}^{(c)}(t, x)-\bar{V}_{\pi}^{(c)}(t, x), \\
V_{\pi}^{(c)}(t, x) & =V_{\pi}^{(c)}\left(t_{j_{0}-1}, x\right), t \in\left[t_{j_{0}-1}, t_{j_{0}}\right), j_{0} \in\left\{n_{0}, n_{0}-1, \ldots, 1\right\}, \\
\bar{V}_{\pi}^{(c)}(t, x) & =\bar{V}_{\pi}^{(c)}\left(t_{j_{0}-1}, x\right), t \in\left[t_{j_{0}-1}, t_{j_{0}}\right) .
\end{aligned}
$$

for each $c \in\{0,1, \ldots, M\}$.
The proof of Theorem 3.1 will be provided in Section 5.

## 4 Proof of Theorem 2.1

We first prove three lemmas. Then, by combining these lemmas, we can provide a proof for the theorem.

### 4.1 Three Lemmas and Their Proofs

Lemma 4.1 Under the conditions of Theorem 2.1, consider a tuplet for each fixed $x \in D$,

$$
\begin{equation*}
(U(\cdot, x), \bar{U}(\cdot, x)) \in \mathcal{Q}_{\mathcal{F}}^{2}([0, T] \times D) \tag{4.1}
\end{equation*}
$$

Then, there exists another tuplet $(V(\cdot, x), \bar{V}(\cdot, x))$ such that

$$
\begin{equation*}
V(t, x)=H(x)+\int_{t}^{T} \mathcal{L}(s, x, U, \cdot) d s+\int_{t}^{T}(\mathcal{J}(s, x, U, \cdot)-\bar{V}(s, x)) d W(s) \tag{4.2}
\end{equation*}
$$

where $V$ is a $\left\{\mathcal{F}_{t}\right\}$-adapted càdlàg process, $\bar{V}$ is the corresponding predictable process. Furthermore, for each $x \in D$,

$$
\begin{align*}
& E\left[\int_{0}^{T}\|V(t, x)\|^{2} d t\right]<\infty  \tag{4.3}\\
& E\left[\int_{0}^{T}\|\bar{V}(t, x)\|^{2} d t\right]<\infty \tag{4.4}
\end{align*}
$$

Proof. For each fixed $x \in D$ and a tuplet $(U(\cdot, x), \bar{U}(\cdot, x))$ as stated in (4.1), it follows from conditions (2.11)-(2.15) that

$$
\begin{equation*}
\mathcal{L}(\cdot, x, U, \cdot) \in L_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q}\right)\right), \mathcal{J}(\cdot, x, U, \cdot) \in L_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q \times d}\right)\right) \tag{4.5}
\end{equation*}
$$

Now, consider $\mathcal{L}$ and $\mathcal{J}$ in (4.5) as two new starting $\mathcal{L}(\cdot, x, 0, \cdot)$ and $\mathcal{J}(\cdot, x, 0, \cdot)$. Then, by the Martingale representation theorem (see, e.g., Theorem 43 in page 186 of Protter [28]), we know that there is a unique predictable process $\bar{V}(\cdot, x)$ which is square-integrable for each $x \in D$ in the sense of (4.4) such that

$$
\begin{align*}
\hat{V}(t, x) & \equiv E\left[H(x)+\int_{0}^{T} \mathcal{L}(s, x, U, \cdot) d s+\int_{0}^{T} \mathcal{J}(s, x, U, \cdot) d W(s) \mid \mathcal{F}_{t}\right]  \tag{4.6}\\
& =\hat{V}(0, x)+\int_{0}^{t} \bar{V}(s, x) d W(s) .
\end{align*}
$$

Hence, we have,

$$
\begin{align*}
\hat{V}(0, x) & =\hat{V}(T, x)-\int_{0}^{T} \bar{V}(s, x) d W(s)  \tag{4.7}\\
& =H(x)+\int_{0}^{T} \mathcal{L}(s, x, U, \cdot) d s+\int_{0}^{T}(\mathcal{J}(s, x, U, \cdot)-\bar{V}(s, x)) d W(s) .
\end{align*}
$$

Furthermore, owing to the Corollary in page 8 of Protter [28], $\hat{V}(\cdot, x)$ can be taken as a càdlàg process. Next, define a process $V$ given by

$$
\begin{equation*}
V(t, x)=E\left[H(x)+\int_{t}^{T} \mathcal{L}(s, x, U, \cdot) d s+\int_{t}^{T} \mathcal{J}(s, x, U, \cdot) d W(s) \mid \mathcal{F}_{t}\right] \tag{4.8}
\end{equation*}
$$

Then, it follows from (2.12)-(2.14) and simple calculation that $V(\cdot, x)$ is square-integrable in the sense of (4.3). Furthermore, by (4.6)-(4.8), we know that

$$
\begin{equation*}
V(t, x)=\hat{V}(t, x)-\int_{0}^{t} \mathcal{L}(s, x, U, \cdot) d s-\int_{0}^{t} \mathcal{J}(s, x, U, \cdot) d W(s) \tag{4.9}
\end{equation*}
$$

which indicates that $V(\cdot, x)$ is a càdlàg process. Now, for a given tuplet $(U(\cdot, x), \bar{U}(\cdot, x))$, it follows from (4.6)-(4.7) and (4.9) that the corresponding tuplet $(V(\cdot, x), \bar{V}(\cdot, x))$ satisfies the equation (4.2) as stated in the lemma. Thus, we know that

$$
\begin{equation*}
V(t, x) \equiv V(0, x)-\int_{0}^{t} \mathcal{L}(s, x, U, \cdot) d s-\int_{0}^{t}(\mathcal{J}(s, x, U, \cdot)-\bar{V}(s, x)) d W(s) \tag{4.10}
\end{equation*}
$$

Hence, we complete the proof of Lemma 4.1.
Lemma 4.2 Under the conditions of Theorem 2.1, consider a tuplet as in (4.1) for each fixed $x \in D$ and define $V(t, x)$ and $\bar{V}(t, x)$ by (4.2). Then, $\left(V^{(c)}(\cdot, x)\right.$, $\left.\bar{V}^{(c)}(\cdot, x)\right)$ for each $c \in\{0,1, \ldots$,$\} exists a.s. and satisfies$

$$
\begin{align*}
V_{\left(i_{1} \ldots i_{p}\right)}^{(c)}(t, x)= & H_{\left(i_{1} \ldots i_{p}\right)}^{(c)}(x)+\int_{t}^{T} \mathcal{L}_{\left(i_{1} \ldots i_{p}\right)}^{(c)}(s, x, U, \cdot) d s  \tag{4.11}\\
& +\int_{t}^{T}\left(\mathcal{J}_{\left(i_{1} \ldots i_{p}\right)}^{(c)}(s, x, U, \cdot)-\bar{V}_{\left(i_{1} \ldots i_{p}\right)}^{(c)}(s, x)\right) d W(s),
\end{align*}
$$

where $i_{1}+\ldots+i_{p}=c$ and $i_{l} \in\{0,1, \ldots, c\}$ with $l \in\{1, \ldots, p\}$. Furthermore, $V_{\left(i_{1} \ldots i_{p}\right)}^{(c)}$ for each $c \in\{0,1, \ldots\}$ is a $\left\{\mathcal{F}_{t}\right\}$-adapted càdlàg process, and $\bar{V}_{\left(i_{1} \ldots i_{p}\right)}^{(c)}$ is the corresponding predictable processes. Both of them are square-integrable in the senses of (4.3)-(4.4).

Proof. First, we prove the claim in the lemma to be true for $c=1$. To do so, for each given $t \in[0, T], x \in D$, and $(U(t, x), \bar{U}(t, x))$ as in the lemma, let

$$
\begin{equation*}
\left(V_{(l)}^{(1)}(t, x), \bar{V}_{(l)}^{(1)}(t, x)\right) \tag{4.12}
\end{equation*}
$$

be defined by using (4.2), where $\mathcal{L}$ and $\mathcal{J}$ are replaced by their first-order partial derivatives $\mathcal{L}_{(l)}^{(1)}$ and $\mathcal{J}_{(l)}^{(1)}$ in terms of $x_{l}$ with $l \in\{1, \ldots, p\}$. Then, we can prove that the tuplet defined in (4.12) for each $l$ is indeed the required first-order partial derivative of $(V, \bar{V})$ that is defined by using (4.2) for the given $(U, \bar{U})$.

In fact, for each $f \in\{U, \bar{U}, V, \bar{V}\}$, sufficiently small positive constant $\delta$, and $l \in\{1, \ldots, p\}$, define

$$
\begin{equation*}
f_{(l), \delta}(t, x) \equiv f\left(t, x+\delta e_{l}\right) \tag{4.13}
\end{equation*}
$$

where $e_{l}$ is the unit vector whose $l$ th component is one and others are zero. Furthermore, let

$$
\begin{equation*}
\Delta f_{(l), \delta}^{(1)}(t, x)=\frac{f_{(l), \delta}(t, x)-f(t, x)}{\delta}-f_{(l)}^{(1)}(t, x) \tag{4.14}
\end{equation*}
$$

for each $f \in\{U, \bar{U}, V, \bar{V}\}$. In addition, let

$$
\begin{align*}
\Delta \mathcal{I}_{(l), \delta}^{(1)}(s, x, U)= & \frac{1}{\delta}\left(\mathcal{I}\left(s, x+\delta e_{l}, U\left(s, x+\delta e_{l}\right), \cdot\right)-\mathcal{I}(s, x, U(s, x), \cdot)\right)  \tag{4.15}\\
& -\mathcal{I}_{(l)}^{(1)}(s, x, U(s, x), \cdot)
\end{align*}
$$

for each $\mathcal{I} \in\{\mathcal{L}, \mathcal{J}\}$, and let $\operatorname{Tr}(A)$ denote the trace of the matrix $A^{\prime} A$ for a given matrix A. Then, by applying (4.10) and the Ito's formula (see, e.g., Theorem 33 in page 81 of Protter [28]) to the function

$$
\zeta\left(\Delta V_{(l), \delta}^{(1)}(t, x)\right) \equiv \operatorname{Tr}\left(\Delta V_{(l), \delta}^{(1)}(t, x)\right) e^{2 \gamma t}
$$

for some $\gamma>0$, we see that

$$
\begin{aligned}
& (4.16) \zeta\left(\Delta V_{(l), \delta}^{(1)}(t, x)\right)+\int_{t}^{T} \operatorname{Tr}\left(\Delta \mathcal{J}_{(l), \delta}^{(1)}(s, x, U)-\Delta \bar{V}_{(l), \delta}^{(1)}(s, x)\right) e^{2 \gamma s} d s \\
& \quad=2 \int_{t}^{T}\left(-\gamma \operatorname{Tr}\left(\Delta V_{(l), \delta}^{(1)}(s, x)\right)+\left(\Delta V_{(l), \delta}^{(1)}(s, x)\right)^{\prime}\left(\Delta \mathcal{L}_{(l), \delta}^{(1)}(s, x, U)\right)\right) e^{2 \gamma s} d s-M_{\delta}(t) \\
& \leq\left(-2 \gamma+\frac{3 K_{D, 1}^{2}}{\hat{\gamma}}\right) \int_{t}^{T} \operatorname{Tr}\left(\Delta V_{(l), \delta}^{(1)}(s, x)\right) e^{2 \gamma s} d s+\hat{\gamma} \int_{t}^{T}\left\|\Delta \mathcal{L}_{(l), \delta}^{(1)}(s, x, U)\right\|^{2} e^{2 \gamma s} d s-M_{\delta}(t) \\
& =\hat{\gamma} \int_{t}^{T}\left\|\Delta \mathcal{L}_{(l), \delta}^{(1)}(s, x, U)\right\|^{2} e^{2 \gamma s} d s-M_{\delta}(t)
\end{aligned}
$$

if, in the last equality, we take

$$
\begin{equation*}
\hat{\gamma}=\frac{3 K_{D, 1}^{2}}{2 \gamma}>0 \tag{4.17}
\end{equation*}
$$

where $K_{D, 1}$ is defined in (2.11)-(2.14) and $M_{\delta}(t)$ is a martingale given by

$$
2 \sum_{j=1}^{d} \int_{t}^{T}\left(\Delta V_{(l), \delta}^{(1)}(s, x)\right)^{\prime}\left(\Delta\left(\mathcal{J}_{j}\right)_{(l), \delta}^{(1)}(s, x, U)-\Delta\left(\bar{V}_{j}\right)_{(l), \delta}^{(1)}(s, x)\right) e^{2 \gamma s} d W_{j}(s) .
$$

Next, by Lemma 1.3 in Peskir and Shiryaev [27], there is a sequence of $\left\{\delta_{n}, n=1,2, \ldots\right\} \subset$ $[0, \sigma]$ for each $t \in[0, T]$ and $\sigma>0$ such that

$$
\begin{align*}
& E\left[\text { ess } \sup _{0 \leq \delta \leq \sigma} \zeta\left(\Delta V_{(l), \delta}^{(1)}(t, x)\right)\right]  \tag{4.18}\\
= & E\left[\text { ess } \sup _{\left\{\delta_{n}: 0 \leq \delta_{n} \leq \sigma, n=1,2, \ldots\right\}} \zeta\left(\Delta V_{(l), \delta_{n}}^{(1)}(t, x)\right)\right] \\
= & \lim _{n \rightarrow \infty} E\left[\zeta\left(\Delta V_{(l), \delta_{n}}^{(1)}(t, x)\right)\right] \\
\leq & \hat{\gamma} \lim _{n \rightarrow \infty} E\left[\int_{t}^{T}\left\|\Delta \mathcal{L}_{(l), \delta_{n}}^{(1)}(s, x, U)\right\|^{2} e^{2 \gamma s} d s\right]-\lim _{n \rightarrow \infty} E\left[M_{\delta_{n}}(t)\right] \\
\leq & \hat{\gamma} E\left[\int_{t}^{T} \text { ess } \sup _{0 \leq \delta \leq \sigma}\left\|\Delta \mathcal{L}_{(l), \delta}^{(1)}(s, x, U)\right\|^{2} e^{2 \gamma s} d s\right]
\end{align*}
$$

where "esssup" denotes the essential supremum. Furthermore, the first inequality in (4.18) is owing to (4.16). Thus, by the Lebesgue's dominated convergence theorem, we have

$$
\begin{align*}
& \lim _{\sigma \rightarrow 0} E\left[\text { ess } \sup _{0 \leq \delta \leq \sigma} \zeta\left(\Delta V_{(l), \delta}^{(1)}(t, x)\right)\right]  \tag{4.19}\\
\leq & \hat{\gamma} E\left[\int_{t}^{T} \lim _{\sigma \rightarrow 0} \text { ess } \sup _{0 \leq \delta \leq \sigma}\left\|\Delta \mathcal{L}_{(l), \delta}^{(1)}(s, x, U)\right\|^{2} e^{2 \gamma s} d s\right],
\end{align*}
$$

where we have used the following fact owing to the mean-value theorem and the conditions stated in (2.12),

$$
\left\|\Delta \mathcal{L}_{(l), \delta}^{(1)}(t, x, U)\right\| \leq 2 K_{D, 1}\left(\|U\|_{C^{k+1}(D, q)}+\|\bar{U}\|_{C^{m+1}(D, q d)}\right) .
$$

Thus, it follows from (4.19) and the Fatou's lemma that, for any sequence $\sigma_{n}$ satisfying $\sigma_{n} \rightarrow 0$ along $n \in \mathcal{N}$, there is a subsequence $\mathcal{N}^{\prime} \subset \mathcal{N}$ such that

$$
\begin{equation*}
\text { ess } \sup _{0 \leq \delta \leq \sigma_{n}} \zeta\left(\Delta V_{(l), \delta}^{(1)}(t, x)\right) \rightarrow 0 \text { along } n \in \mathcal{N}^{\prime} \text { a.s.. } \tag{4.20}
\end{equation*}
$$

Therefore, we know that the first-order derivative of $V$ with respect to $x_{l}$ for each $l \in\{1, \ldots, p\}$ exists and equals $V_{(l)}^{(1)}(t, x)$ a.s. for each $t \in[0, T]$ and $x \in D$. Furthermore, it is $\left\{\mathcal{F}_{t}\right\}$-adapted.

Now, by applying the similar proof as used in (4.18), we have

$$
\begin{align*}
& \lim _{\sigma \rightarrow 0} E\left[\int_{t}^{T} \text { ess } \sup _{0 \leq \delta \leq \sigma} \operatorname{Tr}\left(\Delta \mathcal{J}_{(l), \delta}^{(1)}(s, x, U)-\Delta \bar{V}_{(l), \delta}^{(1)}(s, x)\right) e^{2 \gamma s} d s\right]  \tag{4.21}\\
\leq & \hat{\gamma} E\left[\int_{t}^{T} \lim _{\sigma \rightarrow 0} \text { ess } \sup _{0 \leq \delta \leq \sigma}\left\|\Delta \mathcal{L}_{(l), \delta}^{(1)}(s, x, U)\right\|^{2} e^{2 \gamma s} d s\right] .
\end{align*}
$$

Hence, it follows from (4.20) and (4.21) that

$$
\lim _{\delta \rightarrow 0} \Delta \bar{V}_{(l), \delta}^{(1)}(t, x)=\lim _{\delta \rightarrow 0} \Delta \mathcal{J}_{(l), \delta}^{(1)}(t, x, U)=0 . \quad \text { a.s. }
$$

Thus, we know that the first-order derivative of $\bar{V}$ in terms of $x_{l}$ for each $l \in\{1, \ldots, p\}$ exists and equals $\bar{V}_{(l)}^{(1)}(t, x)$ a.s. for every $t \in[0, T]$ and $x \in D$. Furthermore, it is a $\left\{\mathcal{F}_{t}\right\}$-predictable process.

Second, supposing that $\left(V^{(c-1)}(t, x), \bar{V}^{(c-1)}(t, x)\right)$ associated with a given $(U(t, x), \bar{U}(t, x))$ $\in \mathcal{Q}_{\mathcal{F}}^{2}([0, T])$ exists for any given $c \in\{1,2, \ldots\}$. Then, we can prove that

$$
\begin{equation*}
\left(V^{(c)}(t, x), \bar{V}^{(c)}(t, x)\right) \tag{4.22}
\end{equation*}
$$

exists for the given $c$. In doing so, for any fixed nonnegative integer numbers $i_{1}, \ldots, i_{p}$ satisfying $i_{1}+\ldots+i_{p}=c-1$ for the given $c \in\{1,2, \ldots\}$, any $f \in\{V, \bar{V}\}$, any $l \in\{1, \ldots, p\}$, and any small enough $\delta>0$, we define

$$
\begin{equation*}
f_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right), \delta}^{(c-1)}(t, x) \equiv f_{\left(i_{1} \ldots i_{p}\right)}^{(c-1)}\left(t, x+\delta e_{l}\right), \tag{4.23}
\end{equation*}
$$

which corresponds to $\mathcal{I}_{\left(i_{1} \ldots i_{p}\right)}^{(c-1)}\left(s, x+\delta e_{l}, U\left(s, x+\delta e_{l}\right), \cdot\right)$ with $\mathcal{I} \in\{\mathcal{L}, \mathcal{J}\}$ via (4.2), where the differential operators $\mathcal{L}$ and $\mathcal{J}$ are replaced by their $(c-1)$ th-order partial derivatives $\mathcal{L}_{\left(i_{1} \ldots i_{p}\right)}^{(c-1)}$ and $\mathcal{J}_{\left(i_{1} \ldots i_{p}\right)}^{(c-1)}$. Similarly, let

$$
\left(V_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right)}^{(c)}(t, x), \bar{V}_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right)}^{(c)}(t, x)\right)
$$

be defined by using (4.2), where $\mathcal{L}$ and $\mathcal{J}$ are replaced by their $c$ th-order partial derivatives $\mathcal{L}_{i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}}^{(c)}$ and $\mathcal{J}_{i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}}^{(c)}$ corresponding to a given $t, x, U(t, x), \bar{U}(t, x)$. Furthermore, set

$$
\begin{equation*}
\Delta f_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right), \delta}^{(c)}(t, x)=\frac{f_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right), \delta}^{(c-1)}(t, x)-f_{\left(i_{1} \ldots i_{p}\right)}^{(c-1)}(t, x)}{\delta}-f_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right)}^{(c)}(t, x) \tag{4.24}
\end{equation*}
$$

for each $f \in\{U, \bar{U}, V, \bar{V}\}$, and let

$$
\begin{align*}
& \Delta \mathcal{I}_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right), \delta}^{(c)}(t, x, U)  \tag{4.25}\\
= & \frac{1}{\delta}\left(\mathcal{I}_{\left(i_{1} \ldots i_{p}\right)}^{(c-1)}\left(t, x+\delta e_{l}, U\left(t, x+\delta e_{l}\right), \cdot\right)-\mathcal{I}_{\left(i_{1} \ldots i_{p}\right)}^{(c-1)}(s, x, U(s, x), \cdot)\right) \\
& -\mathcal{I}_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right)}^{(c)}(s, x, U(s, x), \cdot)
\end{align*}
$$

for $\mathcal{I} \in\{\mathcal{L}, \mathcal{J}\}$. Then, by the Itô's formula and repeating the procedure as used in the second step, we know that

$$
\left(V_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right)}^{(c)}(t, x), \bar{V}_{\left(i_{1} \ldots\left(i_{l}+1\right) \ldots i_{p}\right)}^{(c)}(t, x)\right)
$$

exist for the given $c \in\{1,2, \ldots\}$ and all $l \in\{1, \ldots, p\}$. Thus, we know that the claim in (4.22) is correct.

Third, by the induction method in terms of $c \in\{1,2, \ldots\}$, we know that the claims stated in the lemma are right. Therefore, we complete the proof of Lemma 4.2.

To state our next lemma, we let $D_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q}\right)\right)$ be the set of $R^{q}$-valued $\left\{\mathcal{F}_{t}\right\}$ adapted and square integrable càdlàg processes as in (2.7). Furthermore, for any given number sequence $\gamma=\left\{\gamma_{c}, c=0,1,2, \ldots\right\}$ with $\gamma_{c} \in R$, define $\mathcal{M}_{\gamma}^{D}[0, T]$ to be the following Banach space (see, e.g., the similar explanation as used in Yong and Zhou [34], and Situ [30])

$$
\begin{equation*}
\mathcal{M}_{\gamma}^{D}[0, T]=D_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q}\right)\right) \times L_{\mathcal{F}, p}^{2}\left([0, T], C^{\infty}\left(D, R^{q \times d}\right)\right) \tag{4.26}
\end{equation*}
$$

endowed with the norm: for any given $(U, \bar{U}) \in \mathcal{M}_{\gamma}^{D}[0, T]$,

$$
\begin{equation*}
\|(U, \bar{U})\|_{\mathcal{M}_{\gamma}^{D}}^{2} \equiv \sum_{c=0}^{\infty} \xi(c)\|(U, \bar{U})\|_{\mathcal{M}_{\gamma, c}^{D},}^{2}, \tag{4.27}
\end{equation*}
$$

where, without loss of generality, we assume that $m=k$ in (1.1) and

$$
\begin{equation*}
\|(U, \bar{U})\|_{\mathcal{M}_{\gamma c, c}^{D}}^{2}=E\left[\sup _{0 \leq t \leq T}\|U(t)\|_{C^{c}(D, q)}^{2} e^{2 \gamma_{c} t}\right]+E\left[\int_{0}^{T}\|\bar{U}(t)\|_{C^{c}(D, q d)}^{2} e^{2 \gamma_{c} t} d t\right] \tag{4.28}
\end{equation*}
$$

Then, we have the following lemma.
Lemma 4.3 Under the conditions of Theorem 2.1, all the claims in the theorem are true.
Proof. First, by using (4.2), we can define the following map,

$$
\Xi:(U(\cdot, x), \bar{U}(\cdot, x)) \rightarrow(V(\cdot, x), \bar{V}(\cdot, x)) .
$$

Then, based on the norm defined in (4.27), we can show that $\Xi$ forms a contraction mapping in $\mathcal{M}_{\gamma}^{D}[0, T]$. In fact, for $i \in\{1,2, \ldots\}$, consider the following sequence of processes,

$$
\begin{aligned}
& \left(U^{i}(\cdot, x), \bar{U}^{i}(\cdot, x)\right) \in \mathcal{M}_{\gamma}^{D}[0, T], \\
& \left(U^{i+1}(\cdot, x), \bar{U}^{i+1}(\cdot, x)\right)=\Xi\left(U^{i}(\cdot, x), \bar{U}^{i}(\cdot, x)\right) .
\end{aligned}
$$

Furthermore, define

$$
\Delta f^{i}=f^{i+1}-f^{i} \quad \text { with } \quad f \in\{U, \bar{U}\}
$$

and take

$$
\begin{equation*}
\zeta\left(\Delta U^{i}(t, x)\right)=\operatorname{Tr}\left(\Delta U^{i}(t, x)\right) e^{2 \gamma_{0} t} \tag{4.29}
\end{equation*}
$$

Then, by using (2.11) and the similar argument as used in proving (4.16), we know that, for a $\gamma_{0}>0$ and each $i \in\{2,3, \ldots\}$,

$$
\begin{align*}
& \zeta\left(\Delta U^{i}(t, x)\right)+\int_{t}^{T} \operatorname{Tr}\left(\Delta \mathcal{J}\left(s, x, U^{i}, U^{i-1}\right)-\Delta \bar{U}^{i}(s, x)\right) e^{2 \gamma_{0} s} d s  \tag{4.30}\\
& \leq \hat{\gamma}_{0} \int_{t}^{T}\left\|\Delta \mathcal{L}\left(s, x, U^{i}, U^{i-1}\right)\right\|^{2} e^{2 \gamma_{0} s} d s-M^{i}(t) \\
& \leq \hat{\gamma}_{0} K_{a, 0} N^{i-1}(t)-M^{i}(t),
\end{align*}
$$

where $K_{a, 0}$ is some nonnegative constant depending only on $K_{D, 0}$. For the last inequality in (4.30), we have taken

$$
\begin{equation*}
\hat{\gamma}_{0}=\frac{3 K_{D, 0}^{2}}{2 \gamma_{0}}>0 . \tag{4.31}
\end{equation*}
$$

Furthermore, $N^{i-1}(t)$ appeared in (4.30) is given by

$$
\begin{equation*}
N^{i-1}(t)=\int_{t}^{T}\left(\left\|\Delta U^{i-1}(s)\right\|_{C^{k}(D, q)}^{2}+\left\|\Delta \bar{U}^{i-1}(s)\right\|_{C^{k}(D, q d)}^{2}\right) e^{2 \gamma_{0} s} d s \tag{4.32}
\end{equation*}
$$

and $M^{i}(t)$ is a martingale of the following form,

$$
\begin{equation*}
M^{i}(t)=-2 \sum_{j=1}^{d} \int_{t}^{T}\left(\left(\Delta U^{i}\right)(s, x)\right)^{\prime}\left(\Delta \mathcal{J}_{j}\left(s, x, U^{i}, U^{i-1}\right)-\left(\Delta \bar{U}^{i}\right)_{j}(s, x)\right) e^{2 \gamma_{0} s} d W_{j}(s) \tag{4.33}
\end{equation*}
$$

Then, by applying (4.30)-(4.33) and the martingale properties related to stochastic integral, we have

$$
\begin{align*}
& E\left[\left\|\Delta U^{i}(t, x)\right\|^{2} e^{2 \gamma_{0} t}+\int_{t}^{T} \operatorname{Tr}\left(\Delta \mathcal{J}\left(s, x, U^{i}, U^{i-1}\right)-\Delta \bar{U}^{i}(s, x)\right) e^{2 \gamma_{0} s} d s\right.  \tag{4.34}\\
& \leq \hat{\gamma}_{0}(T+1) K_{a, 0}\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma_{0}, k}^{D}}^{2} .
\end{align*}
$$

Thus, by using (4.33) and the Burkholder-Davis-Gundy's inequality (see, e.g., Theorem 48 in page 193 of Protter [28]), we have,

$$
\begin{align*}
& \text { 35) } \quad E\left[\sup _{0 \leq t \leq T}\left|M^{i}(t)\right|\right]  \tag{4.35}\\
& \leq \\
& \leq 4 \sum_{j=1}^{d} E\left[\sup _{0 \leq t \leq T}\left|\int_{0}^{t}\left(\Delta U^{i}(s, x)\right)^{\prime}\left(\Delta \mathcal{J}_{j}\left(s, x, U^{i}, U^{i-1}\right)-\left(\Delta \bar{U}^{i}\right)_{j}(s, x)\right) e^{2 \gamma_{0} s} d W_{j}(s)\right|\right] \\
& \leq \\
& \leq K_{b, 0} \sum_{j=1}^{d} E\left[\left(\int_{0}^{T}\left\|\Delta U^{i}(s, x)\right\|^{2}\left\|\left(\Delta \mathcal{J}^{i}\right)_{j}\left(s, x, U^{i}, U^{i-1}\right)-\left(\Delta \bar{U}^{i}\right)_{j}(s, x)\right\|^{2} e^{4 \gamma_{0} s} d s\right)^{\frac{1}{2}}\right] \\
& \leq \\
& \leq K_{b, 0} E\left[\left(\sup _{0 \leq t \leq T}\left\|\Delta U^{i}(t, x)\right\|^{2} e^{2 \gamma_{0} t}\right)^{\frac{1}{2}}\right.
\end{align*}
$$

$$
\begin{aligned}
& \quad\left(\sum_{j=1}^{d}\left(\int_{0}^{T}\left\|\Delta \mathcal{J}_{j}\left(s, x, U^{i}, U^{i-1}\right)-\left(\Delta \bar{U}^{i}\right)_{j}(s, x)\right\|^{2} e^{2 \gamma_{0} s} d s\right)^{\frac{1}{2}}\right] \\
& \leq \\
& \frac{1}{2} E\left[\sup _{0 \leq t \leq T}\left\|\Delta U^{i}(t, x)\right\|^{2} e^{2 \gamma_{0} t}\right] \\
& \\
& +d K_{b, 0}^{2} E\left[\left(\int_{0}^{T} \operatorname{Tr}\left(\Delta \mathcal{J}\left(s, x, U^{i}, U^{i-1}\right)-\left(\Delta \bar{U}^{i}\right)(s, x)\right) e^{2 \gamma_{0} s} d s\right)\right] \\
& \leq \\
& \frac{1}{2} E\left[\sup _{0 \leq t \leq T}\left\|\Delta U^{i}(t, x)\right\|_{C^{0}(q)}^{2} e^{2 \gamma_{0} t}\right]+\hat{\gamma}_{0}(T+1) d K_{a, 0} K_{b, 0}^{2}\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma_{0}, k}^{D}}^{2}
\end{aligned}
$$

where $K_{b, 0}$ is some nonnegative constant depending only on $K_{D, 0}$ and $T$. The last inequality of (4.35) is owing to (4.34). Therefore, by using (4.30)-(4.35), we know that

$$
\begin{align*}
& E\left[\sup _{0 \leq t \leq T}\left\|\Delta U^{i}(t)\right\|_{C^{0}(q)}^{2} e^{2 \gamma_{0} t}\right]  \tag{4.36}\\
& \leq 2\left(1+d K_{b, 0}^{2}\right) K_{a, 0} \hat{\gamma}_{0}(T+1)\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma_{0}, k}^{D}}^{2}
\end{align*}
$$

Furthermore, by using (4.30) and (2.12), we know that, for each $i \in\{3,4, \ldots\}$,

$$
\begin{align*}
& E\left[\int_{t}^{T} \operatorname{Tr}\left(\Delta \bar{U}^{i}(s, x)\right) e^{2 \gamma_{0} s} d s\right]  \tag{4.37}\\
\leq & 2 E\left[\int_{t}^{T} \operatorname{Tr}\left(\Delta \mathcal{J}\left(s, x, U^{i}, U^{i-1}\right)-\Delta \bar{U}^{i}(s, x)\right) e^{2 \gamma_{0} s} d s\right] \\
& +2 E\left[\int_{t}^{T} \operatorname{Tr}\left(\Delta \mathcal{J}\left(s, x, U^{i}, U^{i-1}\right)\right) e^{2 \gamma_{0} s} d s\right] \\
\leq & 2 \hat{\gamma}_{0} K_{C, 0}\left(\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma_{0}, k}^{D}}^{2}+\left\|\left(\Delta U^{i-2}, \Delta \bar{U}^{i-2}\right)\right\|_{\mathcal{M}_{\gamma_{0}, k}^{D}}^{2}\right),
\end{align*}
$$

where $K_{C, 0}$ is some nonnegative constant depending only on $K_{D, 0}$ and $T$. Hence, by (4.30), (4.36)-(4.37), and the fact that all functions and norms used in this paper are continuous in terms of $x$, we have that

$$
\begin{align*}
& \left\|\left(\Delta U^{i}, \Delta \bar{U}^{i}\right)\right\|_{\mathcal{M}_{\gamma_{0}, 0}^{D}}^{2}  \tag{4.38}\\
& \leq \hat{\gamma}_{0} K_{d, 0}\left(\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma_{0}, k}^{D}}^{2}+\left\|\left(\Delta U^{i-2}, \Delta \bar{U}^{i-2}\right)\right\|_{\mathcal{M}_{\gamma_{0}, k}^{D}}^{2}\right)
\end{align*}
$$

where $K_{d, 0}$ is some nonnegative constant depending only on $K_{D, 0}$ and $T$.
Next, by using Lemma 4.2 and the similar construction as used in (4.29), we can define

$$
\begin{equation*}
\zeta\left(\Delta U^{c, i}(t, x)\right) \equiv \operatorname{Tr}\left(\Delta U^{c, i}(t, x)\right) e^{2 \gamma_{c} t} \tag{4.39}
\end{equation*}
$$

for each $c \in\{1,2, \ldots\}$, where

$$
\left.\left.\left.\Delta U^{c, i}(t, x)\right)=\left(\Delta U^{(0), i}(t, x)\right), \Delta U^{(1), i}(t, x)\right), \ldots, \Delta U^{(c), i}(t, x)\right)^{\prime}
$$

Thus, by using the Itô's formula and the similar discussion for (4.38), we have that

$$
\begin{align*}
& \left\|\left(\Delta U^{i}, \Delta \bar{U}^{i}\right)\right\|_{\mathcal{M}_{\gamma c, c}}^{2}  \tag{4.40}\\
\leq & \hat{\gamma}_{c} K_{d, c}\left(\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma c, k+c}}^{2}+\left\|\left(\Delta U^{i-2}, \Delta \bar{U}^{i-2}\right)\right\|_{\mathcal{M}_{\gamma, k+c}^{D}}^{2}\right) \\
\leq & \frac{\delta}{\left((c+1)^{10}(c+2)^{10} \ldots(c+k)^{10}\right)(\eta(c+1) \eta(c+2) \ldots \eta(c+k))} \\
& \left(\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma_{k+c}, k+c}^{D}}^{2}+\left\|\left(\Delta U^{i-2}, \Delta \bar{U}^{i-2}\right)\right\|_{\mathcal{M}_{\gamma_{k+c}, k+c}^{D}}^{D}\right),
\end{align*}
$$

where, for the last inequality of (4.40), we have taken the number sequence $\gamma$ such that $\gamma_{0}<\gamma_{1}<\ldots$ and

$$
\hat{\gamma}_{c} K_{d, c}\left((c+1)^{10}(c+2)^{10} \ldots(c+k)^{10}\right)(\eta(c+1) \eta(c+2) \ldots \eta(c+k)) \leq \delta
$$

for some $\delta>0$ such that $2 \sqrt{e^{k} \delta}$ is sufficiently small. Hence, we know that

$$
\begin{equation*}
\left\|\left(\Delta U^{i}, \Delta \bar{U}^{i}\right)\right\|_{\mathcal{M}_{\gamma}^{D}}^{2} \leq e^{k} \delta\left(\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma}^{D}}^{2}+\left\|\left(\Delta U^{i-2}, \Delta \bar{U}^{i-2}\right)\right\|_{\mathcal{M}_{\gamma}^{D}}^{2}\right) . \tag{4.41}
\end{equation*}
$$

Owing to $\left(a^{2}+b^{2}\right)^{1 / 2} \leq a+b$ for $a, b \geq 0$, we can conclude that

$$
\begin{equation*}
\left\|\left(\Delta U^{i}, \Delta \bar{U}^{i}\right)\right\|_{\mathcal{M}_{\gamma}^{D}} \leq \sqrt{e^{k} \delta}\left(\left\|\left(\Delta U^{i-1}, \Delta \bar{U}^{i-1}\right)\right\|_{\mathcal{M}_{\gamma}^{D}}+\left\|\left(\Delta U^{i-2}, \Delta \bar{U}^{i-2}\right)\right\|_{\mathcal{M}_{\gamma}^{D}}\right) . \tag{4.42}
\end{equation*}
$$

Thus, it follows from (4.42) that

$$
\begin{align*}
\sum_{i=3}^{\infty}\left\|\left(\Delta U^{i}, \Delta \bar{U}^{i}\right)\right\|_{\mathcal{M}_{\gamma}^{D}} & \leq \frac{\sqrt{e^{k} \delta}}{1-2 \sqrt{e^{k} \delta}}\left(2\left\|\left(\Delta U^{2}, \Delta \bar{U}^{2}\right)\right\|_{\mathcal{M}_{\gamma}^{D}}+\left\|\left(\Delta U^{1}, \Delta \bar{U}^{1}\right)\right\|_{\mathcal{M}_{\gamma}^{D}}\right)  \tag{4.43}\\
& <\infty .
\end{align*}
$$

Therefore, by using (4.43), we see that $\left(U^{i}, \bar{U}^{i}\right)$ with $i \in\{1,2, \ldots\}$ forms a Cauchy sequence in $\mathcal{M}_{\gamma}^{D}[0, T]$. Hence, there is some $(U, \bar{U})$ such that

$$
\begin{equation*}
\left(U^{i}, \bar{U}^{i}\right) \rightarrow(U, \bar{U}) \text { as } i \rightarrow \infty \text { in } \mathcal{M}_{\gamma}^{D}[0, T] . \tag{4.44}
\end{equation*}
$$

Finally, by using (4.44) and the similar procedure as used for Theorem 5.2.1 in pages 68-71 of $\emptyset$ ksendal [22], the proof of Lemma 4.3 is completed.

### 4.2 Proof of Theorem 2.1

By combining Lemma 4.1-Lemma 4.3, we can reach a proof for Theorem 2.1.

## 5 Proof of Theorem 3.1

To prove the theorem, we first develop new fundamental theory for random field based Malliavin Calculus in subsections 5.1-5.5. Then, based on this newly developed theory, we provide a proof for Theorem 3.1 in subsections 5.6-5.7.

### 5.1 Basic Properties of Random Field Based Malliavin Calculus

Lemma 5.1 The unbounded operator defined in (3.22) is closable from $L^{2}\left(\Omega, C^{\infty}\left(D, R^{q}\right)\right)$ to $L_{\alpha, 2}^{2}\left(\Omega,\left(C^{\infty}(D, H)\right)^{q}\right)$ with $\alpha \in\{1,2\}$.

Proof. First, we consider the case that $\alpha=1$. Let $\left\{F^{i}: i \in\{1,2, \ldots\}\right\}$ be a sequence of smooth random variables, which converges to zero along $i \in\{1,2, \ldots\}$ in $L^{2}\left(\Omega, C^{\infty}\left(D, R^{q}\right)\right)$. Thus, we can conclude that $F_{r, i_{1} \ldots i_{p}}^{(c), i}(x) \rightarrow 0$ along $i \in\{1,2, \ldots\}$ in the usual mean-square sense for each $x \in D, r \in\{1, \ldots, q\}, c \in\{0,1, \ldots\}$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$. In the meanwhile, we suppose that the corresponding sequence related to Malliavin derivatives converges to some $\eta$ in $L_{1,2}^{2}\left(\Omega,\left(C^{\infty}(D, H)\right)^{q}\right)$, which implies that $D F_{r, i_{1} \ldots i_{p}}^{(c), i}(x) \rightarrow \eta_{r, i_{1} \ldots i_{p}}^{(c)}(x)$ along $i \in\{1,2, \ldots\}$ in the usual mean-square sense for each $x \in D, r \in\{1, \ldots, q\}, c \in\{0,1, \ldots\}$, and $\left(i_{1}, \ldots, i_{p}\right) \in$ $\mathcal{I}^{c}$. Then, it follows from the proof of Proposition 1.2 .1 in page 26 of Nualart [20] that $\eta_{r, i_{1} \ldots i_{p}}^{(c)}(x)=0$ for each $x \in D, r \in\{1, \ldots, q\}, c \in\{0,1, \ldots\}$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$. Thus, we know that $\eta=0$. Hence, by the definition of the closable operator (see, e.g., page 77 of Yosida [35]), we conclude that the claim in the lemma is true if $\alpha=1$.

Second, we consider the case that $\alpha=2$. By combining the above discussion and the proof used for Exercise 1.2.3 in page 34 of Nualart [20], we know that the claim for $\alpha=2$ is also true.

Lemma 5.2 Consider each $F \in \mathcal{D}_{\infty}^{1,2}$. Then, for each $c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$, and $x \in D$, we have

$$
\begin{equation*}
E\left[D_{t} F_{i_{1} \ldots i_{p}}^{(c)}(x) \mid \mathcal{F}_{t}\right] \in\left(L^{2}\left([0, T] \times \Omega, C^{\infty}\left(D, R^{q}\right)\right)\right)^{d} \tag{5.1}
\end{equation*}
$$

and furthermore,

$$
\begin{equation*}
F_{i_{1} \ldots i_{p}}^{(c)}(x)=E\left[F_{i_{1} \ldots i_{p}}^{(c)}(x)\right]+\int_{0}^{T} E\left[D_{t} F_{i_{1} \ldots i_{p}}^{(c)}(x) \mid \mathcal{F}_{t}\right] d W(t) . \tag{5.2}
\end{equation*}
$$

Proof. For each $F \in \mathcal{D}_{\infty}^{1,2}$ and $r \in\{1, \ldots, q\}$, we have the following calculation,

$$
\begin{align*}
& E\left[\int_{0}^{T}\left\|E\left[D_{t} F_{r, i_{1} \ldots i_{p}}^{(c)}(x) \mid \mathcal{F}_{t}\right]\right\|_{C^{\infty}(D, d)}^{2} d t\right]  \tag{5.3}\\
\leq & \int_{0}^{T} E\left[\left\|D_{t} F_{r, i_{1} \ldots i_{p}}^{(c)}(x)\right\|_{C^{\infty}(D, d)}^{2}\right] d t \\
= & E\left[\int_{0}^{T}\left\|D_{t} F_{r, i_{1} \ldots i_{p}}^{(c)}(x)\right\|_{C^{\infty}(D, d)}^{2} d t\right] .
\end{align*}
$$

Thus, it follows from (5.3) that the claim in (5.1) is true. Furthermore, owing to the Clark-Haussmann-Ocone formula (see, e.g., Aase et al. [1]), we know that (5.2) holds.

Lemma 5.3 Let $Z \in L_{\mathcal{F}, p}^{2}\left([t, T], C^{\infty}\left(D, R^{d}\right)\right)$ with a fixed $t \in[0, T]$ replacing $t=0$ in the previous discussion be such that $F \in \mathcal{D}_{\infty}^{1,2}$ with $F$ defined by

$$
\begin{equation*}
F_{i_{1} \ldots i_{p}}^{(c)}(t, x)=\int_{t}^{T} Z_{i_{1} \ldots i_{p}}^{(c)}(s, x) d W(s) \tag{5.4}
\end{equation*}
$$

for each $x \in D, c \in\{0,1, \ldots\}$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$. Then, $Z \in \mathcal{D}_{\infty}^{1,2} \cap L_{1,2}^{2}\left(\Omega,\left(C^{\infty}(D, H)\right)^{d}\right)$. Furthermore, for each $j \in\{1, \ldots, d\}$,

$$
\mathcal{D}_{\theta}^{j} F_{i_{1} \ldots i_{p}}^{(c)}(t, x)= \begin{cases}\int_{t}^{T} \mathcal{D}_{\theta}^{j} Z_{i_{1} \ldots i_{p}}^{(c)}(s, x) d W(s) & \text { if } \theta \leq t,  \tag{5.5}\\ Z_{i_{1} \ldots i_{p}}^{(c), j}(\theta, x)+\int_{\theta}^{T} \mathcal{D}_{\theta}^{j} Z_{i_{1} \ldots i_{p}}^{(c)}(s, x) d W(s) & \text { if } \theta>t\end{cases}
$$

where $Z_{i_{1} \ldots i_{p}}^{(c), j}$ is the $j$ th component of $Z_{i_{1} \ldots i_{p}}^{(c)}$.
Proof. First of all, if $Z \in \mathcal{D}_{\infty}^{1,2} \cap L_{1,2}^{2}\left(\Omega,\left(C^{\infty}(D, H)\right)^{d}\right)$ and $F$ is defined by (5.4) for each fixed $t \in[0, T]$, it follows from Proposition 3.4 in Nualart and Pardoux [21] that $F \in \mathcal{D}_{\infty}^{1,2}$ and the claim in (5.5) holds for each $x \in D, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$, and $j \in\{1, \ldots, d\}$. Furthermore, owing to the Ito's isometry, we have

$$
\begin{equation*}
\|\mid F\|_{1,2}^{\infty, 2}=\sum_{v=0}^{\infty} \xi(v)\|F\|_{1,2}^{v, 2} \tag{5.6}
\end{equation*}
$$

where

$$
\|\mid F\| \|_{1,2}^{v, 2}=E\left[\int_{t}^{T} \Lambda_{v}\left\|Z_{r, i_{1} \ldots i_{p}}^{(c)}(s, x)\right\|_{C^{v}(D, d)}^{2} d s+\int_{t}^{T} \int_{t}^{T} \Lambda_{v}\left\|\mathcal{D}_{\theta} Z_{r, i_{1} \ldots i_{p}}^{(c)}(s, x)\right\|_{C^{v}(D, d d)}^{2} d \theta d s\right],
$$

and $\Lambda_{v}$ is defined in (3.26). Therefore, by the similar argument as used in the proof of Lemma 2.3 of Pardoux and Peng [24], it suffices to show that the following set for each fixed $t \in[0, T]$

$$
\begin{equation*}
\left\{F \text { satisfying (5.4) with } Z \in \mathcal{D}_{\infty}^{1,2} \cap L_{1,2}^{2}\left(\Omega,\left(C^{\infty}(D, H)\right)^{d}\right)\right\} \tag{5.7}
\end{equation*}
$$

is dense in $\mathcal{D}_{\infty}^{1,2} \cap L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(D, R^{d}\right)\right)$. In fact, it is a direct conclusion of Lemma 5.1 and the fact that the set defined in (5.7) contains the following set for each fixed $t \in[0, T]$ owing to Lemma 5.2,

$$
\begin{aligned}
& \left\{F \in \mathcal{S} \cap L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(D, R^{d}\right)\right) \text { with } E\left[F_{i_{1} \ldots i_{p}}^{(c)}(t, x)\right]=0\right. \\
& \text { for each } \left.x \in D c \in\{1,2, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}\right\} .
\end{aligned}
$$

Hence, we complete the proof of the lemma.
Now, for each $t \in\left[t_{j_{0}-1}, t_{j_{0}}\right), x \in D$, and $v \in\{0,1, \ldots, M\}$, we consider the following B-SPDE,

$$
\begin{align*}
V_{i_{1} \ldots i_{p}}^{(v)}(t, x)= & H_{i_{1} \ldots i_{p}}^{(v)}(x)+\int_{t}^{T} \mathcal{L}_{i_{1} \ldots i_{p}}^{(v)}(s, x, V, \bar{V}) d s  \tag{5.8}\\
& +\int_{t}^{T}\left(\mathcal{J}_{i_{1} \ldots i_{p}}^{(v)}(s, x, V)-\bar{V}_{i_{1} \ldots i_{p}}^{(v)}(s, x)\right) d W(s),
\end{align*}
$$

where $i_{1}+\ldots+i_{p}=v$ and $i_{l} \in\{0,1, \ldots, v\}$ with $l \in\{1, \ldots, p\}$. Then, we have the following lemma.

Lemma 5.4 Under conditions (3.29) and (2.11)-(2.14), there is a unique adapted and squareintegrable solution $\left(V_{i_{1} \ldots i_{p}}^{(v)}(t, x), \bar{V}_{i_{1} \ldots i_{p}}^{(v)}(t, x)\right)$ to the B-SPDE in (5.8).

Proof. The lemma is a direct conclusion of Theorem 2.1.
Remark 5.1 Note that, since the structures of the B-SPDEs displayed in (5.8) are the same for all $v \in\{0,1, \ldots, M\}$, we only consider the case that $v=0$ in the rest of this subsection, i.e., the equation in (1.1). Furthermore, for the time-inhomogeneous B-SPDE in (5.8), we can introduce an additional B-SPDE through

$$
\begin{equation*}
V^{0}(t, x)=T-\int_{t}^{T} d s \tag{5.9}
\end{equation*}
$$

Obviously, $\left(V^{0}(t, x), \bar{V}^{0}(t, x)\right)$ with $V^{0}(t, x)=t$ and $\bar{V}^{0}(t, x)=(0, \ldots, 0)=\hat{0}$ (ad-dimensional zero row vector) is the unique solution to the B-SPDE in (5.9). Then, by combining (5.9) and (1.1), we can get a $(q+1)$-dimensional B-SPDE,

$$
\begin{equation*}
U(t, x)=\tilde{H}(x)+\int_{t}^{T} \tilde{\mathcal{L}}(x, U) d s+\int_{t}^{T}(\tilde{\mathcal{J}}(x, U)-\bar{U}(s, x)) d W(s), \tag{5.10}
\end{equation*}
$$

where

$$
\begin{aligned}
\tilde{H}(x) & =\left(T, H(x)^{\prime}\right)^{\prime}, \\
U(t, x) & =\left(V^{0}(t, x), V(t, x)^{\prime}\right)^{\prime}, \\
\bar{U}(t, x) & =\left(\bar{V}^{0}(t, x)^{\prime}, \bar{V}(t, x)^{\prime}\right)^{\prime}, \\
\overline{\mathcal{L}}(x, U) & =\left(-1, \mathcal{L}(x, U)^{\prime}\right)^{\prime}, \\
\bar{J}(x, U) & =\left(\hat{0}^{\prime}, \mathcal{J}(x, U)^{\prime}\right)^{\prime} .
\end{aligned}
$$

Thus, without loss of generality and to be simple for notations, we only consider the timehomogeneous case in (1.1) in the rest of this section, i.e., the case corresponding to $\mathcal{L}(s, x, V)=$ $\mathcal{L}(x, V)$ and $\mathcal{J}(s, x, V)=\mathcal{J}(x, V)$.

### 5.2 B-SPDE with Malliavin Derivative Terminal Condition

First, consider a properly chosen number sequence $\gamma=\left\{\gamma_{c}, c=0,1,2, \ldots\right\}$ satisfying $0<\gamma_{0}<$ $\gamma_{1}<\ldots$ such that the discussions for Theorem 2.1 and Subsections 5.2-5.3 are meaningful, which will be elaborated during the subsequent proof. Second, we redefine the space in (4.26) as follows,

$$
\begin{equation*}
\mathcal{N}_{\gamma}^{D}[0, T]=D_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q \times d}\right)\right) \times L_{\mathcal{F}, p}^{2}\left([0, T], C^{\infty}\left(D, R^{q \times d \times d}\right)\right) \tag{5.11}
\end{equation*}
$$

endowed with the norm similarly defined as in (4.27)-(4.28). Then, we have the following lemma.

Lemma 5.5 Under conditions as required in Theorem 3.1 and with Remark 5.1, if $(V(t, x), \bar{V}(t, x))$ $\in \mathcal{Q}_{\mathcal{F}}^{2}([0, T] \times D)$ is an adapted solution to (1.1), then the system of the following B-SPDEs has a unique square-integrable adapted solution $\left(V_{i_{1} \ldots i_{p}}^{(c), \theta}(t, x), \bar{V}_{i_{1} \ldots i_{p}}^{(c), \theta}(t, x)\right)$, i.e., each component $\left(V_{i_{1} \ldots i_{p}}^{j,(c), \theta}(t, x), \bar{V}_{i_{1} \ldots i_{p}}^{j,(c), \theta}(t, x)\right)$ satisfies ,

$$
\begin{align*}
V_{i_{1} \ldots i_{p}}^{j,(c), \theta}(t, x)= & D_{\theta}^{j} H_{i_{1} \ldots i_{p}}^{(c)}(x)  \tag{5.12}\\
& +\int_{t}^{T} \sum_{l=0}^{k} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}}^{(c+1)}(x, V) V_{j_{1} \ldots j_{p}}^{j,(c+l), \theta}(s, x) d s \\
& +\int_{t}^{T} \sum_{l=0}^{m} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, \bar{v}_{j_{1} \ldots j_{p}}}^{(c+1)}(x, V) \bar{V}_{j_{1} \ldots j_{p}}^{j,(c+l), \theta}(s, x) d s \\
& +\int_{t}^{T} \sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{J}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c+1)}(x, V) V_{j_{1} \ldots j_{p}}^{j,(c+l), \theta}(s, x) d W(s) \\
& -\int_{t}^{T} \bar{V}_{i_{1} \ldots i_{p}}^{j,(c), \theta}(s, x) d W(s)
\end{align*}
$$

for $j \in\{1, \ldots, d\}, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, t \in[\theta, T]$, and $x \in D$. Furthermore, for each $\delta \in\left(0, \frac{2}{3}\right)$, there is a number sequence $\gamma_{0}<\gamma_{1}<\cdots$ such that

$$
\begin{equation*}
\int_{0}^{T}\left\|\left(V^{\theta}, \bar{V}^{\theta}\right)\right\|_{\mathcal{N}_{\gamma}^{D}[\theta, T]}^{2} d \theta<\frac{1}{1-\delta}\|H\|_{1,2}^{\infty, 2}+\frac{\delta T^{2}}{1-\delta}<\infty \tag{5.13}
\end{equation*}
$$

Proof. First, we note that it follows from Theorem 2.1 and its proof that there is a unique adapted solution $(V(t, x), \bar{V}(t, x)) \in \mathcal{Q}_{\mathcal{F}}^{2}([0, T] \times D)$ to (1.1). Thus, we know that there is no explosion time for the process $(V(t, x), \bar{V}(t, x))$ over time interval $[0, T]$. Furthermore, for each $x \in D, V(\cdot, x)$ is a càdlàg process and $\bar{V}(\cdot, x)$ is its corresponding predictable process. Then, it follows from Theorem 3 in page 4 of Protter [28] and Remark 5.1 in page 21 of Ikeda and Watanabe [15] that

$$
\begin{equation*}
\tau_{w} \equiv T \wedge \inf \left\{t>0,\|V(t)\|_{C^{\infty}(D, q)}+\|\bar{V}(t)\|_{C^{\infty}(D, q d)}>w\right\} \tag{5.14}
\end{equation*}
$$

is a sequence of nondecreasing $\left\{\mathcal{F}_{t}\right\}$-stopping times and satisfies $\tau_{w} \rightarrow T$ a.s. as $w \rightarrow \infty$ along $w \in\{0,1, \ldots\}$.

Now, for all $j \in\{1, \ldots, d\}, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, t \in\left[\theta, \tau_{w}\right]$, and $x \in D$, we define the following system of B-SPDEs,

$$
\begin{align*}
V_{i_{1} \ldots i_{p}}^{j,(c), \theta}(t, x)= & E\left[D_{\theta}^{j} H_{i_{1} \ldots i_{p}}^{(c)}(x) \mid \mathcal{F}_{\tau_{w}}\right]  \tag{5.15}\\
& +\int_{t \wedge \tau_{w}}^{\tau_{w}} \sum_{l=0}^{k} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}}^{(c+1)}(x, V) V_{j_{1} \ldots j_{p}}^{j,(c+l), \theta}(s, x) d s \\
& +\int_{t \wedge \tau_{w}}^{\tau_{w}} \sum_{l=0}^{m} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, \bar{v}_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c+1)}(x, V) \bar{V}_{j_{1} \ldots j_{p}}^{j,(c+l), \theta}(s, x) d s
\end{align*}
$$

$$
\begin{aligned}
& +\int_{t \wedge \tau_{w}}^{\tau_{w}} \sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{J}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(x, V) V_{j_{1} \ldots j_{p}}^{j,(c+l), \theta}(s, x) d W(s)} \\
& -\int_{t \wedge \tau_{w}}^{\tau_{w}} \bar{V}_{i_{1} \ldots i_{p}}^{j,(c), \theta}(s, x) d W(s) .
\end{aligned}
$$

Then, over the random time interval $\left[\theta, \tau_{w}\right]$, the B-SPDEs in (5.15) satisfy conditions (2.11) and (2.14). Thus, by slightly generalizing the discussions in proving Theorem 2.1 and Yong and Zhou [34], we know that (5.15) has a unique adapted solution $\left(V^{\theta, w}, \bar{V}^{\theta, w}\right)$ in $\mathcal{N}_{\gamma}^{D}\left[\theta, \tau_{w}\right]$. Furthermore, the solution has the following infinite-dimensional vector form,

$$
\begin{equation*}
\left\{\left(V_{r, i_{1} \ldots i_{p}}^{(c), \theta, w}(t, x), \bar{V}_{r, i_{1} \ldots i_{p}}^{(c), \theta w}(t, x)\right), r \in\{1, \ldots, q\}, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}\right\} . \tag{5.16}
\end{equation*}
$$

Thus, by (3.29), (2.13)-(2.14), the Itô's formula, and the similar technique in the proof for the claims in (4.38) and (4.40), we know that

$$
\left.\begin{array}{rl} 
& \left\|\left(V^{\theta, w}, \bar{V}^{\theta, w}\right)\right\|_{\mathcal{N}_{\gamma c, c}^{D}\left[\theta, \tau_{w}\right]}^{2}  \tag{5.17}\\
\leq & E\left[\bar{\Lambda}_{c}\left\|D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)}\right\|_{C^{\infty}(D, q)}^{2}\right]+\hat{\gamma}_{c} K_{d, c}^{1}\left((T-\theta) \delta_{0 c}+\left\|\left(V^{\theta, w}, \bar{V}^{\theta, w}\right)\right\|_{\mathcal{N}_{\gamma c, c+2 k}^{D}}^{2}\left[\theta, \tau_{w}\right]\right.
\end{array}\right)
$$

for each $c \in\{0,1, \ldots\}$. The notation $\bar{\Lambda}_{c}$ in (5.17) is defined by

$$
\begin{equation*}
\bar{\Lambda}_{c}=\max _{v \in\{0,1, \ldots, c\}} \max _{\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}}, \tag{5.18}
\end{equation*}
$$

and $\delta_{0 c}$ is defined in (2.13)-(2.14). Furthermore, $K_{d, c}^{1}$ is some nonnegative constant depending only on $c, T$ and the region $D$, which satisfies $K_{d, c}^{1} \geq K_{d, c}$ (that is used in (4.40)). In addition, $\hat{\gamma}_{c}$ is a nonnegative constant depending on $\gamma_{c}$ and can be arbitrarily chosen by suitably managing the number sequence $\gamma_{0}<\gamma_{1}<\cdots$ such that

$$
\begin{equation*}
\hat{\gamma}_{c} K_{d, c}^{1}\left((c+1)^{10}(c+2)^{10} \ldots(c+2 k)^{10}\right)(\eta(c+1) \eta(c+2) \ldots \eta(c+2 k)) e^{2 k} \leq \delta \tag{5.19}
\end{equation*}
$$

for some constant $\delta \in(0,2 / 3)$. Therefore, we have

$$
\begin{aligned}
& \left\|\left(V^{\theta, w}, \bar{V}^{\theta, w}\right)\right\|_{\mathcal{N}_{\gamma}^{D}\left[\theta, \tau_{w}\right]} \\
\leq & \sum_{c=1}^{\infty} \xi(c) E\left[\bar{\Lambda}_{c}\left\|D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)}\right\|_{C^{\infty}(D, q)}^{2}\right]+\delta T+\delta\left\|\left(V^{\theta, w}, \bar{V}^{\theta, w}\right)\right\|_{\mathcal{N}_{\gamma}^{D}\left[\theta, \tau_{w}\right]}
\end{aligned}
$$

Thus, it follows from conditions (3.29)-(3.31) that

$$
\begin{align*}
\left\|\left(V^{\theta, w}, \bar{V}^{\theta, w}\right)\right\|_{\mathcal{N}_{\gamma}^{D}\left[\theta, \tau_{w}\right]}^{2} & \leq \frac{1}{1-\delta} \sum_{c=1}^{\infty} \xi(c) E\left[\bar{\Lambda}_{c}\left\|D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)}\right\|_{C^{\infty}(D, q)}^{2}\right]+\frac{\delta T}{1-\delta}  \tag{5.20}\\
& <\infty
\end{align*}
$$

Second, we use $\Pi^{\theta, w}(t, x) \equiv\left(V^{\theta, w}(t, x), \bar{V}^{\theta, w}(t, x)\right)$ for $t \leq \tau_{w}$ and $x \in D$ to denote the unique adapted solution to the system in (5.15) for each $w \in\{1,2, \ldots\}$. Then, it follows from the Ito's formula, conditions (2.11)-(2.14), and the similar proof for (5.20) that

$$
\begin{align*}
& \left\|\Pi^{\theta, w_{1}}-\Pi^{\theta, w_{2}}\right\|_{\mathcal{N}_{\gamma}^{D}[\theta, T]}^{2}  \tag{5.21}\\
\leq & \frac{1}{1-\delta} \sum_{c=1}^{\infty} \xi(c) E\left[\bar{\Lambda}_{c}\left\|E\left[D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)} \mid \mathcal{F}_{\tau_{w_{1}}}\right]-E\left[D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)} \mid \mathcal{F}_{\tau_{w_{2}}}\right]\right\|_{C^{\infty}(D, q)}^{2}\right] \\
& +\frac{\delta}{1-\delta} E\left[\tau_{w_{2}}-\tau_{w_{1}}\right] \\
\leq & \frac{1}{1-\delta} \sum_{c=1}^{\infty} \xi(c)\left(E\left[\bar{\Lambda}_{c}\left\|E\left[D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)} \mid \mathcal{F}_{\tau_{w_{1}}}\right]-D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)}\right\|_{C^{\infty}(D, q)}^{2}\right]\right. \\
& \left.+E\left[\bar{\Lambda}_{c}\left\|E\left[D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)} \mid \mathcal{F}_{\tau_{w_{1}}}\right]-D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)}\right\|_{C^{\infty}(D, q)}^{2}\right]\right)+\frac{\delta}{1-\delta} E\left[\tau_{w_{2}}-\tau_{w_{1}}\right] \\
\rightarrow & 0
\end{align*}
$$

as $w_{1}, w_{2} \rightarrow \infty$ along $w_{1}, w_{2} \in\{1,2, \ldots\}$. Note that the last claim of (5.21) follows from (3.29)-(3.31) and the Martingale convergence theorem (see, e.g., Page 8 of Protter [28]). Furthermore, in the proof of (5.21), we also used the fact that

$$
\begin{equation*}
V^{\theta, w}(t, x)=E\left[D_{\theta} H_{i_{1} \ldots i_{p}}^{(v)}(x) \mid \mathcal{F}_{\tau_{w_{1}}}\right] \quad \text { for each } t \in\left[\tau_{w}, T\right] . \tag{5.22}
\end{equation*}
$$

Thus, from (5.21), we know that $\left\{\Pi^{\theta, w}, w \in\{1,2, \ldots\}\right\}$ is a cauchy sequence in $\mathcal{N}_{\gamma}^{D}[\theta, T]$. Hence, there is a $\Pi^{\theta} \in \mathcal{N}_{\gamma}^{D}[\theta, T]$ such that

$$
\begin{equation*}
\Pi^{\theta, w} \rightarrow \Pi^{\theta} \text { as } w \rightarrow \infty . \tag{5.23}
\end{equation*}
$$

In addition, we claim that $\Pi^{\theta}$ is the unique square-integrable adapted solution to the system of B-SPDEs in (5.12).

In fact, since $\Pi^{\theta, w}$ is a solution satisfying (5.15) for each $w \in\{1,2, \ldots\}$, it follows from the Ito's isometry, Holder's inequality, the similar ideas as used for (5.21) and the proof of Theorem 5.1.2 in page 68 of $\emptyset$ ksendal [22] that $\Pi^{\theta}$ is a square-integrable adapted solution to the system of B-SPDEs in (5.12). Next, suppose that $\Pi_{1}^{\theta}$ and $\Pi_{2}^{\theta}$ are two required solutions to the system in (5.12). Then, $\Pi_{1}^{\theta}-\Pi_{2}^{\theta}$ is a square-integrable adapted solution to the system in (5.12) with terminal value 0 . Thus, $\Pi_{1}^{\theta}-\Pi_{2}^{\theta}$ is the unique square-integrable adapted solution to the system in (5.15) with terminal value 0 over each random interval $\left[0, \tau_{w}\right]$ for $w \in\{1,2, \ldots\}$. Since $\tau_{w} \rightarrow T$ as $w \rightarrow \infty$, we know that $\Pi_{1}^{\theta}=\Pi_{2}^{\theta}$ a.s.

Finally, it follows from (5.23) and (5.20) that the claim in (5.13) is true. Thus, we complete the proof of Lemma 5.5.

### 5.3 First-Order Malliavin Derivative Based B-SPDE

First, let $L_{\alpha, 2}^{\infty, 2}\left([0, T] \times \Omega,\left(C^{\infty}(D, H)\right)^{q}\right)$ represent the set of $H^{q}$-valued progressively measurable processes $\{\zeta(t, x, \omega), 0 \leq t \leq T, \omega \in \Omega\}$ for each $x \in D$ such that

- For a.e. $t \in[0, T], \zeta(t, \cdot, \cdot) \in \mathcal{D}_{\infty}^{\alpha, 2}$;
- $(t, x, \omega) \rightarrow \mathcal{D} \zeta_{i_{1} \ldots i_{p}}^{(c)}(t, x, \omega) \in\left(L^{2}\left([0, T] \times \Omega, C^{\infty}\left(D, R^{q}\right)\right)\right)^{d}$ admits a progressively measurable version for each $x \in D, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$ if $\alpha=1$. In addition, $(t, x, \omega) \rightarrow \mathcal{D D} \zeta_{i_{1} \ldots i_{p}}^{(c)}(t, x, \omega) \in\left(L^{2}\left([0, T] \times \Omega, C^{\infty}\left(D, R^{q}\right)\right)\right)^{d \times d}$ also admits a progressively measurable version if $\alpha=2$;
- The following norm is defined,

$$
\left\|\left|\zeta \| | _ { \alpha , 2 } ^ { \infty , 2 } = \sum _ { v = 0 } ^ { \infty } \xi ( v ) \| \zeta \zeta \left\|\|_{\alpha, 2}^{v, 2}<\infty\right.\right.\right.
$$

where, for each $v \in\{0,1, \ldots\}$,

$$
\begin{array}{ll}
\|\mid \zeta\| \|_{1,2}^{v, 2}= & E\left[\int_{0}^{T} \Lambda_{v}\left\|\zeta_{r, i_{1} \ldots i_{p}}^{(c)}(t)\right\|_{C^{v}(D, 1)}^{2} d t+\int_{0}^{T} \int_{0}^{T} \Lambda_{v}\left\|\mathcal{D}_{\theta_{1}} \zeta_{r, i_{1} \ldots i_{p}}^{(c)}(t)\right\|_{C^{v}(D, d)}^{2} d \theta_{1} d t\right], \\
\|\zeta\|_{2,2}^{v, 2}= & \|\zeta\|_{1,2}^{v, 2}+E\left[\int_{0}^{T} \int_{0}^{T} \int_{0}^{T} \Lambda_{v} \| \mathcal{D}_{\theta_{2}} \mathcal{D}_{\left.\theta_{1} \zeta_{r, i_{1} \ldots i_{p}}^{(c)}(t) \|_{C^{v}(D, d \times d)}^{2} d \theta_{1} d \theta_{2} d t\right] .} .\right.
\end{array}
$$

Then, we have the following lemma.
Lemma 5.6 Under conditions as required in Theorem 3.1 and with Remark 5.1, if $(V(t, x), \bar{V}(t, x))$ $\in \mathcal{Q}_{\mathcal{F}}^{2}([0, T] \times D)$ is the adapted solution to (1.1), then

$$
(V(t, x), \bar{V}(t, x)) \in L_{1,2}^{\infty, 2}\left([0, T] \times \Omega,\left(C^{\infty}(D, H)\right)^{q \times q \times d}\right) .
$$

Furthermore, for each $1 \leq j \leq d$ and $x \in D$, a version of the infinite-dimensional vector process

$$
\left\{\left(D_{\theta}^{j} V_{i_{1} \ldots i_{p}}^{(c)}(t, x), D_{\theta}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(t, x)\right): 0 \leq \theta, t \leq T, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}\right\}
$$

is the solution of the following system of Malliavin derivative based B-SPDEs under random environment,

$$
\begin{aligned}
(5.24) D_{\theta}^{j} V_{i_{1} \ldots i_{p}}^{(c)}(t, x)= & D_{\theta}^{j} H_{i_{1} \ldots i_{p}}^{(c)}(x) \\
& +\int_{t}^{T} \sum_{l=0}^{k} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c)}(x, V) D_{\theta}^{j} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d s \\
& +\int_{t}^{T} \sum_{l=0}^{m} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, \bar{v}_{j_{1} \ldots j_{p}}^{(c+1)}}(x, V) D_{\theta}^{j} \bar{V}_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d s \\
& +\int_{t}^{T} \sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{J}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(x)}(x, V) D_{\theta}^{j} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d W(s) \\
& -\int_{t}^{T} D_{\theta}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(s, x) d W(s),
\end{aligned}
$$

where $j_{1}, \ldots, j_{p}$ are nonnegative integers, and for $0 \leq t<\theta \leq T$,

$$
\begin{equation*}
D_{\theta}^{j} V_{i_{1} \ldots i_{p}}^{(c)}(t, x)=0, \quad D_{\theta}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(t, x)=0 . \tag{5.25}
\end{equation*}
$$

In addition, let " $=d$ " denote "equal in distribution", then

$$
\begin{align*}
& \left\{\bar{V}_{i_{1} \ldots i_{p}}^{(c)}(t, x), t \in[0, T], c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, x \in D\right\}  \tag{5.26}\\
={ }^{d} & \left\{D_{t} V_{i_{1} \ldots i_{p}}^{(c)}(t, x)+\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}(x, V), t \in[0, T], c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, x \in D\right\} .
\end{align*}
$$

Proof. First, it follows from Theorem 2.1 and its proof that there is a unique adapted solution $(V(t, x), \bar{V}(t, x)) \in \mathcal{Q}_{\mathcal{F}}^{2}([0, T] \times D)$ to (1.1). Furthermore, it can be approximated by a sequence of $\left(V^{i}(t, x), \bar{V}^{i}(t, x)\right) \in \mathcal{M}_{\gamma}^{D}[0, T]$ with $i \in\{0,1, \ldots\}$, satisfying,

$$
\begin{align*}
V^{0}(t, x)= & \bar{V}^{0}(t, x)=0  \tag{5.27}\\
V_{i_{1} \ldots i_{p}}^{(c), i+1}(t, x)= & H_{i_{1} \ldots i_{p}}^{(c)}(x)+\int_{t}^{T} \mathcal{L}_{i_{1} \ldots i_{p}}^{(c)}\left(x, V^{i}\right) d s  \tag{5.28}\\
& +\int_{t}^{T}\left(\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}\left(x, V^{i}\right)-\bar{V}_{i_{1} \ldots i_{p}}^{(c), i+1}(s, x)\right) d W(s)
\end{align*}
$$

for all $t \in[0, T], x \in D$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$.
Now, by induction in terms of $i \in\{0,1, \ldots\}$, we can show that

$$
\left(V^{i}(t, x), \bar{V}^{i}(t, x)\right) \in L_{1,2}^{\infty, 2}\left([0, T] \times \Omega,\left(C^{\infty}(D, H)\right)^{q \times q \times d}\right)
$$

Equivalently, if

$$
\left(V^{i}(t, x), \bar{V}^{i}(t, x)\right) \in L_{1,2}^{\infty, 2}\left([0, T] \times \Omega,\left(C^{\infty}(D, H)\right)^{q \times q \times d}\right)
$$

for any $i \in\{0,1, \ldots\}$, we need to prove that

$$
\left(V^{i+1}(t, x), \bar{V}^{i+1}(t, x)\right) \in L_{1,2}^{\infty, 2}\left([0, T] \times \Omega,\left(C^{\infty}(D, H)\right)^{q \times q \times d}\right) .
$$

In fact, since

$$
\begin{equation*}
H(x)+\int_{t}^{T} \mathcal{L}\left(x, V^{i}\right) d s \in \mathcal{D}_{\infty}^{1,2} \tag{5.29}
\end{equation*}
$$

it follows from Lemma 5.2 and Lemma 5.3 that

$$
\begin{equation*}
V^{i+1}(t, x)=E\left[H(x)+\int_{t}^{T} \mathcal{L}\left(x, V^{i}\right) d s \mid \mathcal{F}_{t}\right] \in \mathcal{D}_{\infty}^{1,2} \tag{5.30}
\end{equation*}
$$

Thus, by using (5.28)-(5.30) and Lemma 5.3, we know that

$$
\mathcal{J}\left(x, V^{i}\right)-\bar{V}^{i+1}(s, x) \in \mathcal{D}_{\infty}^{1,2}
$$

Hence, by chain rule for Malliavin calculus, we have that

$$
\bar{V}^{i+1}(s, x) \in \mathcal{D}_{\infty}^{1,2}
$$

Therefore, for each $0 \leq \theta \leq t$ and $j \in\{1, \ldots, d\}$, it follows from chain rule for Malliavin calculus and Lemma 5.3 that

$$
\begin{align*}
D_{\theta}^{j} V_{i_{1} \ldots i_{p}}^{(c), i+1}(t, x)= & D_{\theta}^{j} H_{i_{1} \ldots i_{p}}^{(c)}(x)  \tag{5.31}\\
& +\int_{t}^{T} \sum_{l=0}^{k} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c+1)}\left(x, V^{i}\right) D_{\theta}^{j} V_{j_{1} \ldots j_{p}}^{(c+l), i}(s, x) d s \\
& +\int_{t}^{T} \sum_{l=0}^{m} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, \bar{v}_{j_{1} \ldots j_{p}}^{(c+1)}}\left(x, V^{i}\right) D_{\theta}^{j} \bar{V}_{j_{1} \ldots j_{p}}^{(c+l), i}(s, x) d s \\
& +\int_{t}^{T} \sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{J}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c+1)}\left(x, V^{i}\right) D_{\theta}^{j} V_{j_{1} \ldots j_{p}}^{(c+l), i}(s, x) d W(s) \\
& -\int_{t}^{T} D_{\theta}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c), i+1}(s, x) d W(s) .
\end{align*}
$$

Furthermore, it follows from the proof of Lemma 4.1 that
$\left\{\left(D_{\theta}^{j} V_{i_{1} \ldots i_{p}}^{(c), i+1}(t, x), D_{\theta}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c), i+1}(t, x)\right), i \in\{1,2, \ldots\}, j \in\{1, \ldots, d\}, c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}\right\}$
is the unique adapted solution to the system in (5.31). In addition, it follows from the similar proof of Lemma 4.2 that this solution is continuous with respect to $x \in D$.

Next, we show that $\left(V^{i}(t, x), \bar{V}^{i}(t, x)\right)$ converges in $L_{1,2}^{\infty, 2}\left([0, T] \times \Omega,\left(C^{\infty}(D, H)\right)^{q \times q \times d}\right)$. In particular, we have the convergence of their Malliavin derivatives as $i \rightarrow \infty$ as follows,

$$
\begin{align*}
& \left\{\left(D_{\theta} V_{i_{1} \ldots i_{p}}^{(c), i}(t, x), D_{\theta} \bar{V}_{i_{1} \ldots i_{p}}^{(c), i}(t, x)\right), c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, t \in[\theta, T], x \in D\right\}  \tag{5.32}\\
& \rightarrow\left\{\left(V_{i_{1} \ldots i_{p}}^{(c), \theta}(t, x), \bar{V}_{i_{1} \ldots i_{p}}^{(c), \theta}(t, x)\right), c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, t \in[\theta, T], x \in D\right\},
\end{align*}
$$

where each component $\left(V_{i_{1} \ldots i_{p}}^{j,(c), \theta}(t, x), \bar{V}_{i_{1} \ldots i_{p}}^{j,(c), \theta}(t, x)\right)$ of the limit $\left(V_{i_{1} \ldots i_{p}}^{(c), \theta}(t, x), \bar{V}_{i_{1} \ldots i_{p}}^{(c), \theta}(t, x)\right)$ with $j \in\{1, \ldots, d\}$ is the unique adapted solution to the B-SSPDEs in (5.12) for all $c \in\{0,1, \ldots\}$, $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, t \in[\theta, T]$, and $x \in D$ owing to Lemma 5.5.

Now, by applying conditions (3.29), (2.11)-(2.14), the Itô's formula, and the similar technique used in the proof of Lemma 4.3, we have that

$$
\begin{equation*}
\left\|\left(\mathcal{D}_{\theta} V^{i+1}-V^{\theta}, \mathcal{D}_{\theta} \bar{V}^{i+1}-\bar{V}^{\theta}\right)\right\|_{\mathcal{N}_{\gamma_{c}, c}}^{2} \leq \hat{\gamma}_{c} K_{d, c}^{1} E\left[\int_{\theta}^{T}\left(\alpha^{i}(s)+\beta^{i}(s)\right) e^{2 \gamma_{c} s} d s\right] \tag{5.33}
\end{equation*}
$$

for all $c \in\{0,1, \ldots\}$ with each given $i \in\{0,1, \ldots\}$. The notation $K_{d, c}^{1}$ is some nonnegative constant depending only on $c, D$, and $T, \hat{\gamma}_{c}$ is taken and explained as in (5.19). The functions $\alpha^{i}(s)$ and $\beta^{i}(s)$ are respectively given by

$$
\begin{aligned}
\alpha^{i}(s)= & \left(1+\left\|V^{i}(s)\right\|_{C^{c+k+1}(D, q)}^{2}\right)\left\|\mathcal{D}_{\theta} V^{i}(s)-V^{\theta}(s)\right\|_{C^{c+k+1}(D, q d)}^{2} \\
& +\left(1+\left\|\bar{V}^{i}(s)\right\|_{C^{c+k+1}(D, q d)}^{2}\right)\left\|\mathcal{D}_{\theta} \bar{V}^{i}(s)-\bar{V}^{\theta}(s)\right\|_{C^{c+k+1}(D, q d d)}^{2} \\
\beta^{i}(s)= & \left\|V^{\theta}(s)\right\|_{C^{c+k+1}(D, q d)}^{2}\left(1+\left\|V^{i}(s)-V(s)\right\|_{C^{c+k+1}(D, q)}^{2}\right) \\
& +\left\|\bar{V}^{\theta}(s)\right\|_{C^{c+k+1}(D, q d d)}^{2}\left(1+\left\|\bar{V}^{i}(s)-\bar{V}(s)\right\|_{C^{c+k+1}(D, q d)}^{2}\right)
\end{aligned}
$$

Thus, by (5.13), (5.33), and the fact that $|a b| \leq \frac{1}{2}\left(a^{2}+b^{2}\right)$ for any two real numbers $a$ and $b$, we have that

$$
\begin{align*}
& \int_{0}^{T}\left\|\left(\mathcal{D}_{\theta} V^{i+1}-V^{\theta}, \mathcal{D}_{\theta} \bar{V}^{i+1}-\bar{V}^{\theta}\right)\right\|_{\mathcal{N}_{\gamma}^{D}[\theta, T]}^{2} d \theta  \tag{5.34}\\
\leq & \delta T\left\|\left(V^{i}-V, \bar{V}^{i}-\bar{V}\right)\right\|_{\mathcal{N}_{\gamma}^{D}[\theta, T]}^{2}+\delta T\|(V, \bar{V})\|_{\mathcal{N}_{\gamma}^{D}}^{2}+\frac{3 \delta}{2} \int_{0}^{T}\left\|\left(V^{\theta}, \bar{V}^{\theta}\right)\right\|_{\mathcal{N}_{\gamma}^{D}[\theta, T]}^{2} d \theta \\
& +\frac{3 \delta}{2} E\left[\int_{0}^{T}\left\|\left(\mathcal{D}_{\theta} V^{i}-V^{\theta}, \mathcal{D}_{\theta} \bar{V}^{i}-\bar{V}^{\theta}\right)\right\|_{\mathcal{N}_{\gamma}^{D}[\theta, T]}^{2} d \theta\right] \\
\leq & \left(\delta+\ldots+\delta^{i}\right) K_{1}+\delta^{i} \int_{0}^{T}\left\|\left(\mathcal{D}_{\theta} V^{0}-V^{\theta}, \mathcal{D}_{\theta} \bar{V}^{0}-\bar{V}^{\theta}\right)\right\|_{\mathcal{N}_{\gamma}^{D}[\theta, T]}^{2} d \theta \\
\leq & \frac{\delta K_{1}}{1-\delta}+\delta^{i} K_{2}
\end{align*}
$$

where $K_{1}$ and $K_{2}$ are some nonnegative constants. Since $e^{2 \gamma_{c} t}>1$ for all $c \in\{0,1, \ldots\}$, we know that

$$
\begin{align*}
& \sum_{v=0}^{\infty} \xi(v) E\left[\int_{0}^{T} \int_{0}^{T} \Lambda_{v}\left\|\left(\mathcal{D}_{\theta} V^{i+1}-V^{\theta}, \mathcal{D}_{\theta} \bar{V}^{i+1}-\bar{V}^{\theta}\right)\right\|_{C^{v}(D, q d \times q d d)}^{2} d \theta d t\right]  \tag{5.35}\\
& \leq \frac{\delta K_{1}}{1-\delta}+\delta^{i} K_{2} \\
& \rightarrow 0
\end{align*}
$$

by letting $i \rightarrow \infty$ first and $\delta \rightarrow 0$ second since $\delta \in(0,1)$. Thus, by (5.35) and the factor that $e^{2 \gamma_{c} t}>1$ again, we have

$$
\begin{equation*}
\left\|\mid\left(V^{i}, \bar{V}^{i}\right)-(V, \bar{V})\right\|_{1,2}^{\infty, 2} \rightarrow 0 \text { as } i \rightarrow \infty . \tag{5.36}
\end{equation*}
$$

Thus, we know that $\left(V^{i}, \bar{V}^{i}\right)$ with Malliavin derivative $\left(\mathcal{D}_{\theta} V^{i}, \mathcal{D}_{\theta} \bar{V}^{i}\right)$ converges to ( $V, \bar{V}$ ) with Malliavin derivative $\left(V^{\theta}, \bar{V}^{\theta}\right)$ in $L_{1,2}^{\infty, 2}\left([0, T] \times \Omega,\left(C^{\infty}(D, H)\right)^{q \times q \times d}\right)$ as $i \rightarrow \infty$. Hence, a version of the following infinite-dimensional vector process

$$
\left\{\left(D_{\theta}^{j} V_{i_{1} \ldots i_{p}}^{(c)}(t, x), D_{\theta}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(t, x)\right): 0 \leq \theta, t \leq T, c \in\{0,1, \ldots\}, j \in\{1, \ldots, d\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}\right\}
$$

is given by (5.24).
Finally, for the considered version, the claims in (5.25) of Lemma 5.6 are follows from the fact that $(V, \bar{V})$ is an adapted solution to the B-SPDE displayed in (1.1) and Corollary 1.2.1 in page 34 and its related remark in page 42 of Nualart [20]. Furthermore, the claims in (5.26) are justified as follows. Since, for $t \leq u$, we have that

$$
\begin{align*}
V_{i_{1} \ldots i_{p}}^{(c)}(u, x)= & V_{i_{1} \ldots i_{p}}^{(c)}(t, x)-\int_{t}^{u} \mathcal{L}_{i_{1} \ldots i_{p}}^{(c)}(x, V) d s  \tag{5.37}\\
& -\int_{t}^{u}\left(\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}(x, V)-\bar{V}_{i_{1} \ldots i_{p}}^{(c)}(s, x)\right) d W(s)
\end{align*}
$$

for all $x \in D, c \in\{0,1, \ldots\}$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$. Then, it follows from Lemma 5.3 that, for $j \in\{1, \ldots, d\}$ and $t<\theta \leq u$,

$$
\begin{align*}
D_{\theta}^{j} V_{i_{1} \ldots i_{p}}^{(c)}(u, x)= & \bar{V}_{i_{1} \ldots i_{p}}^{(c), j}(\theta, x)-\mathcal{J}_{i_{1} \ldots i_{p}}^{(c), j}(x, V)  \tag{5.38}\\
& -\int_{\theta}^{u} \sum_{l=0}^{k} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c+1)}(x, V) D_{\theta}^{j} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d s \\
& -\int_{\theta}^{u} \sum_{l=0}^{m} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, \bar{v}_{j_{1} \ldots j_{p}}^{(c+1)}}(x, V) D_{\theta}^{j} \bar{V}_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d s \\
& -\int_{\theta}^{u} \sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{J}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c)}(x, V) D_{\theta}^{j} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d W(s) \\
& +\int_{\theta}^{u} D_{\theta}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(s, x) d W(s) .
\end{align*}
$$

Thus, by taking $\theta=u$ in (5.38), we know that the claims in (5.26) are true. Hence, we complete the proof of Lemma 5.6.

### 5.4 Second-Order Marlliavin Derivative Based B-SPDE

First, we use $\theta_{1}$ to replace $\theta$ in (5.24). Second, for each $j \in\{1, \ldots, d\}, c \in\{0,1, \ldots\}$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$, we define

$$
\begin{align*}
\overline{\mathcal{L}}_{i_{1} \ldots i_{p}}^{(c+1)}\left(x, V, D_{\theta_{1}}^{j} V\right)= & \sum_{l=0}^{k} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}}^{(c+1)}(x, V) D_{\theta_{1}}^{j} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x)  \tag{5.39}\\
& +\sum_{l=0}^{m} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{L}_{i_{1} \ldots i_{p}, \bar{v}_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c+1)}(x, V) D_{\theta_{1}}^{j} \bar{V}_{j_{1} \ldots j_{p}}^{(c+l)}(s, x), \\
\overline{\mathcal{J}}_{i_{1} \ldots i_{p}}^{(c+1)}\left(x, V, D_{\theta_{1}}^{j} V\right)= & \sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{J}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}^{(c+}(x, V) D_{\theta_{1}}^{j} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) . \tag{5.40}
\end{align*}
$$

Then, we can obtain the following system of B-SPDEs for each $\bar{j} \in\{1, \ldots, d\}$ by taking Malliavin derivatives on both sides of the equation in (5.24),

$$
\begin{align*}
& D_{\theta_{2}}^{\bar{j}} D_{\theta_{1}}^{j} V_{i_{1} \ldots i_{p}}^{(c)}(t, x)  \tag{5.41}\\
= & D_{\theta_{2}}^{\bar{j}} D_{\theta_{1}}^{j} H_{i_{1} \ldots i_{p}}^{(c)}(x) \\
& +\int_{t}^{T} \sum_{l=0}^{k} \sum_{j_{1}+\ldots+j_{p}=c+l} \overline{\mathcal{L}}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+2)}}\left(x, V, D_{\theta_{1}}^{j} V\right) D_{\theta_{2}}^{\bar{j}} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d s \\
& +\int_{t}^{T} \sum_{l=0}^{k} \sum_{j_{1}+\ldots+j_{p}=c+l} \overline{\mathcal{L}}_{i_{1} \ldots i_{p},\left(D_{\theta_{1}}^{j} v\right)_{j_{1} \ldots j_{p}}^{(c+2)}\left(x, V, D_{\theta_{1}}^{j} V\right) D_{\theta_{2}}^{\bar{j}} D_{\theta_{1}}^{j} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d s} \\
& +\int_{t}^{T} \sum_{l=0}^{m} \sum_{j_{1}+\ldots+j_{p}=c+l} \overline{\mathcal{L}}_{i_{1} \ldots i_{p},\left(D_{\theta_{1}}^{j} \bar{v}\right)_{j_{1} \ldots j_{p}}^{(c+2)}}\left(x, V, D_{\theta_{1}}^{j} V\right) D_{\theta_{2}}^{\bar{j}} \bar{V}_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d s
\end{align*}
$$

$$
\begin{aligned}
& +\int_{t}^{T} \sum_{l=0}^{m} \sum_{j_{1}+\ldots+j_{p}=c+l} \overline{\mathcal{L}}_{i_{1} \ldots i_{p},\left(D_{\theta_{1}}^{j} \overline{v_{j_{1}} \ldots j_{p}}\right.}^{(c+2)}\left(x, V, D_{\theta_{1}}^{j} V\right) D_{\theta_{2}}^{\bar{j}} D_{\theta_{1}}^{j} \bar{V}_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d s \\
& +\int_{t}^{T} \sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \overline{\mathcal{J}}_{i_{1} \ldots i_{p},\left(D_{\theta_{1}}^{j} v\right)_{j_{1} \ldots j_{p}}^{(c+2)}\left(x, V, D_{\theta_{1}}^{j} V\right) D_{\theta_{2}}^{\bar{j}} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d W(s)}^{+\int_{t}^{T} \sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \overline{\mathcal{J}}_{i_{1} \ldots i_{p},\left(D_{\theta_{1}}^{j} v\right)_{j_{1} \ldots j_{p}}^{(c+2)}}\left(x, V, D_{\theta_{1}}^{j} V\right) D_{\theta_{2}}^{\bar{j}} D_{\theta_{1}}^{j} V_{j_{1} \ldots j_{p}}^{(c+l)}(s, x) d W(s)} \\
& -\int_{t}^{T} D_{\theta_{2}}^{\bar{j}} D_{\theta_{1}}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(s, x) d W(s) .
\end{aligned}
$$

Furthermore, consider a properly chosen number sequence $\gamma=\left\{\gamma_{c}, c=0,1,2, \ldots\right\}$ satisfying $0<\gamma_{0}<\gamma_{1}<\ldots$ such that the discussions for Theorem 2.1, Subsections 5.2-5.3, and the following Lemma 5.7 are meaningful, which can be elaborated similar to the previous proof in Subsection 5.2. Then, we can define the space

$$
\begin{equation*}
\mathcal{O}_{\gamma}^{D}[0, T]=D_{\mathcal{F}}^{2}\left([0, T], C^{\infty}\left(D, R^{q \times d \times d}\right)\right) \times L_{\mathcal{F}, p}^{2}\left([0, T], C^{\infty}\left(D, R^{q \times d \times d \times d}\right)\right) \tag{5.42}
\end{equation*}
$$

endowed with the norm similarly defined as in (4.27)-(4.28). Thus, we have the following lemma.

Lemma 5.7 Under conditions as required in Theorem 3.1 and with Remark 5.1, if $(V(t, x), \bar{V}(t, x))$ $\in \mathcal{Q}_{\mathcal{F}}^{2}([0, T] \times D)$ is the adapted solution to (1.1), then,

$$
(V(t, x), \bar{V}(t, x)) \in L_{2,2}^{\infty, 2}\left([0, T] \times \Omega,\left(C^{\infty}(D, H)\right)^{q \times q \times d}\right) .
$$

Furthermore, for $x \in D$, a version of the following infinite-dimensional vector process

$$
\left\{\left(D_{\theta_{2}} D_{\theta_{1}} V_{i_{1} \ldots i_{p}}^{(c)}(t, x), D_{\theta_{2}} D_{\theta_{1}} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(t, x)\right): 0 \leq \theta_{1}, \theta_{2}, t \leq T, c \in\{0,1, \ldots\}, \quad\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}\right\}
$$

is given by the system in (5.41). In addition, for $0 \leq t<\theta_{1} \wedge \theta_{2} \leq T$ and $1 \leq \bar{j}, j \leq d$,

$$
\begin{equation*}
D_{\theta_{2}}^{\bar{j}} D_{\theta_{1}}^{j} V_{i_{1} \ldots i_{p}}^{(c)}(t, x)=0, \quad D_{\theta_{2}}^{\bar{j}} D_{\theta_{1}}^{j} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(t, x)=0, \tag{5.43}
\end{equation*}
$$

and

$$
\begin{align*}
& \left\{D_{t} \bar{V}_{i_{1} \ldots i_{p}}^{(c)}(t, x), t \in[0, T], c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, x \in D\right\}  \tag{5.44}\\
& ={ }^{d}\left\{D_{t} D_{t} V_{i_{1} \ldots i_{p}}^{(c)}(t, x)+\sum_{l=0}^{n} \sum_{j_{1}+\ldots+j_{p}=c+l} \mathcal{J}_{i_{1} \ldots i_{p}, v_{j_{1} \ldots j_{p}}^{(c+1)}}(x, V) D_{t} V_{j_{1} \ldots j_{p}}^{(c+l)}(t, x),\right. \\
& \left.\quad t \in[0, T], c \in\{0,1, \ldots\},\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}, x \in D\right\} .
\end{align*}
$$

Proof. Let

$$
\begin{aligned}
L(t) \equiv & \|V(t)\|_{C^{\infty}(D, q)}+\|\bar{V}(t)\|_{C^{\infty}(D, q d)} \\
& +\left\|D_{\theta_{1}} V(t)\right\|_{C^{\infty}(D, q d)}+\left\|D_{\theta_{1}} \bar{V}(t)\right\|_{C^{\infty}(D, q d d)} \\
& +\left\|D_{\theta_{2}} V(t)\right\|_{C^{\infty}(D, q d)}+\left\|D_{\theta_{2}} \bar{V}(t)\right\|_{C^{\infty}(D, q d d)}
\end{aligned}
$$

Then, similar to (5.14), we define a sequence of nondecreasing $\left\{\mathcal{F}_{t}\right\}$-stopping times along $w \in\{0,1, \ldots\}$ as follows,

$$
\begin{equation*}
\tau_{w} \equiv T \wedge \inf \{t>0, L(t)>w\}, \tag{5.45}
\end{equation*}
$$

which satisfies $\tau_{w} \rightarrow T$ a.s. as $w \rightarrow \infty$. Thus, by the similar arguments as used in the proofs of Lemmas 5.5-5.6, we can provide a proof for Lemma 5.7.

### 5.5 Priori Estimates

Lemma 5.8 Under conditions as required in Theorem 3.1 and with Remark 5.1, if $\left(V^{i}(t, x)\right.$, $\left.\bar{V}^{i}(t, x)\right)$ for each $i \in\{1,2\}$ is the unique adapted solution to equation (1.1) with terminal condition $H^{i}(x)$, then,

$$
\begin{align*}
\left\|\left(V^{i}, \bar{V}^{i}\right)\right\|_{\mathcal{M}_{\gamma}^{D}[0, T]}^{2} & \leq \bar{C}\left(1+\left\|H^{i}\right\|_{L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(D, R^{q}\right)\right)}^{2}\right),  \tag{5.46}\\
\left\|\left(V^{2}, \bar{V}^{2}\right)-\left(V^{1}, \bar{V}^{1}\right)\right\|_{\mathcal{M}_{\gamma}^{D}[s, t]}^{2} & \leq \bar{C}\left\|H^{2}-H^{1}\right\|_{L_{\mathcal{F}_{T}}^{2}\left(\Omega, C^{\infty}\left(C, R^{q}\right)\right)}^{2} \tag{5.47}
\end{align*}
$$

for some nonnegative constant $\bar{C}$ only depending on the terminal time $T$, the region $D$. Furthermore, for each $c \in\{0,1, \ldots\}$ and any $s, t \in[0, T]$ with $s \leq t$, we have

$$
\begin{align*}
E\left[\left\|V^{i}(t)-V^{i}(s)\right\|_{C^{c}(D, q)}^{2}\right] & \leq C(t-s)  \tag{5.48}\\
E\left[\left\|\bar{V}^{i}(t)-\bar{V}^{i}(s)\right\|_{C^{c}(D, q d)}^{2}\right] & \leq C(t-s) \tag{5.49}
\end{align*}
$$

for some nonnegative constant $C$ only depending on the terminal time $T$, the region $D$, and the terminal random variable.

Proof. By applying the Itô's formula and the similar proof as used for (5.13), we know that the claims in (5.46)-(5.47) are true. Now, consider the B-SPDE (1.1) over $[s, t]$ with terminal condition $V(t, x)$. Then, by (2.13)-(2.14), (5.26), (5.23) and (5.20), we know that

$$
\begin{align*}
& E\left[\left\|V^{i}(t)-V^{i}(s)\right\|_{C^{c}(D, q)}^{2}\right]  \tag{5.50}\\
\leq & C_{1} \int_{s}^{t}\left(1+\left\|V^{i}(r)\right\|_{C^{k+c}(D, q)}^{2}+\left\|D_{r} V^{i}(r)\right\|_{C^{k+c}(D, q d)}^{2}\right) d r \\
\leq & C_{2}(t-s)\left(\left\|\left(V^{i}, \bar{V}^{i}\right)\right\|_{\mathcal{M}_{\gamma}^{D}[0, T]}^{2}+\left\|\left(D V^{i}, D \bar{V}^{i}\right)\right\|_{\mathcal{N}_{\gamma}^{D}[0, T]}^{2}\right) \\
\leq & C_{3}(t-s),
\end{align*}
$$

where $C_{1}$ and $C_{2}$ are some nonnegative constants.

Furthermore, by using (5.26), (5.44) in Lemma 5.7, and the similar argument as used in (5.50), we know that

$$
\begin{aligned}
& E\left[\left\|\bar{V}^{i}(t)-\bar{V}^{i}(s)\right\|_{C^{c}(D, q d)}^{2}\right] \\
\leq & C_{3} E\left[\left\|V^{i}(t)-V^{i}(s)\right\|_{C^{c}(D, q)}^{2}+\left\|D_{t} V^{i}(t)-D_{s} V^{i}(s)\right\|_{C^{c}(D, q d)}^{2}\right] \\
\leq & C_{4} \int_{s}^{t}\left(1+\left\|V^{i}(r)\right\|_{C^{k+c}(D, q)}^{2}+\left\|D_{r} V^{i}(r)\right\|_{C^{k+c}(D, q d)}^{2}+\left\|D_{r} D_{r} V^{i}(r)\right\|_{C^{k+c}(D, q d d)}^{2}\right) d r \\
\leq & C_{5}(t-s)\left(\left\|\left(V^{i}, \bar{V}^{i}\right)\right\|_{\mathcal{M}_{\gamma}^{D}[0, T]}^{2}+\left\|\left(D V^{i}, D \bar{V}^{i}\right)\right\|_{\mathcal{N}_{\gamma}^{D}[0, T]}^{2}+\left\|\left(D D V^{i}, D D \bar{V}^{i}\right)\right\|_{\mathcal{O}_{\gamma}^{D}[0, T]}^{2}\right) \\
\leq & C_{6}(t-s),
\end{aligned}
$$

where $C_{4}-C_{6}$ are some nonnegative constants. Finally, take $C=\max \left\{C_{3}, C_{6}\right\}$ sure that both (5.48) and (5.49) are true. Hence, we complete the proof of Lemma 5.8.

### 5.6 Representation Formulas

Concerning Algorithm 3.1, we first define the following quantities as $j_{0}$ decreases from $n_{0}$ to 1 for each $c \in\{0,1, \ldots, M\}$ and $x \in D$,

$$
\begin{align*}
V_{i_{1} \ldots i_{p}, \pi, 0}^{(c)}\left(t_{n_{0}}, x\right) \equiv & H_{i_{1} \ldots i_{p}}^{(c)}(x), \bar{V}_{i_{1} \ldots i_{p}, \pi, 0}^{(c)}\left(t_{n_{0}}, x\right)=0  \tag{5.51}\\
V_{i_{1} \ldots i_{p}, \pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right) \equiv & E\left[V_{i_{1} \ldots i_{p}}^{(c)}\left(t_{j_{0}}, x\right)+\mathcal{L}_{i_{1} \ldots i_{p}}^{(c)}\left(t_{j_{0}}, x, V\left(t_{j_{0}}, x\right)\right) \Delta_{j_{0}}^{\pi} \mid \mathcal{F}_{t_{j_{0}-1}}\right],  \tag{5.52}\\
\bar{V}_{i_{1} \ldots i_{p}, \pi, 0}^{(c)}\left(t_{j_{0}-1}\right) \equiv & \frac{1}{\Delta_{j_{0}}^{\pi}} E\left[V_{i_{1} \ldots i_{p}}^{(c)}\left(t_{j_{0}}, x\right) \Delta^{\pi} W_{j_{0}} \mid \mathcal{F}_{t_{j_{0}-1}}\right]  \tag{5.53}\\
& +E\left[\mathcal{L}_{i_{1} \ldots i_{p}}^{(c)}\left(t_{j_{0}}, x, V\left(t_{j_{0}}, x\right)\right) \Delta^{\pi} W_{j_{0}} \mid \mathcal{F}_{t_{j_{0}-1}}\right] \\
& +\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}\left(t_{j_{0}-1}, x, V_{\pi}\left(t_{j_{0}-1}, x\right)\right) .
\end{align*}
$$

Then, we consider the following iterative procedure,

$$
\begin{align*}
V_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}(t, x)= & H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)  \tag{5.54}\\
& +\int_{t}^{t_{j_{0}}}\left(\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}\left(s, x, V_{\pi, 1}(s, x)\right)-\bar{V}_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}(s, x)\right) d W(s)
\end{align*}
$$

for each $t \in\left[t_{j_{0}-1}, t_{j_{0}}\right)$ and $x \in D$, where

$$
\begin{align*}
& H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)=V_{i_{1} \ldots i_{p}}\left(t_{j_{0}}, x\right)+\mathcal{L}_{i_{1} \ldots i_{p}}^{(c)}\left(t_{j_{0}}, x, V\left(t_{j_{0}}, x\right)\right) \Delta_{j_{0}}^{\pi},  \tag{5.55}\\
& H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{n_{0}}, x\right)=H_{i_{1} \ldots i_{p}}^{(c)}(x) \tag{5.56}
\end{align*}
$$

Note that the equation displayed in (5.54) for each $j_{0} \in\left\{n_{0}, n_{0}-1, \ldots, 1\right\}$ is a B-SPDE with terminal value $H_{i_{1}, \ldots, i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)$. Then, we have the following lemma.

Lemma 5.9 Under conditions (3.29) and (2.11)-(2.14), there is a unique adapted and squareintegrable solution $\left(V_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}(t, x), \bar{V}_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}(t, x)\right)$ to the B-SPDE in (5.54) over $t \in\left[t_{j_{0}-1}, t_{j_{0}}\right)$ and $x \in D$ for each $c \in\{0,1, \ldots, M\}$ and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$. Moreover, we have

$$
\begin{align*}
& V_{i_{1} \ldots i_{p}, \pi, 0}^{(c)}(t, x)=V_{i_{1} \ldots . i_{p}, \pi, 1}^{(c)}(t, x), \quad t \in\left[t_{j_{0}-1}, t_{j_{0}}\right), x \in D,  \tag{5.57}\\
& \bar{V}_{i_{1} \ldots i_{p}, \pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)  \tag{5.58}\\
= & \frac{1}{\Delta_{j_{0}}^{\pi}} E\left[\int_{t_{j_{0}-1}}^{t_{j_{0}}}\left(\bar{V}_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}(s, x)-\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}\left(s, x, V_{\pi, 1}(s, x)\right)\right) d s \mid \mathcal{F}_{t_{j_{0}-1}}\right] \\
& +\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}\left(t_{j_{0}-1}, x, V_{\pi, 1}\left(t_{j_{0}-1}, x\right)\right) .
\end{align*}
$$

Proof. Without loss of generality, we only consider the case that $c=0$. Owing to conditions (3.29), (2.11)-(2.14), and Theorem 2.1, we know that there is a unique adapted and meansquare integrable solution $\left(V_{\pi, 1}(t, x), \bar{V}_{\pi, 1}(t, x)\right)$ to the B-SPDE in (5.54) over $t \in\left[t_{j_{0}-1}, t_{j_{0}}\right)$ and $x \in D$ for each $j_{0} \in\left\{n_{0}, n_{0}-1, \ldots, 1\right\}$. Then, by taking the conditional expectations on both sides of (5.54) at each time $t$ and the independent increment property of the Brownian motion, we know that the claim in (5.57) is true by a backward induction method in terms of $j_{0}=n_{0}, n_{0}-1, \ldots, 1$.

Now, it follows from Lemma 5.6 that $H_{\pi, 1}\left(t_{j_{0}}, x\right) \in \mathcal{D}_{\infty}^{1,2} \bigcap L_{\mathcal{F}_{t_{j}}}^{2}\left(\Omega, C^{\infty}\left(D, R^{q}\right)\right)$ for each $j_{0} \in\left\{n_{0}, n_{0}-1, \ldots, 1\right\}$. Then, for each $c \in\{0,1, \ldots\}, x \in D$, and $\left(i_{1}, \ldots, i_{p}\right) \in \mathcal{I}^{c}$, it follows from Lemma 5.2 that

$$
\begin{equation*}
H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)=E\left[H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)\right]+\int_{0}^{t_{j o}} E\left[D_{t} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{t}\right] d W(t) \tag{5.59}
\end{equation*}
$$

with

$$
E\left[\left|E\left[D_{t} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{t}\right]\right|^{2} d t\right]<\infty
$$

Thus, we know that

$$
\begin{align*}
E\left[H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)\right]= & E\left[H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{t_{j_{0}-1}}\right]  \tag{5.60}\\
& -\int_{0}^{t_{j_{0}-1}} E\left[D_{s} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{s}\right] d W(s) .
\end{align*}
$$

Hence, it follows from (5.59) and (5.60) that

$$
\begin{align*}
H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)= & E\left[H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{t_{j_{0}-1}}\right]  \tag{5.61}\\
& +\int_{t_{j_{0}-1}}^{t_{j_{0}}} E\left[D_{s} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{s}\right] d W(s) .
\end{align*}
$$

Now, for any set $A \in \mathcal{F}_{t_{j_{0}-1}}$, we have

$$
\begin{equation*}
E\left[H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \Delta^{\pi} W_{j_{0}} I_{A}\right]=E\left[H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \int_{t_{j_{0}-1}}^{t_{j_{0}}} I_{A} d W(s)\right] \tag{5.62}
\end{equation*}
$$

$$
\begin{aligned}
& =E\left[\int_{t_{j_{0}-1}}^{t_{j_{0}}} I_{A} D_{s} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) d s\right] \\
& =E\left[\int_{t_{j_{0}-1}}^{t_{j_{0}}} I_{A} E\left[D_{s} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{s}\right] d s\right] \\
& =E\left[I_{A} \int_{t_{j_{0}-1}}^{t_{j_{0}}} E\left[D_{s} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{s}\right] d s\right],
\end{aligned}
$$

where the second equality is obtained from the Malliavin integration by parts formula (see, e.g., Theorem A.3.9 in page 283 of Biagini et al. [5]) and the third equality follows from the tower property for conditional expectations owing to the square integrability and the Fubini's theorem. Hence, it follows from the definition of conditional expectation that

$$
\begin{equation*}
E\left[H_{i_{1} \ldots i_{p, \pi}, 1}^{(c)}\left(t_{j_{0}}, x\right) \Delta^{\pi} W_{j_{0}} \mid \mathcal{F}_{t_{j_{0}-1}}\right]=E\left[\int_{t_{j_{0}-1}}^{t_{j_{0}}} E\left[D_{s} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{s}\right] d s \mid \mathcal{F}_{t_{j_{0}-1}}\right] . \tag{5.63}
\end{equation*}
$$

Therefore we have

$$
\begin{align*}
& \mid E\left[\int _ { t _ { j _ { 0 } - 1 } } ^ { t _ { j _ { 0 } } } \left(E\left[D_{s} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{s}\right]\right.\right.  \tag{5.64}\\
& \left.\left.\quad-\left(\bar{V}_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}(s, x)-\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}\left(s, x, V_{\pi, 1}(s, x)\right)\right)\right) d s \mid \mathcal{F}_{t_{j_{0}-1}}\right] \mid \\
\leq & \left(E \left[\int _ { t _ { j _ { 0 } - 1 } } ^ { t _ { j _ { 0 } } } \left(E\left[D_{s} H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{s}\right]\right.\right.\right. \\
& \left.\left.\left.\quad-\left(\bar{V}_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}(s, x)-\mathcal{J}_{i_{1} \ldots i_{p}}^{(c)}\left(s, x, V_{\pi, 1}(s, x)\right)\right)\right)^{2} d s \mid \mathcal{F}_{t_{j_{0}-1}}\right]\right)^{1 / 2} \\
= & \left(E \left[\left(H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)-E\left[H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right) \mid \mathcal{F}_{t_{j_{0}-1}}\right]\right.\right.\right. \\
\quad & \left.\left.\left.\quad-\left(H_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}}, x\right)-V_{i_{1} \ldots i_{p}, \pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)\right)\right)^{2} \mid \mathcal{F}_{t_{j_{0}-1}}\right]\right)^{1 / 2} \\
= & 0,
\end{align*}
$$

where the first inequality in (5.64) follows from the Hölder's and Jensen's inequalities; the first equality in (5.64) follows from (5.54), (5.61), and the Itô's isometry; the second inequality in (5.64) follows from (5.54) and (5.57). Thus, it follows from (5.63)-(5.64), (5.53), and (5.57) that (5.58) is true. Hence, we finish the proof of Lemma 5.9.

### 5.7 Proof of Theorem 3.1

First, we note that the convention given in Remark 5.1 will be employed in the following proof. Then, for each $t \in\left[t_{j_{0}-1}, t_{j_{0}}\right), x \in \mathcal{X}$, and $c \in\{0,1, \ldots, M\}$, we can obtain that

$$
\begin{align*}
E\left[\left\|\Delta V^{(c)}(t, x)\right\|^{2}\right] \leq & 5 E\left[\left\|V^{(c)}(t, x)-V^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]  \tag{5.65}\\
& +5 E\left[\left\|V^{(c)}\left(t_{j_{0}-1}, x\right)-V_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]
\end{align*}
$$

$$
\begin{aligned}
& +5 E\left[\left\|V_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)-V_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right] \\
& +5 E\left[\left\|V_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)-V_{\pi, 1}^{(c)}(t, x)\right\|^{2}\right] \\
& +5 E\left[\left\|V_{\pi, 1}^{(c)}(t, x)-V_{\pi}^{(c)}(t, x)\right\|^{2}\right]
\end{aligned}
$$

which implies that there is some nonnegative constant $K_{0}$ such that

$$
\begin{equation*}
E\left[\left\|\Delta V^{(c)}(t, x)\right\|^{2}\right] \leq K_{0} \pi \tag{5.66}
\end{equation*}
$$

In fact, for the first and fourth terms on the right-hand side of (5.65), it follows from (5.48) in Lemma 5.8 that there is some nonnegative constant $K_{1}$ such that

$$
\begin{align*}
& E\left[\left\|V^{(c)}(t, x)-V^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right] \leq K_{1} \pi  \tag{5.67}\\
& E\left[\left\|V_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)-V_{\pi, 1}^{(c)}(t, x)\right\|^{2}\right] \leq K_{1} \pi \tag{5.68}
\end{align*}
$$

For the third term on the right-hand side of (5.65), it follows from (5.57) in Lemma 5.9 that

$$
\begin{equation*}
E\left[\left\|V_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)-V_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]=0 \leq K_{1} \pi \tag{5.69}
\end{equation*}
$$

For the second term on the right-hand side of (5.65), it follows from (5.51)-(5.52), (2.11)(2.12), the Jensen's inequality, and (5.48)-(5.49) that

$$
\begin{align*}
& E\left[\left\|V^{(c)}\left(t_{j_{0}-1}, x\right)-V_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]  \tag{5.70}\\
\leq & \bar{K}_{2} \int_{t_{j_{0}-1}}^{t_{j_{0}}}\left(\left\|V(s)-V\left(t_{j_{0}}\right)\right\|_{C^{k+c}(D, q)}^{2}+\left\|\bar{V}(s)-\bar{V}\left(t_{j_{0}}\right)\right\|_{C^{k+c}(D, q d)}^{2}\right) d s \\
\leq & K_{2} \pi
\end{align*}
$$

where $\bar{K}_{2}$ and $K_{2}$ are some nonnegative constants.
For the last term on the right-hand side of (5.65), it follows from (5.57) in Lemma 5.9, Lemma 5.8, and Taylor's Theorem that

$$
\begin{align*}
& E\left[\left\|V_{\pi, 1}^{(c)}(t, x)-V_{\pi}^{(c)}(t, x)\right\|^{2}\right]  \tag{5.71}\\
\leq & E\left[\left\|V_{\pi, 0}^{(c)}(t, x)-V^{(c)}(t, x)\right\|^{2}\right]+E\left[\left\|V^{(c)}(t, x)-V^{(c)}(t, \xi(x))\right\|^{2}\right] \\
\leq & \bar{K}_{2} \int_{t}^{t_{j_{0}}}\left(\left\|V(s)-V\left(t_{j_{0}}\right)\right\|_{C^{k+c}(D, q)}^{2}+\left\|\bar{V}(s)-\bar{V}\left(t_{j_{0}}\right)\right\|_{C^{k+c}(D, q d)}^{2}\right) d s \\
& +\bar{K}_{3} \pi E\left[\left\|V^{(c+1)}\left(t, \xi_{1}(x)\right)\right\|^{2}\right] \\
\leq & K_{3} \pi
\end{align*}
$$

where $\bar{K}_{3}$ and $K_{3}$ are some nonnegative constants, $\xi(x)$ and $\xi_{1}(x)$ along each sample path are in some small neighborhoods centered at $x$. Therefore, it follows from (5.67)-(5.71) that the claim in (5.66) is true.

Furthermore, for each $t \in\left[t_{j_{0}-1}, t_{j_{0}}\right), x \in \mathcal{X}$, and $c \in\{0,1, \ldots, M\}$, we have that

$$
\begin{align*}
E\left[\left\|\Delta \bar{V}^{(c)}(t, x)\right\|^{2}\right] \leq & 5 E\left[\left\|\bar{V}^{(c)}(t, x)-\bar{V}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]  \tag{5.72}\\
& +5 E\left[\left\|\bar{V}^{(c)}\left(t_{j_{0}-1}, x\right)-\bar{V}_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right] \\
& +5 E\left[\left\|\bar{V}_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)-\bar{V}_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right] \\
& +5 E\left[\left\|\bar{V}_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)-\bar{V}_{\pi, 1}^{(c)}(t, x)\right\|^{2}\right] \\
& +5 E\left[\left\|\bar{V}_{\pi, 1}^{(c)}(t, x)-\bar{V}_{\pi}^{(c)}(t, x)\right\|^{2}\right],
\end{align*}
$$

which implies that there is some nonnegative constant $\bar{K}_{0}$ such that

$$
\begin{equation*}
E\left[\left\|\Delta \bar{V}^{(c)}(t, x)\right\|^{2}\right] \leq \bar{K}_{0} \pi . \tag{5.73}
\end{equation*}
$$

In fact, for the first and fourth terms on the right-hand side of (5.72), it follows from (5.49) in Lemma 5.8 that there is some nonnegative constant $\kappa_{1}$ such that

$$
\begin{align*}
& E\left[\left\|\bar{V}^{(c)}(t, x)-\bar{V}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right] \leq \kappa_{1} \pi,  \tag{5.74}\\
& E\left[\left\|\bar{V}_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)-\bar{V}_{\pi, 1}^{(c)}(t, x)\right\|^{2}\right] \leq \kappa_{1} \pi . \tag{5.75}
\end{align*}
$$

For the third term on the right-hand side of (5.72), note that

$$
\begin{equation*}
E\left[\left(X-E\left[X \mid \mathcal{F}_{t_{j_{0}-1}}\right]\right)^{2}\right] \leq E\left[(X-Y)^{2}\right] \tag{5.76}
\end{equation*}
$$

for any two $L_{\mathcal{F}_{t_{j_{0}}}}^{2}(P)$-integrable random variables $X$ and $Y$. Then, it follows from (5.58) in Lemma 5.9, the Hölder's inequality, (5.76), (5.75), (2.12), and (5.68) that

$$
\begin{align*}
& E\left[\left\|\bar{V}_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)-\bar{V}_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]  \tag{5.77}\\
& \leq \frac{2}{\Delta_{j_{0}}^{\pi}} E\left[\int_{t_{j_{0}-1}}^{t_{j_{0}}} \|\left(\bar{V}_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)-\mathcal{J}^{(c)}\left(x, V_{\pi, 1}\left(t_{j_{0}-1}, x\right)\right)\right.\right. \\
& \quad-\left(\bar{V}_{\pi, 1}^{(c)}(s, x)-\mathcal{J}^{(c)}\left(x, V_{\pi, 1}(s, x)\right) \|^{2} d s\right] \\
& +\frac{2}{\Delta_{j_{0}}^{\pi}} E\left[\int_{t_{j_{0}-1}}^{t_{j_{0}}} \|\right.
\end{align*} \quad \|\left(\bar{V}_{\pi, 1}^{(c)}(s, x)-\mathcal{J}^{(c)}\left(x, V_{\pi, 1}(s, x)\right) .\right.
$$

$$
\begin{array}{ll}
\leq \frac{4}{\Delta_{j_{0}}^{\pi}} E\left[\int_{t_{j_{0}-1}}^{t_{j_{0}}}\right. & \|\left(\bar{V}_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)-\mathcal{J}^{(c)}\left(x, V_{\pi, 1}\left(t_{j_{0}-1}, x\right)\right)\right. \\
& -\left(\bar{V}_{\pi, 1}^{(c)}(s, x)-\mathcal{J}^{(c)}\left(x, V_{\pi, 1}(s, x)\right) \|^{2} d s\right] \\
\leq \kappa_{1} \pi
\end{array}
$$

for some nonnegative constant $\kappa_{1}$.
For the second term on the right-hand side of (5.72), it follows from (5.77), the special form of the terminal variable in (5.54), Lemmas 5.7-5.8, and the proof of the first three terms in (5.65) that

$$
\begin{align*}
& E\left[\left\|\bar{V}^{(c)}\left(t_{j_{0}-1}, x\right)-\bar{V}_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]  \tag{5.78}\\
\leq & 2\left(E\left[\left\|\bar{V}^{(c)}\left(t_{j_{0}-1}, x\right)-\bar{V}_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]+E\left[\left\|\bar{V}_{\pi, 1}^{(c)}\left(t_{j_{0}-1}, x\right)-\bar{V}_{\pi, 0}^{(c)}\left(t_{j_{0}-1}, x\right)\right\|^{2}\right]\right) \\
\leq & 2 \int_{t_{j_{0}-1}}^{t_{j_{0}}} E\left[\left\|\bar{V}^{(c)}(s, x)-\bar{V}_{\pi, 1}^{(c)}(s, x)\right\|^{2}\right] d s+\bar{\kappa}_{2} \int_{t_{j_{0}-1}}^{t_{j_{0}}} E\left[\left\|V^{(c)}(s, x)-V_{\pi, 1}^{(c)}(s, x)\right\|^{2}\right] d s \\
& +\bar{\kappa}_{2} \pi \\
\leq & \kappa_{2} \pi
\end{align*}
$$

where $\kappa_{2}$ and $\bar{\kappa}_{2}$ are some nonnegative constants.
For the last term on the right-hand side of (5.72), it follows from Lemma 5.8, Taylor's Theorem, and the proof in (5.78) that

$$
\begin{align*}
& E\left[\left\|\bar{V}_{\pi, 1}^{(c)}(t, x)-\bar{V}_{\pi}^{(c)}(t, x)\right\|^{2}\right]  \tag{5.79}\\
\leq & E\left[\left\|\bar{V}_{\pi, 1}^{(c)}(t, x)-\bar{V}^{(c)}(t, x)\right\|^{2}\right]+E\left[\left\|\bar{V}^{(c)}(t, x)-\bar{V}^{(c)}(t, \xi(x))\right\|^{2}\right] \\
\leq & \bar{\kappa}_{2} \pi+\bar{\kappa}_{3} \pi E\left[\left\|V^{(c+1)}\left(t, \xi_{1}(x)\right)\right\|^{2}\right] \\
\leq & \kappa_{3} \pi
\end{align*}
$$

where $\bar{\kappa}_{3}$ and $\kappa_{3}$ are some nonnegative constants, $\xi(x)$ and $\xi_{1}(x)$ along each sample path are in some small neighborhoods centered at $x$. Therefore, it follows from (5.74)-(5.79) that the claim in (5.73) is true.

Finally, the claim in (3.32) for Algorithm 3.1 follows from (5.66) and (5.73). Hence, we finish the proof of Theorem 3.1.

## References

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