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Decision-making with distance measures and induced aggregation operators

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Abstract: We present a new decision-making approach that uses distance measures and induced aggregation operators. We introduce the induced ordered weighted averaging distance (IOWAD) operator, a new aggregation operator that extends the OWA operator by using distance measures and a reordering of the arguments that depends on order-inducing variables. The main advantage of the IOWAD is that it provides a parameterized family of distance aggregation operators between the maximum and the minimum distance based on a complex reordering process that reflects a complex attitudinal character of the decision-maker. We study some of its main properties and particular cases. We develop an application in a decision-making problem regarding the selection of investments. We see that the main advantage of this approach in decision-making is that it is able to provide a more complete picture of the decision process, so the decision-maker is able to select the alternative most in accordance with his interests.

Keywords: Decision-making, OWA operator, Hamming distance, Induced aggregation operators.

JEL Classification: C44, C49, D81, D89.

Resumen: Se presenta un nuevo modelo para la toma de decisiones basado en el uso de medidas de distancia y de operadores de agregación inducidos. Se introduce la distancia media ponderada ordenada inducida (IOWAD). Es un nuevo operador de agregación que extiende el operador OWA a través del uso de distancias y un proceso de reordenación de los argumentos basado en variables de ordenación inducidas. La principal ventaja el operador IOWAD es la posibilidad de utilizar una familia parametrizada de operadores de agregación entre la distancia individual máxima y la mínima. Se estudian algunas de sus principales propiedades y algunos casos particulares. Se desarrolla un ejemplo numérico en un problema de toma de decisiones sobre selección de inversiones. Se observa que la principal ventaja de este modelo en la toma de decisiones es la posibilidad de mostrar una visión más completa del proceso, de forma que el decisor está capacitado para seleccionar la alternativa que está más cerca de sus intereses.

1. Introduction

In the literature, we find a wide range of methods for decision-making (Alonso et al. 2008, Bustince et al. 2008; Canós and Liern 2008; Figueira et al. 2005; Gil-Aluja 1998; Merigó 2008; Merigó and Casanovas 2007; Merigó and Gil-Lafuente 2007; 2008a; 2008b; 2009; Xu 2008b; Xu 2008c; Xu and Yager 2008; Zarghami et al. 2008). A very useful technique is the Hamming distance (Hamming 1950) and more generally all distance measures (Gil-Aluja 1998; Karayiannis 2000; Kaufmann 1975; Merigó 2008; Merigó and Gil-Lafuente 2007; 2008; 2008b; Szmidt and Kacprzyk 2000). The main advantage of using distance measures in decision-making is that we can compare the alternatives of the problem with some ideal result (Gil-Aluja 1998): the alternative with a closest result to the ideal is the optimal choice.

Usually, when using distance measures in decision-making, we normalize it by using the arithmetic mean or the weighted average (WA), obtaining the normalized Hamming distance (NHD) and the weighted Hamming distance (WHD) respectively. However, it would sometimes be interesting to consider the possibility of parameterizing the results from the maximum distance to the minimum distance. It would therefore be useful to use the ordered weighted averaging (OWA) operator (Yager 1988). The OWA operator is a very useful technique for aggregating the information providing a parameterized family of aggregation operators which includes the maximum, the minimum and the average, among others (Ahn and Park 2008; Beliakov et al. 2007; Chiclana et al. 2007; Emrouznejad 2008; Liu 2008; 2009; Merigó 2008; Merigó and Gil-Lafuente 2008a; 2008b; 2009; Xu 2005; 2008a; Yager 1993; 1996; 2007). The use of the OWA operator in different types of distance measures has been studied in (Karayiannis 2000; Merigó 2008; Merigó and Gil-Lafuente 2008a; 2008b). For other developments of the OWA operator, see (Beliakov et al. 2007; Calvo et al. 2002; Chiclana et al. 2004; 2007; Merigó 2008; Merigó and

Casanovas 2007; Merigó and Gil-Lafuente 2008b; Wang 2008; Yager 2002; 2003; 2008; Yager and Kacprzyk 1997).

An interesting extension of the OWA operator is the induced OWA (IOWA) operator (Yager and Filev 1999). The difference is that the reordering step is not developed with the values of the arguments but could be induced by another mechanism such that the ordered position of the arguments depends upon the values of their associated order-inducing variables. In recent years, the IOWA operator has received increasing attention, see (Chiclana et al. 2004; 2007; Merigó 2008; Merigó and Casanovas 2007; Merigó and Gil-Lafuente 2009; Yager 2003).

The aim of this paper is to present the use of the induced OWA (IOWA) operator in decision-making with distance measures. This will enable us to formulate a more general model by using order-inducing variables in the reordering process. To do so, we will introduce a new aggregation operator: the induced ordered weighted averaging distance (IOWAD) operator. The IOWAD operator is an aggregation operator that provides a parameterized family of distance aggregation operators which ranges from the minimum to the maximum distance. The main advantage of the IOWAD operator is that it is able to deal with complex attitudinal characters (or complex degrees of orness) in the decision process. Therefore, we are able to deal with more complex situations that are closer to the real world.

For example, important business decisions are usually taken by the board of directors of the company. Thus, the decision involves the attitudinal character of a group of persons which has to be coordinated in one simple decision according to their interests. Obviously, the attitudinal character of this example is much more complex than simply using the degree of optimism (degree of orness) of the company. Note that in this example, we could analyze the attitudinal character (degree of orness) in group decision-making problems, but the real analysis would be much more complex. A good method for analyzing this problem would be the use of order-inducing variables.

We study some basic properties of the IOWAD operator and we consider a wide range of particular cases such as the NHD, the WHD, the ordered weighted averaging distance (OWAD) operator, the median-IOWAD, the Olympic-IOWAD, the centered-IOWAD, and so on. We see that each particular case is useful for some special situation according to the interests of the decision-maker. Depending on the particular type used, the results may differ. Note that it is possible to generalize this aggregation operator by using generalized means following the ideas of (Merigó and Gil-Lafuente 2009).

We also present an application of the new approach in a decision-making problem regarding the selection of investments. The main advantage of this model is that it gives a more complete view of the decision problem because it considers a wide range of distance aggregation operators according to the interests of the decision-maker. Note also that the IOWAD operator is applicable to a wide range of situations such as fuzzy set theory, operational research, statistics, etc. In decision-making problems it is also applicable to different problems in contexts such as strategic decision-making, human resource management, product management, financial management, etc.

This paper is organized as follows. In Section 2, we briefly review some basic concepts to be used throughout the paper. In Section 3, we present the IOWAD operator. Section 4 analyzes different families of IOWAD operators. In Section 5 we present a method for decision-making with the IOWAD operator in investment selection. Section 6 develops a numerical example of the new approach. Finally, in Section 7 we summarize the main conclusions of the paper.

2. Preliminaries

In this Section, we briefly describe the OWA operator, the induced OWA operator and the Hamming distance.

2.1. The Hamming distance

The Hamming distance (Hamming 1950) is a very useful technique for calculating the differences between two elements, two sets, etc. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, etc. For two sets A and B, it can be defined as follows.

Definition 1. A normalized Hamming distance of dimension n is a mapping d_H: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ such that:

$$d_H(A, B) = \left(\frac{1}{n} \sum_{i=1}^{n} | a_i - b_i | \right) \tag{1}$$

where a_i and b_i are the ith arguments of the sets A and B respectively.

Sometimes, when normalizing the Hamming distance we prefer to give different weights to each individual distance. The distance is then known as the weighted Hamming distance. It can be defined as follows.

Definition 2. A weighted Hamming distance of dimension n is a mapping d_{WH}: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ which has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$. Then:

$$d_{WH}(A, B) = \left(\sum_{i=1}^{n} w_i \mid a_i - b_i \mid \right) \tag{2}$$

where a_i and b_i are the ith arguments of the sets A and B respectively.

Note that it is possible to generalize this definition to all the real numbers by using $R^n \times R^n \rightarrow R$. For the formulation used in fuzzy set theory, see for example (Gil-Aluja 1998; Kaufmann 1975; Merigó 2008; Szmidt and Kacprzyk 2000).

2.2. The OWA operator

The OWA operator (Yager 1988) provides a parameterized family of aggregation operators that include the maximum, the minimum and the average criteria as special cases. It can be defined as follows.

Definition 3. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ which has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, according to the following formula:

$$\text{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{3}$$

where b_j is the jth largest of the a_i.

From a generalized perspective of the reordering step, it is possible to distinguish between the descending OWA (DOWA) operator and the ascending OWA (AOWA) operator. Note that this distinction in the reordering step is relevant in a wide range of problems, especially in situations where the highest argument is the best result and situations where the lowest argument is the best result. The OWA operator is a mean operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent (Yager 1988).

The OWA operator aggregates the information according to the attitudinal character (or degree of orness) of the decision-maker (Yager 1988). The attitudinal character is represented according to the following formula:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} \right) \tag{4}$$

Different families of OWA operators are found by using different manifestations in the weighting vector such as the maximum, the minimum and the average criteria. For more information on other families, see (Ahn and Park 2008; Beliakov 2005; Beliakov et al. 2007; Emrouznejad 2008; Liu 2008; 2009; Merigó 2008; Xu 2005; 2008; Yager 1993; 1996; 2007).

2.3. The induced OWA operator

The IOWA operator (Yager and Filev 1999) is an extension of the OWA operator. Its main difference is that the reordering step is not carried out with the values of the arguments a_i, but with order inducing variables that reflects a more complex reordering process. The IOWA operator also includes as particular cases the maximum, the minimum and the average criteria. It can be defined as follows.

Definition 4. An IOWA operator of dimension n is a mapping IOWA: $R^n \rightarrow R$ which has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, then:

$$\text{IOWA}(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{5}$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable.

Note that it is possible to distinguish between the Descending IOWA (DIOWA) operator and the Ascending IOWA (AIOWA) operator (Merigó and Gil-Lafuente 2009). The IOWA operator is also monotonic, bounded, idempotent and commutative (Yager and Filev 1999).

3. The induced ordered weighted averaging distance operator

The IOWAD operator is a distance measure that uses the IOWA operator in the normalization process of the Hamming distance. The reordering of the individual distances is developed with order-inducing variables. It can be defined as follows for two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Definition 5. An IOWAD operator of dimension n is a mapping IOWAD: $R^n \times R^n \rightarrow R$ which has an associated weighting vector W such that $w_j \in [0, 1]$ and the sum of the weights is 1, according to the following formula:

$$\text{IOWAD}(\langle u_1, x_1, y_1 \rangle, \langle u_2, x_2, y_2 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \sum_{j=1}^{n} w_j b_j \tag{6}$$

where b_j is the $|x_i - y_i|$ value of the IOWAD triplet $\langle u_i, x_i, y_i \rangle$ having the jth largest u_i, u_i is the order inducing variable and $|x_i - y_i|$ is the argument variable represented in the form of individual distances.

A fundamental aspect of the IOWAD operator is the reordering of the arguments based upon order-inducing variables. That is, rather than being associated with a specific argument, as in the case with the usual Hamming distance, the weights are associated with the position given by the order-inducing variables. This reordering introduces nonlinearity into an otherwise linear process.

If D is a vector corresponding to the ordered arguments b_j, we shall call this the ordered argument vector, and W^T is the transpose of the weighting vector; then the IOWAD operator can be represented as follows:

$$\mathrm{IOWAD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = W^T D \tag{7}$$

From a generalized perspective of the reordering step it is possible to distinguish between descending (DIOWAD) and ascending (AIOWAD) orders. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DIOWAD and w^*_{n-j+1} the jth weight of the AIOWAD operator.

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the IOWAD operator can be expressed as:

$$\mathrm{IOWAD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \frac{1}{W}\sum_{j=1}^{n} w_j b_j \tag{8}$$

Note that $\mathrm{IOWAD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_\mathrm{n}, x_n, y_\mathrm{n}\rangle) = 0$ if and only if $x_i = y_i$ for all $i \in [1, n]$. Note also that $\mathrm{IOWAD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_\mathrm{n}, x_n, y_\mathrm{n}\rangle) = \mathrm{IOWAD}(\langle u_1, y_1, x_1\rangle, \langle u_2, y_2, x_2\rangle, \ldots, \langle u_\mathrm{n}, y_n, x_\mathrm{n}\rangle)$.

The IOWAD operator is commutative, monotonic, bounded and idempotent. These properties can be proved with the following theorems.

Theorem 1 (Commutativity). *Assume f is the IOWAD operator, then*:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) \tag{9}$$

where $(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$ *is any permutation of the arguments* $(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle)$.

Proof. Let

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{10}$$

$$f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) = \sum_{j=1}^{n} w_j e_j \tag{11}$$

Since $(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$ is a permutation of $(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle)$, we have $|x_i - y_i| = |c_i - d_i|$, for all i, and then

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) \quad \blacksquare$$

Theorem 2 (Monotonicity). *Assume f is the IOWAD operator, if* $|x_i - y_i| \geq |c_i - d_i|$, *for all* i_i, *then*:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \geq f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) \tag{12}$$

Proof. Let

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{13}$$

$$f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) = \sum_{j=1}^{n} w_j e_j \tag{14}$$

Since $|x_i - y_i| \geq |c_i - d_i|$, for all i, then

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \geq f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) \quad \blacksquare$$

Theorem 3 (Bounded). *Assume f is the IOWAD operator, then*:

$$\min\{|x_i - y_i|\} \leq f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq \max\{|x_i - y_i|\} \tag{15}$$

Proof. Let $\max\{|x_i - y_i|\} = c$, and $\min\{|x_i - y_i|\} = d$, then

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \leq \sum_{j=1}^{n} w_j c = c \sum_{j=1}^{n} w_j \tag{16}$$

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \geq \sum_{j=1}^{n} w_j d = d \sum_{j=1}^{n} w_j \tag{17}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq c \tag{18}$$

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \geq d \tag{19}$$

Therefore,

$$\min\{|x_i - y_i|\} \leq f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq \max\{|x_i - y_i|\} \quad \blacksquare$$

Theorem 4 (Idempotency). *Assume f is the IOWAD operator, if $|x_i - y_i| = a$, for all i, then*:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = a \tag{20}$$

Proof. Since $|x_i - y_i| = a$, for all i, we have

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j = \sum_{j=1}^{n} w_j a = a \sum_{j=1}^{n} w_j \tag{21}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = a \qquad \blacksquare$$

Another interesting issue to consider is the use of different measures for characterizing the weighting vector. For example, we could consider the entropy of dispersion (Yager 1988), the balance operator (Yager 1996) and the divergence of W (Yager 2002). The entropy of dispersion is defined as follows:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \tag{22}$$

For the balance operator, we obtain:

$$BAL(W) = \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1}\right) w_j \tag{23}$$

And for the divergence of W:

$$DIV(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2 \tag{24}$$

Another interesting issue is the problem of ties in the reordering process of the order-inducing variables. In order to solve this problem, we recommend the policy explained in (Yager and Filev 1999) regarding the replacement of the tied arguments by their average. Note that, in this case, it would mean that we are replacing the tied arguments by their normalized Hamming distance.

Note that in the analysis of the order-inducing variables of the IOWAD operator, the values used can be drawn from any space, the only requirement being that they should have linear ordering. Therefore, it is possible to use different kinds of attributes for the order-inducing variables which permit us, for example, to mix numbers with words in the aggregations. Note also that in some situations it is possible to use the implicit lexicographic ordering associated with words such as the ordering of words in dictionaries (Yager and Filev 1999).

4. Families of IOWAD operators

By choosing a different manifestation of the weighting vector, we are able to obtain different types of IOWAD operators such as the normalized Hamming distance (NHD), the weighted Hamming distance (WHD), the ordered weighted averaging distance (OWAD) operator, the step-IOWAD, the window-IOWAD, the median-IOWAD, the olympic-IOWAD, the centered-IOWAD, etc.

Remark 1: For example, the maximum distance, the minimum distance, the step-IOWAD, the NHD, the WHD and the OWAD, are obtained as follows.

- The maximum distance is found if $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$.
- The minimum distance if $w_n = 1$ and $w_j = 0$, for all $j \neq n$.
- More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we obtain the step-IOWAD operator.

- The NHD is formed when $w_j = 1/n$, for all i.
- The WHD is obtained when the ordered position of i is the same as j.
- The OWAD is found if the ordered position of u_i is the same as the ordered position of the values of the $|x_i - y_i|$, for all i.

Remark 2: Another particular case is the olympic-IOWAD operator. It is found when $w_1 = w_n = 0$, and for all others $w_{j*} = 1/(n - 2)$. Note that if $n = 3$ or $n = 4$, the olympic-IOWAD becomes the median-IOWAD and if $m = n - 2$ and $k = 2$, it becomes the window-IOWAD.

Remark 3: Following (Liu 2009), it is possible to present a general form of the olympic-IOWAD operator considering that $w_j = 0$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$. Note that if $k = 1$, then, this general form becomes the usual olympic-IOWAD. If $k = (n - 1)/2$, then it becomes the median-IOWAD operator.

Remark 4: Note that it is also possible to present the opposite case of the general olympic-IOWAD operator. In this case, $w_j = (1/2k)$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and $w_j = 0$, for all others, where $k < n/2$. Note that if $k = 1$, then, we obtain the opposite case of the median-IOWAD.

Remark 5: Another interesting family is the S-IOWAD operator based on the S-OWA operator (Yager 1993). It can be subdivided in three classes: the "orlike", the "andlike" and the generalized S-IOWAD operator. The generalized S-IOWAD operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-IOWAD operator becomes the “andlike” S-IOWAD and if $\beta = 0$, it becomes the “orlike” S-IOWAD. Also note that if $\alpha + \beta = 1$, we obtain the induced Hurwicz distance criteria.

Remark 6: A further family that could be used is the centered-IOWAD operator, based on (Yager 2007). An IOWAD operator can be defined as a centered aggregation operator if it is symmetric, strongly decaying and inclusive.

- It is symmetric if $w_j = w_{j+n-1}$.
- It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$.
- It is inclusive if $w_j > 0$.

Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. (softly decaying centered-IOWAD operator). Another particular situation of the centered-IOWAD appears if we remove the third condition (non-inclusive centered-IOWAD).

Remark 7: Using a similar methodology, we can develop many other families of IOWAD operators. For more information, see (Ahn and Park 2008; Beliakov et al. 2007; Chiclana et al. 2007; Emrouznejad 2008; Liu 2008; 2009; Merigó 2008; Merigó and Gil-Lafuente 2008a; 2008b; 2009; Xu 2005; 2008a; Yager 1993; 1996).

5. Decision-making with the IOWAD operator

The IOWAD operator is applicable in a wide range of situations in contexts such as decision-making, statistics, engineering, etc. In this paper, we will consider a decision-making application in the selection of investments. The main motivation for using the IOWAD operator in the selection of investments is because the decision-maker wants to take the decision according to a complex attitudinal character. This can be useful in many situations, for example, when the board of directors of a company wants to take a decision. Obviously, the attitudinal character of the board of directors is very complex because it involves the decisions of different persons and their interests may be different.

The process to follow in the selection of investments with the IOWAD operator is similar to the process developed in (Merigó and Gil-Lafuente 2007; 2008a; 2008c), with the difference that we are now considering a problem of investments. The five steps to follow can be summarized as follows:

Step 1: Analysis and determination of the significant characteristics of the available investments for the company. Theoretically, it will be represented as follows: $C = \{C_1, C_2, \ldots, C_i, \ldots, C_n\}$, where C_i is the ith characteristic of the investment and we suppose a limited number n of required characteristics.

Step 2: Fixation of the ideal levels of each characteristic in order to form the ideal investment.

Table 1. Ideal investment

	C_1	C_2	…	C_i	…	C_n
$P =$	μ_1	μ_2	…	μ_i	…	μ_n

where P is the ideal investment expressed by a fuzzy subset, C_i is the ith characteristic to consider and $\mu_i \in [0, 1]$; $i = 1, 2, \ldots, n$, is a number between 0 and 1 for the ith characteristic.

Step 3: Fixation of the real level of each characteristic for all the investments considered.

Table 2. Available alternatives

	C_1	C_2	…	C_i	…	C_n
$P_k =$	$\mu_1^{(k)}$	$\mu_2^{(k)}$	…	$\mu_i^{(k)}$	…	$\mu_n^{(k)}$

with $k = 1, 2, \ldots, m$; where P_k is the kth investment expressed by a fuzzy subset, C_i is the ith characteristic to consider and $\mu_i^{(k)} \in [0, 1]$; $i = 1, \ldots, n$, is a number between 0 and 1 for the ith characteristic of the kth investment.

Step 4: Comparison between the ideal investment and the different alternatives considered using the IOWAD operator. In this step, the objective is to express numerically the distance between the ideal investment and the different alternatives considered. Note that it is possible to consider a wide range of IOWAD operators such as those described in Section 3 and 4.

Step 5: Adoption of decisions according to the results found in the previous steps. Finally, we take the decision about which investment to select. Obviously, our decision will be the investment with the best results according to the particular type of IOWAD operator used.

6. Illustrative example

In the following, we present a brief illustrative example of the new approach in a decision-making problem regarding investment selection.

Assume a decision-maker wants to invest some money in a company. After analyzing the market he considers four possible alternatives.

1) Invest in a chemical company called A_1.
2) Invest in a food company called A_2.
3) Invest in a computer company called A_3.
4) Invest in a car company called A_4.

After careful review of the information, the decision-maker establishes the following general information about the investments. He has summarized the information of the investments in five general characteristics $C = \{C_1, C_2, C_3, C_4, C_5\}$.

- C_1: Benefits in the short term.
- C_2: Benefits in the mid term.
- C_3: Benefits in the long term.
- C_4: Risk of the investment.
- C_5: Other factors.

The results are shown in Table 3. Note that the results are valuations (numbers) between 0 and 1.

Table 3. Characteristics of the investments.

	C_1	C_2	C_3	C_4	C_5
A_1	0.8	0.7	0.5	0.6	0.8
A_2	0.9	0.5	0.7	0.7	0.7
A_3	0.8	0.6	0.9	0.6	0.8
A_4	0.4	0.8	0.9	0.6	0.7

In accordance with his objectives, the decision-maker establishes the following ideal investment. The results are shown in Table 4.

Table 4. Ideal investment.

	C_1	C_2	C_3	C_4	C_5
I	0.9	0.8	1	0.7	0.9

With this information, it is possible to develop different methods based on the IOWAD operator for selecting an investment. In this example, we will consider the maximum distance, the minimum distance, the NHD, the WHD, the step-IOWAD, the induced Hurwicz distance criteria (α = 0.4), the OWAD, the AOWAD, the IOWAD, the AIOWAD, the median and the olympic-IOWAD operator. We will assume the following weighting vector W = (0.1, 0.2, 0.2, 0.2, 0.3). The results are shown in Table 5 and 6.

Table 5. Aggregated results 1.

	Maximum	Minimum	NHD	WHD	Step (k=2)	Hurwicz
A_1	0.5	0.1	0.18	0.14	0.1	0.26
A_2	0.3	0	0.16	0.18	0.3	0.12
A_3	0.2	0.1	0.12	0.12	0.2	0.1
A_4	0.4	0	0.16	0.19	0.2	0.28

Table 6. Aggregated results 2.

	OWAD	AOWAD	IOWAD	AIOWAD	Median	Olympic
A_1	0.14	0.22	0.14	0.22	0.1	0.1
A_2	0.13	0.19	0.18	0.14	0	0.2
A_3	0.11	0.13	0.12	0.12	0.1	0.13
A_4	0.12	0.20	0.19	0.13	0.1	0.1

As we can see, for most of the cases, the best alternative is A_3 because it seems to be the one with the lowest distance from the ideal investment. However, for some particular situations, another choice may be optimal. Therefore, it is interesting to establish an ordering of the investments for each particular case.

Table 7. Ordering of the investments

	Ordering		Ordering
Maximum	$A_3 \succ A_2 \succ A_4 \succ A_1$	OWAD	$A_3 \succ A_4 \succ A_2 \succ A_1$
Minimum	$A_2 = A_4 \succ A_1 = A_3$	AOWAD	$A_3 \succ A_2 \succ A_4 \succ A_1$
NHD	$A_3 \succ A_2 = A_4 \succ A_1$	IOWAD	$A_3 \succ A_1 \succ A_2 \succ A_4$
WHD	$A_3 \succ A_1 \succ A_2 \succ A_4$	AIOWAD	$A_3 \succ A_4 \succ A_2 \succ A_1$
Step-IOWAD	$A_1 \succ A_3 = A_4 \succ A_2$	Median	$A_2 \succ A_1 = A_3 = A_4$
Hurwicz	$A_3 \succ A_2 \succ A_1 \succ A_4$	Olympic	$A_1 = A_4 \succ A_3 \succ A_2$

As we can see, depending on the particular type of distance aggregation operator used, the results may differ and may lead to different decisions.

7. Conclusions

We have presented a decision-making approach that uses distance measures and induced aggregation operators. This approach is based on the use of the IOWAD operator. It is an extension of the OWA operator that uses the Hamming distance and order-inducing variables in the reordering process. The main advantage of this operator is that it is able to consider complex attitudinal characters in the decision process. This is a key feature in decision-making because there may be many factors that affect the decision-makers' decisions including the fact that the decision-maker is often a group of persons such as the board of directors of a company.

We have analysed an application of the new approach in a decision-making problem regarding the selection of investments. We have seen that this approach gives a more complete information of the decision problem because it is able to consider a wide range of scenarios depending on the interests of the decision-maker. Moreover, by using order-inducing variables, it is possible to consider different scenarios according to complex attitudinal characters. We have also seen that, depending on the particular type of aggregation operator used, the results may lead to different decisions.

In future research, we intend to develop further extensions of this approach by using other characteristics in the decision process such as uncertain information (interval numbers, fuzzy numbers, linguistic variables, etc.), generalized aggregation operators, etc. We will also consider other decision-making problems and other applications.

References

Ahn, B.S. and H. Park (2008) "Least-squared ordered weighted averaging operator weights", International Journal of Intelligent Systems, 23, 33-49.

Alonso, S., F.J. Cabrerizo, F. Chiclana, F. Herrera and E. Herrera-Viedma (2008) "An Interactive Decision Support System Based on Consistency Criteria", Journal of Multiple-Valued Logic & Soft Computing, 14, 371-386.

Beliakov, G. (2005) "Learning Weights in the Generalized OWA Operators", Fuzzy Optimization and Decision Making, 4, 119-130.

Beliakov, G., A. Pradera and T. Calvo (2007) "Aggregation Functions: A guide for practitioners", Springer-Verlag, Berlin.

Bustince, H., F. Herrera, and J. Montero (2008) "Fuzzy Sets and their Extensions: Representation, Aggregation and Models", Springer, Berlin.

Calvo, T., G. Mayor and R. Mesiar (2002) "Aggregation Operators: New Trends and Applications", Physica-Verlag, New York.

Canós, L. and V. Liern (2008) "Soft computing-based aggregation methods for human resource management", European Journal of Operational Research, 189, 669-681.

Chiclana, F., F. Herrera, E. Herrera-Viedma and S. Alonso (2004) “Induced ordered weighted geometric operators and their use in the aggregation of multiplicative preference relations”, International Journal of Intelligent Systems, 19, 233-255.

Chiclana, F., E. Herrera-Viedma, F. Herrera and S. Alonso (2007) “Some induced ordered weighted averaging operators and their use for solving group decision-making problems based on fuzzy preference relations”, European Journal of Operational Research, 182, 383-399.

Emrouznejad, A. (2008) “MP-OWA: The most preferred OWA operator”, Knowledge-Based Systems, 21, 847-851.

Figueira, J., S. Greco and M. Ehrgott (2005) “Multiple criteria decision analysis: state of the art surveys”, Springer. Boston.

Gil-Aluja, J. (1998) “The interactive management of human resources in uncertainty”, Kluwer Academic Publishers, Dordrecht.

Hamming, R.W. (1950) “Error-detecting and error-correcting codes”, Bell Systems Technical Journal, 29, 147-160.

Karayiannis, N. (2000) “Soft Learning Vector Quantization and Clustering Algorithms Based on Ordered Weighted Aggregation Operators”, IEEE Transactions on Neural Networks, 11, 1093-1105.

Kaufmann, A. (1975) “Introduction to the theory of fuzzy subsets”, Academic Press, New York.

Liu, X. (2008) “A general model of parameterized OWA aggregation with given orness level”, International Journal of Approximate Reasoning, 48, 598-627.

Liu, X. (2010) “Parameterized OWA operator determination with optimization criteria: A general method”, IEEE Transactions on Fuzzy Systems, (to be published).

Merigó, J.M. (2008) “Nuevas extensiones a los operadores OWA y su aplicación en los métodos de decisión”, PhD Thesis (In Spanish), Department of Business Administration, University of Barcelona.

Merigó, J.M. and M. Casanovas (2009) “Induced aggregation operators in decision making with the Dempster-Shafer belief structure”, International Journal of Intelligent Systems, 24, 934-954.

Merigó, J.M. and A.M. Gil-Lafuente (2007): “Unification point in methods for the selection of financial products”, Fuzzy Economic Review, 13, 35-50, 2007.

Merigó, J.M. and A.M. Gil-Lafuente (2008a) “Using the OWA operator in the Minkowski distance”, International Journal of Computer Science, 3, 149-157.

Merigó, J.M. and A.M. Gil-Lafuente (2008b) “The generalized adequacy coefficient and its application in strategic decision making”, Fuzzy Economic Review, 13, 17-36.

Merigó, J.M. and A.M. Gil-Lafuente (2009) “The induced generalized OWA operator”, Information Sciences, 179, 729-741.

Szmidt, E. and J. Kacprzyk (2000) “Distances between intuitionistic fuzzy sets”, Fuzzy Sets and Systems, 114, 505-518.

Wang, X. (2008) “Fuzzy number intuitionistic fuzzy arithmetic aggregation operators”, International Journal of Fuzzy Systems, 10, 104-111.

Xu, Z.S. (2005) “An Overview of Methods for Determining OWA Weights”, International Journal of Intelligent Systems, 20, 843-865.

Xu, Z.S. (2008a) “Dependent uncertain ordered weighted averaging operators”, Information Fusion, 9, 310-316.

Xu, Z.S. (2008b) “On multi-period multi-attribute decision making”, Knowledge-Based Systems, 21, 164-171.

Xu, Z.S. (2008c) “Group decision making based on multiple types of linguistic preference relations”, Information Sciences, 178, 452-467.

Xu, Z.S. and R.R. Yager (2008) "Dynamic intuitionistic fuzzy multi-attribute decision making", International Journal of Approximate Reasoning, 48, 246-262.

Yager, R.R. (1988) "On Ordered Weighted Averaging Aggregation Operators in Multi-Criteria Decision Making", IEEE Transactions on Systems, Man and Cybernetics B, 18, 183-190.

Yager, R.R. (1993) "Families of OWA operators", Fuzzy Sets and Systems, 59, 125-148.

Yager, R.R. (1996) "Constrained OWA Aggregation", Fuzzy Sets and Systems, 81, 89-101.

Yager, R.R. (2002) "Heavy OWA Operators", Fuzzy Optimization and Decision Making, 1, 379-397.

Yager, R.R. (2003) "Induced aggregation operators", Fuzzy Sets and Systems, 137, 59-69.

Yager, R.R. (2007) "Centered OWA operators", Soft Computing, 11, 631-639.

Yager, R.R. (2008) "Using trapezoids for representing granular objects: Applications to learning and OWA aggregation", Information Sciences, 178, 363-380.

Yager, R.R. and D.P. Filev (1999) "Induced ordered weighted averaging operators", IEEE Transactions on Systems, Man and Cybernetics B, 29, 141-150.

Yager, R.R. and J. Kacprzyk (1997) "The Ordered Weighted Averaging Operators: Theory and Applications", Kluwer Academic Publishers, Norwell, MA.

Zarghami, M., F. Szidarovszky, and R. Ardakanian (2008) "A fuzzy-stochastic OWA model for robust multi-criteria decision making", Fuzzy Optimization and Decision Making, 7, 11-15.