# L1 scheme for solving an inverse problem subject to a fractional diffusion equation * 

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#### Abstract

This paper considers the temporal discretization of an inverse problem subject to a time fractional diffusion equation. Firstly, the convergence of the L1 scheme is established with an arbitrary sectorial operator of spectral angle $<\pi / 2$, that is the resolvent set of this operator contains $\{z \in \mathbb{C} \backslash\{0\}:|\operatorname{Arg} z|<\theta\}$ for some $\pi / 2<\theta<\pi$. The relationship between the time fractional order $\alpha \in(0,1)$ and the constants in the error estimates is precisely characterized, revealing that the L1 scheme is robust as $\alpha$ approaches 1 . Then an inverse problem of a fractional diffusion equation is analyzed, and the convergence analysis of a temporal discretization of this inverse problem is given. Finally, numerical results are provided to confirm the theoretical results.


Keywords: fractional diffusion equation, L1 scheme, convergence, inverse problem.

## 1 Introduction

Let $0<T<\infty$ and let $\Omega \subset \mathbb{R}^{d}(d=1,2,3)$ be a bounded domain with Lipschitz continuous boundary. Assume that $\mathcal{A}$ is the realization of a second-order partial differential operator with homogeneous Dirichlet boundary condition in $L^{2}(\Omega)$. We consider the following fractional diffusion equation:

$$
\begin{equation*}
\mathrm{D}_{0+}^{\alpha} y(t)-\mathcal{A} y(t)=f(t), \quad 0<t \leqslant T, \quad \text { with } y(0)=0 \tag{1}
\end{equation*}
$$

where $0<\alpha<1, \mathrm{D}_{0+}^{\alpha}$ is a Riemann-Liouville fractional differential operator of order $\alpha$, and $f$ is a given function.

The L1 scheme is one of the most popular numerical methods for fractional diffusion equations. Lin and Xu [18] analyzed the L1 scheme for the fractional diffusion equation and obtained the temporal accuracy $O\left(\tau^{2-\alpha}\right)$ with $0<\alpha<1$, where $\tau$ denotes the time step size. Sun and Wu [35] proposed the L1 scheme

[^0]and derived temporal accuracy $O\left(\tau^{3-\alpha}\right)$ with $1<\alpha<2$ for the fractional wave equation. The analysis in the above two papers both assume that the underlying solution is sufficiently smooth. However, Jin et al. [9] proved that the L1 scheme is of only first-order temporal accuracy for fractional diffusion equations with non-vanishing initial value, and Jin et al. [13, Lemma 4.2] derived only first-order temporal accuracy for an inhomogeneous fractional equation. This phenomenon is caused by the well-known fact that the solution of a fractional diffusion equation generally has singularity in time no matter how smooth the data are, and it indicates that numerical analysis without regularity restrictions on the solution is important for the fractional diffusion equation. Recently, Yan et al. [41] proposed a modified L1 scheme for a fractional diffusion equation, which has $(2-\alpha)$-order temporal accuracy. For the L1 scheme with nonuniform grids, we refer the reader to $[34,17]$; we also note that analyzing the L1 scheme with nonuniform grids for a fractional diffusion equation with nonsmooth initial value remains to be an open problem.

Although the sectorial operator is considered, the theoretical results in [9, 41] can not be applied to a fractional diffusion equation with an arbitrary sectorial operator, since they require the spectral angle of the sectorial operator not to be greater than $\pi / 4$ (cf. [9, Remark 3.8]), that is the resolvent set of this operator must contain $\{z \in \mathbb{C} \backslash\{0\}:|\operatorname{Arg} z|<3 \pi / 4\}$. In our work, the analysis is suitable for an arbitrary sectorial operator with spectral angle $<\pi / 2$.

As the fractional diffusion equation is an extension of the normal diffusion equation, the solution of a fractional diffusion equation will naturally converge to the solution of a normal diffusion equation as $\alpha \rightarrow 1-$, and hence the L1 scheme is expected to be robust as $\alpha \rightarrow 1-$. Recently, Huang et al. [6] obtained an $\alpha$ robust error estimate for a multi-term fractional diffusion problem. However, to our best knowledge, the $\alpha$-robust convergence of the L1 scheme with an arbitrary sectorial operator is not available in the literature. Here we note that the constants in the error estimates in $[22,10,9,41]$ all depend on $\alpha$ and that the constants in the error estimates in [14] will clearly blow up as $\alpha \rightarrow 1-$. This motivates us to develop new techniques to analyze the convergence of the L1 scheme with an arbitrary sectorial operator and to investigate the robustness of the L1 scheme as $\alpha \rightarrow 1-$.

The theory of inverse problems for differential equations has been extensively developed within the framework of mathematical physics. One important class of inverse problems for parabolic equations is to reconstruct the source term, the initial value or the boundary conditions from the value of the solution at the final time; see [32, 33]. The time fractional diffusion equation is an extension of the normal diffusion equation, widely used to model the physical phenomena with memory effect. Hence, this paper considers the source term identification of a time fractional diffusion equation, based on the value of the solution at the final time. For the related theoretical results, we refer the reader to [7, $19,28,37,38,39]$ and the references therein. We apply the famous Tikhonov regularization technique to this inverse problem and establish the convergence of its temporal discretization that uses the L1 scheme.

The main contributions of this paper are as follows:

1. the convergence of the L1 scheme for solving time fractional diffusion equations with an arbitrary sectorial operator of spectral angle $<\pi / 2$ is established;
2. the constants in the derived error estimates will not blow up as $\alpha \rightarrow 1-$, which shows that the L1 scheme is robust as $\alpha \rightarrow 1$-;
3. the convergence analysis of a temporally discrete inverse problem subject to a fractional diffusion equation is provided.

Moreover, a feature of the error estimates in this paper is that they immediately derive the corresponding error estimates of the backward Euler scheme, by passing to the limit $\alpha \rightarrow 1-$.

Before concluding this section, we would also like to mention two important algorithms for solving fractional diffusion equations. The first algorithm uses the convolution quadrature proposed by Lubich [20, 21]. Lubich et al. [22, 1] firstly used the convolution quadrature to design numerical methods for fractional diffusion-wave equations, and then Jin et al. [10, 11] further developed these algorithms. The second algorithm employs the Galerkin methods to discretize the time fractional operators, which was firstly developed by McLean and Mustapha [25, 30, 31, 29].

The rest of the paper is organized as follows. Section 2 introduces some conventions, the definitions of $\mathcal{A}$ and $\mathcal{A}^{*}$, the Riemann-Liouville fractional operators and the mild solution theory of linear fractional diffusion equations. Section 3 derives the convergence of the L1 scheme. Section 4 investigates an inverse problem of a fractional diffusion equation and establishes the convergence of a temporally discrete inverse problem. Finally, Section 5 performs three numerical experiments to verify the theoretical results.

## 2 Preliminaries

Throughout this paper, we will use the following conventions: for each linear vector space, the scalars are the complex numbers; $H_{0}^{1}(\Omega)$ is a standard complexvalued Sobolev space, and $H^{-1}(\Omega)$ is the usual dual space of $H_{0}^{1}(\Omega) ; \mathcal{L}\left(L^{2}(\Omega)\right)$ is the space of all bounded linear operators on $L^{2}(\Omega)$; for a Banach space $\mathcal{B}$, we use $\langle\cdot, \cdot\rangle_{\mathcal{B}}$ to denote a duality paring between $\mathcal{B}^{*}$ (the dual space of $\mathcal{B}$ ) and $\mathcal{B}$; for a Lebesgue measurable subset $\mathcal{D} \subset \mathbb{R}^{l}, 1 \leqslant l \leqslant 4,\langle p, q\rangle_{\mathcal{D}}$ means the integral $\int_{\mathcal{D}} p \bar{q}$, where $\bar{q}$ is the conjugate of $q$; for a function $v$ defined on $(0, T)$, by $v(t-)$, $0<t \leqslant T$, we mean the limit $\lim _{s \rightarrow t-} v(s)$; the notations $c_{\times}, d_{\times}, C_{\times}$mean some positive constants and their values may differ at each occurrence. In addition, for any $0<\theta<\pi$, define

$$
\begin{align*}
& \Sigma_{\theta}:=\{z \in \mathbb{C} \backslash\{0\}:-\theta<\operatorname{Arg} z<\theta\}  \tag{2}\\
& \Gamma_{\theta}:=\{z \in \mathbb{C} \backslash\{0\}:|\operatorname{Arg} z|=\theta\} \cup\{0\}  \tag{3}\\
& \Upsilon_{\theta}:=\left\{z \in \Gamma_{\theta}:-\pi \leqslant \Im z \leqslant \pi\right\} \tag{4}
\end{align*}
$$

where $\Gamma_{\theta}$ and $\Upsilon_{\theta}$ are so oriented that the negative real axis is to their left. For the integral $\int_{\Gamma_{\theta}} v \mathrm{~d} z$ or $\int_{\Upsilon_{\theta}} v \mathrm{~d} z$, if $v$ has singularity or is not defined at the origin, then $\Gamma_{\theta}$ or $\Upsilon_{\theta}$ should be deformed so that the origin is to its left; for example, $\Gamma_{\theta}$ is deformed to

$$
\{z \in \mathbb{C}:|z|>\epsilon,|\operatorname{Arg} z|=\theta\} \cup\{z \in \mathbb{C}:|z|=\epsilon,|\operatorname{Arg} z| \leqslant \theta\}
$$

where $0<\epsilon<\infty$.

Riemann-Liouville fractional calculus operators. Assume that $-\infty \leqslant$ $a<b \leqslant \infty$ and $X$ is a Banach space. For any $\gamma>0$, define

$$
\begin{array}{ll}
\left(\mathrm{D}_{a+}^{-\gamma} v\right)(t):=\frac{1}{\Gamma(\gamma)} \int_{a}^{t}(t-s)^{\gamma-1} v(s) \mathrm{d} s, & \text { a.e. } t \in(a, b), \\
\left(\mathrm{D}_{b-}^{-\gamma} v\right)(t):=\frac{1}{\Gamma(\gamma)} \int_{t}^{b}(s-t)^{\gamma-1} v(s) \mathrm{d} s, & \text { a.e. } t \in(a, b),
\end{array}
$$

for all $v \in L^{1}(a, b ; X)$, where $\Gamma(\cdot)$ is the gamma function. In addition, let $\mathrm{D}_{a+}^{0}$ and $\mathrm{D}_{b-}^{0}$ be the identity operator on $L^{1}(a, b ; X)$. For $j-1<\gamma \leqslant j, j \in \mathbb{N}_{>0}$, define

$$
\begin{aligned}
\mathrm{D}_{a+}^{\gamma} v & :=\mathrm{D}^{j} \mathrm{D}_{a+}^{\gamma-j} v, \\
\mathrm{D}_{b-}^{\gamma} v & :=(-\mathrm{D})^{j} \mathrm{D}_{b-}^{\gamma-j} v,
\end{aligned}
$$

for all $v \in L^{1}(a, b ; X)$, where D is the first-order differential operator in the distribution sense.
Definitions of $\mathcal{A}$ and $\mathcal{A}^{*}$. Let $\mathcal{A}: H_{0}^{1}(\Omega) \rightarrow H^{-1}(\Omega)$ be a second-order partial differential operator of the form

$$
\mathcal{A} v:=\sum_{i, j=1}^{d} \frac{\partial}{\partial_{x_{i}}}\left(a_{i j}(x) \frac{\partial}{\partial x_{j}} v\right)+b(x) \cdot \nabla v+c(x) v, \quad \forall v \in H_{0}^{1}(\Omega),
$$

where $a_{i j} \in L^{\infty}(\Omega), b \in\left[L^{\infty}(\Omega)\right]^{d}$ and $c \in L^{\infty}(\Omega)$ are real-valued. Assume that $\mathcal{A}: H_{0}^{1}(\Omega) \rightarrow H^{-1}(\Omega)$ is a sectorial operator satisfying that

$$
\left\{\begin{array}{l}
\rho(\mathcal{A}) \supset \Sigma_{\omega_{0}}  \tag{5a}\\
\|R(z, \mathcal{A})\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \leqslant \mathcal{M}_{0}|z|^{-1} \quad \forall z \in \Sigma_{\omega_{0}}, \\
\langle\mathcal{A} v, v\rangle_{H_{0}^{1}(\Omega)} \leqslant 0, \quad \forall v \in H_{0}^{1}(\Omega)
\end{array}\right.
$$

where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}, \pi / 2<\omega_{0}<\pi, R(z, \mathcal{A}):=(z-\mathcal{A})^{-1}$, and $\mathcal{M}_{0}$ is a positive constant. Define the adjoint operator $\mathcal{A}^{*}: H_{0}^{1}(\Omega) \rightarrow H^{-1}(\Omega)$ of $\mathcal{A}$ by that

$$
\mathcal{A}^{*} v:=\sum_{i, j=1}^{d} \frac{\partial}{\partial_{x_{j}}}\left(a_{i j}(x) \frac{\partial}{\partial x_{i}} v\right)-\nabla \cdot(b(x) v)+c(x) v, \quad \forall v \in H_{0}^{1}(\Omega) .
$$

It is evident that

$$
\langle\mathcal{A} v, w\rangle_{H_{0}^{1}(\Omega)}=\overline{\left\langle\mathcal{A}^{*} w, v\right\rangle_{H_{0}^{1}(\Omega)}} \quad \text { for all } v, w \in H_{0}^{1}(\Omega)
$$

Solutions of the fractional diffusion equation. For any $t>0$, define

$$
\begin{equation*}
E(t):=\frac{1}{2 \pi i} \int_{\Gamma_{\omega_{0}}} e^{t z} R\left(z^{\alpha}, \mathcal{A}\right) \mathrm{d} z \tag{6}
\end{equation*}
$$

By (5b), it is evident that $E$ is an $\mathcal{L}\left(L^{2}(\Omega)\right)$-valued analytic function on $(0, \infty)$. Moreover, a direct computation gives the following two estimates (cf. Jin et al. [10]): for any $t>0$,

$$
\begin{align*}
\|E(t)\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} & \leqslant C_{\omega_{0}, \mathcal{M}_{0}} t^{\alpha-1}  \tag{7}\\
\left\|E^{\prime}(t)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} & \leqslant C_{\omega_{0}, \mathcal{M}_{0}} t^{\alpha-2} \tag{8}
\end{align*}
$$

For any $g \in L^{1}\left(0, T ; L^{2}(\Omega)\right)$, we call

$$
\begin{equation*}
(S g)(t):=(E * g)(t)=\int_{0}^{t} E(t-s) g(s) \mathrm{d} s, \quad \text { a.e. } 0<t \leqslant T \tag{9}
\end{equation*}
$$

the mild solution to the following fractional diffusion equation

$$
\begin{equation*}
\left(\mathrm{D}_{0+}^{\alpha}-\mathcal{A}\right) w=g, \quad \text { with } w(0)=0 \tag{10}
\end{equation*}
$$

where the symbol $*$ denotes the convolution.
If $g=v \delta_{0}$ with $v \in L^{2}(\Omega)$ and $\delta_{0}$ being the Dirac measure (in time) concentrated at $t=0$, then we call

$$
\begin{equation*}
\left(S\left(v \delta_{0}\right)\right)(t):=E(t) v, \quad 0<t \leqslant T \tag{11}
\end{equation*}
$$

the mild solution to equation (10). Symmetrically, for any $g \in L^{1}\left(0, T ; L^{2}(\Omega)\right)$, we call

$$
\begin{equation*}
\left(S^{*} g\right)(t):=\int_{t}^{T} E^{*}(s-t) g(s) \mathrm{d} s, \quad \text { a.e. } 0<t<T \tag{12}
\end{equation*}
$$

the mild solution to the following backward fractional diffusion equation:

$$
\begin{equation*}
\left(\mathrm{D}_{T-}^{\alpha}-\mathcal{A}^{*}\right) w=g, \quad \text { with } w(T)=0 . \tag{13}
\end{equation*}
$$

If $g=v \delta_{T}$ with $v \in L^{2}(\Omega)$ and $\delta_{T}$ being the Dirac measure (in time) concentrated at $t=T$, then we call

$$
\begin{equation*}
\left(S^{*}\left(v \delta_{T}\right)\right)(t):=E^{*}(T-t) v, \quad 0<t \leqslant T \tag{14}
\end{equation*}
$$

the mild solution to equation (13). The above $E^{*}$ is defined by

$$
\begin{equation*}
E^{*}(t):=\frac{1}{2 \pi i} \int_{\Gamma_{\omega_{0}}} e^{t z} R\left(z^{\alpha}, \mathcal{A}^{*}\right) \mathrm{d} z, \quad t>0 . \tag{15}
\end{equation*}
$$

Similarly to (7), (8), for any $t>0$, we have

$$
\begin{align*}
&\left\|E^{*}(t)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} t^{\alpha-1}  \tag{16}\\
&\left\|\left(E^{*}\right)^{\prime}(t)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} t^{\alpha-2} \tag{17}
\end{align*}
$$

Evidently, for any $t>0, E^{*}(t)$ is the adjoint operator of $E(t)$ in the sense that

$$
\begin{equation*}
\langle E(t) v, w\rangle_{\Omega}=\left\langle v, E^{*}(t) w\right\rangle_{\Omega} \quad \forall v, w \in L^{2}(\Omega) \tag{18}
\end{equation*}
$$

Remark 2.1. By (7), a routine calculation (cf. [2, Theorem 2.6]) yields that

$$
\begin{equation*}
\|S g\|_{C\left([0, T] ; L^{2}(\Omega)\right)} \leqslant C_{\alpha, q, \omega_{0}, \mathcal{M}_{0}, T}\|g\|_{L^{q}\left(0, T ; L^{2}(\Omega)\right)} \tag{19}
\end{equation*}
$$

for all $g \in L^{q}\left(0, T ; L^{2}(\Omega)\right)$ with $q>1 / \alpha$.
Remark 2.2. For the above solution theory of fractional diffusion equations, we refer the reader to [22, 26, 10].

The L1 scheme. Let $J \in \mathbb{N}_{>0}$ and define $t_{j}:=j \tau$ for each $j=0,1,2, \ldots, J$, where $\tau:=T / J$. Define $b_{j}:=j^{1-\alpha} / \Gamma(2-\alpha)$ for each $j \in \mathbb{N}$. Assume that $g \in L^{1}\left(0, T ; H^{-1}(\Omega)\right)$. Applying the L1 scheme [18] to problem (10) yields the following discretization: seek $\left\{W_{j}\right\}_{j=1}^{J} \subset H_{0}^{1}(\Omega)$ such that, for any $1 \leqslant k \leqslant J$,

$$
\begin{equation*}
b_{1} W_{k}+\sum_{j=1}^{k-1}\left(b_{k-j+1}-2 b_{k-j}+b_{k-j-1}\right) W_{j}-\tau^{\alpha} \mathcal{A} W_{k}=\tau^{\alpha-1} \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t \tag{20}
\end{equation*}
$$

in $H^{-1}(\Omega)$, where $W_{j}, 1 \leqslant j \leqslant J$, is an approximation of $w\left(t_{j}\right)$. Symmetrically, applying the L1 scheme to problem (13) yields the following discretization: seek $\left\{\mathcal{W}_{j}\right\}_{j=1}^{J} \subset H_{0}^{1}(\Omega)$ such that, for any $1 \leqslant k \leqslant J$,

$$
\begin{equation*}
b_{1} \mathcal{W}_{k}+\sum_{j=k+1}^{J}\left(b_{j-k+1}-2 b_{j-k}+b_{j-k-1}\right) \mathcal{W}_{j}-\tau^{\alpha} \mathcal{A}^{*} \mathcal{W}_{k}=\tau^{\alpha-1} \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t \tag{21}
\end{equation*}
$$

in $H^{-1}(\Omega)$. For each $1 \leqslant j \leqslant J$, we will use $S_{\tau, j} g$ and $S_{\tau, j}^{*} g$ to denote the above $W_{j}$ and $\mathcal{W}_{j}$, respectively, that is

$$
\begin{equation*}
S_{\tau, j} g:=W_{j}, \quad S_{\tau, j}^{*} g:=\mathcal{W}_{j} \tag{22}
\end{equation*}
$$

In addition, for each $1 \leqslant j \leqslant J$, we define

$$
\begin{equation*}
\mathcal{S}_{\tau, j}\left(v \delta_{0}\right):=\mathcal{S}_{\tau, j}\left(v \widehat{\delta}_{0}\right), \quad \mathcal{S}_{\tau, j}^{*}\left(v \delta_{T}\right):=\mathcal{S}_{\tau, j}^{*}\left(v \widehat{\delta}_{T}\right) \tag{23}
\end{equation*}
$$

where $v \in H^{-1}(\Omega)$ and

$$
\begin{align*}
& \widehat{\delta}_{0}(t):= \begin{cases}\tau^{-1} & \text { if } 0<t<t_{1} \\
0 & \text { if } t_{1}<t<T\end{cases}  \tag{24}\\
& \widehat{\delta}_{T}(t):= \begin{cases}0 & \text { if } 0<t<t_{J-1} \\
\tau^{-1} & \text { if } t_{J-1}<t<T\end{cases} \tag{25}
\end{align*}
$$

## 3 Convergence of the L1 scheme

Theorem 3.1. Let $0<\alpha<1$. Let $S g$ and $S_{\tau, j} g$ be defined by (9) and (22), respectively. Then we have the following estimates:

1. For any $g \in L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$,

$$
\begin{equation*}
\max _{1 \leqslant j \leqslant J}\left\|(S g)\left(t_{j}\right)-S_{\tau, j} g\right\|_{L^{2}(\Omega)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(\frac{1}{\alpha}+\frac{1-J^{\alpha-1}}{1-\alpha}\right)\|g\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)} \tag{26}
\end{equation*}
$$

2. For any $v \in L^{2}(\Omega)$,

$$
\begin{align*}
& \max _{1 \leqslant j \leqslant J} j^{2-\alpha}\left\|S\left(v \delta_{0}\right)\left(t_{j}\right)-S_{\tau, j}\left(v \delta_{0}\right)\right\|_{L^{2}(\Omega)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha-1}\|v\|_{L^{2}(\Omega)}  \tag{27}\\
& \sum_{j=1}^{J}\left\|S\left(v \delta_{0}\right)-S_{\tau, j}\left(v \delta_{0}\right)\right\|_{L^{1}\left(t_{j-1}, t_{j} ; L^{2}(\Omega)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(\frac{1}{\alpha}+\frac{1-J^{\alpha-1}}{1-\alpha}\right)\|v\|_{L^{2}(\Omega)} \tag{28}
\end{align*}
$$

Remark 3.1. Assume that $g \in L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$. Passing to the limit $\alpha \rightarrow 1-$ in (20) and (26) yields that, for the parabolic equation

$$
w^{\prime}-\mathcal{A} w=g, \quad \text { with } w(0)=0
$$

and the corresponding backward Euler scheme

$$
\left\{\begin{array}{l}
W_{0}=0 \\
W_{k}-W_{k-1}-\tau \mathcal{A} W_{k}=\int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t, \quad 1 \leqslant k \leqslant J
\end{array}\right.
$$

one has the error estimate, noting that $\lim _{\alpha \rightarrow 1} \frac{1-J^{\alpha-1}}{1-\alpha}=\ln J$,

$$
\max _{1 \leqslant j \leqslant J}\left\|w\left(t_{j}\right)-W_{j}\right\|_{L^{2}(\Omega)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}}(1+\ln J) \tau\|g\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)}
$$

Remark 3.2. Let us consider the following time fractional diffusion equation

$$
\mathrm{D}_{0+}^{\alpha}\left(y-y_{0}\right)(t)-\mathcal{A} y(t)=0, \quad 0<t \leqslant T, \quad \text { with } y(0)=y_{0}
$$

where $y_{0} \in L^{2}(\Omega)$ is given. Applying the L1 scheme to this equation yields the following discretization: seek $\left\{W_{j}\right\}_{j=1}^{J} \subset H_{0}^{1}(\Omega)$ such that, for any $1 \leqslant k \leqslant J$,

$$
b_{1} W_{k}+\sum_{j=1}^{k-1}\left(b_{k-j+1}-2 b_{k-j}+b_{k-j-1}\right) W_{j}-\tau^{\alpha} \mathcal{A} W_{k}=\tau^{\alpha-1}\left(b_{k}-b_{k-1}\right) y_{0}
$$

in $H^{-1}(\Omega)$. Following the proof of [9, Theorem 3.1], we can use the technical results in Subsection 3.1 to derive that, for any $1 \leqslant j \leqslant J$,

$$
\left\|y\left(t_{j}\right)-W_{j}\right\|_{L^{2}(\Omega)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau t_{j}^{-1}\left\|y_{0}\right\|_{L^{2}(\Omega)}
$$

The main task of the rest of this section is to prove the above theorem.

### 3.1 Some technical results

Define the discrete Laplace transform of $\left\{b_{j}\right\}_{j=1}^{\infty}$ by that

$$
\widehat{b}(z):=\sum_{j=1}^{\infty} b_{j} e^{-j z}, \quad z \in \Sigma_{\pi / 2}
$$

By the analytic continuation technique, $\widehat{b}$ has an analytic continuation (cf. [27, Equation (21)])

$$
\begin{equation*}
\widehat{b}(z)=\frac{1}{2 \pi i} \int_{-\infty}^{(0+)} \frac{e^{w-z}}{1-e^{w-z}} w^{\alpha-2} \mathrm{~d} w, \quad z \in \Sigma_{\pi} \tag{29}
\end{equation*}
$$

where $\int_{-\infty}^{(0+)}$ means an integral on a piecewise smooth and non-self-intersecting path enclosing the negative real axis and orienting counterclockwise, and 0 and $\{z+2 k \pi i \neq 0: k \in \mathbb{Z}\}$ lie on the different sides of this path. Define

$$
\begin{equation*}
\psi(z):=\left(e^{z}-1\right)^{2} \widehat{b}(z), \quad z \in \Sigma_{\pi} \tag{30}
\end{equation*}
$$

For $z=x+i y \in \mathbb{C} \backslash(-\infty, 0]$, we have that (cf. [9, Equation (3.7)])
$\Re\left(e^{-z} \psi(z)\right)=\frac{\sin (\pi(1-\alpha))}{\pi} \int_{0}^{\infty} \frac{s^{\alpha-2}\left(1-e^{-s}\right)\left(1+e^{-2 x-s}-e^{-x-s} \cos y-e^{-x} \cos y\right)}{1-2 e^{-x-s} \cos y+e^{-2 x-2 s}} \mathrm{~d} s$.

Lemma 3.1. For any $L>0$, we have

$$
\begin{equation*}
\sup _{0<\alpha<1} \sup _{\substack{z \in \Sigma, \pi \\-L \leqslant \Re z \leqslant 0 \\-\pi \leqslant \Im z \leqslant \pi}}\left|\widehat{b}(z)-z^{\alpha-2}\right|=C_{L} \tag{32}
\end{equation*}
$$

Proof. For any $z \in \Sigma_{\pi}$ satisfying $-L \leqslant \Re z \leqslant 0$ and $0 \leqslant \Im z \leqslant \pi$, by (29), Cauchy's integral theorem and the residue theorem we obtain

$$
\begin{aligned}
\widehat{b}(z)= & z^{\alpha-2}+\frac{1}{2 \pi i} \int_{-\infty-i \pi}^{1-i \pi} \frac{e^{w-z}}{1-e^{w-z}} w^{\alpha-2} \mathrm{~d} w+\frac{1}{2 \pi i} \int_{1-i \pi}^{1+i 3 \pi / 2} \frac{e^{w-z}}{1-e^{w-z}} w^{\alpha-2} \mathrm{~d} w \\
& +\frac{1}{2 \pi i} \int_{1+i 3 \pi / 2}^{-\infty+i 3 \pi / 2} \frac{e^{w-z}}{1-e^{w-z}} w^{\alpha-2} \mathrm{~d} w \\
= & z^{\alpha-2}+G(\alpha, z)
\end{aligned}
$$

A routine calculation verifies that $G$ is continuous on

$$
[0,1] \times\{\xi \in \mathbb{C}:-L \leqslant \Re \xi \leqslant 0,0 \leqslant \Im \xi \leqslant \pi\}
$$

and so

$$
\sup _{0<\alpha<1} \sup _{\substack{-L \leqslant \Re z \leqslant 0 \\ 0 \leqslant \Im z \leqslant \pi}}|G(\alpha, z)|=C_{L}
$$

It follows that

$$
\sup _{\substack { 0<\alpha<1 \\
\begin{subarray}{c}{z \in \Sigma_{\pi} \\
-L \leqslant \Im z \leqslant 0 \\
0 \leqslant \Im z \leqslant \pi{ 0 < \alpha < 1 \\
\begin{subarray} { c } { z \in \Sigma _ { \pi } \\
- L \leqslant \Im z \leqslant 0 \\
0 \leqslant \Im z \leqslant \pi } }\end{subarray}} \sup _{\substack{ \\
}}\left|\widehat{b}(z)-z^{\alpha-2}\right|=C_{L}
$$

Similarly,

$$
\sup _{\substack{0<\alpha<1\\}} \sup _{\substack{z \in \Sigma_{\pi} \\-L \leqslant \Re z \leqslant 0 \\-\pi \leqslant \Im z \leqslant 0}}\left|\widehat{b}(z)-z^{\alpha-2}\right|=C_{L}
$$

Combining the above two estimates proves (32) and hence this lemma.
Lemma 3.2. For any $0<\delta<\pi$ and $L>0$, we have

$$
\begin{align*}
\inf _{0<\alpha<1} \inf _{\delta \leqslant y \leqslant \pi} \Re\left(e^{-i y} \psi(i y)\right) & =C_{\delta},  \tag{33}\\
\sup _{0<\alpha<1} \sup _{\substack{-L \leqslant \Re z \leqslant 0 \\
\delta \leqslant \Im z \leqslant \pi}}\left|\frac{\mathrm{~d}}{\mathrm{~d} z}\left(e^{-z} \psi(z)\right)\right| & =C_{\delta, L} . \tag{34}
\end{align*}
$$

Proof. For any $\delta \leqslant y \leqslant \pi$, we have, by (31) with $z=0+i y$,

$$
\begin{aligned}
& \Re\left(e^{-i y} \psi(i y)\right)=\frac{\sin (\pi(1-\alpha))}{\pi} \int_{0}^{\infty} \frac{s^{\alpha-2}\left(1-e^{-2 s}\right)(1-\cos y)}{1-2 e^{-s} \cos y+e^{-2 s}} \mathrm{~d} s \\
& >\frac{\sin (\pi(1-\alpha))}{\pi}(1-\cos \delta) \int_{0}^{\infty} \frac{s^{\alpha-2}\left(1-e^{-2 s}\right)}{1+2 e^{-s}+e^{-2 s}} \mathrm{~d} s \\
& =\frac{\sin (\pi(1-\alpha))}{\pi}(1-\cos \delta)\left[\int_{0}^{1} \frac{s^{\alpha-2}\left(1-e^{-2 s}\right)}{1+2 e^{-s}+e^{-2 s}} \mathrm{~d} s+\int_{1}^{\infty} \frac{s^{\alpha-2}\left(1-e^{-2 s}\right)}{1+2 e^{-s}+e^{-2 s}} \mathrm{~d} s\right] .
\end{aligned}
$$

In view of the two simple estimates

$$
\begin{aligned}
\int_{0}^{1} \frac{s^{\alpha-2}\left(1-e^{-2 s}\right)}{1+2 e^{-s}+e^{-2 s}} \mathrm{~d} s & >\int_{0}^{1} \frac{s^{\alpha-2}\left(e^{-2 s} 2 s\right)}{4} \mathrm{~d} s=\int_{0}^{1} \frac{s^{\alpha-1}\left(e^{-2 s}\right)}{2} \mathrm{~d} s \\
& >\int_{0}^{1} \frac{s^{\alpha-1}\left(e^{-2}\right)}{2} \mathrm{~d} s=\frac{e^{-2}}{2 \alpha}
\end{aligned}
$$

and

$$
\int_{1}^{\infty} \frac{s^{\alpha-2}\left(1-e^{-2 s}\right)}{1+2 e^{-s}+e^{-2 s}} \mathrm{~d} s>\int_{1}^{\infty} s^{\alpha-2} \frac{1-e^{-2}}{4} \mathrm{~d} s=\frac{1-e^{-2}}{4(1-\alpha)}
$$

we then obtain, for any $\delta \leqslant y \leqslant \pi$,

$$
\Re\left(e^{-i y} \psi(i y)\right) \geqslant \frac{\sin (\pi(1-\alpha))}{\pi}(1-\cos \delta)\left(\frac{e^{-2}}{2 \alpha}+\frac{1-e^{-2}}{4(1-\alpha))}\right) \geqslant C_{\delta} .
$$

This implies inequality (33).
Now let us prove (34). For any $z \in \mathbb{C}$ satisfying $\delta \leqslant \Im z \leqslant \pi$, using the residue theorem yields, by (29), that

$$
\begin{equation*}
\widehat{b}(z)=\sum_{k=-\infty}^{\infty}(z+2 k \pi i)^{\alpha-2}, \tag{35}
\end{equation*}
$$

and hence

$$
\widehat{b}^{\prime}(z)=(\alpha-2) \sum_{k=-\infty}^{\infty}(z+2 k \pi i)^{\alpha-3} .
$$

A simple calculation then gives

$$
\sup _{0<\alpha<1} \sup _{\substack{-L \leqslant \Re z \leqslant 0 \\ \delta \leqslant \Im z \leqslant \pi}}\left|e^{-z}\left(e^{z}-1\right)^{2} \widehat{b}^{\prime}(z)\right|=C_{\delta, L} .
$$

In addition, Lemma 3.1 implies

$$
\sup _{0<\alpha<1} \sup _{\substack{-L \leqslant \Re z \leqslant 0 \\ \delta \leqslant \Im z \leqslant \pi}}\left|\left(e^{z}-e^{-z}\right) \widehat{b}(z)\right|=C_{\delta, L}
$$

Consequently, (34) follows from the equality

$$
\frac{\mathrm{d}}{\mathrm{~d} z}\left(e^{-z} \psi(z)\right)=\left(e^{z}-e^{-z}\right) \widehat{b}(z)+e^{-z}\left(e^{z}-1\right)^{2} \widehat{b^{\prime}}(z), \quad z \in \Sigma_{\pi}
$$

This completes the proof.
Lemma 3.3. Assume that $\pi / 2<\theta_{0}<\pi$. Then there exists $\pi / 2<\theta^{*} \leqslant \theta_{0}$ depending only on $\theta_{0}$ such that

$$
\begin{equation*}
e^{-z} \psi(z) \in \Sigma_{\theta_{0}} \quad \text { for all } z \in \Sigma_{\theta^{*}} \text { with }-\pi \leqslant \Im z \leqslant \pi \tag{36}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|e^{-z} \psi(z)\right| \geqslant C_{\theta_{0}}|z|^{\alpha} \quad \text { for all } z \in \Upsilon_{\theta^{*}} \backslash\{0\} \tag{37}
\end{equation*}
$$

Proof. Step 1. By (31), a simple calculation gives

$$
\Re\left(e^{-z} \psi(z)\right)>0 \text { for all } z \in D_{1}
$$

so that

$$
\begin{equation*}
e^{-z} \psi(z) \in \Sigma_{\pi / 2} \quad \text { for all } z \in D_{1} \tag{38}
\end{equation*}
$$

where

$$
D_{1}=\{z \in \mathbb{C}: \Re z \geqslant 0,0 \leqslant \Im z \leqslant \pi, z \neq 0\}
$$

Step 2. From (30) and (32) we conclude that there exists a continuous function $G$ on $(0,1) \times D_{2}$, such that

$$
\begin{equation*}
e^{-z} \psi(z)=z^{\alpha}(1+z G(\alpha, z)) \quad \forall z \in D_{2} \tag{39}
\end{equation*}
$$

and that

$$
\sup _{0<\alpha<1} \sup _{z \in D_{2}}|G(\alpha, z)|=C_{\theta_{0}}
$$

where

$$
D_{2}:=\left\{\xi \in \mathbb{C} \backslash\{0\}: \pi / 2 \leqslant \operatorname{Arg}(\xi) \leqslant \theta_{0}, 0<\Im \xi \leqslant \pi\right\}
$$

Hence, there exists $0<\epsilon_{0}<\pi$, depending only on $\theta_{0}$, such that

$$
\begin{aligned}
& |\operatorname{Arg}(1+z G(\alpha, z))| \leqslant\left(\theta_{0}-\pi / 2\right) / 2 \quad \text { and } \quad\left|e^{-z} \psi(z)\right| \geqslant C_{\theta_{0}}|z|^{\alpha} \\
& \text { for all } z \in \Sigma_{\theta_{0}} \backslash \Sigma_{\pi / 2} \text { with } 0<\Im z \leqslant \epsilon_{0} .
\end{aligned}
$$

Since

$$
\begin{aligned}
& \operatorname{Arg}\left(e^{-z} \psi(z)\right)=\operatorname{Arg}\left(z^{\alpha}(1+z G(\alpha, z))\right) \quad(\text { by }(39)) \\
= & \alpha \operatorname{Arg}(z)+\operatorname{Arg}(1+z G(\alpha, z))
\end{aligned}
$$

it follows that

$$
\begin{align*}
& e^{-z} \psi(z) \in \Sigma_{\theta_{0}} \text { and }\left|e^{-z} \psi(z)\right| \geqslant C_{\theta_{0}}|z|^{\alpha}  \tag{40}\\
& \text { for all } z \in \Sigma_{\left(\theta_{0}+\pi / 2\right) / 2} \backslash \Sigma_{\pi / 2} \text { with } 0<\Im z \leqslant \epsilon_{0}
\end{align*}
$$

Step 3. Note that $\epsilon_{0}$ is a constant depending only on $\theta_{0}$. By (33) we have

$$
\inf _{0<\alpha<1} \inf _{\substack{\Re z=0 \\ \epsilon_{0} \leqslant \Im z \leqslant \pi}} \Re\left(e^{-z} \psi(z)\right)=C_{\theta_{0}} .
$$

From (34) we then conclude that there exists $0<\epsilon_{1}<\pi$, depending only on $\theta_{0}$, such that

$$
\inf _{0<\alpha<1}^{\substack{-\epsilon_{1} \leqslant \Re z \leqslant 0 \\ \epsilon_{0} \leqslant \Im z \leqslant \pi}} \mid \inf \Re\left(e^{-z} \psi(z)\right)=C_{\theta_{0}}>0 .
$$

It follows that

$$
\begin{align*}
& e^{-z} \psi(z) \in \Sigma_{\pi / 2} \text { and }\left|e^{-z} \psi(z)\right| \geqslant C_{\theta_{0}} \text { for all }  \tag{41}\\
& z \in \Sigma_{\left(\theta_{0}+\pi / 2\right) / 2} \backslash \Sigma_{\pi / 2} \text { with }-\epsilon_{1} \leqslant \Re z \leqslant 0 \text { and } \epsilon_{0} \leqslant \Im z \leqslant \pi
\end{align*}
$$

Letting $\theta^{*}:=\pi / 2+\arctan \left(\epsilon_{1} / \pi\right)$, by (38), (40) and (41) we obtain that

$$
\begin{equation*}
e^{-z} \psi(z) \in \Sigma_{\theta_{0}} \quad \text { for all } z \in \Sigma_{\theta^{*}} \text { with } 0 \leqslant \Im z \leqslant \pi \tag{42}
\end{equation*}
$$

and that

$$
\begin{equation*}
\left|e^{-z} \psi(z)\right| \geqslant C_{\theta_{0}}|z|^{\alpha} \text { for all } z \in \Upsilon_{\theta^{*}} \text { with } 0<\Im z \leqslant \pi . \tag{43}
\end{equation*}
$$

Step 4. By the fact that

$$
\overline{e^{-z} \psi(z)}=e^{-\bar{z}} \psi(\bar{z}) \quad \text { for all } z \in \Sigma_{\pi},
$$

using (42) and (43) proves (36) and (37), respectively. This completes the proof.

By (30) and Lemma 3.1, a routine calculation gives the following lemma.
Lemma 3.4. Assume that $\pi / 2<\theta<\pi$. Then

$$
\begin{equation*}
\left|\psi(z)-z^{\alpha}\right| \leqslant C_{\theta}|z|^{\alpha+1} \tag{44}
\end{equation*}
$$

for all $z \in \Upsilon_{\theta} \backslash\{0\}$.
Remark 3.3. In Lemma 3.3, we prove that for any given $\theta_{0} \in(\pi / 2, \pi)$, we can show that $e^{-z} \psi(z) \in \Sigma_{\theta_{0}}$ for $z \in \Sigma_{\theta^{*}}$ with some $\pi / 2<\theta^{*} \leqslant \theta_{0}$. Therefore our error estimates hold for any elliptic operator $\mathcal{A}$ where the resolvent set of $\mathcal{A}$ lies in $\Sigma_{\theta_{0}}$. The techniques used in the proof of Lemma 3.3 are new and may be extended to consider the error estimates for the higher order L-type schemes. Let us recall some available approach in literature for proving Lemma 3.3. In Jin et al. [9] the authors use the following steps to show $e^{-z} \psi(z) \in \Sigma_{\theta_{0}}$ :

Step 1. Let $z \in\left\{z: \operatorname{Arg}(z)=\theta^{*}=\pi / 2\right\}$ and prove that $e^{-z} \psi(z) \in \Sigma_{\theta_{0}}$ for some suitable $\theta_{0} \in(\pi / 2, \pi)$.

Step 2. By the continuity of $e^{-z} \psi(z)$ with respect to $\theta^{*}$, one may claim that $e^{-z} \psi(z) \in \Sigma_{\theta_{0}}$ also for $\theta^{*} \in(\pi / 2, \pi)$ for $\theta^{*}$ sufficiently close to $\pi / 2$.

By using this approach, Jin et al. [9] show that $\theta_{0}=3 \pi / 4-\epsilon$, with $\epsilon>0$, which implies that this approach do not work for the elliptic operator $\mathcal{A}$ where the resolvent set of $\mathcal{A}$ lies in $\Sigma_{\theta_{0}}$ with $\theta_{0}<3 \pi / 4$. It seems also very difficult to prove the similar results as in Lemma 3.3 for the higher order L-type scheme by using the approach in [9]. Therefore the new techniques developed in the proof of Lemma 3.3 may open a door to consider the numerical analysis for high order L-type schemes for solving time fractional partial differential equations.

### 3.2 Proof of Theorem 3.1

By Lemma 3.3, there exists $\pi / 2<\omega^{*} \leqslant \omega_{0}$, depending only on $\omega_{0}$, such that

$$
\begin{equation*}
e^{-z} \psi(z) \in \Sigma_{\omega_{0}} \text { for all } z \in \Sigma_{\omega^{*}} \text { with }-\pi \leqslant \operatorname{Im} z \leqslant \pi \tag{45}
\end{equation*}
$$

and that

$$
\begin{equation*}
\left|e^{-z} \psi(z)\right| \geqslant C_{\omega_{0}}|z|^{\alpha} \quad \text { for all } z \in \Upsilon_{\omega^{*}} \backslash\{0\} \tag{46}
\end{equation*}
$$

Define

$$
\begin{equation*}
\mathcal{E}(t):=\tau^{-1} \mathcal{E}_{\lfloor t / \tau\rfloor}, \quad t>0 \tag{47}
\end{equation*}
$$

where $\lfloor\cdot\rfloor$ is the floor function and

$$
\begin{equation*}
\mathcal{E}_{j}:=\frac{1}{2 \pi i} \int_{\Upsilon_{\omega^{*}}} e^{j z} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) \mathrm{d} z, \quad j \in \mathbb{N} . \tag{48}
\end{equation*}
$$

Note that (5a) and (45) guarantee that the above $\mathcal{E}_{j}$ is well defined, and we recall that $\psi$ is defined by (30).

Lemma 3.5. For any $g \in L^{1}\left(0, T ; L^{2}(\Omega)\right)$, we have

$$
\begin{equation*}
S_{\tau, j} g=\int_{0}^{t_{j}} \mathcal{E}\left(t_{j}-t\right) g(t) \mathrm{d} t \quad \forall 1 \leqslant j \leqslant J \tag{49}
\end{equation*}
$$

Proof. Since the techniques used in this proof are standard in the theory of Laplace transform, we only provide a brief proof; see [27, 8, 41] for more details. Extend $g$ to $(T, \infty)$ by zero and define $t_{j}:=j \tau$ for each $j>J$. Define $\left\{W_{k}\right\}_{k=1}^{\infty} \subset H_{0}^{1}(\Omega)$ by that, for any $k \geqslant 1$,

$$
\begin{equation*}
b_{1} W_{k}+\sum_{j=1}^{k-1}\left(b_{k-j+1}-2 b_{k-j}+b_{k-j-1}\right) W_{j}-\tau^{\alpha} \mathcal{A} W_{k}=\tau^{\alpha-1} \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t \tag{50}
\end{equation*}
$$

in $H^{-1}(\Omega)$. By definition,

$$
\begin{equation*}
S_{\tau, j} g=W_{j}, \quad \forall 1 \leqslant j \leqslant J \tag{51}
\end{equation*}
$$

The rest of this proof is divided into three steps.
Step 1. We prove that the following discrete Laplace transform of $\left\{W_{k}\right\}_{k=1}^{\infty}$ is analytic on $\Sigma_{\pi / 2}$ :

$$
\begin{equation*}
\widehat{W}(z):=\sum_{k=1}^{\infty} e^{-k z} W_{k}, \quad z \in \Sigma_{\pi / 2} \tag{52}
\end{equation*}
$$

Note first that we can assume that $g \in L^{\infty}\left(0, \infty ; L^{2}(\Omega)\right)$. Since

$$
\sup _{a>0}\|g\|_{0 H^{-\alpha / 2}\left(0, a ; L^{2}(\Omega)\right)}<\infty
$$

by the techniques to prove (75) and (78) we can obtain

$$
\sup _{k \geqslant 1}\left\|W_{k}\right\|_{L^{2}(\Omega)}<\infty .
$$

Therefore, it is evident that $\widehat{W}$ is analytic on $\Sigma_{\pi / 2}$.
Step 2. Let us prove that, for any $1 \leqslant j \leqslant J$,

$$
\begin{equation*}
W_{j}=\sum_{k=1}^{J} \frac{\tau^{-1}}{2 \pi i} \int_{1-\pi i}^{1+\pi i} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) e^{(j-k) z} \mathrm{~d} z \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t . \tag{53}
\end{equation*}
$$

Multiplying both sides of (50) by $e^{-k z}$ and summing over $k$ from 1 to $\infty$, we obtain

$$
\left(\left(e^{z}-2+e^{-z}\right) \widehat{b}(z)-\tau^{\alpha} \mathcal{A}\right) \widehat{W}(z)=\tau^{\alpha-1} \sum_{k=1}^{\infty} \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t e^{-k z}, \quad \forall z \in \Sigma_{\pi / 2}
$$

which, together with (30), yields

$$
\begin{equation*}
\left(e^{-z} \psi(z)-\tau^{\alpha} \mathcal{A}\right) \widehat{W}(z)=\tau^{\alpha-1} \sum_{k=1}^{\infty} \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t e^{-k z}, \quad \forall z \in \Sigma_{\pi / 2} \tag{54}
\end{equation*}
$$

Hence, from (5a), (45) and the fact $\left.g\right|_{(T, \infty)}=0$, it follows that

$$
\begin{aligned}
\widehat{W}(z) & =\tau^{-1} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) \sum_{k=1}^{\infty} \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t e^{-k z} \\
& =\tau^{-1} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) \sum_{k=1}^{J} \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t e^{-k z}
\end{aligned}
$$

for all $z \in \Sigma_{\pi / 2}$ with $-\pi \leqslant \operatorname{Im} z \leqslant \pi$. Therefore, (53) follows from the equality

$$
W_{j}=\frac{1}{2 \pi i} \int_{1-\pi i}^{1+\pi i} \widehat{W}(z) e^{j z} \mathrm{~d} z
$$

which is evident by (52).
Step 3. By Cauchy's integral theorem, we have, for any $a>1$, when $k \geqslant j+1$,

$$
\begin{align*}
& \left\|\int_{1-\pi i}^{1+\pi i} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) e^{(j-k) z} \mathrm{~d} z\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \\
= & \left\|\int_{a-\pi i}^{a+\pi i} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) e^{(j-k) z} \mathrm{~d} z\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \\
\leqslant & \mathcal{M}_{0} e^{(j-k) a} \int_{a-\pi i}^{a+\pi i} \frac{|d z|}{\tau^{-\alpha}\left|e^{-z} \psi(z)\right|} \quad(\text { by }(5 \mathrm{~b})) . \tag{55}
\end{align*}
$$

Since (31) implies

$$
\left|e^{-z} \psi(z)\right| \geqslant C_{\alpha} \quad \text { for all } z \in \mathbb{C} \text { with } \Re z \geqslant 1,
$$

passing to the limit $a \rightarrow \infty$ in (55) yields

$$
\int_{1-\pi i}^{1+\pi i} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) e^{(j-k) z} \mathrm{~d} z=0, \quad \text { for } k \geqslant j+1
$$

Thus from (53) we obtain

$$
\begin{aligned}
W_{j} & =\sum_{k=1}^{j} \frac{\tau^{-1}}{2 \pi i} \int_{1-\pi i}^{1+\pi i} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) e^{(j-k) z} \mathrm{~d} z \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t \\
& =\sum_{k=1}^{j} \mathcal{E}_{j-k} \int_{t_{k-1}}^{t_{k}} g(t) \mathrm{d} t=\int_{0}^{t_{j}} \mathcal{E}\left(t_{j}-t\right) g(t) \mathrm{d} t .
\end{aligned}
$$

Here we have used the equality

$$
\int_{1-\pi i}^{1+\pi i} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) e^{(j-k) z} \mathrm{~d} z=\int_{\Upsilon \omega^{*}} R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) e^{(j-k) z} \mathrm{~d} z
$$

which can be easily verified by Cauchy's integral theorem. By (51), this proves (49) and thus completes the proof.

Remark 3.4. In (49), we use the piecewise kernel function $\mathcal{E}(t)$ to express the discrete solution $S_{\tau, j} g$, which is different from the discrete solution expression in literature [9, 41], where the authors assumed that the function $g$ has more
regularities at 0 and has the Taylor expansion at 0 and then applied the convolution techniques for obtaining the discrete solution. In our paper, we only assume that $g \in L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$ and we did not use the convolution techniques for obtaining the discrete solutions as in [9, 41]. One may use the similar idea to consider more general function $g$; for example, $g$ is a stochastic Wiener process $g=\frac{d W(t)}{d t}$, where $W$ is the Hilbert space valued cylindrical Wiener process.
Lemma 3.6. For any $z \in \Upsilon_{\omega^{*}} \backslash\{0\}$,

$$
\begin{equation*}
\left\|e^{z} R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right)-R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}}|z|^{1-\alpha} \tau^{\alpha} \tag{56}
\end{equation*}
$$

Proof. We have

$$
\begin{aligned}
& e^{z} R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right)-R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) \\
= & \left(\tau^{-\alpha}\left(\psi(z)-z^{\alpha}\right)+\left(1-e^{z}\right) \mathcal{A}\right) R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right) R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) \\
= & \mathbb{I}_{1}+\mathbb{I}_{2}
\end{aligned}
$$

where

$$
\begin{aligned}
& \mathbb{I}_{1}:=\tau^{-\alpha}\left(\psi(z)-z^{\alpha}\right) R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right) R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right) \\
& \mathbb{I}_{2}:=\left(1-e^{z}\right) \mathcal{A} R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right) R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right)
\end{aligned}
$$

Note that (5b), (45) and (46) imply

$$
\begin{align*}
\left\|R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} & \leqslant C_{\mathcal{M}_{0}}|z|^{-\alpha} \tau^{\alpha}  \tag{57}\\
\left\|R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} & \leqslant C_{\omega_{0}, \mathcal{M}_{0}}|z|^{-\alpha} \tau^{\alpha} \tag{58}
\end{align*}
$$

By (44), (57) and (58) we have

$$
\left\|\mathbb{I}_{1}\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}}|z|^{1-\alpha} \tau^{\alpha}
$$

Since

$$
\begin{aligned}
& \left\|\mathcal{A} R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right) R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \\
= & \left\|\left(\tau^{-\alpha} z^{\alpha} R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right)-I\right) R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \\
\leqslant & C_{\omega_{0}, \mathcal{M}_{0}}|z|^{-\alpha} \tau^{\alpha} \quad(\text { by }(57) \text { and }(58)),
\end{aligned}
$$

we obtain

$$
\left\|\mathbb{I}_{2}\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}}|z|^{1-\alpha} \tau^{\alpha}
$$

Combining the above estimates of $\mathbb{I}_{1}$ and $\mathbb{I}_{2}$ proves (56) and hence this lemma.

Lemma 3.7. For any $1 \leqslant j \leqslant J$,

$$
\begin{equation*}
\left\|E\left(t_{j}\right)-\mathcal{E}\left(t_{j}-\right)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha-1} j^{\alpha-2} \tag{59}
\end{equation*}
$$

Proof. Inserting $t=t_{j}$ into (6) yields

$$
E\left(t_{j}\right)=\frac{1}{2 \pi i} \int_{\Gamma_{\omega^{*}}} e^{t_{j} z} R\left(z^{\alpha}, \mathcal{A}\right) \mathrm{d} z=\frac{\tau^{-1}}{2 \pi i} \int_{\Gamma_{\omega^{*}}} e^{j z} R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right) \mathrm{d} z
$$

so that from (47) and (48) it follows that

$$
E\left(t_{j}\right)-\mathcal{E}\left(t_{j}-\right)=\mathbb{I}_{1}+\mathbb{I}_{2}
$$

where

$$
\begin{aligned}
& \mathbb{I}_{1}:=\frac{\tau^{-1}}{2 \pi i} \int_{\Gamma_{\omega^{*}} \backslash \Upsilon_{\omega^{*}}} e^{j z} R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right) \mathrm{d} z \\
& \mathbb{I}_{2}:=\frac{\tau^{-1}}{2 \pi i} \int_{\Upsilon_{\omega^{*}}} e^{(j-1) z}\left(e^{z} R\left(\tau^{-\alpha} z^{\alpha}, \mathcal{A}\right)-R\left(\tau^{-\alpha} e^{-z} \psi(z), \mathcal{A}\right)\right) \mathrm{d} z
\end{aligned}
$$

For $\mathbb{I}_{1}$, we have, by (5b),

$$
\begin{aligned}
& \left\|\mathbb{I}_{1}\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} \leqslant C_{\mathcal{M}_{0}} \tau^{-1} \int_{\pi / \sin \omega^{*}}^{\infty} e^{j \cos \omega^{*} r}\left(\tau^{\alpha} r^{-\alpha}\right) \mathrm{d} r \\
\leqslant & C_{\mathcal{M}_{0}} \tau^{\alpha-1} \int_{\pi / \sin \omega^{*}}^{\infty} e^{j \cos \omega^{*} r} r^{-\alpha} \mathrm{d} r \\
\leqslant & C_{\mathcal{M}_{0}} \tau^{\alpha-1} \int_{\pi / \sin \omega^{*}}^{\infty} e^{j \cos \omega^{*} r} r^{1-\alpha} \mathrm{d} r \quad(\text { since } r \text { is lower bounded }) \\
\leqslant & C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha-1} j^{\alpha-2} e^{j \pi \cot \omega^{*}} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha-1} j^{\alpha-2}
\end{aligned}
$$

For $\mathbb{I}_{2}$, by (56) we obtain

$$
\begin{aligned}
\left\|\mathbb{I}_{2}\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} & \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{-1} \int_{0}^{\pi / \sin \omega^{*}} e^{(j-1) \cos \omega^{*} r} r\left(\tau^{\alpha} r^{-\alpha}\right) \mathrm{d} r \\
& \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha-1} \int_{0}^{\pi / \sin \omega^{*}} e^{(j-1) \cos \omega^{*} r} r^{1-\alpha} \mathrm{d} r \\
& \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha-1} \int_{0}^{\pi / \sin \omega^{*}} e^{j \cos \omega^{*} r} r^{1-\alpha} \mathrm{d} r \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha-1} j^{\alpha-2}
\end{aligned}
$$

Combining the above estimates of $\mathbb{I}_{1}$ and $\mathbb{I}_{2}$ yields (59) and thus concludes the proof.

Lemma 3.8. We have

$$
\begin{equation*}
\|E-\mathcal{E}\|_{L^{1}\left(0, T ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}}\left(\frac{1}{\alpha}+\frac{1-J^{\alpha-1}}{1-\alpha}\right) \tau^{\alpha} \tag{60}
\end{equation*}
$$

Proof. By (7) we have

$$
\begin{equation*}
\left\|E-E\left(t_{1}\right)\right\|_{L^{1}\left(0, t_{1} ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha} \alpha^{-1} \tag{61}
\end{equation*}
$$

and a straightforward calculation gives, by (8),

$$
\begin{align*}
& \sum_{j=2}^{J}\left\|E-E\left(t_{j}\right)\right\|_{L^{1}\left(t_{j-1}, t_{j} ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)} \leqslant \tau\left\|E^{\prime}\right\|_{L^{1}\left(t_{1}, T ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)} \\
\leqslant & C_{\omega_{0}, \mathcal{M}_{0}} \tau \int_{t_{1}}^{T} t^{\alpha-2} \mathrm{~d} t=C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(1-J^{\alpha-1}\right)(1-\alpha)^{-1} . \tag{62}
\end{align*}
$$

It follows that

$$
\sum_{j=1}^{J}\left\|E-E\left(t_{j}\right)\right\|_{L^{1}\left(t_{j-1}, t_{j} ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(\alpha^{-1}+\left(1-J^{\alpha-1}\right)(1-\alpha)^{-1}\right)
$$

Further we have, by Lemma 3.7,

$$
\begin{aligned}
\sum_{j=1}^{J} \tau\left\|E\left(t_{j}\right)-\mathcal{E}\left(t_{j}-\right)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)} & \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha} \sum_{j=1}^{J} j^{\alpha-2} \\
& \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(1-J^{\alpha-1}\right)(1-\alpha)^{-1}
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
& \|E-\mathcal{E}\|_{L^{1}\left(0, T ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)} \\
\leqslant & \sum_{j=1}^{J}\left(\left\|E-E\left(t_{j}\right)\right\|_{L^{1}\left(t_{j-1}, t_{j} ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)}+\tau\left\|E\left(t_{j}\right)-\mathcal{E}\left(t_{j}-\right)\right\|_{L^{1}\left(t_{j-1}, t_{j} ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)}\right) \\
\leqslant & C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(\alpha^{-1}+\left(1-J^{\alpha-1}\right)(1-\alpha)^{-1}\right) .
\end{aligned}
$$

This proves (60) and hence this lemma.
Finally, we are in a position to conclude the proof of Theorem 3.1 as follows. By (9) and (49) we have

$$
\max _{1 \leqslant j \leqslant J}\left\|(S g)\left(t_{j}\right)-S_{\tau, j} g\right\|_{L^{2}(\Omega)} \leqslant\|E-\mathcal{E}\|_{L^{1}\left(0, T ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)}\|g\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)}
$$

so that (26) follows from (60). By (47) we see that $\mathcal{E}$ is piecewise constant, and then by (23), (49) and (24) we obtain $S_{\tau, j}\left(v \delta_{0}\right)=\mathcal{E}\left(t_{j}-\right) v, 1 \leqslant j \leqslant J$. Hence, a straightforward computation yields, by (11),

$$
\begin{aligned}
& \max _{1 \leqslant j \leqslant J} j^{2-\alpha}\left\|S\left(v \delta_{0}\right)\left(t_{j}\right)-S_{\tau, j}\left(v \delta_{0}\right)\right\|_{L^{2}(\Omega)} \leqslant \max _{1 \leqslant j \leqslant J} j^{2-\alpha}\left\|E\left(t_{j}\right)-\mathcal{E}\left(t_{j}-\right)\right\|_{\mathcal{L}\left(L^{2}(\Omega)\right)}\|v\|_{L^{2}(\Omega)}, \\
& \sum_{j=1}^{J}\left\|S\left(v \delta_{0}\right)-S_{\tau, j}\left(v \delta_{0}\right)\right\|_{L^{1}\left(t_{j-1}, t_{j} ; L^{2}(\Omega)\right)} \leqslant\|E-\mathcal{E}\|_{L^{1}\left(0, T ; \mathcal{L}\left(L^{2}(\Omega)\right)\right)}\|v\|_{L^{2}(\Omega)}
\end{aligned}
$$

Therefore, (27), (28) follow from (59), (60), respectively. This completes the proof of Theorem 3.1.

## 4 An inverse problem of a fractional diffusion equation

### 4.1 Continuous problem

We consider reconstructing the source term of a fractional diffusion equation from the value of the solution at a fixed time; more precisely, the task is to seek a suitable source $f$ to ensure that the solution of problem (1) achieves a given value $y_{d}$ at the final time $T$. Applying the well-known Tikhonov regularization technique to this inverse problem yields the following minimization problem:

$$
\begin{equation*}
\min _{\substack{\left.u \in U_{\mathrm{ad}}(\Omega)\right)}} J(y, u):=\frac{1}{2}\left\|y(T)-y_{d}\right\|_{L^{2}(\Omega)}^{2}+\frac{\nu}{2}\|u\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)}^{2} \tag{63}
\end{equation*}
$$

subject to the state equation

$$
\begin{equation*}
\left(\mathrm{D}_{0+}^{\alpha}-\mathcal{A}\right) y=u, \quad \text { with } y(0)=0 \tag{64}
\end{equation*}
$$

where $y_{d} \in L^{2}(\Omega), \nu>0$ is a regularization parameter, and

$$
U_{\mathrm{ad}}:=\left\{v \in L^{2}\left(0, T ; L^{2}(\Omega)\right): u_{*} \leqslant v \leqslant u^{*} \text { a.e. in } \Omega \times(0, T)\right\},
$$

with $u_{*}$ and $u^{*}$ being two given constants.
Remark 4.1. We refer the reader to [32, 33] for the inverse problems of parabolic partial differential equations, and refer the reader to [36, Chapter 3] for the linear-quadratic parabolic control problems.

We call $u \in U_{\text {ad }}$ a mild solution to problem (63) if $u$ solves the following minimization problem:

$$
\begin{equation*}
\min _{u \in U_{\mathrm{ad}}} J(u):=\frac{1}{2}\left\|(S u)(T)-y_{d}\right\|_{L^{2}(\Omega)}^{2}+\frac{\nu}{2}\|u\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)}, \tag{65}
\end{equation*}
$$

where we recall that $S$ is defined by (9).
Lemma 4.1. Assume that $g \in L^{q}\left(0, T ; L^{2}(\Omega)\right)$ with $q>1 / \alpha$. Then

$$
\begin{equation*}
\langle(S g)(T), v\rangle_{\Omega}=\left\langle g, S^{*}\left(v \delta_{T}\right)\right\rangle_{\Omega \times(0, T)} \tag{66}
\end{equation*}
$$

for all $v \in L^{2}(\Omega)$.
Proof. By (9) and (19), $S g \in C\left([0, T] ; L^{2}(\Omega)\right)$ and

$$
(S g)(T)=\int_{0}^{T} E(T-t) g(t) \mathrm{d} t
$$

so that

$$
\langle(S g)(T), v\rangle_{\Omega}=\left\langle\int_{0}^{T} E(T-t) g(t) \mathrm{d} t, v\right\rangle_{\Omega}=\int_{0}^{T}\langle E(T-t) g(t), v\rangle_{\Omega} \mathrm{d} t .
$$

Because (18) implies

$$
\langle E(T-t) g(t), v\rangle_{\Omega}=\left\langle g(t), E^{*}(T-t) v\right\rangle_{\Omega}, \quad \text { a.e. } t \in(0, T),
$$

it follows that

$$
\langle(S g)(T), v\rangle_{\Omega}=\int_{0}^{T}\left\langle g(t), E^{*}(T-t) v\right\rangle_{\Omega} \mathrm{d} t=\left\langle g, S^{*}\left(v \delta_{T}\right)\right\rangle_{\Omega \times(0, T)} \quad(\text { by }(14))
$$

namely, (66) holds indeed. This completes the proof.
Assume that $q>1 / \alpha$ and $q \geqslant 2$. By (19), $(S \cdot)(T)$ is a bounded linear operator form $L^{q}\left(0, T ; L^{2}(\Omega)\right)$ to $L^{2}(\Omega)$. Clearly, $J$ in (65) is a strictly convex functional on $L^{q}\left(0, T ; L^{2}(\Omega)\right)$, and $U_{\text {ad }}$ is a convex, bounded and closed subset of $L^{q}\left(0, T ; L^{2}(\Omega)\right)$. By Lemma 4.1, a routine argument (cf. [36, Theorems 2.14 and 2.21]) yields the following theorem.

Theorem 4.1. Problem (65) admits a unique mild solution $u \in U_{a d}$, and the following first-order optimality condition holds:

$$
\left\{\begin{array}{l}
y=S u  \tag{67a}\\
p=S^{*}\left(\left(y(T)-y_{d}\right) \delta_{T}\right), \\
\langle p+\nu u, v-u\rangle_{\Omega \times(0, T)} \geqslant 0 \quad \text { for all } v \in U_{a d}
\end{array}\right.
$$

Remark 4.2. Assume that $u, y$ and $p$ are defined in Theorem 4.1. By (67c) we have $u=f(p)$, where

$$
f(r):= \begin{cases}u_{*} & \text { if } r>-\nu u_{*},  \tag{68}\\ r & \text { if }-\nu u^{*} \leqslant r \leqslant-\nu u_{*}, \\ u^{*} & \text { if } r<-\nu u^{*} .\end{cases}
$$

Noting that $f$ is Lipschitz continuous with Lipschitz constant $1 / \nu$, we obtain

$$
u^{\prime}(t)=f^{\prime}(p(t)) p^{\prime}(t) \text { in } L^{2}(\Omega), \quad \text { a.e. } 0<t<T,
$$

and hence $\left\|u^{\prime}(t)\right\|_{L^{2}(\Omega)} \leqslant \nu^{-1}\left\|p^{\prime}(t)\right\|_{L^{2}(\Omega)}$, a.e. $0<t<T$. It follows from (67b), (14) and (17) that
$\left\|u^{\prime}(t)\right\|_{L^{2}(\Omega)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \nu^{-1}(T-t)^{\alpha-2}\left(\|y(T)\|_{L^{2}(\Omega)}+\left\|y_{d}\right\|_{L^{2}(\Omega)}\right), \quad$ a.e. $0<t<T$.
Since (67a), (9), (7) and the fact $u \in U_{a d}$ imply

$$
\begin{equation*}
\|y(T)\|_{L^{2}(\Omega)} \leqslant C_{u_{*}, u^{*}, \omega_{0}, \mathcal{M}_{0}, T, \Omega} \alpha^{-1} \tag{69}
\end{equation*}
$$

we conclude therefore that

$$
\begin{equation*}
\left\|u^{\prime}(t)\right\|_{L^{2}(\Omega)} \leqslant C_{u_{*}, u^{*}, \omega_{0}, \mathcal{M}_{0}, T, \Omega} \nu^{-1}(T-t)^{\alpha-2}\left(\alpha^{-1}+\left\|y_{d}\right\|_{L^{2}(\Omega)}\right), \quad \text { a.e. } 0<t<T \text {. } \tag{70}
\end{equation*}
$$

Remark 4.3. Let $u_{\nu}$ be the mild solution of problem (65). A standard argument yields that there exits $y_{T} \in L^{2}(\Omega)$ such that

$$
\begin{equation*}
\left\|\left(S u_{\nu}\right)(T)-y_{T}\right\|_{L^{2}(\Omega)} \leqslant C_{u_{*}, u^{*}, T, \Omega} \sqrt{\nu} \tag{71}
\end{equation*}
$$

Since $U_{a d}$ is a convex, bounded and closed subset of $L^{q}\left(0, T ; L^{2}(\Omega)\right), q>1 / \alpha$, there exist $u_{0} \in U_{a d}$ and a decreasing sequence $\left\{\nu_{n}\right\}_{n=0}^{\infty} \subset(0, \infty)$ with limit zero such that

$$
\lim _{n \rightarrow \infty} u_{\nu_{n}}=u_{0} \quad \text { weakly in } L^{q}\left(0, T ; L^{2}(\Omega)\right)
$$

As $(S \cdot)(T)$ is a bounded linear operator from $L^{q}\left(0, T ; L^{2}(\Omega)\right)$ to $L^{2}(\Omega)$, we have that $\left(S u_{\nu_{n}}\right)(T)$ converges to $\left(S u_{0}\right)(T)$ weakly in $L^{2}(\Omega)$ as $n \rightarrow \infty$, so that (71) implies $\left(S u_{0}\right)(T)=y_{T}$. Furthermore, a trivial calculation yields that $u_{0}$ is a mild solution of problem (63) with $\nu=0$.

### 4.2 Temporally discrete problem

Define

$$
W_{\tau}:=\left\{V \in L^{\infty}\left(0, T ; H_{0}^{1}(\Omega)\right): V \text { is constant on }\left(t_{j-1}, t_{j}\right) \quad \forall 1 \leqslant j \leqslant J\right\}
$$

For any $g \in W_{\tau}^{*}$, define $S_{\tau} g \in W_{\tau}$ and $S_{\tau}^{*} g \in W_{\tau}$, respectively, by that

$$
\begin{align*}
\left\langle\mathrm{D}_{0+}^{\alpha} S_{\tau} g, V\right\rangle_{\Omega \times(0, T)}-\left\langle\mathcal{A} S_{\tau} g, V\right\rangle_{L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)} & =\langle g, V\rangle_{W_{\tau}},  \tag{72}\\
\left\langle\left(\mathrm{D}_{T-}^{\alpha} S_{\tau}^{*} g, V\right\rangle_{\Omega \times(0, T)}-\left\langle\mathcal{A}^{*} S_{\tau}^{*} g, V\right\rangle_{L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)}\right. & =\langle g, V\rangle_{W_{\tau}}, \tag{73}
\end{align*}
$$

for all $V \in W_{\tau}$. By (88) we have that

$$
\begin{equation*}
\left\langle S_{\tau} f, g\right\rangle_{\Omega \times(0, T)}=\left\langle f, S_{\tau}^{*} g\right\rangle_{\Omega \times(0, T)} \quad \forall f, g \in L^{1}\left(0, T ; L^{2}(\Omega)\right) \tag{74}
\end{equation*}
$$

A direct calculation yields that (cf. [12, Remark 3]), for any $g \in W_{\tau}^{*}$,

$$
\begin{equation*}
\left(S_{\tau} g\right)\left(t_{j}-\right)=S_{\tau, j} g \quad \forall 1 \leqslant j \leqslant J \tag{75}
\end{equation*}
$$

Hence, from Theorem 3.1, we readily conclude the following two estimates: for any $g \in L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$,

$$
\begin{equation*}
\left\|(S g)(T)-\left(S_{\tau} g\right)(T-)\right\|_{L^{2}(\Omega)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(\frac{1}{\alpha}+\frac{1-J^{\alpha-1}}{1-\alpha}\right)\|g\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)} \tag{76}
\end{equation*}
$$

for any $v \in L^{2}(\Omega)$,

$$
\begin{equation*}
\left\|S\left(v \delta_{0}\right)-S_{\tau}\left(v \widehat{\delta}_{0}\right)\right\|_{L^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(\frac{1}{\alpha}+\frac{1-J^{\alpha-1}}{1-\alpha}\right)\|v\|_{L^{2}(\Omega)} \tag{77}
\end{equation*}
$$

Furthermore, we have the following stability estimate.
Lemma 4.2. Assume that $g \in{ }_{0} H^{-\alpha / 2}\left(0, T ; L^{2}(\Omega)\right)$. Then, for any $1 \leqslant j \leqslant J$,

$$
\begin{equation*}
\left\|\left(S_{\tau} g\right)\left(t_{j}-\right)\right\|_{L^{2}(\Omega)} \leqslant C_{\alpha} \tau^{(\alpha-1) / 2}\|g\|_{0 H^{-\alpha / 2}\left(0, T ; L^{2}(\Omega)\right)} \tag{78}
\end{equation*}
$$

Proof. We only prove (78) with $j=J$, since the other cases $1 \leqslant j<J$ can be proved analogously. Let $v:=\left(S_{\tau} g\right)\left(t_{j}-\right)$. We have

$$
\begin{aligned}
& \|v\|_{L^{2}(\Omega)}^{2}=\left\langle v \widehat{\delta}_{T}, S_{\tau} g\right\rangle_{\Omega \times(0, T)} \\
= & \left\langle\mathrm{D}_{T-}^{\alpha / 2} \mathrm{D}_{T-}^{-\alpha / 2}\left(v \widehat{\delta}_{T}\right), S_{\tau} g\right\rangle_{\Omega \times(0, T)} \\
= & \left\langle\mathrm{D}_{T-}^{-\alpha / 2}\left(v \widehat{\delta}_{T}\right), \mathrm{D}_{0+}^{\alpha / 2} S_{\tau} g\right\rangle_{\Omega \times(0, T)} \quad(\text { by }(88)) \\
\leqslant & \left\|\mathrm{D}_{0+}^{\alpha / 2} S_{\tau} g\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|\mathrm{D}_{T-}^{-\alpha / 2}\left(v \widehat{\delta}_{T}\right)\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)},
\end{aligned}
$$

where we recall that $\widehat{\delta}_{T}$ is defined by (25). Since inserting $V:=S_{\tau} g$ into (72) yields, by (88), (89) and (5c), that

$$
\left\|\mathrm{D}_{0+}^{\alpha / 2} S_{\tau} g\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C_{\alpha}\|g\|_{0 H^{-\alpha / 2}\left(0, T ; L^{2}(\Omega)\right)}
$$

it follows that

$$
\|v\|_{L^{2}(\Omega)}^{2} \leqslant C_{\alpha}\|g\|_{0 H^{-\alpha / 2}\left(0, T ; L^{2}(\Omega)\right)}\left\|\mathrm{D}_{T-}^{-\alpha / 2}\left(v \widehat{\delta}_{T}\right)\right\|_{L^{2}(\Omega)} .
$$

It suffices, therefore, to prove

$$
\begin{equation*}
\left\|\mathrm{D}_{T-}^{-\alpha / 2}\left(v \widehat{\delta}_{T}\right)\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C_{\alpha} \tau^{(\alpha-1) / 2}\|v\|_{L^{2}(\Omega)} \tag{79}
\end{equation*}
$$

To this end, we note that

$$
\begin{aligned}
& \left\|\mathrm{D}_{T-}^{-\alpha / 2}\left(v \widehat{\delta}_{T}\right)\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)}^{2} \\
= & \left(\frac{\|v\|_{L^{2}(\Omega)}}{\Gamma(\alpha / 2)}\right)^{2} \tau^{-2} \int_{0}^{T}\left|\int_{t}^{T}(s-t)^{\alpha / 2-1} \widehat{\delta}_{T}(s) \mathrm{d} s\right|^{2} \mathrm{~d} t \\
= & \left(\frac{\|v\|_{L^{2}(\Omega)}}{\Gamma(\alpha / 2)}\right)^{2} \tau^{-2}\left(\mathbb{I}_{1}+\mathbb{I}_{2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
& \mathbb{I}_{1}:=\int_{0}^{T-\tau}\left|\int_{T-\tau}^{T}(s-t)^{\alpha / 2-1} \mathrm{~d} s\right|^{2} \mathrm{~d} s \mathrm{~d} t, \\
& \mathbb{I}_{2}:=\int_{T-\tau}^{T}\left|\int_{t}^{T}(s-t)^{\alpha / 2-1} \mathrm{~d} s\right|^{2} \mathrm{~d} t .
\end{aligned}
$$

A straightforward calculation gives

$$
\begin{aligned}
\mathbb{I}_{1} & =4 / \alpha^{2} \int_{0}^{T-\tau}\left((T-t)^{\alpha / 2}-(T-\tau-t)^{\alpha / 2}\right)^{2} \mathrm{~d} t \\
& =4 / \alpha^{2} \tau^{1+\alpha} \int_{0}^{T / \tau}\left(s^{\alpha / 2}-(s-1)^{\alpha / 2}\right)^{2} \mathrm{~d} s \\
& <4 / \alpha^{2} \tau^{1+\alpha} \int_{0}^{\infty}\left(s^{\alpha / 2}-(s-1)^{\alpha / 2}\right)^{2} \mathrm{~d} s=C_{\alpha} \tau^{1+\alpha}
\end{aligned}
$$

and

$$
\mathbb{I}_{2}=4 / \alpha^{2} \int_{T-\tau}^{T}(T-t)^{\alpha} \mathrm{d} t=C_{\alpha} \tau^{1+\alpha}
$$

Combining the above estimates of $\mathbb{I}_{1}$ and $\mathbb{I}_{2}$ proves (79) and hence this lemma.

Remark 4.4. We note that if the temporal grid is nonuniform, then (72) is not equivalent to the L1 scheme for fractional diffusion equations. For the numerical analysis of (72) with nonuniform temporal grid, we refer the reader to [15, 16].

Following the idea in [4], we consider the following temporally discrete problem:

$$
\begin{equation*}
\min _{U \in U_{\mathrm{ad}}} J_{\tau}(U):=\frac{1}{2}\left\|\left(S_{\tau} U\right)(T-)-y_{d}\right\|_{L^{2}(\Omega)}^{2}+\frac{\nu}{2}\|U\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)}^{2} \tag{80}
\end{equation*}
$$

Note that $U_{\text {ad }}$ is a convex, bounded and closed subset of $L^{2}\left(0, T ; L^{2}(\Omega)\right)$ and that $\left(S_{\tau} \cdot\right)(T-)$ is, by (78), a bounded linear operator from $L^{2}\left(0, T ; L^{2}(\Omega)\right)$ to $L^{2}(\Omega)$. Hence, applying [36, Theorems 2.14 and 2.21 ] to problem (80) yields the following theorem.

Theorem 4.2. Problem (80) admits a unique solution $U \in U_{a d}$, and the following optimality condition holds:

$$
\left\{\begin{array}{l}
Y=S_{\tau} U  \tag{81a}\\
P=S_{\tau}^{*}\left(\left(Y(T-)-y_{d}\right) \widehat{\delta}_{T}\right) \\
\langle P+\nu U, V-U\rangle_{\Omega \times(0, T)} \geqslant 0 \quad \text { for all } V \in U_{a d}
\end{array}\right.
$$

where $\widehat{\delta}_{T}$ is defined by (25).

Theorem 4.3. Let $u$ and $y$ be defined in Theorem 4.1, and let $U$ and $Y$ be defined in Theorem 4.2. Then

$$
\begin{align*}
& \|(y-Y)(T-)\|_{L^{2}(\Omega)}+\sqrt{\nu}\|u-U\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} \\
\leqslant & C_{y_{d}, u_{*}, u^{*}, \omega_{0}, \mathcal{M}_{0}, T, \Omega}\left(\frac{1}{\alpha}+\left(\frac{1-J^{\alpha-1}}{1-\alpha}\right)^{1 / 2}+\frac{1-J^{\alpha-1}}{1-\alpha} \tau^{\alpha / 2}\right) \tau^{\alpha / 2} \tag{82}
\end{align*}
$$

Proof. Since the idea of of this proof is standard (cf. [5, Theorem 3.4]), we only provide a brief proof. Let us first prove that

$$
\begin{align*}
& \left\|(S u)(T)-\left(S_{\tau} U\right)(T-)\right\|_{L^{2}(\Omega)}+\sqrt{\nu}\|u-U\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} \\
& \leqslant C_{u_{*}, u^{*}, \Omega}\left\|S^{*}\left(\left(y(T)-y_{d}\right) \delta_{T}\right)-S_{\tau}^{*}\left(\left(y(T)-y_{d}\right) \widehat{\delta}_{T}\right)\right\|_{L^{1}\left(0, T ; L^{2}(\Omega)\right)}^{1 / 2} \\
& \quad+2\left\|(S u)(T)-\left(S_{\tau} u\right)(T-)\right\|_{L^{2}(\Omega)} \tag{83}
\end{align*}
$$

By (67c) and (81c), we have

$$
\begin{array}{r}
\left\langle S^{*}\left(\left(y(T)-y_{d}\right) \delta_{T}\right)+\nu u, U-u\right\rangle_{\Omega \times(0, T)} \geqslant 0 \\
\left\langle S_{\tau}^{*}\left(\left(Y(T-)-y_{d}\right) \widehat{\delta}_{T}\right)+\nu U, u-U\right\rangle_{\Omega \times(0, T)} \geqslant 0,
\end{array}
$$

so that

$$
\begin{equation*}
\nu\|u-U\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)}^{2} \leqslant \mathbb{I}_{1}+\mathbb{I}_{2}, \tag{84}
\end{equation*}
$$

where

$$
\begin{aligned}
& \left.\mathbb{I}_{1}:=\left\langle S^{*}\left(\left(y(T)-y_{d}\right) \delta_{T}\right)-S_{\tau}^{*}\left(y(T)-y_{d}\right) \widehat{\delta}_{T}\right), U-u\right\rangle_{\Omega \times(0, T)}, \\
& \mathbb{I}_{2}:=\left\langle S_{\tau}^{*}\left((y(T)-Y(T-)) \widehat{\delta}_{T}\right), U-u\right\rangle_{\Omega \times(0, T)}
\end{aligned}
$$

It is clear that

$$
\mathbb{I}_{1} \leqslant C_{u_{*}, u^{*}, \Omega}\left\|S^{*}\left(\left(y(T)-y_{d}\right) \delta_{T}\right)-S_{\tau}^{*}\left(\left(y(T)-y_{d}\right) \widehat{\delta}_{T}\right)\right\|_{L^{1}\left(0, T ; L^{2}(\Omega)\right)}
$$

by the fact that $u, U \in U_{\text {ad }}$. A straightforward computation yields

$$
\begin{aligned}
\mathbb{I}_{2} & =\left\langle(y(T)-Y(T-)) \widehat{\delta}_{T}, S_{\tau}(U-u)\right\rangle_{\Omega \times(0, T)} \quad(\text { by }(74)) \\
& =\left\langle y(T)-Y(T-),\left(S_{\tau}(U-u)\right)(T-)\right\rangle_{\Omega} \quad(\text { by }(25)) \\
& =\left\langle(S u)(T)-\left(S_{\tau} U\right)(T-),\left(S_{\tau}(U-u)\right)(T-)\right\rangle_{\Omega} \quad(\text { by }(67 \mathrm{a}) \text { and }(81 \mathrm{a})) \\
& =\left\langle(S u)(T)-\left(S_{\tau} u\right)(T-),\left(S_{\tau}(U-u)\right)(T-)\right\rangle_{\Omega}-\left\|\left(S_{\tau}(u-U)\right)(T-)\right\|_{L^{2}(\Omega)}^{2} \\
& \leqslant \frac{1}{2}\left\|(S u)(T)-\left(S_{\tau} u\right)(T-)\right\|_{L^{2}(\Omega)}^{2}-\frac{1}{2}\left\|\left(S_{\tau}(u-U)\right)(T-)\right\|_{L^{2}(\Omega)}^{2} \\
& \leqslant\left\|(S u)(T)-\left(S_{\tau} u\right)(T-)\right\|_{L^{2}(\Omega)}^{2}-\frac{1}{2}\left\|(S u)(T)-\left(S_{\tau} U\right)(T-)\right\|_{L^{2}(\Omega)}^{2}
\end{aligned}
$$

Combining (84) and the above estimates of $\mathbb{I}_{1}$ and $\mathbb{I}_{2}$ gives (83).
Then, by the symmetric version of (77) we obtain

$$
\begin{aligned}
& \left\|S^{*}\left(\left(y(T)-y_{d}\right) \delta_{T}\right)-S_{\tau}^{*}\left(\left(y(T)-y_{d}\right) \widehat{\delta}_{T}\right)\right\|_{L^{1}\left(0, T ; L^{2}(\Omega)\right)} \\
\leqslant & C_{\omega_{0}, \mathcal{M}_{0}} \tau^{\alpha}\left(\frac{1}{\alpha}+\frac{1-J^{\alpha-1}}{1-\alpha}\right)\left\|y(T)-y_{d}\right\|_{L^{2}(\Omega)}
\end{aligned}
$$

so that (69) implies

$$
\begin{align*}
& \left\|S^{*}\left(\left(y(T)-y_{d}\right) \delta_{T}\right)-S_{\tau}^{*}\left(\left(y(T)-y_{d}\right) \widehat{\delta}_{T}\right)\right\|_{L^{1}\left(0, T ; L^{2}(\Omega)\right)} \\
\leqslant & C_{u_{*}, u^{*}, \omega_{0}, \mathcal{M}_{0}, T, \Omega} \tau^{\alpha}\left(\frac{1}{\alpha}+\frac{1-J^{\alpha-1}}{1-\alpha}\right)\left(1 / \alpha+\left\|y_{d}\right\|_{L^{2}(\Omega)}\right) \\
\leqslant & C_{y_{d}, u_{*}, u^{*}, \omega_{0}, \mathcal{M}_{0}, T, \Omega}\left(\frac{1}{\alpha^{2}}+\frac{1-J^{\alpha-1}}{1-\alpha}\right) \tau^{\alpha} . \tag{85}
\end{align*}
$$

We obtain from (76) that

$$
\begin{equation*}
\left\|(S u)(T)-\left(S_{\tau} u\right)(T-)\right\|_{L^{2}(\Omega)} \leqslant C_{u_{*}, u^{*}, \omega_{0}, \mathcal{M}_{0}, \Omega}\left(\frac{1}{\alpha}+\frac{1-J^{\alpha-1}}{1-\alpha}\right) \tau^{\alpha} \tag{86}
\end{equation*}
$$

Finally, combining (83), (85) with (86) gives

$$
\begin{aligned}
& \left\|(S u)(T)-\left(S_{\tau} U\right)(T-)\right\|_{L^{2}(\Omega)}+\sqrt{\nu}\|u-U\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} \\
\leqslant & C_{y_{d}, u_{*}, u^{*}, \omega_{0}, \mathcal{M}_{0}, T, \Omega}\left(\frac{1}{\alpha}+\left(\frac{1-J^{\alpha-1}}{1-\alpha}\right)^{1 / 2}+\frac{1-J^{\alpha-1}}{1-\alpha} \tau^{\alpha / 2}\right) \tau^{\alpha / 2}
\end{aligned}
$$

which, together with (67a) and (81a), implies (82). This completes the proof.

Remark 4.5. Let $y_{T}$ be defined in Remark 4.3. Combining (71) and (82) yields

$$
\begin{aligned}
& \left\|y_{T}-Y(T-)\right\|_{L^{2}(\Omega)} \\
\leqslant & C_{y_{d}, u_{*}, u^{*}, \omega_{0}, \mathcal{M}_{0}, T, \Omega}\left(\sqrt{\nu}+\left(\frac{1}{\alpha}+\left(\frac{1-J^{\alpha-1}}{1-\alpha}\right)^{1 / 2}+\frac{1-J^{\alpha-1}}{1-\alpha} \tau^{\alpha / 2}\right) \tau^{\alpha / 2}\right) .
\end{aligned}
$$

## 5 Numerical experiments

This section performs three numerical experiments in one dimensional space to verify the theoretical results, in the following settings: $T=0.1 ; \Omega=(0,1)$; $\mathcal{A}=\Delta$; the space is discretized by a standard Galerkin finite element method, with the space
$\mathcal{V}_{h}:=\left\{v_{h} \in H_{0}^{1}(0,1): v_{h}\right.$ is linear on $\left((m-1) / 2^{10}, m / 2^{10}\right)$ for all $\left.1 \leqslant m \leqslant 2^{10}\right\}$.
Experiment 1. The purpose of this experiment is to verify (27) and (28). We set $v(x):=x^{-0.49}, 0<x<1$, and let

$$
\begin{aligned}
e_{T} & :=\left\|S_{\tau, J}\left(v \delta_{0}\right)-S_{\tau^{*}, J^{*}}\left(v \delta_{0}\right)\right\|_{L^{2}(\Omega)}, \\
e_{l 1} & :=\sum_{j=1}^{J^{*}} T / J^{*}\left\|S_{\tau,\left\lceil j J / J^{*}\right\rceil}\left(v \delta_{0}\right)-S_{\tau^{*}, j}\left(v \delta_{0}\right)\right\|_{L^{2}(\Omega)},
\end{aligned}
$$

where $J^{*}:=2^{15}, \tau^{*}=T / J^{*}$, and $\lceil\cdot\rceil$ is the ceiling function. Table 1 shows that $e_{T} /\left(\tau^{\alpha-1} J^{\alpha-2}\right)$ will not blow up as $\alpha \rightarrow 1-$, which agrees well with (27). The numerical results in Figure 1 illustrate that $e_{T}$ is close to $O(\tau)$, and this also

Table 1: $e_{T} /\left(\tau^{\alpha-1} J^{\alpha-2}\right)$ of Experiment 1.

| $\alpha$ | $J=2^{7}$ | $J=2^{8}$ | $J=2^{9}$ |
| :---: | :---: | :---: | :---: |
| 0.90 | $5.35 \mathrm{e}-3$ | $5.19 \mathrm{e}-3$ | $5.03 \mathrm{e}-3$ |
| 0.95 | $5.13 \mathrm{e}-3$ | $4.90 \mathrm{e}-3$ | $4.74 \mathrm{e}-3$ |
| 0.99 | $4.37 \mathrm{e}-3$ | $4.10 \mathrm{e}-3$ | $3.94 \mathrm{e}-3$ |
| 0.999 | $4.10 \mathrm{e}-3$ | $3.82 \mathrm{e}-3$ | $3.66 \mathrm{e}-3$ |

agrees well with (27). The numerical results in Figure 2 demonstrate that $e_{l 1}$ is close to $O\left(\tau^{\alpha}\right)$, and this is in good agreement with (28).


Figure 1: $e_{T}$ of numerical Example 1. Figure 2: $e_{l 1}$ of numerical Example 1.
Experiment 2. The purpose of this experiment is to verify (26). To this end, we set

$$
f(t, x):=x^{-0.49}, \quad 0<t<T, \quad 0<x<1
$$

and define

$$
e_{\infty}:=\max _{1 \leqslant j \leqslant J}\left\|S_{\tau, j} f-S_{\tau^{*},\left\lceil j J^{*} / J\right\rceil} f\right\|_{L^{2}(\Omega)}
$$

where $J^{*}=2^{15}$ and $\tau^{*}=T / J^{*}$. The numerical results in Figure 3 shows that $e_{\infty}$ is close to $O\left(\tau^{\alpha}\right)$, which is in good agreement with (26).


Figure 3: $e_{\infty}$ of numerical Example 2.

Experiment 3. The purpose of this experiment is to verify Theorem 4.3, in the following settings: $a=0 ; b=10 ; \nu=10 ; y_{d}: \equiv 1$. Discretization (80) is solved by the following iteration algorithm (cf. [5, Algorithm 3.2]):

$$
\begin{aligned}
& U_{0}:=0 \\
& U_{j}=f\left(S_{\tau}^{*}\left(\left(\left(S_{\tau} U_{j-1}\right)(T-)-y_{d}\right) \widehat{\delta}_{T}\right)\right), \quad 1 \leqslant j \leqslant k
\end{aligned}
$$

where $f$ is defined by (68) and $k$ is large enough such that

$$
\left\|U_{k}-U_{k-1}\right\|_{L^{\infty}\left(0, T ; L^{\infty}(\Omega)\right)}<10^{-12} .
$$

The "Error" in Figure 4 means

$$
\left\|Y(T-)-Y^{*}(T-)\right\|_{L^{2}(\Omega)}+\left\|U-U^{*}\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)}
$$

where $U^{*}$ and $Y^{*}$ are the numerical solutions with $J=2^{15}$. The theoretical convergence rate $O\left(\tau^{\alpha / 2}\right)$ is observed in Table 4.


Figure 4: Numerical results of numerical Example 3.

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## 6 Appendix: Properties of the fractional calculus operators

Assume that $-\infty<a<b<\infty$. Define

$$
\begin{aligned}
& { }_{0} H^{1}\left(a, b ; L^{2}(\Omega)\right):=\left\{v \in H^{1}\left(a, b ; L^{2}(\Omega)\right): v(a)=0\right\}, \\
& { }^{0} H^{1}\left(a, b ; L^{2}(\Omega)\right):=\left\{v \in H^{1}\left(a, b ; L^{2}(\Omega)\right): v(b)=0\right\},
\end{aligned}
$$

where $H^{1}\left(a, b ; L^{2}(\Omega)\right)$ is a standard vector valued Sobolev space. For each $0<$ $\beta<1$, define

$$
\begin{aligned}
& { }_{0} H^{\beta}\left(a, b ; L^{2}(\Omega)\right):=\left(L^{2}\left(a, b ; L^{2}(\Omega)\right),{ }_{0} H^{1}\left(a, b ; L^{2}(\Omega)\right)\right)_{\beta, 2}, \\
& { }^{0} H^{\beta}\left(a, b ; L^{2}(\Omega)\right):=\left(L^{2}\left(a, b ; L^{2}(\Omega)\right),{ }^{0} H^{1}\left(a, b ; L^{2}(\Omega)\right)\right)_{\beta, 2},
\end{aligned}
$$

where $(\cdot, \cdot)_{\beta, 2}$ means the interpolation space defined by the $K$-method (cf. [23]). In addition, we use ${ }_{0} H^{-\beta}\left(a, b ; L^{2}(\Omega)\right)$ and ${ }^{0} H^{-\beta}\left(a, b ; L^{2}(\Omega)\right)$ to denote the dual spaces of ${ }^{0} H^{\beta}\left(a, b ; L^{2}(\Omega)\right)$ and ${ }_{0} H^{\beta}\left(a, b ; L^{2}(\Omega)\right)$, respectively.

Assume that $0<\gamma<1 / 2$. For any $v \in{ }_{0} H^{\gamma}\left(a, b ; L^{2}(\Omega)\right)$,

$$
\begin{equation*}
C_{0}\|v\|_{0 H^{\gamma}\left(a, b ; L^{2}(\Omega)\right)} \leqslant\left\|\mathrm{D}_{a+}^{\gamma} v\right\|_{L^{2}\left(a, b ; L^{2}(\Omega)\right)} \leqslant C_{1}\|v\|_{0 H^{\gamma}\left(a, b ; L^{2}(\Omega)\right)} \tag{87}
\end{equation*}
$$

where $C_{0}$ and $C_{1}$ are two positive constants depending only on $\gamma$. For any $v \in{ }_{0} H^{\gamma}\left(a, b ; L^{2}(\Omega)\right)$ and $w \in{ }^{0} H^{\gamma}\left(a, b ; L^{2}(\Omega)\right)$,

$$
\begin{equation*}
\left\langle\mathrm{D}_{a+}^{2 \gamma} v, w\right\rangle_{0_{H^{\gamma}\left(a, b ; L^{2}(\Omega)\right)}}=\left\langle\mathrm{D}_{a+}^{\gamma} v, \mathrm{D}_{b-}^{\gamma} w\right\rangle_{\Omega \times(0, T)}=\overline{\left\langle\mathrm{D}_{b-}^{2 \gamma} w, v\right\rangle_{0 H^{\gamma}\left(a, b ; L^{2}(\Omega)\right)}} . \tag{88}
\end{equation*}
$$

For any $v \in{ }_{0} H^{\gamma}\left(a, b ; L^{2}(\Omega)\right)$,

$$
\begin{equation*}
\cos (\gamma \pi)\left\|\mathrm{D}_{a+}^{\gamma} v\right\|_{L^{2}\left(a, b ; L^{2}(\Omega)\right)}^{2} \leqslant\left\langle\mathrm{D}_{a+}^{\gamma} v, \mathrm{D}_{b-}^{\gamma} v\right\rangle_{\Omega \times(0, T)} \leqslant \sec (\gamma \pi)\left\|\mathrm{D}_{a+}^{\gamma} v\right\|_{L^{2}\left(a, b ; L^{2}(\Omega)\right)}^{2} \tag{89}
\end{equation*}
$$

For the proof of (87) we refer the reader to [24, Section 3], and, for the proofs of (88) and (89), we refer the reader to [3].


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