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Research paper

Dirichlet problem on the half-line for a forced Burgers equation with time-variable coefficients and exactly solvable models

Şirin A. Büyükaşık*, Aylin Bozacı

Department of Mathematics, Izmir Institute of Technology, Urla, Izmir 35430, Turkey

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ABSTRACT

We consider a forced Burgers equation with time-variable coefficients and solve the initial-boundary value problem on the half-line $0 < x < \infty$ with inhomogeneous Dirichlet boundary condition imposed at $x = 0$. Solution of this problem is obtained in terms of a corresponding second order ordinary differential equation and a second kind singular Volterra type integral equation. As an application of the general results, we introduce three different Burgers type models with specific damping, diffusion and forcing coefficients and construct classes of exactly solvable models. The Burgers problems with smooth time-dependent boundary data and an initial profile with pole type singularity have exact solutions with moving singularity. For each model we provide the solutions explicitly and describe the dynamical properties of the singularities depending on the time-variable coefficients and the given initial and boundary data.

1. Introduction

Burgers equation is a nonlinear partial differential equation (PDE), which appears in description of many physical phenomena. J.M. Burgers considered it as a simplified model of Navier–Stokes equation and turbulence [1,2], Hopf and Cole discussed Burgers equation in the context of gas dynamics [3,4], and Lighthill in acoustics [5]. Then, many other application areas have appeared, such as Burgers-KPZ turbulence [6,7], stochastic processes, Polyakov problem and stretched vortices in hydrodynamic flows, [8,9].

From mathematical point of view, the standard Burgers equation $V_t + VV_x = \nu Vxx$ is a C-integrable model, due to the Cole-Hopf linearization transform [3,4], which converts Burgers equation to a linear heat equation. This transform is special also in the sense that the initial condition (IC) for Burgers equation (BE) transforms directly to initial condition for the corresponding heat equation (HE). Then one can easily write and analyze solutions of the Burgers initial value problem (IVP) on the whole real line $-\infty < x < \infty$. However, for initial-boundary value problems (IBVP's) posed on the infinite half-line $0 < x < \infty$ the situation is different. Depending on the type of the boundary conditions solving the problem is not always straightforward. Long time ago, Rodin in his work [10] discussed the IBVP for a standard Burgers equation on the half-line $0 < x < \infty$, with Dirichlet boundary condition (BC) imposed at $x = 0$, and showed that to find a solution of this IBVP, one must first solve a corresponding second-kind linear Volterra type integral equation. Since in general its solution requires

* Corresponding author.
E-mail address: sirinatilgan@iyte.edu.tr (Ş. A. Büyükaşık).

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approximation techniques, Rodin discussed a technique for obtaining closed form solutions of Burgers equation on the half-line by "sacrificing" the initial data, but retaining the exact boundary condition. By this approach he re-obtained some well-known solutions, and then Sachdev in [11] enlarged the solution class of these problems. Later, in [12] the weak limit of solutions to IBVP for Burgers equation on the quarter plane $x > 0, t > 0$ was addressed, and in [13] for Riemann type initial and boundary conditions the small viscosity limit and the large time behavior of the solutions was investigated. Almost at the same time, Calogero and De Lillo introduced a "generalized" Cole-Hopf transform for the Dirichlet Burgers problem on the half-line [14]. In a following article [15] for Burgers equation with forcing of the type $F(t)\delta(x)$ it was shown that the presence of a Dirac delta function in the forcing term implies that the problem can be viewed as connection of two problems on the infinite semilines $x < 0$ and $x > 0$ respectively, showing that semiline and delta forcing problems are closely related.

Recently, Fokas and De Lillo used the unified transform method to solve the Dirichlet problem for standard BE on the half-line [16]. After the introduction of the unified transform method (Fokas method) [17], there is a renewed interest in IBVP's for linear and non-linear PDE's on the infinite half-line. As known, using traditional techniques solution of an IBVP is usually expressed as an integral transform, and for non-homogeneous boundary data the solution is not uniformly convergent. However, the Fokas method yields expressions, which are always uniformly convergent at the boundary and therefore suitable for numerical computations.

As an integrable model, we recall that standard Burgers equation has different type of exact solutions such as shock and multi-shock traveling waves, rational solutions, triangular waves, N-shaped waves and periodic ones. Among them, rational type solutions form an important class. Indeed, zeros of the polynomial solutions of the heat problem lead to pole singularities for the Burgers rational solutions. Then, the motion of these singularities corresponds formally to the motion of one-dimensional particles interacting via two-body potentials and the corresponding many body problems are integrable, [18,19]. For recent work on the pole dynamics one can see [20], and for classes of rational type solutions of the Burgers hierarchy one can see [21]. Analysis of dynamics of complex pole singularities and spatial analyticity properties of the solution to Burgers equation was addressed in [22].

Here, we would like to notice that all research mentioned above was mainly done for the "standard" viscous Burgers equation. In the present work, we consider Burgers equation both with forcing term and with time-variable coefficients. As known, forced boundary value problems are natural in describing physical phenomena, since an external force acting on the system usually can be introduced not only as an inhomogeneous term in the equation, but also in the form of an inhomogeneous boundary condition. For recent discussion on the presence of forcing terms in Burgers equation and possible applications in description of standing waves in resonators, nonlinear acoustics and Burgulence one can see the work of Rudenko and Hedberg [23] and references cited therein. On the other side, recently there is an increasing interest in evolution equations with time variable coefficients, which are more realistic for better understanding the complicated nature of various nonlinear phenomena. This is because variable coefficients are able to reflect the slowly varying inhomogeneities of media and non-uniformities of boundaries, as investigated in [24,25].

Therefore, motivated by the above discussions and our previous work [26], the question about exact solutions of IBVP's for forced Burgers equations with time-variable coefficients on the half-line appears naturally. Precisely, in [26] we considered a Burgers equation of the form

$$U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - \omega^2(t)x, \quad t > t_0, \tag{1}$$

where $\Gamma(t) = \dot{\mu}(t)/\mu(t)$ is the damping coefficient, $d(t) = 1/2\mu(t)$ is the diffusion (or viscosity) coefficient, and $f(x,t) = -\omega^2(t)x$ is the forcing term which is linear in the space variable x. In [26] we investigated the pure IVP with initial condition $U(x,t_0) = F(x)$ on the whole line $-\infty < x < \infty$. Linearization and exact solvability of a generalization of this problem was given by Schulze–Halberg in [27], and very recently in [28] authors solved the Cauchy problem for Burgers hierarchy and forced Burgers equation with general time-dependent coefficients, using the Green's function of the corresponding linear problem.

In the present work, we study an IBVP for Burgers Eq. (1) on the half-line $0 < x < \infty$ with initial condition $U(x,t_0) = F(x)$ and Dirichlet boundary condition $U(0,t) = D(t)$, $t_0 < t < T$. In context of diffusion processes defined on a half-line domain $0 < x < \infty$, such as semi-infinite rod, Dirichlet BC imposed at x=0 can model a situation, where the concentration of the substance at the origin is prescribed. Depending on the phenomena being studied, concentration at the origin could be set to be zero ($D(t) = 0$), which corresponds to a homogeneous Dirichlet BC. On the other hand, if concentration at $x = 0$ is specified as some non-zero constant or it varies with time we speak about inhomogeneous Dirichlet BC. In context of fluid dynamics, an example of Dirichlet BC is the no-slip condition for viscous fluids, which states that at a solid boundary, the fluid will have zero velocity relative to the boundary. In general, Dirichlet boundary condition can be used to control directly the solution at the boundary of the semi-infinite domain.

The aim of this work is to provide a solution method for the Dirichlet IBVP of BE (1) on the infinite half-line, to introduce exactly solvable models and construct exact solutions. For this, in Section 2, we show that the Burgers IBVP can be transformed to a linear heat problem with Robin BC at $x = 0$. Then, we obtain the analytical solution of the Burgers IBVP in terms of two independent solutions to a second order ordinary differential equation (ODE) and a second-kind Volterra type integral equation with weakly singular kernel. Both the ODE and the Volterra integral equation are linear, but with time-dependent coefficients and due to this, they rarely admit exact solutions. However, exact solutions are always of considerable interest and as an application of the general results derived in Section 2, in next Sections 3, 4 and 5 we introduce three different Burgers type models with specific damping, diffusion and forcing coefficients. For each model, we construct special classes of exact solutions. Burgers solutions satisfying smooth IC and homogeneous Dirichlet BC are derived from solutions of the associated heat problem with Neumann BC, and they are smooth on the domain $x > 0$, $t > t_0$. More interesting type of exact Burgers solutions are constructed by imposing special initial and boundary conditions. These solutions have moving singularity on the real domain, due to pole type singularity in the initial profile. We investigate how time-dependent coefficients effect the propagation of the initial singularities, how their time-evolution is related with the given initial and boundary data, and then illustrate their dynamics explicitly. Section 6 includes discussion of the results and some concluding remarks.

2. Solution of the Dirichlet Burgers problem on the half-line

We consider an IBVP for the forced Burgers equation with time-variable coefficients given by

$$\begin{cases} U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - \omega^2(t)x, \quad 0 < x < \infty, \ \ 0 \le t_0 < t < T, \\ U(x, t_0) = F(x), \quad 0 < x < \infty, \\ U(0, t) = D(t), \quad t_0 < t < T, \end{cases} \tag{2}$$

where $U = U(x, t)$ is a real-valued function, $\mu(t) > 0$ is continuously differentiable on $[t_0, T)$ and $\omega^2(t)$ is a real-valued continuous function on $[t_0, T)$. Here, subscripts denote partial derivatives and dot denotes ordinary derivative with respect to time t. We assume that the initial function $F(x)$ does not grow too fast as $x \to \infty$, and the boundary $D(t)$ is sufficiently smooth. Consider also a corresponding second order linear ODE

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \omega^2(t)r = 0, \quad t_0 \le t < T, \tag{3}$$

and let $r_1(t) > 0$ and $r_2(t)$ be two independent solutions of (3) on $[t_0, T)$, satisfying the initial conditions, respectively

$$r_1(t_0) = r_0 \neq 0, \ \dot{r}_1(t_0) = 0; \qquad r_2(t_0) = 0, \ \dot{r}_2(t_0) = r_0/\mu(t_0). \tag{4}$$

Now, defining the functions

$$\eta(x, t) \equiv \frac{r_0}{r_1(t)}x, \qquad \tau(t) \equiv \frac{r_2(t)}{r_1(t)} = r_0^2 \int_{t_0}^{t} \frac{d\xi}{\mu(\xi)r_1^2(\xi)}, \quad t_0 \le t < T, \tag{5}$$

and denoting the heat kernel and the Neumann heat kernel, respectively by

$$K(\eta, \tau) \equiv \exp\left(-\left(\eta^2/2\tau\right)\right)/\sqrt{2\pi\tau}, \quad N(\eta, \xi; \tau) \equiv K(\eta - \xi, \tau) + K(\eta + \xi, \tau), \tag{6}$$

we prove the following proposition.

Proposition 1. *Solution of the IBVP (2) is of the form*

$$U(x, t) = \frac{\dot{r}_1(t)}{r_1(t)}x - \left(\frac{r_0}{\mu(t)r_1(t)}\right)\frac{\int_0^\infty \partial_\eta N(\eta(x, t), \xi; \tau(t))F_0(\xi)d\xi - \int_0^{\tau(t)} \partial_\eta K(\eta(x, t), \tau(t) - \tau')Q(\tau')d\tau'}{\int_0^\infty N(\eta(x, t), \xi; \tau(t))F_0(\xi)d\xi - \int_0^{\tau(t)} K(\eta(x, t), \tau(t) - \tau')Q(\tau')d\tau'}, \tag{7}$$

where

$$F_0(\xi) = \exp\left(-\mu_0 \int^{\xi} F(x)dx\right), \quad \mu_0 = \mu(t_0), \tag{8}$$

and the function $Q(\tau)$ is obtained by solving the second-kind Volterra integral equation

$$Q(\tau) = q(\tau) + d_0(\tau) \int_0^{\tau} \frac{Q(\tau')}{\sqrt{2\pi(\tau - \tau')}}d\tau', \ \ \tau > 0, \tag{9}$$

with inhomogeneous term $q(\tau)$ and coefficient $d_0(\tau)$, respectively given by

$$q(\tau) = -2d_0(\tau) \int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}F_0(\xi)d\xi, \qquad d_0(\tau) = \frac{r_1(t(\tau))\mu(t(\tau))D(t(\tau))}{r_0}, \tag{10}$$

and $\tau = \tau(t), \ t \in [t_0, T) \Leftrightarrow t = t(\tau), \ \ \tau \in [0, \tau(T))$.

Proof. First, we show that the IBVP (2) has solution of the form

$$U(x, t) = \frac{\dot{r}_1(t)}{r_1(t)}x + \frac{r_0}{\mu(t)r_1(t)}V(\eta(x, t), \tau(t)), \tag{11}$$

where $V(\eta, \tau)$ satisfies the IBVP for the standard Burgers equation with Dirichlet BC

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ V(\eta, 0) = \mu_0 U(\eta, t_0), & 0 < \eta < \infty, \\ V(0, \tau) = [\mu(t(\tau)) r_1(t(\tau))/r_0] U(0, t(\tau)), & 0 < \tau < \tau(T). \end{cases} \tag{12}$$

Indeed, in [26] it was found that the forced Burgers equation with specific time-variable coefficients in (2) has solution of the form (11), where the functions $\eta(x, t)$, and $\tau(t)$ are as defined in (5). Then, using (11) the initial condition $U(x, t_0)$ gives the initial condition $V(\eta, 0) = \mu_0 U(\eta, t_0) \equiv \mu_0 F(\eta)$. On the other hand, we notice that continuity of $\mu(t) > 0$ and $r_1^2(t) > 0$ for $t \in [t_0, T)$, imply that $\tau(t)$ defined in (5) is strictly increasing continuous function on $[t_0, T)$ and thus its inverse $t(\tau)$ exists for $\tau \in [0, \tau(T))$. Then, Dirichlet boundary condition $U(0, t) = D(t)$ transforms to Dirichlet boundary condition in (12), and IBVP (2) for the FBE transforms to the IBVP (12).

Second, using Cole-Hopf transform $V = -\varphi_\eta/\varphi$ it is not difficult to show that the IBVP (2) has solution of the form

$$U(x, t) = \frac{\dot{r}_1(t)}{r_1(t)} x - \frac{r_0}{\mu(t) r_1(t)} \frac{\varphi_\eta(\eta(x, t), \tau(t))}{\varphi(\eta(x, t), \tau(t))}, \tag{13}$$

where $\varphi(\eta, \tau)$ satisfies the IBVP for the standard heat equation with Robin BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ \varphi(\eta, 0) = \exp\left[-\mu_0 \int^\eta U(x, t_0) dx\right], & 0 < \eta < \infty, \\ [\mu(t(\tau)) r_1(t(\tau)) U(0, t(\tau))] \varphi(0, \tau) + r_0 \varphi_\eta(0, \tau) = 0, & 0 < \tau < \tau(T). \end{cases} \tag{14}$$

Thus, solving the IBVP (2) for FBE reduces to the problem of solving IBVP (14) for heat equation with Robin BC. Formally, we can write solution of the heat IBVP (14) using two approaches: the Neumann boundary approach and Dirichlet boundary approach, [29] and [10]. (For Dirichlet boundary approach one can assume temporary we know $\varphi(0, \tau) = H(\tau)$.) Here, we use the Neumann boundary approach and assume temporary we know $\varphi_\eta(0, \tau) = Q(\tau)$. Then, the following IBVP with Neumann BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ \varphi(\eta, 0) = \exp\left[-\mu_0 \int^\eta U(x, t_0) dx\right], & 0 < \eta < \infty, \\ \varphi_\eta(0, \tau) = Q(\tau), & 0 < \tau < \tau(T), \end{cases} \tag{15}$$

has solution

$$\varphi(\eta, \tau) = \int_0^\infty N(\eta, \xi; \tau) F_0(\xi) d\xi - \int_0^\tau K(\eta, \tau - \tau') Q(\tau') d\tau', \tag{16}$$

where $F_0(\xi) = \varphi(\xi, 0)$. It follows that

$$\varphi(0, \tau) = 2\int_0^\infty K(\xi, \tau) F_0(\xi) d\xi - \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau - \tau')}} d\tau'.$$

Substituting $\varphi(0, \tau)$ and $\varphi_\eta(0, \tau) = Q(\tau)$ into the Robin BC of (14) gives,

$$Q(\tau) = \frac{r_1(t(\tau)) \mu(t(\tau)) U(0, t(\tau))}{r_0} \left(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau - \tau')}} d\tau' - 2\int_0^\infty K(\xi, \tau) F_0(\xi) d\xi \right),$$

which is exactly the second-kind singular Volterra type integral Eq. (9) for the unknown function $Q(\tau)$. Then, with $Q(\tau)$ determined by this integral equation the function (16) is solution of the heat problem (14). Therefore, Cole-Hopf transform $V = -\varphi_\eta/\varphi$ gives the solution of the IBVP (12), that is

$$V(\eta, \tau) = -\frac{\int_0^\infty \partial_\eta N(\eta, \xi, \tau) F_0(\xi) d\xi - \int_0^\tau \partial_\eta K(\eta, \tau - \tau') Q(\tau') d\tau'}{\int_0^\infty N(\eta, \xi, \tau) F_0(\xi) d\xi - \int_0^\tau K(\eta, \tau - \tau') Q(\tau') d\tau'},$$

which substituted back in (11) gives the solution (7) of the IBVP (2). □

As a summary, we can say that the Burgers IBVP (2) transforms to a heat IBVP with inhomogeneous Robin BC, and therefore reduces to the problem of solving a second order ODE (3) and a second-kind singular Volterra type integral Eq. (9). We note that, parameters of the ODE are determined by the Burgers equation, while the integral equation depends also on the initial and boundary data. In the present work, we are interested in obtaining exact and explicit solutions of the IBVP (2). Clearly, this will require specific dependence on the variable coefficients, and special initial and boundary data. Then, using the results given by Proposition 1, in next sections we construct some exactly solvable models and discuss their properties.

3. Model 1

The first model, which we introduce is an IBVP for a forced Burgers equation with constant coefficients $\mu(t)=1,\ \omega^2(t)=-\omega_0^2,\ \omega_0>0$, and defined by

$$\begin{cases} U_t+UU_x=\frac{1}{2}U_{xx}+\omega_0{}^2x, & 0<x<\infty,\ \ t>0,\\ U(x,0)=F(x), & 0<x<\infty,\\ U(0,t)=D(t), & t>0. \end{cases} \tag{17}$$

The corresponding second order ODE is $\ddot{r}(t)-\omega_0^2r(t)=0$ and it has solutions

$$r_1(t)=r_0\cosh(\omega_0t),\quad r_2(t)=(r_0/\omega_0)\sinh(\omega_0t),\qquad 0\le t<\infty$$

satisfying the required IC's (4). For $r_0>0$ both solutions are positive, increasing and tending to infinity as $t\to\infty$. According to Proposition 1, solution of the Burgers problem (17) is given by

$$U(x,t)=\omega_0\tanh(\omega_0t)x-sech(\omega_0t)\frac{\varphi_\eta(\eta(x,t),\tau(t))}{\varphi(\eta(x,t),\tau(t))}, \tag{18}$$

where

$$\eta(x,t)=sech(\omega_0t)x,\quad \tau(t)=\tanh(\omega_0t)/\omega_0, \tag{19}$$

and $\varphi(\eta,\ \tau)$ satisfies the IBVP for the heat equation with Robin BC

$$\begin{cases} \varphi_\tau=\frac{1}{2}\varphi_{\eta\eta}, & 0<\eta<\infty,\ \ 0<\tau<1/\omega_0,\\ \varphi(\eta,0)=\exp\left[-\int^\eta U(x,0)dx\right], & 0<\eta<\infty,\\ U(0,t(\tau))\varphi(0,\tau)+\sqrt{1-(\omega_0\tau)^2}\varphi_\eta(0,\tau)=0, & 0<\tau<1/\omega_0. \end{cases} \tag{20}$$

In that model, $\tau(t)$ is positive, strictly increasing, bounded above function and its inverse is $t(\tau)=\tanh^{-1}(\omega_0\tau)/\omega_0,\ 0<\tau<1/\omega_0$. Then, the heat problem (20) has solution of the form

$$\varphi(\eta,\tau)=\int_0^\infty\left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}+e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi F(x)dx}d\xi-\int_0^\tau\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau',$$

where $Q(\tau)$ is found by solving the Volterra integral equation

$$Q(\tau)=D(t(\tau))\left[\frac{1}{\sqrt{1-(\omega_0\tau)^2}}\right]\left(\int_0^\tau\frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'-2\int_0^\infty\left(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi F(x)dx}d\xi\right). \tag{21}$$

Clearly, for arbitrary initial and boundary data solving the integral equation may require different approaches. In what follows, we introduce two special classes of Burgers IBVP's.

Model 1.1 Consider the following class of Burgers problems (17) with special initial condition depending on $m=0,1,2,\dots,$ and homogeneous Dirichlet boundary condition, that is

$$\begin{cases} U_t+UU_x=\frac{1}{2}U_{xx}+\omega_0{}^2x, & 0<x<\infty,\ \ t>0,\\ U(x,0)=-\frac{m}{x}, & 0<x<\infty,\\ U(0,t)=0, & t>0. \end{cases} \tag{22}$$

It reduces (without loss of generality for a suitable integration constant) to an IBVP for the heat equation, with homogeneous Neumann BC

$$\begin{cases} \varphi_\tau=\frac{1}{2}\varphi_{\eta\eta}, & 0<\eta<\infty,\ \ 0<\tau<1/\omega_0,\\ \varphi(\eta,0)=\eta^m, & 0<\eta<\infty,\\ \varphi_\eta(0,\tau)=0, & 0<\tau<1/\omega_0, \end{cases} \tag{23}$$

whose solutions depending on m, can be easily expressed in terms of the functions

$$h_p^-(\eta,\tau)=\int_0^\infty\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^pd\xi, \tag{24}$$

$$h_p^+(\eta,\tau)=\int_0^\infty\frac{e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^pd\xi. \tag{25}$$

These functions are well-known solutions of the heat equation for $-\infty<\eta<\infty$, and are positive for $0<\eta<\infty,\ \tau>0$, see for example [11,30–32]. Then, the heat problem (23) and hence the Burgers IBVP have the following solutions:

(a) For even powers, i.e. $m = 2p,\ p = 0, 1, 2, \ldots,$ solution of (23) is $\varphi(\eta,\ \tau) \equiv H_{2p}(\eta,\ \tau)$, where

$$H_{2p}(\eta,\tau) = h^-_{2p}(\eta,\tau) + h^+_{2p}(\eta,\tau) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \xi^{2p} d\xi,$$

are the even Kampe de Feriet polynomials (KFP). The first few KFP in explicit form are:

$H_0(\eta,\tau) = 1,\ H_1(\eta,\tau) = \eta,\ H_2(\eta,\tau) = \eta^2 + \tau,\ H_3(\eta,\tau) = \eta^3 + 3\eta\tau,\ H_4(\eta,\tau) = \eta^4 + 6\eta^2\tau + 3\tau^2$. Then, Burgers problem (22) has solution $U(x,\ t) \equiv U_{2p}(x,\ t)$, where

$$U_{2p}(x,t) = \omega_0 \tanh(\omega_0 t)x - sech(\omega_0 t)\frac{2pH_{2p-1}(sech(\omega_0 t)x, \tanh(\omega_0 t)/\omega_0)}{H_{2p}(sech(\omega_0 t)x, \tanh(\omega_0 t)/\omega_0)}, \tag{26}$$

and we used that $\partial_\eta H_p(\eta,\tau) = pH_{p-1}(\eta,\tau)$, for all $p = 1, 2, \ldots$. These solutions are smooth on the real domain $x > 0,\ t > 0$, and are in fact known solutions, [26]. Note that the odd KFP's defined by $H_{2p+1}(\eta,\tau) = h^-_{2p+1}(\eta,\tau) - h^+_{2p+1}(\eta,\tau)$ do not satisfy the Neumann BC $\varphi_\eta(0,\tau) = 0$, and therefore are not solutions of IBVP (23).

(b) For odd powers, i.e. $m = 2p + 1,\ p = 0, 1, 2, \ldots,$ solution of the heat problem (23) is

$$\varphi_{2p+1}(\eta,\tau) = h^-_{2p+1}(\eta,\tau) + h^+_{2p+1}(\eta,\tau), \tag{27}$$

and the corresponding solution of the Burgers problem (22) becomes

$$U_{2p+1}(x,t) = \omega_0 \tanh(\omega_0 t)x - (2p+1)sech(\omega_0 t)\left[\frac{h^-_{2p}(\eta(x,t),\tau(t)) - h^+_{2p}(\eta(x,t),\tau(t))}{h^-_{2p+1}(\eta(x,t),\tau(t)) + h^+_{2p+1}(\eta(x,t),\tau(t))}\right], \tag{28}$$

where we used that $\partial_\eta[h^-_p(\eta,\tau) + h^+_p(\eta,\tau)] = p[h^-_{p-1}(\eta,\tau) - h^+_{p-1}(\eta,\tau)]$, for each $p = 1, 2, \ldots$. These solutions are also smooth for $x > 0,\ t > 0$, and as an example we write the solutions for $p = 0$ and $p = 1$ explicitly,

$$U_1(x,t) = \omega_0 \tanh(\omega_0 t)x - sech(\omega_0 t)\frac{Erf\left[\frac{\eta(x,t)}{\sqrt{2\tau(t)}}\right]}{\sqrt{\frac{2\tau(t)}{\pi}}e^{-\frac{\eta^2(x,t)}{2\tau(t)}} + \eta(x,t)Erf\left[\frac{\eta(x,t)}{\sqrt{2\tau(t)}}\right]},$$

$$\begin{aligned} U_3(x,t) &= \omega_0 \tanh(\omega_0 t)x \\ &- 3sech(\omega_0 t)\frac{\sqrt{\frac{2\tau(t)}{\pi}}\eta(x,t)\sqrt{\tau(t)}e^{-\frac{\eta^2(x,t)}{2\tau(t)}} + \left(\eta^2(x,t) + \tau(t)\right)Erf\left[\frac{\eta(x,t)}{\sqrt{2\tau(t)}}\right]}{\sqrt{\frac{2\tau(t)}{\pi}}\left(\eta^2(x,t) + 2\tau(t)\right)e^{-\frac{\eta^2(x,t)}{2\tau(t)}} + \left(\eta^3(x,t) + 3\eta(x,t)\tau(t)\right)Erf\left[\frac{\eta(x,t)}{\sqrt{2\tau(t)}}\right]}, \end{aligned}$$

where $Erf(x)$ is the error function $Erf(x) = (2/\sqrt{\pi})\int_0^x e^{-y^2}dy$, and $\eta(x,\ t),\ \tau(t)$ are as defined in (19). Since for $x > 0$ we have

$$\lim_{t\to\infty} \eta(x,t) = 0, \qquad \lim_{t\to\infty} \tau(t) = 1/\omega_0,$$

we note that for above exact solutions

$$\lim_{t\to\infty} U(x,t) = \omega_0 x, \quad 0 < x < \infty$$

and the limiting function $U^*(x) = \omega_0 x$ is a steady-state satisfying $UU_x = \frac{1}{2}U_{xx} + {\omega_0}^2 x$ on the interval $0 < x < \infty$ with boundary data $U(0) = 0$. Thus, in the long-time limit the system becomes stable with U proportional to x.

Model 1.2 Now, we study a class of Burgers problems with special rational type initial condition and an inhomogeneous Dirichlet boundary condition,

$$\begin{cases} U_t + UU_x = \frac{1}{2}U_{xx} + {\omega_0}^2 x, & 0 < x < \infty,\ t > 0, \\ U(x,0) = -\frac{(2m+1)[x-2m/D_0]}{x[x-(2m+1)/D_0]}, & 0 < x < \infty, \\ U(0,t) = D_0 sech(\omega_0 t), & t > 0, \end{cases} \tag{29}$$

where $D_0 > 0$ is a constant parameter, and $m = 0, 1, 2, \ldots$. The initial profile has simple zero at $x = 2m/D_0$, and pole type singularity at $x = (2m + 1)/D_0$, for $x > 0$. On the other side, the boundary data is same for all $m = 0, 1, 2, \ldots,$ and it is time-dependent, smooth with $U(0,\ t) \to 0$ as $t \to \infty$. Here, parameter D_0 can be used to control the relation between the initial and boundary data. That is, when the strength D_0 of the BC increases, the initial singularity becomes closer to the boundary $x = 0$, and conversely, when parameter D_0 is small and close to zero the initial singularity is away from the boundary $x = 0$.

Burgers models (29) reduce to heat problems with polynomial type initial data and Robin BC as follows

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty,\ 0 < \tau < 1/\omega_0, \\ \varphi(\eta,0) = \left(\eta^{2m+1} - \frac{(2m+1)}{D_0}\eta^{2m}\right), & 0 < \eta < \infty, \\ D_0\varphi(0,\tau) + \varphi_\eta(0,\tau) = 0, & 0 < \tau < 1/\omega_0. \end{cases} \tag{30}$$

Solution of (30), according to (16), is given by $\varphi(\eta, \tau) \equiv \varphi_m(\eta, \tau)$, where

$$\varphi_m(\eta, \tau) = \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \right) \left(\xi^{2m+1} - \frac{(2m+1)}{D_0} \xi^{2m} \right) d\xi$$
$$- \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}} Q_m(\tau') d\tau', \tag{31}$$

and $Q_m(\tau)$ is solution of the following Abel integral equation of the second kind,

$$Q_m(\tau) = D_0 \left[\int_0^\tau \frac{Q_m(\tau')d\tau'}{\sqrt{2\pi(\tau-\tau')}} - 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\xi^{2m+1} - \frac{(2m+1)}{D_0} \xi^{2m} \right) d\xi \right]. \tag{32}$$

Using that, the integral equation has solution $Q_m(\tau) = 1.3.5\ldots(2m+1)\tau^m$, and substituting it into (31), as expected solution of the heat problem (30) becomes

$$\varphi_m(\eta, \tau) = H_{2m+1}(\eta, \tau) - \frac{(2m+1)}{D_0} H_{2m}(\eta, \tau), \tag{33}$$

which is a special linear superposition of two Kampe de Feriet polynomials.

For $m = 0, 1, 2$ solutions (33) are explicitly written as

$$\varphi_0(\eta, \tau) = H_1(\eta, \tau) - \frac{1}{D_0} H_0(\eta, \tau) = \eta - 1/D_0,$$
$$\varphi_1(\eta, \tau) = H_3(\eta, \tau) - \frac{3}{D_0} H_2(\eta, \tau) = (\eta^3 + 3\eta\tau) - \frac{3}{D_0}\left(\eta^2 + \tau\right),$$
$$\varphi_2(\eta, \tau) = H_5(\eta, \tau) - \frac{5}{D_0} H_4(\eta, \tau) = \left(\eta^5 + 10\eta^3\tau + 15\eta\tau^2\right) - \frac{5}{D_0}\left(\eta^4 + 6\eta^2\tau + 3\tau^2\right).$$

Note that IC in (30) has only one simple real zero $\eta = (2m+1)/D_0$ for $D_0 > 0$, $\eta > 0$ and each $m = 0, 1, 2, \ldots$. Then, the corresponding heat solution (33) has a zero for $\eta > 0$, $\tau > 0$, which propagates along the semiline $0 < \eta < \infty$ during the evolution process and its position can be described by a continuous function $\eta = \chi_m(\tau)$, satisfying $\chi_m(0) = (2m+1)/D_0$ and

$$\varphi_m(\chi_m(\tau), \tau) = 0, \quad m = 0, 1, 2, \ldots. \tag{34}$$

The corresponding solution $U(x, t) \equiv U_m(x, t)$ for the Burgers IBVP (29) becomes

$$U_m(x, t) = \omega_0 \tanh(\omega_0 t)x - sech(\omega_0 t) \frac{\partial_\eta \varphi_m(\eta(x, t), \tau(t))}{\varphi_m(\eta(x, t), \tau(t))}, \tag{35}$$

where $\eta(x, t)$, $\tau(t)$ are as defined in Eq. (19). Clearly, $U_m(x, t)$ has discontinuity of infinite type at points where $\varphi_m(\eta(x, t), \tau(t)) = 0$. It follows that $U_m(x, t)$ has singularity for $x > 0, t > 0$, whose position with respect to time is described by the function $x = x_m(t)$, where

$$x_m(t) = \frac{r_1(t)}{r_0} \chi_m(\tau(t)) = \frac{r_1(t)}{r_0} \chi_m\left(\frac{r_2(t)}{r_1(t)}\right), \quad t > 0. \tag{36}$$

According to this, we say that $U_m(x, t)$ is solution of BE with moving singularity $x_m(t)$, if it satisfies BE on the domain $\{(x, t): x \in (0, \infty) \backslash \{x_m(t)\}, t \in (t_0, T]\}$, and $|U_m(x, t)| \to \infty$ as $x \to x_m(t)$ for every $t \in [t_0, T]$.

Notice that, if $\tau(t) \to \tilde{\tau}$ for $t \to \infty$, where $\tilde{\tau} > 0$ is some constant, then by continuity of function χ_m, one has $\lim_{t\to\infty} \chi_m(\tau(t)) = \chi(\tilde{\tau})$. In that case, Eq. (36) shows that the behavior of the singularity as $t \to \infty$ is essentially determined by the properties of function $r_1(t)$. In particular, for solution (35) of this model, one has

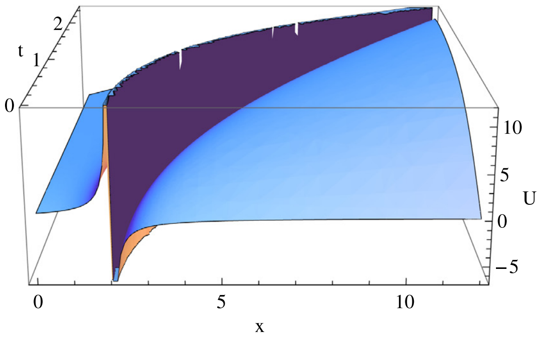
$$x_m(t) = \cosh(\omega_0 t) \chi_m(\tanh(\omega_0 t)/\omega_0), \quad t > 0,$$

which implies that $x_m(t) \to \infty$ as $t \to \infty$. It means that, for each $m = 0, 1, 2, \ldots$ the singularity of solution (35) moves away from the boundary $x = 0$, when time increases.

As an example, first we discuss Burgers problem (29) for $m = 0$. In that case, the initial profile is $U_0(x, 0) = -1/(x - 1/D_0)$, $x > 0$ and it has discontinuity at $x = 1/D_0$. The boundary condition is $U_0(0, t) = D_0 sech(\omega_0 t)$, $t > 0$, which shows that at point $x = 0$ and at time $t = 0$ the value of U_0 can be set to be a positive constant D_0, and that at later times the value of U_0 at point $x = 0$ will continuously decrease and approach zero. For these initial and boundary data, the explicit form of Burgers solution is

$$U_0(x, t) = \omega_0 \tanh(\omega_0 t)x - \frac{D_0 sech(\omega_0 t)}{D_0 sech(\omega_0 t)x - 1}, \tag{37}$$

and from the denumerator it is easy to see that $U_0(x, t)$ will have discontinuity at $x = \cosh(\omega_0 t)/D_0$, $t > 0$. It means that position of the singularity is continuously changing and monotone increasing function of time and that the initial singularity



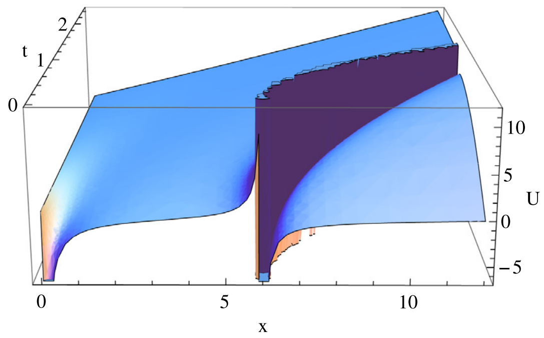


Fig. 1. Model 1.2 (a) Solution $U_0(x, t)$ given by Eq. (37) for $\omega_0 = 1,\ D_0 = 0.5$. (b) Solution $U_1(x, t)$ given by Eq. (38) for $\omega_0 = 1,\ D_0 = 0.5$.

located at $x = 1/D_0$ will move away from the boundary with increasing speed $v(t) = (\omega_0/D_0)\sinh(\omega_0 t)$ as $t \to \infty$. In Fig. 1(a) one can see how singularity changes with time and solution grows unboundedly near the singularity points.

For $m = 1$, the initial condition is $U_1(x, 0) = -3(x - 2/D_0)/x(x - 3/D_0)$, $x > 0$, which is discontinuous at $x = 3/D_0$ and boundary data is same as for $m = 0$. Now, the initial and boundary data are incompatible at $x = 0, t = 0$. However, for $x > 0, t > 0$ the corresponding Burgers solution becomes

$$\begin{aligned} U_1(x,t) &= \omega_0 \tanh(\omega_0 t)x \\ &- \frac{3\omega_0 D_0 x^2 - 6\omega_0\cosh(\omega_0 t)x + 3D_0\sinh(\omega_0 t)\cosh(\omega_0 t)}{\omega_0 D_0 x^3 - 3\omega_0\cosh(\omega_0 t)x^2 + 3D_0\sinh(\omega_0 t)\cosh(\omega_0 t)x - 3\sinh(\omega_0 t)\cosh^2(\omega_0 t)}, \end{aligned} \tag{38}$$

and in Fig. 1(b) it is given for $\omega_0 = 1,\ D_0 = 0.5$. We see that it is everywhere smooth except at discontinuity points created by the propagation of the initial singularity located at $x = 6$. Precisely, in this example position of the singularity is given by $x_1(t) = \cosh(t)\chi_1(\tanh(t))$, $t > 0$, where function χ_1 is explicitly found as

$$\chi_1(\tau) = 2 + \sqrt[3]{8 + \sqrt{(\tau - 4)^3 + 64}} + \sqrt[3]{8 - \sqrt{(\tau - 4)^3 + 64}}. \tag{39}$$

Since $\chi_1(\tau)$ is a positive and continuous function for $\tau > 0$, then for the singularity position we have $x_1(t) \to \infty$ as $t \to \infty$, and one can see its motion in Fig. 1(b).

4. Model 2

Now, we study the IBVP for a forced Burgers equation

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \frac{\gamma^2}{4}x, & 0 < x < \infty, \quad t > 0, \\ U(x,0) = F(x), & 0 < x < \infty, \\ U(0,t) = D(t), & t > 0, \end{cases} \tag{40}$$

with constant damping $\Gamma(t) = \gamma > 0$, exponentially decaying diffusion coefficient $1/2\mu(t) = (1/2)e^{-\gamma t}$, and $\omega_0^2 = (\gamma^2/4)$. We notice that, when parameter $\gamma > 0$ increases damping and forcing coefficients become larger, while diffusion coefficient gets smaller and goes to zero, when $t \to \infty$. The corresponding linear ODE is

$$\ddot{r}(t) + \gamma\dot{r}(t) + \frac{\gamma^2}{4}r(t) = 0, \quad t \geq 0, \tag{41}$$

and it has two independent solutions satisfying the IC's (4)

$$r_1(t) = r_0 e^{-\frac{\gamma t}{2}}\left(1 + \frac{\gamma}{2}t\right), \quad r_2(t) = r_0 t e^{-\gamma t/2}, \qquad t \geq 0. \tag{42}$$

In this model, for $r_0 > 0$ both solutions are positive and approaching zero, when $t \to \infty$. Then Burgers problem (40) has solution of the form

$$U(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1 + \frac{\gamma}{2}t}x\right) - \left(\frac{e^{-\gamma t/2}}{1 + \frac{\gamma}{2}t}\right)\frac{\varphi_\eta(\eta(x,t),\tau(t))}{\varphi(\eta(x,t),\tau(t))},$$

where

$$\eta(x,t) = \frac{e^{\gamma t/2}}{(1 + \gamma t/2)}x, \quad \tau(t) = \frac{t}{(1 + \gamma t/2)}, \quad t \geq 0, \tag{43}$$

$\tau(t)$ being positive, strictly increasing, bounded above for $t \geq 0$, with inverse $t(\tau) = \tau/(1-\gamma\tau/2)$ and $\varphi(\eta, \tau)$ satisfies the IBVP for the heat equation

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \ \ 0 < \tau < 2/\gamma, \\ \varphi(\eta, 0) = \exp\left[-\int^{\eta} U(x,0)dx\right], \quad 0 < \eta < \infty, \\ \left[(\frac{2}{2-\gamma\tau})e^{(\frac{\gamma\tau}{2-\gamma\tau})}U(0,t(\tau))\right]\varphi(0,\tau) + \varphi_\eta(0,\tau) = 0, \quad 0 < \tau < 2/\gamma. \end{cases} \tag{44}$$

Solution of the heat problem is formally given by (16), where $Q(\tau)$ is found by solving the integral equation

$$Q(\tau) = \left[\frac{2e^{\frac{\gamma\tau}{2-\gamma\tau}}}{(2-\gamma\tau)}U(0,t(\tau))\right]\left(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}e^{-\int^\xi F(x)dx}d\xi\right).$$

In what follows we shall again consider two concrete models. The first one is when $U(0,t(\tau)) = 0$, which leads to homogeneous Neumann boundary condition for the heat problem, and second one is when the boundary condition is chosen so that $U(0,t(\tau)) = D_0(2-\gamma\tau)/(2e^{\frac{\gamma\tau}{2-\gamma\tau}})$.

Model 2.1 Let the Burgers problem (40) with rational initial condition and homogeneous Dirichlet boundary condition be given

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \frac{\gamma^2}{4}x, \quad 0 < x < \infty, \ \ t > 0, \\ U(x,0) = \frac{-m}{x}, \quad 0 < x < \infty, \\ U(0,t) = 0, \quad t > 0. \end{cases} \tag{45}$$

It reduces to IBVP for the heat equation described in (23), with only difference that in this problem we have time domain $0 < \tau < 2/\gamma$. Thus, solutions are found in a similar way as follows.

(a) For $m = 2p, \ p = 0, 1, 2, \ldots,$ solution of the heat problem is the even Kampe de Feriet polynomial $\varphi_{2p}(\eta, \tau) = H_{2p}(\eta, \tau)$, and therefore solution of the Burgers problem (45) is of the form

$$U_{2p}(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\gamma t/2}x\right) - \left[\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right]\frac{2pH_{2p-1}\left(\frac{e^{\gamma t/2}}{1+\gamma t/2}x, \frac{t}{1+\gamma t/2}\right)}{H_{2p}\left(\frac{e^{\gamma t/2}}{1+\gamma t/2}x, \frac{t}{1+\gamma t/2}\right)}. \tag{46}$$

(b) For $m = 2p+1, \ p = 0, 1, 2, \ldots,$ solution of the heat problem again can be written as in (27) and the corresponding solution of the Burgers problem (45) becomes

$$\begin{aligned} U_{2p+1}(x,t) &= -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\gamma t/2}x\right) \\ &\quad - (2p+1)\left[\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right]\left[\frac{h^-_{2p}(\eta(x,t),\tau(t)) - h^+_{2p}(\eta(x,t),\tau(t))}{h^-_{2p+1}(\eta(x,t),\tau(t)) + h^+_{2p+1}(\eta(x,t),\tau(t))}\right], \end{aligned} \tag{47}$$

where $\eta(x, t)$ and $\tau(t)$ are as defined in (43). In that model, for $x > 0$ we have

$$\lim_{t\to\infty} \eta(x,t) = \infty, \quad \lim_{t\to\infty} \tau(t) = 2/\gamma,$$

and the long-time behavior of Burgers solution $U(x, t)$ is described by

$$\lim_{t\to\infty} U(x,t) = -\gamma x/2,$$

where the limiting function $U^*(x) = -\gamma x/2$ satisfies the equation $\gamma U + UU_x = -\gamma^2 x/4$ on the interval $0 < x < \infty$ with boundary condition $U(0) = 0$.

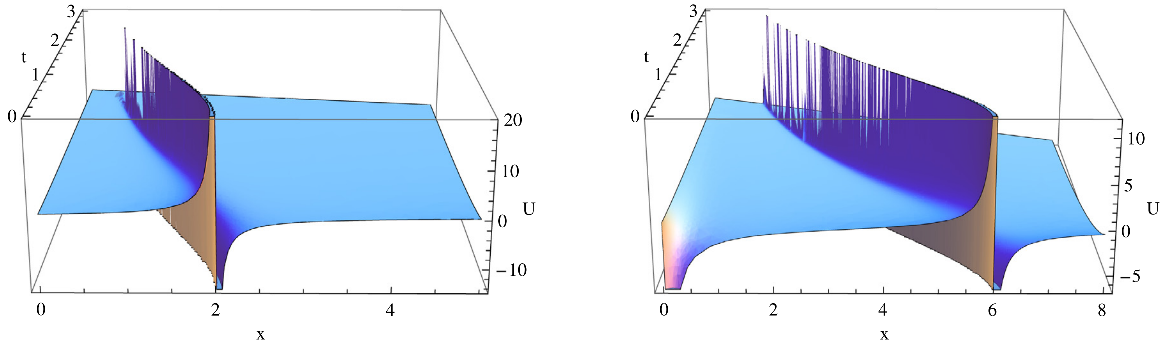
Model 2.2 Now we study the IBVP

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \frac{\gamma^2}{4}x, \quad 0 < x < \infty, \ \ t > 0, \\ U(x,0) = -\frac{(2m+1)[x-2m/D_0]}{x[x-(2m+1)/D_0]}, \quad 0 < x < \infty, \\ U(0,t) = D_0 e^{-\gamma t/2}/(1+\gamma t/2), \quad t > 0, \end{cases} \tag{48}$$

where $D_0 > 0$ and $m = 0, 1, 2, \ldots$. Here, the initial condition is same with that in Model 1.2 and has singularity depending on $m = 0, 1, 2, \ldots$. The boundary data has changed according to the time-variable coefficients, but we still have $U(0, t) \to 0$ as $t \to \infty$. As before, parameter $D_0 > 0$ can be used to correlate and fix the place of the initial singularity and the strength of the BC.

Burgers problem (48) reduces to the heat problem (30), with $0 < \tau < 2/\gamma$, and therefore (48) has solution

$$U_m(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\gamma t/2}x\right) - \left(\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right)\frac{\partial_\eta \varphi_m(\eta(x,t),\tau(t))}{\varphi_m(\eta(x,t),\tau(t))},$$



(a) (b)

Fig. 2. Model 2.2 (a) Solution $U_0(x, t)$ given by (50) with $D_0 = 0.5$ and $\gamma = 2$. (b) Solution $U_1(x, t)$ given by (51) with $D_0 = 0.5$ and $\gamma = 2$.

where $\varphi_m(\eta, \tau)$ is given by (33) and $\eta(x, t)$, $\tau(t)$ are as defined in (43). According to (34) heat function $\varphi_m(\eta, \tau)$ has moving zero $\eta = \chi_m(\tau)$, and it follows that for $x > 0, t > 0$, Burgers solution $U_m(x, t)$ has moving singularity described by

$$x_m(t) = (1 + \gamma t/2)e^{-\gamma t/2}\,\chi_m\left(\frac{t}{1 + \gamma t/2}\right), \quad t > 0. \tag{49}$$

Here, we have $x_m(t) \to 0$ as $t \to \infty$, showing that singularity initially located at $x = (2m+1)/D_0$ approaches the boundary $x = 0$, when time increases and by changing parameter $\gamma > 0$, one can control the speed at which singularity approaches the boundary. More precisely, when γ becomes larger, diffusion coefficient and boundary data go faster to zero and singularity goes faster to $x = 0$.

As an example, we investigate Burgers problem (48) for $m = 0$. The initial profile is $U_0(x, 0) = -1/(x - 1/D_0)$, $x > 0$ with discontinuity at $x = 1/D_0$, and we have smooth boundary condition $U_0(0, t) = D_0 e^{-\gamma t/2}/(1 + \gamma t/2)$, $t > 0$. Since initial and boundary data are compatible, at time $t = 0$ the value of U_0 at point $x = 0$ can be fixed to be a constant $D_0 > 0$, and boundary data shows that at later times the value of U_0 at point $x = 0$ will smoothly decrease and approach zero. Under these conditions Burgers solution becomes

$$U_0(x, t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1 + \gamma t/2}x\right) - \frac{D_0 e^{-\gamma t/2}}{D_0 e^{\gamma t/2}x - (1 + \gamma t/2)}. \tag{50}$$

Here, the singularity motion is described by the monotone decreasing function $x = (1/D_0)(1 + \gamma t/2)e^{-\gamma t/2}$, and it means that the singularity initially located at $x = 1/D_0$ will move along the semiline $0 < x < \infty$ continuously approaching the boundary point $x = 0$, when time increases. This behavior can be seen in Fig. 2(a) for $D_0 = 0.5$ and $\gamma = 2$.

Similarly, for $m = 1$, we have initial data $U_1(x, 0) = -3(x - 2/D_0)/x(x - 3/D_0)$, $x > 0$, which is discontinuous at $x = 3/D_0$, and boundary data is same as for case $m = 0$. Then, Burgers solution becomes

$$U_1(x, t) = -\left(\frac{\left(\frac{\gamma}{2}\right)^2 t}{1 + \gamma t/2}x\right) - \frac{3D_0 e^{\gamma t/2}x^2 - 6\left(1 + \frac{\gamma t}{2}\right)x + 3D_0\left(1 + \frac{\gamma t}{2}\right)e^{-\gamma t/2}t}{D_0 e^{3\gamma t/2}x^3 - 3\left(1 + \frac{\gamma t}{2}\right)e^{\gamma t}x^2 + 3D_0\left(1 + \frac{\gamma t}{2}\right)te^{\gamma t/2}x - 3\left(1 + \frac{\gamma t}{2}\right)^2 t}, \tag{51}$$

and its behavior for $D_0 = 0.5$ and $\gamma = 2$ is illustrated in Fig. 2(b). For these parameters, solution $U_1(x, t)$ has moving singularity described by $x_1(t) = (1 + t)e^{-t}\,\chi_1(t/(1 + t))$, $t > 0$, where χ_1 is defined by (39). It follows that singularity smoothly approaches the boundary $x = 0$ during the evolution process, as shown in Fig. 2(b).

5. Model 3

Consider the Burgers IBVP

$$\begin{cases} U_t + \frac{\alpha}{t}U + UU_x = \frac{1}{2t^\alpha}U_{xx} - \frac{(\alpha-1)^2}{4}\frac{1}{t^2}x, & 0 < x < \infty, \quad t > 1, \\ U(x, 1) = F(x), & 0 < x < \infty, \\ U(0, t) = D(t), & t > 1, \end{cases} \tag{52}$$

with coefficients $\Gamma(t) = \alpha/t$, $\mu(t) = 1/t^\alpha$, and $\omega^2(t) = (\alpha - 1)^2/4t^2$, all depending on time $t > 1$ and parameter $\alpha > 1$. The corresponding linear ODE is of Cauchy–Euler type

$$\ddot{r} + \frac{\alpha}{t}\dot{r} + \frac{(\alpha - 1)^2}{4}\frac{1}{t^2}r = 0, \tag{53}$$

and has solutions

$$r_1(t) = r_0\left[1 + \frac{(\alpha-1)}{2}\ln t\right]t^{-\frac{(\alpha-1)}{2}}, \quad r_2(t) = r_0 t^{-\frac{(\alpha-1)}{2}}\ln t, \quad t \geq 1, \tag{54}$$

where $r_1(t)$ satisfies $r_1(1) = r_0$, $\dot{r}_1(1) = 0$, and $r_2(t)$ satisfies $r_2(1) = 0$, $\dot{r}_2(1) = r_0$. For $r_0 > 0$ and $\alpha > 1$ these solutions are positive and approaching zero, when $t \to \infty$. For larger parameters $\alpha > 1$, diffusion coefficient goes faster to zero, and so do solutions of the ODE. The Burgers problem (52) has solution of the form

$$U(x,t) = -\frac{(\alpha-1)^2\ln t}{4t(1+\frac{(\alpha-1)}{2}\ln t)}x - \left[\frac{t^{-\frac{(\alpha+1)}{2}}}{1+\frac{(\alpha-1)}{2}\ln t}\right]\frac{\varphi_\eta(\eta(x,t),\tau(t))}{\varphi(\eta(x,t),\tau(t))},$$

where

$$\eta(x,t) = \left(\frac{t^{\frac{\alpha-1}{2}}}{1+\frac{(\alpha-1)}{2}\ln t}\right)x, \quad \tau(t) = \frac{\ln t}{1+\frac{(\alpha-1)}{2}\ln t}, \quad t \geq 1, \tag{55}$$

$\tau(t)$ again being positive, strictly increasing, bounded above with $t(\tau) = \exp(\frac{2\tau}{2-(\alpha-1)\tau})$, and $\varphi(\eta, \tau)$ satisfies

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < 2/(\alpha-1), \\ \varphi(\eta, 0) = \exp\left[-\int^\eta U(x,0)dx\right], \quad 0 < \eta < \infty, \\ \left[\frac{2}{2-(\alpha-1)\tau}\exp\left[\frac{(\alpha+1)\tau}{2-(\alpha-1)\tau}\right]U(0,t(\tau))\right]\varphi(0,\tau) + \varphi_\eta(0,\tau) = 0, \quad 0 < \tau < 2/(\alpha-1). \end{cases} \tag{56}$$

Then, solution of this problem is given by (16), where $Q(\tau)$ is found by solving the integral equation

$$Q(\tau) = \left[\frac{\exp\left[\frac{(\alpha+1)\tau}{2-(\alpha-1)\tau}\right]}{1-\frac{(\alpha-1)}{2}\tau}U(0,t(\tau))\right]\left(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}e^{-\int^\xi F(x)dx}d\xi\right).$$

As before, we study two exactly solvable models.

Model 3.1 The first model is

$$\begin{cases} U_t + \frac{\alpha}{t}U + UU_x = \frac{1}{2t^\alpha}U_{xx} - \frac{(\alpha-1)^2}{4}\frac{1}{t^2}x, \quad 0 < x < \infty, \quad t > 1, \\ U(x,1) = \frac{-m}{x}, \quad 0 < x < \infty, \\ U(0,t) = 0, \quad t > 1, \end{cases} \tag{57}$$

and it reduces to IBVP (23), with time domain $0 < \tau < 2/(\alpha-1)$.

(a) For $m = 2p$, $p = 0, 1, 2, \ldots$, solution of Burgers problem (57) is of the form

$$U_{2p}(x,t) = -\frac{(\alpha-1)^2\ln t}{4t\left(1+\frac{(\alpha-1)}{2}\ln t\right)}x - (2p)\left[\frac{t^{-\frac{(\alpha+1)}{2}}}{1+\frac{(\alpha-1)}{2}\ln t}\right]\frac{H_{2p-1}(\eta(x,t),\tau(t))}{H_{2p}(\eta(x,t),\tau(t))}, \tag{58}$$

where $H_{2p}(\eta,\tau) = \varphi_{2p}(\eta,\tau)$ is the even Kampe de Feriet polynomial solution of heat problem (23).

(b) For $m = 2p+1$, $p = 0, 1, 2, \ldots$, solution of the heat problem again can be written as in (27) and the corresponding solution of the Burgers problem (57) becomes

$$\begin{aligned} U_{2p+1}(x,t) = &-\frac{(\alpha-1)^2\ln t}{4t\left(1+\frac{(\alpha-1)}{2}\ln t\right)}x \\ &- (2p+1)\left[\frac{t^{-\frac{(\alpha+1)}{2}}}{1+\frac{(\alpha-1)}{2}\ln t}\right]\frac{h^-_{2p}(\eta(x,t),\tau(t)) - h^+_{2p}(\eta(x,t),\tau(t))}{h^-_{2p+1}(\eta(x,t),\tau(t)) + h^+_{2p+1}(\eta(x,t),\tau(t))}. \end{aligned} \tag{59}$$

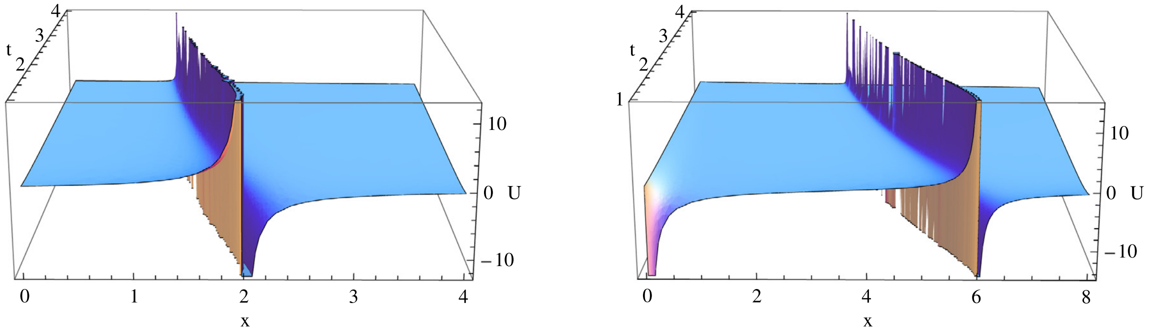
Since we have,

$$\lim_{t\to\infty}\eta(x,t) = \infty, \quad \lim_{t\to\infty}\tau(t) = \frac{2}{\alpha-1},$$

for the solutions of Model 3.1 above, the long-time behavior is given by $\lim_{t\to\infty}U(x,t) = 0$, showing that after sufficient time has elapsed the system becomes stable with $U = 0$.

Model 3.2 Next model is of the form

$$\begin{cases} U_t + \frac{\alpha}{t}U + UU_x = \frac{1}{2t^\alpha}U_{xx} - \frac{(\alpha-1)^2}{4}\frac{1}{t^2}x, \quad 0 < x < \infty, \quad t > 1, \\ U(x,1) = -\frac{(2m+1)[x-2m/D_0]}{x[x-(2m+1)/D_0]}, \quad 0 < x < \infty, \\ U(0,t) = D_0\frac{2}{2+(\alpha-1)\ln t}t^{-(\alpha+1)/2}, \quad t > 1, \end{cases} \tag{60}$$



(a) (b)

Fig. 3. Model 3.2 (a) Solution $U_0(x, t)$ given by (62) with parameters $D_0 = 0.5$ and $\alpha = 3$. (b) Solution $U_1(x, t)$ given by (63) with $D_0 = 0.5$ and $\alpha = 3$.

where $D_0 > 0$, $m = 0, 1, 2, \dots,$ $\alpha > 1$. Here, initial condition is imposed at time $t = 1$, and its profile is familiar from the previous models. Boundary data is again a time-dependent smooth function such that $U(0, t) \to 0$ as $t \to \infty$. Then, Burgers IBVP (60) reduces to heat problem described by (30), where $0 < \tau < 2/(\alpha - 1)$, and has solution

$$U_m(x,t) = -\frac{(\alpha-1)^2 \ln t}{4t\left(1+\frac{(\alpha-1)}{2}\ln t\right)}x - \left[\frac{t^{-\frac{(\alpha+1)}{2}}}{1+\frac{(\alpha-1)}{2}\ln t}\right]\frac{\partial_\eta \varphi_m(\eta(x,t),\tau(t))}{\varphi_m(\eta(x,t),\tau(t))},$$

where $\varphi_m(\eta, \tau)$ is given by (33) and $\eta(x, t)$, $\tau(t)$ are defined in (55). For each $m = 0, 1, 2, \dots,$ singularity evolves according to

$$x_m(t) = \left(1+\frac{\alpha-1}{2}\ln t\right)t^{-\left(\frac{\alpha-1}{2}\right)}\chi_m\left(\frac{\ln t}{1+\frac{(\alpha-1)}{2}\ln t}\right), \quad t > 1, \ \alpha > 1, \tag{61}$$

and when $t \to \infty$, we have $x_m(t) \to 0$. Moreover, taking larger values of parameter $\alpha > 1$, leads to increasing the speed of the singularity at which it approaches the boundary $x = 0$.

As an example, Burgers problem (60) for $m = 0$ has solution explicitly written as

$$U_0(x,t) = -\frac{(\alpha-1)^2 \ln t}{4t\left(1+\frac{(\alpha-1)}{2}\ln t\right)}x - \frac{D_0 t^{-\frac{(\alpha+1)}{2}}}{D_0 t^{\frac{\alpha-1}{2}}x - \left(1+\frac{(\alpha-1)}{2}\ln t\right)}, \tag{62}$$

and one can easily see that it has singularity, whose position is changing with time according to

$$x_0(t) = \frac{\left(1+\frac{(\alpha-1)}{2}\ln t\right)}{D_0 t^{\frac{\alpha-1}{2}}}, \quad t > 1, \ \alpha > 1.$$

In Fig. 3a solution (62) is given for $D_0 = 0.5$ and $\alpha = 3$. Therefore, it has singularity initially located at $x = 2$ and one can see that its position at later times smoothly approaches the boundary point $x = 0$, according to $x_0(t) = 2(1 + \ln t)t^{-1}$.

Similarly, Burgers problem (60) for $m = 1$ has solution

$$\begin{aligned} U_1(x,t) = &-\frac{(\alpha-1)^2 \ln t}{4t\left(\frac{(\alpha+1)}{2}\ln t\right)}x \\ &- \frac{3D_0 t^{\frac{a-3}{2}}x^2 - 6t^{-1}\left(\frac{(\alpha+1)}{2}\ln t\right)x + 3D_0 t^{-\frac{a+1}{2}}\left(\frac{(\alpha+1)}{2}\ln t\right)\ln t}{D_0 t^{\frac{3(a-1)}{2}}x^3 - 3t^{a-1}\left(\frac{(\alpha+1)}{2}\ln t\right)x^2 + 3D_0 t^{-\frac{(\alpha-1)}{2}}\ln t\left(\frac{(\alpha+1)}{2}\ln t\right)x - 3\left(\frac{(\alpha+1)}{2}\ln t\right)^2\ln t}, \end{aligned} \tag{63}$$

and for parameters $D_0 = 0.5$ and $\alpha = 3$ it is shown in Fig. 3(b). In that case, solution $U_1(x, t)$ has initial singularity located at point $x = 6$, and its position at later times is given by $x_1(t) = \left((1 + \ln t)t^{-1}\right)\chi_1\left((\ln t)(1 + \ln t)^{-1}\right)$, where again χ_1 is given by (39). It shows that, when time increases singularity moves smoothly along the semiline and monotonically approaches the boundary point $x = 0$, as one can see in Fig. 3(b).

6. Conclusion

In the present work, we investigated an initial-boundary value problem for a variable parametric inhomogeneous Burgers equation defined on the half-line $0 < x < \infty$ for $\le t_0 < t < T$ and satisfying smooth Dirichlet boundary condition imposed at $x = 0$. We determined its solution in terms of a second-order homogeneous ordinary differential equation and a second kind Volterra type integral equation with weakly singular kernel. Since the associated ODE and the integral equations are linear but with time-variable coefficients, they rarely admit exact solutions. However, exactly solvable models are always of primary interest and as an application of our results, we introduced three exactly solvable Burgers type models with different time-variable coefficients. For each model we constructed classes of exact solutions corresponding to certain initial and boundary data.

The class of Burgers problems corresponding to certain smooth initial data and homogeneous Dirichlet boundary condition $U(0,t)=0,$ have smooth solutions on the real domain $x>0, t>0$, obtained in terms of well-known solutions to the Neumann problem for the heat equation.

In general, for Burgers equation defined on the real domain $x\in R, t>0$, if initial condition is smooth, then the corresponding heat equation can not have zeros, and Burgers solution can not develop singularities. On the other side, it is known that one can construct heat solutions which have zeros at time $t=0,$ but they disappear during time-evolution and this corresponds to disappearance of Burgers singularities. Moreover, for real x and complex time t there is a possibility to develop singular Burgers solutions from non-singular initial data. For more details on the Burgers singularities one can see [33,34].

In this work, we introduced a class of Burgers problems with pole type singularity in the initial data and smooth inhomogeneous Dirichlet BC of the form $U(0,t)=D_0/\mu(t)r_1(t)$. These models have exact solutions with moving singularities on the semiline $0<x<\infty$. We studied the influence of the time-variable coefficients and given boundary data on the dynamics of these singularities.

First, we note that in all three models damping, diffusion and forcing coefficients behave differently, but the time-dependent boundary data tends to zero as $t\to\infty$. Then, we can summarize the results as follows.

In Model 1, Burgers equation is undamped, diffusion coefficient is constant and there is a positive forcing term $f_1(x,t)=\omega_0^2 x,$ linear in position variable x. It follows that solutions $r_1(t)$ and $r_2(t)$ of the corresponding ODE are positive, increasing functions of time, which grow unboundedly and at the same rate, when $t\to\infty$. According to this, we have seen that, when time increases singularity moves away from the boundary $x=0$ and its speed depends on the forcing parameter $\omega_0>0$.

In Model 2, damping and forcing coefficients of BE are positive constants in terms of $\gamma>0$, while diffusion coefficient is exponentially decreasing with time, that is we have damping $\Gamma(t)=\gamma,$ diffusion coefficient (or viscosity) $d(t)=e^{-\gamma t}/2$ and negative forcing term $f_2(x,t)=-\gamma^2 x/4$. Solutions of the related ODE are again positive functions of time, both changing at the same rate, but approaching zero when $t\to\infty$. It follows that singularity of the Burgers solution propagates along the semiline and approaches the boundary $x=0,$ when time increases. Here, by taking larger values of parameter $\gamma>0$ one can increase the strength of the damping and forcing terms, while reducing the diffusion and boundary effects. Such changes will be reflected in the speed of the moving singularity.

In Model 3, all coefficients of Burgers equation are positive functions of time and approaching zero when $t\to\infty$. Precisely, we have $\Gamma(t)=\alpha/t,$ diffusion coefficient $d(t)=1/2t^{\alpha}$ and negative forcing term $f_3(x,t)=-((\alpha-1)^2)/4t^2,\ \alpha>1$. The associated ODE has positive solutions $r_1(t), r_2(t)$, both going to zero at the same rate, and it implies that the singularity smoothly approaches the boundary $x=0,$ when $t\to\infty$. In this model, by choosing a larger parameter $\alpha>1$, one can make diffusion coefficient and boundary data go faster to zero, and then singularity will faster approach the boundary during the evolution process.

Finally, we can say that, the exactly solvable models discussed in this work give us an insight into the question how damping, diffusion and forcing coefficients of Burgers equation and also the correlation parameter D_0 of the initial and boundary data can be used to control the dynamics of the singularities. Then, by choosing different coefficients and boundary data one can enlarge the class of exactly solvable models and construct other solutions with singularities that move in time according to a prescribed rule or with other interesting properties. In general, the approach provided here can be used also to investigate IBVP's for forced Burgers equations with time-variable coefficients subject to some other boundary conditions, and this work is under consideration.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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