Unstaggered-staggered solitons on one- and two-dimensional two-component discrete nonlinear Schrödinger lattices

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Abstract

We study coupled unstaggered-staggered soliton pairs emergent from a system of two coupled discrete nonlinear Schrödinger (DNLS) equations with the self-attractive on-site self-phasemodulation nonlinearity, coupled by the repulsive cross-phase-modulation interaction, on 1D and 2D lattice domains. These mixed modes are of a "symbiotic" type, as each component in isolation may only carry ordinary unstaggered solitons. While most work on DNLS systems addressed symmetric on-site-centered fundamental solitons, these models give rise to a variety of other excited states, which may also be stable. The simplest among them are antisymmetric states in the form of discrete twisted solitons, which have no counterparts in the continuum limit. In the extension to 2D lattice domains, a natural counterpart of the twisted states are vortical solitons. We first introduce a variational approximation (VA) for the solitons, and then correct it numerically to construct exact stationary solutions, which are then used as initial conditions for simulations to check if the stationary states persist under time evolution. Twocomponent solutions obtained include (i) 1D fundamental-twisted and twisted-twisted soliton pairs, (ii) 2D fundamental-fundamental soliton pairs, and (iii) 2D vortical-vortical soliton pairs. We also highlight a variety of other transient dynamical regimes, such as breathers and amplitude death. The findings apply to modeling binary Bose-Einstein condensates, loaded in a deep lattice potential, with identical or different atomic masses of the two components, and arrays of bimodal optical waveguides.

keywords: discrete nonlinear Schrödinger equations; unstaggered-staggered lattice; variational

approximation; solitons

1 Introduction

Discrete nonlinear Schrödinger (DNLS) equations provide models for a great variety of physical systems [1]. A well-known implementation of the basic DNLS equation is provided by arrays of transversely coupled optical waveguides, as predicted in [2] and realized experimentally, in various optical settings [3, 4, 5, 6]. A comprehensive review of nonlinear optics in discrete settings was given by Ref. [7]. Another realization of the DNLS equation in provided by Bose-Einstein condensates (BECs) loaded into deep optical-lattice potentials, which split the condensate into a chain of droplets trapped in local potential wells, which are tunnel-coupled across the potential barriers between them [8, 9]. In the tight-binding approximation, this setting is also described by the DNLS version of the Gross-Pitaevskii (GP) equation [10, 11, 12, 13, 14].

One-dimensional (1D) DNLS equations with self-attractive and self-repulsive on-site nonlinearity generate localized modes of *unstaggered* and *staggered* types, respectively. In the latter case, the on-site amplitudes alternate between adjacent sites of the lattice [1]. In the continuum limit, the unstaggered discrete solitons carry over into regular ones, while the staggered solitons correspond to *gap solitons*, which are supported by the combination of self-defocusing nonlinearity and spatially periodic potentials [15, 16, 17].

Many physical settings are modeled by systems of coupled DNLS equations. In optics, they apply to the bimodal propagation of light represented by orthogonal polarizations or different carrier wavelengths. In BEC, coupled GP equations describe binary condensates [18]. Usually, bimodal discrete solitons in two-component systems are considered with a single type of their structure in both components, either unstaggered or staggered, because the self-phase- and cross-phase-modulation (SPM and XPM) terms, acting in each component and coupled nonlinearly, are assumed to have identical signs [1]. Nevertheless, the opposite signs are also possible in BEC, where either of them may be switched by means of the Feshbach resonance [18, 19, 20, 21, 22]. Discrete solitons of the *mixed* type, built as complexes of unstaggered and staggered components, were introduced in Ref. [23], assuming opposite SPM and XPM signs. Earlier, single-component states of a mixed unstaggered-staggered type were investigated in the form of surface modes at an interface between different lattices [24, 25]. In continuum systems, counterparts of mixed modes are represented by *semi-gap* solitons, which are bound states of an ordinary soliton in one component and a gap soliton in the other [26].

The mixed modes reported in Ref. [23] are "symbiotic" ones, as each component in isolation may support solely ordinary unstaggered solitons. The results were obtained in an analytical form, using the variational approximations (VA), and verified by means of numerical methods. It was found that almost all the symbiotic solitons were predicted by the VA accurately, and were stable. Unstable solitons were found only close to boundary of their existence region, where the solitary modes have very broad envelopes, being poorly approximated by the VA.

Most works on DNLS systems concern symmetric on-site-centered fundamental solitons, which represent the ground state of the corresponding model [1], including the unstaggered-staggered solitons [23]. Furthermore, only fundamental solitons represent stationary states in the continuum NLS equation. However, DNLS models give rise to stationary excited states, which may be stable too. The simplest among them are antisymmetric states in the form of discrete *twisted solitons* [27], which have no counterparts in the continuum limit. Once unstaggered-staggered discrete solitons are possible in two-component DNLS systems, it is natural to introduce the twist in the latter setting too, with three different species of such discrete solitons possible, which are *single-twisted*, in either component—staggered or unstaggered one—or *double-twisted*, in both components.

Discrete solitons on two-dimensional (2D) lattices have been studied in a variety of contexts, [28, 1]. Experimental literature highlights the existence of real solitons on 2D optically induced nonlinear photonic lattices [29, 30]. A natural extension of the earlier analysis, performed in the 1D setting [23], is to build two-component unstaggered-staggered complexes in 2D two-component DNLS systems, which may be realized physically in the same physical settings (optics and BEC) as mentioned above, provided that the corresponding waveguiding arrays are built, in the transverse plane, as 2D lattices, or the BEC is loaded into a deep 2D optical-lattice potential. In addition to 2D fundamental discrete solitons which may be naturally expected in the unstaggered-staggered system, one may also look for compound modes in which one or both components are represented by discrete vortex solitons [31], which is the 2D analogue of the 1D twisted solitons. Vortex solitons have previously been found in various optical setups [32, 33, 34], see also recent reviews [35, 36]. Experimental creation of discrete vortex solitons in self-focusing optically induced lattices was reported in Refs. [30, 37].

The remainder of this paper is organized according to the dimension of the lattice domain, with single- and double-twisted two-component unstaggered-staggered solitons on 1D lattices considered in Section 2. The analysis of unstaggered-staggered 2D discrete soliton complexes, consisting of fundamental soliton pairs, along with the more sophisticated fundamental-vortical and vorticalvortical ones, is reported in Section 3. For both dimensions of the lattice, we present the governing DNLS equations and the corresponding Lagrangian. Assuming a decay rate for the solitons' tails as predicted by the linearization of the DNLS equations, we elaborate the variational approximation (VA) for each type of soliton. We then use the VA-produced predictions as an initial guess to obtain the corresponding states in a numerically exact form. This approach is useful, as without an appropriate input the numerical scheme may readily converge to zero or some non-physical state. Furthermore, for stationary states which are stable and symmetric between the components (fundamental-fundamental, twist-twist, or vortical-vortical), the agreement of VA with numerical findings is quite good, whereas in the case of asymmetric pairs of the components (one twisted or vortical, the other being fundamental) the agreement is less accurate. Starting with numerically exact stationary states, we then simulate their evolution in time to determine what states are stable or unstable. Concluding remarks are made in Section 4.

2 One-dimensional coupled unstaggered-staggered modes

In this section, we initiate the analysis by formulating VA, which has proved to be quite efficient in the studies of fundamental discrete solitons in diverse settings, as shown at heuristic [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65] and more rigorous [66, 67] levels. We introduced the DNLS equations and their Lagrangian in subsection 2.1, elaborate the VA in subsection 2.2, and report results for numerically exact stationary solutions and their subsequent temporal evolution in subsection 2.3. We further select stable stationary states in subsection 2.4, and then outline transient regimes related to the evolution of unstable modes in subsection 2.5.

2.1 The DNLS equations

In Ref. [23], a system of coupled DNLS equations for discrete fields ϕ_n and ψ_n was introduced in 1D:

$$i\frac{d}{dt}\phi_n = -\frac{1}{2}\left(\phi_{n+1} + \phi_{n-1} - 2\phi_n\right) - \left(|\phi_n|^2 + \beta |\psi_n|^2\right)\phi_n,$$
(1a)

$$i\frac{d}{dt}\psi_n = -\frac{1}{2M}\left(\psi_{n+1} + \psi_{n-1} - 2\psi_n\right) - \left(|\psi_n|^2 + \beta |\phi_n|^2\right)\psi_n, \tag{1b}$$

where M > 0 is the relative atomic mass of the two species in the case of BEC, or the inverse ratio of the intersite coupling constants in the waveguide array, and $\beta < 0$ is the relative coefficient of the onsite XPM coupling between the fields, assuming that coefficients of the self-attractive SPM nonlinearity for both fields are scaled to be 1.

Solutions with unstaggered ϕ_n and staggered ψ_n onsite-centered components and two chemical potentials, λ and μ , are sought for as

$$\phi_n(t) = e^{-i\lambda t} u_n, \ \psi_n(t) = e^{-i\mu t} \, (-1)^n \, v_n, \tag{2}$$

where real discrete fields u_n and v_n satisfy the following stationary equations,

$$(\lambda - 1) u_n + \frac{1}{2} (u_{n+1} + u_{n-1}) + (u_n^2 + \beta v_n^2) u_n = 0, \qquad (3a)$$

$$\left(\mu - \frac{1}{M}\right)v_n - \frac{1}{2M}\left(v_{n+1} + v_{n-1}\right) + \left(v_n^2 + \beta u_n^2\right)v_n = 0,$$
(3b)

which can be derived from the Lagrangian,

$$L = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \left[-\frac{1}{2} \left(u_{n+1} - u_n \right)^2 + \lambda u_n^2 + \frac{1}{2M} \left(v_{n+1} - v_n \right)^2 + \left(\mu - \frac{2}{M} \right) v_n^2 + \frac{1}{2} u_n^4 + \frac{1}{2} v_n^4 + \beta u_n^2 v_n^2 \right].$$
(4)

In the standard way, the respective equations (3) can be obtained by taking the first variation of (4) in u_n and v_n , respectively.

Note that, in the limit of $M \gg 1$, which is tantamount to the Thomas-Fermi (TF) approximation for discrete equation (3b), this equation allows one to eliminate v_n in favor of u_n ,

$$v_n^2 = -\mu - \beta u_n^2 \,, \tag{5}$$

hence in this case the coupled stationary system (3) reduces to a single equation. In the opposite limit of $M \ll 1$, large constant M^{-1} makes Eq. (3b) close to its continuum counterpart, where nmay be considered as a continuous coordinate, and, accordingly, the broad v(n) component interacts with a narrow strongly discrete one, u_n . Thus, the effective continuous equation for v(n) amounts to

$$\mu v - \frac{1}{2M} \frac{d^2 v}{dn^2} + v^3 + \beta W_u \delta(n) v = 0,$$
(6)

where $\delta(n)$ is the delta-function, and W_u is the norm of the *u*-component, $W_u = \sum_{n=-\infty}^{\infty} u_n^2$. In the same approximation, v_n^2 in Eq. (3a) reduces to a constant, $v_n \approx v$ (n = 0). For $\beta < 0$, which

is considered in this work, Eq. (6) readily gives rise to an exact solution in the form of a pinned soliton,

$$v(n) = \frac{\sqrt{2\mu}}{\sinh\left(\sqrt{2\mu M} \left(|n| + n_0\right)\right)},$$
(7)

with n_0 determined by condition

$$\coth\left(\sqrt{2\mu M}n_0\right) = -\beta W_u \sqrt{M/(2\mu)}.\tag{8}$$

As follows from Eq. (8), this continuum-limit solution exists in the following interval of chemical potential of the staggered component: $0 < \mu < (M/2) (\beta W_u)^2$.

2.2 The variational approximation

In the general case, asymptotic tails of stationary discrete solitons decay at $|n| \to \infty$ as

$$u_n = A e^{-p|n|}, \quad v_n = B e^{-q|n|},$$
(9)

with p and q determined by the linearized limit of Eqs. (3) at $n \to \pm \infty$. Substituting expressions (9) in the linearized equations, we find $\lambda - 1 + \cosh(p) = 0$ and $\mu - \frac{1}{M} - \frac{1}{M} \cosh(q) = 0$. Solving these equations for p and q in terms of λ , μ , and M, we obtain

$$p = \operatorname{arccosh}(1-\lambda) \equiv \ln\left(1-\lambda+\sqrt{-\lambda(2-\lambda)}\right),$$
 (10a)

$$q = \operatorname{arccosh} (M\mu - 1) \equiv \ln \left(M\mu - 1 + \sqrt{M\mu (M\mu - 2)} \right).$$
(10b)

For p and q, given by Eqs. (10) to be real and positive, the allowed ranges of chemical potentials μ and λ are

$$\lambda < 0, \quad \mu > \frac{2}{M}. \tag{11}$$

Fundamental-soliton solutions of Eqs. (1) were constructed in Ref. [23], while in the present section we extend those results to more sophisticated solutions. A new possibility is to look for *twisted* solitons (i.e., spatially-antisymmetric ones). In the single-component DNLS equation, twisted solitons were introduced in Ref. [27]. An initial ansatz for such solitons, with the twisted structure in one or both components, can be taken as

$$u_n = A e^{-p|n|}, \quad v_n = B n e^{-q|n|},$$
 (12)

$$u_n = Ane^{-p|n|}, \quad v_n = Be^{-q|n|},$$
 (13)

$$u_n = Ane^{-p|n|}, \quad v_n = Bne^{-q|n|}, \tag{14}$$

where p and q are produced by the solution of the linearized equations for the tails in the same form of Eq. (10) as above. Note that, if the u-component is twisted, while the v mode is not, then the TF approximation based on Eq. (5) is definitely irrelevant, as it would predict $v_0^2 = -\mu < 0$ at n = 0. We stress that the two composite straight-twisted configurations, corresponding to ansätze (12) and (13), are not equivalent, as in these cases the twisted constituent is created in the unstaggered or staggered component, respectively.

Studies of stability of plane waves in 1D DNLS lattices suggest that these states are often modulationally unstable to small-wavenumber perturbations [68, 69, 70]. Perturbations of unstable

plane waves of this type often result in the emergence of highly localized structures, such as solitons or other solitary waves [68, 71]. In this case, solitons of the form given by Eqs. (12) and (14) are expected to have narrow envelopes, meaning large values of p and q. More broad envelopes are likely to be modulationally unstable against planewave perturbations in the large wavenumber regime. This intuition is in agreement with the results of Ref. [23] for fundamental solitons, and we will later show that the stability of solutions that we find here agrees with those earlier findings, i.e., stable solutions exhibit rapidly decaying tails, while broader soliton envelopes tend to be unstable.

Assuming the presence of twisted unstaggered and straight staggered components, Eq. (13) gives $u_0 = 0$, $u_{-n} = -u_n$, $v_{-n} = v_n$, and $v_0 > v_1 > v_2 > \dots$. For n = 0, the *u*-component of Eq. (3) is identically satisfied, whereas the equation for v yields $\frac{v_1}{v_0} = M\mu - 1 + Mv_0^2$. As the ratio v_1/v_0 must satisfy restriction $0 < v_1/v_0 < 1$ in localized solutions, this implies $\frac{1}{M} - \mu < v_0^2 < \frac{2}{M} - \mu$. However, condition $\mu > 2/M$ from Eq. (11) then results in $v_0^2 < 0$, therefore there may be no real amplitude v_0 generating twisted unstaggered and straight staggered components. This does not mean that other solutions of such a type (perhaps non-stationary ones) do not exist, but solely that solitons do not exist in this form. Thus, we exclude solitons of the form given by Eq. (13) from the consideration, and we will only seek solitons of the form defined by Eqs. (12) and (14). For each of these, we will obtain analytical approximations through the VA, which will be verified by means of numerical methods.

Now, we outline the construction of solitons of the form (12) and (14) by way of the VA. The existence of VA solutions is tied to the existence of a positive solution for squared amplitudes (A^2, B^2) of the respective *ansätze*, as produced by the Euler-Lagrange equations, derived in the framework of VA. In turn, we find that the existence of such positive solutions strongly depends on values the XPM coefficient, $\beta < 0$, and relative mass of the two components, M > 0. For a fixed set of these values, it is possible to determine a subset of the plane (λ, μ) of the chemical potentials of the two components, where positive solutions for (A^2, B^2) can be found.

2.2.1 Fundamental unstaggered - twisted staggered pairs

We start the VA analysis for $ans \ddot{a}tze$ (12). Substituting it in Lagrangian (4) and performing the summation yields the following effective Lagrangian:

$$2L_{\text{eff}} = -A^{2} \tanh(p/2) + \lambda A^{2} \coth p + \frac{A^{4}}{2} \coth(2p) + \frac{B^{2}}{2M} \frac{(\sinh q)^{-1}}{\cosh(q) + 1} \\ + \frac{B^{2}}{2} \left(\mu - \frac{2}{M}\right) \left(\coth^{3} q - \coth q\right) + \frac{B^{4}}{2} \left[\frac{3}{2} \coth^{5}(2q) - \frac{5}{2} \coth^{3}(2q) + \coth(2q)\right] \\ + \frac{\beta A^{2} B^{2}}{2} \left[\coth^{3}(p+q) - \coth(p+q)\right],$$
(15)

which gives rise to the Euler-Lagrange equations,

$$\frac{\partial L_{\text{eff}}}{\partial (A^2)} = \frac{\partial L_{\text{eff}}}{\partial (B^2)} = 0.$$
(16)

They amount to a system of linear equations for A^2 and B^2 :

$$A^{2}\left[\coth(2p)\right] + \frac{\beta}{2}B^{2}\left[\coth^{3}(p+q) - \coth(p+q)\right] = \tanh(p/2) - \lambda \coth p, \qquad (17)$$

$$\frac{\beta}{2}A^{2}\left[\coth^{3}(p+q) - \coth(p+q)\right] + B^{2}\left[\frac{3}{2}\coth^{5}(2q) - \frac{5}{2}\coth^{3}(2q) + \coth(2q)\right]$$
$$= -\frac{1}{2M}\frac{(\sinh q)^{-1}}{\cosh q + 1} - \frac{1}{2}\left(\mu - \frac{2}{M}\right)\left(\coth^{3}q - \coth q\right).$$
(18)

Note that p and q are already determined from (10). The system (17)-(18) is solved for the unknown A and B, which are the initial amplitudes of the variational approximation at n = 0. As the system is linear in A^2 and B^2 , it is sufficient to obtain a solution for the quantities A^2 and B^2 , and these quantities must be positive. We then take the positive root $\sqrt{A^2} = A$, $\sqrt{B^2} = B$ for the initial amplitudes.

Physically relevant solutions to Eqs. (17) and (18), with $A^2 > 0$ and $B^2 > 0$, do not exist for $\beta > 0$, but they may exist if $\beta < 0$, i.e., in the case of the repulsive interaction between the two components, which is the subject of the present work. The linear system of Eqs. (17)-(18) becomes degenerate (with zero determinant) in the case of

$$\coth(2p)\left[\frac{3}{2}\coth^5(2q) - \frac{5}{2}\coth^3(2q) + \coth(2q)\right] = \frac{\beta^2}{4}\left[\coth^3(p+q) - \coth(p+q)\right]^2.$$
(19)

2.2.2 Twisted unstaggered - twisted staggered pairs

Substituting ansatz (14) into Lagrangian (4) and carrying out the summation as above, we obtain the effective Lagrangian in the following form:

$$2L_{\text{eff}} = -\frac{A^2}{2} \frac{(\sinh p)^{-1}}{\cosh p + 1} + \lambda \frac{A^2}{2} \left(\coth^3 p - \coth p \right) + \frac{B^2}{2m} \frac{(\sinh q)^{-1}}{\cosh(q) + 1} + \frac{B^2}{2} \left(\mu - \frac{2}{M} \right) \left(\coth^3 q - \coth q \right) \\ + \frac{A^4}{2} \left[\frac{3}{2} \coth^5(2p) - \frac{5}{2} \coth^3(2p) + \coth(2p) \right] + \frac{B^4}{2} \left[\frac{3}{2} \coth^5(2q) - \frac{5}{2} \coth^3(2q) + \coth(2q) \right] \\ + \beta A^2 B^2 \left[\frac{3}{2} \coth^5(p+q) - \frac{5}{2} \coth^3(p+q) + \coth(p+q) \right].$$

$$(20)$$

It is convenient to define

$$\chi(\alpha) \equiv \frac{3}{2} \coth^5 \alpha - \frac{5}{2} \coth^3 a + \coth \alpha, \qquad (21)$$

which is positive for all $\alpha > 0$. Euler-Lagrange equations (16) following from Lagrangian (20) can be written as

$$A^{2}\chi(2p) + \beta B^{2}\chi(p+q) = \frac{1}{2} \frac{(\sinh p)^{-1}}{\cosh(p) + 1} - \frac{\lambda}{2} \left(\coth^{3}(p) - \coth(p) \right) , \qquad (22)$$

$$\beta A^2 \chi(p+q) A^2 + B^2 \chi(2q) = -\frac{1}{2M} \frac{(\sinh q)^{-1}}{\cosh(q) + 1} - \frac{1}{2} \left(\mu - \frac{2}{M}\right) \left(\coth^3(q) - \coth(q)\right) \,. \tag{23}$$

The system (22)-(23) is again a system for unknown initial amplitudes A and B in the variational approximation. Once again, this system may give rise to physical solutions only in the case of the repulsive XPM interaction, $\beta < 0$. The system of Eqs. (22) and (23) becomes degenerate if

$$\chi(2p)\chi(2q) = \beta^2 \chi^2(p+q), \qquad (24)$$

where χ is defined as per Eq. (21). Note that $\chi(\alpha) \to \infty$ as $\alpha \to +0$ and $\chi(\alpha) \to 0$ as $\alpha \to \infty$, and that χ is a decreasing function in its domain. Therefore, the ratio $\chi(2p)\chi(2q)/\chi(p+q)$ is always positive, and there always exists a value

$$\beta = \overline{\beta} \equiv -\sqrt{\frac{\chi(2p)\chi(2q)}{\chi(p+q)}} < 0, \qquad (25)$$

at which system (22)-(23) is degenerate.

2.3 The numerical approach

Localized steady-state solutions were numerically computed by solving a truncated form of Eq. (3) on a finite lattice. This was implemented with periodic boundary conditions, although we also compared results for fixed homogeneous conditions to ensure the boundaries played no role in the localized solution structure. The nonlinear system was then solved via the Matlab function 'fsolve,' which implements a Newton-like solution procedure (specifically a trust-region dogleg method). Unless otherwise mentioned, these solutions were carried out using $10^3 + 1$ lattice points, though they are nearly identical to solutions on much smaller lattices, hence it is safe to assume that the truncation has not changed the structure of localized solutions. We fixed function and optimality tolerances of 10^{-12} , and always checked that both the vector (L^2) and component (L^{∞}) norms of the objective function, evaluated on the numerical solution, were small enough ($< 10^{-4}$, and typically much smaller). The VA solitons were used as the initial guess. In some cases the resultant numerical solution was close to the original VA prediction, while in other cases, the initial VA solution converged to a soliton where one component is zero. In the latter case, to find nontrivial solutions in both components, we deflated the objective function away from zero (see [72] for a similar technique applied to the 2D GP equation). Specifically, if \mathbf{F} is the objective function (the left-hand side of Eq. (3), we replace it by

$$\left(1 + \frac{1}{\max(|u_n|)}\right) \left(1 + \frac{1}{\max(|v_n|)}\right) \mathbf{F}.$$
(26)

Once an exact stationary state is found, we tested its stability by simulations of their evolution in the framework of Eq. (1). It was thus found that some stationary solutions are unstable, as they do not persist in the course of the evolution, instead breaking apart or evolving into solutions of other kinds. Many solutions which do stably persist, emit a small amount of radiation at the initial stage of the evolution, whilst the solution adjusts to the true steady state. Unstable states tend to gradually break up into radiation, which disperses throughout the domain. We define a solution as stable if both ϕ_n and ψ_n components persist, keeping constant absolute values

To run simulations of the evolution, we set $\phi_n(0) = u_n$ and $\psi_n(0) = v_n$, and evolved Eqs. (1) using the Matlab function '*ode45*', which implements a fourth-order Runge-Kutta scheme, as a variation of an algorithm from Ref. [73]. We fixed absolute and relative tolerances of 10^{-13} . The solutions were consistent with simulations computed with the help of the first-order stiff solver, '*ode15s*,' for the simulation time scales, therefore we do not anticipate accumulations of errors due to round-off or loss of mass for the duration of the simulations. The simulations terminated when reflection from the boundary occurred, to avoid artifacts caused by the reflected waves. Thus, increasing the lattice size, we may increase the simulation time, and from this we conclude that the solutions we claim to be stable will be stable on an infinite lattice.

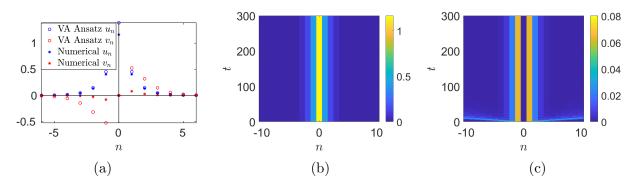


Figure 1: Plots of steady states and time evolution for a stable soliton built of bound fundamental and twisted components. Parameters are $\beta = -10$, $\mu = 2.8$, $\lambda = -0.7$, M = 1. In (a) both the VA-predicted and numerically exact stationary states are plotted. The evolution of the two components is displayed in (b) and (c) for $|\phi_n(t)|$ and $|\psi_n(t)|$, respectively. Component ψ_n shown in (c) initially releases a small amount of radiation, as it adjusts from the initial configuration to the stable stationary state.

2.4 Stable 1D two-component soliton

For $\beta = -0.5 > -1$, changes in the relative-mass parameter M primarily shift regions in the (μ, λ) plane which admit such solutions, but roughly preserve their geometry. In contrast, increasing M for $\beta = -5 < -1$ leads to an increase of the region of admissible localized solutions. For $\beta = -0.5$, the existence regions are similar to those found for fundamental soliton pairs on unstaggered-staggered lattices in Ref. [23] (not shown here in detail).

In fact, most parameter values and initial guesses led to unstable time evolution, but large stability regions for two-component solitons of the twisted-twisted soliton were found at large values of $-\lambda$ and μ (for $\beta = -0.5 > -1$). We have also found stable solitons close to but somewhat different from inputs in the form of Eqs. (12) or (14), such as states with broad multiple-site peaks. In contrast, two-component solitons of the fundamental-twisted type are, generally, less stable, due to fragility of such solutions to time dynamics. Thus, we expect that it may be more difficult to create fundamental-twisted soliton pairs in the experiment, whereas the robust twisted-twisted pairs should be available for a variety of parameter regimes and experimental configurations.

In other cases, we have found that, in the direct time simulations, unstable VA solitons with one or two twisted components spontaneously transform into states with two or one twisted components, respectively. This result highlights the fact that the VA approach is most useful for detecting highly localized waves, yet accurate identification of the structure of such states should be done numerically.

We plot representative stable fundamental-twisted and twisted-twisted soliton pairs in Figs. 1 and 2, respectively. The fundamental-twisted pair shows quantitative disagreement between the VA and exact numerical solution. Again, as the fundamental-twisted pair is not robust in the parameter space, the VA based in the simplest ansatz is not a particularly good fit. In particular, the amplitude of the twisted v_n component, shows quantitative disagreement with the VA prediction, whereas the fundamental components agree with VA quite well. Still, the peaks for both the VA-predicted twisted component and its numerically found counterpart are located at the same sites, and it is the amplitude which is poorly approximated. Simulations of the numerically exact fundamental-

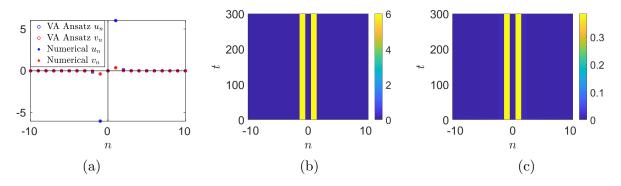


Figure 2: The same as in Fig. 1, but for a stable solitons with components of the twisted-twisted type. The parameters are $\beta = -0.5$, $\mu = 18.9$, $\lambda = -35$, M = 1. In panel (c), component ψ_n initially releases a small amount of radiation, as it adjusts from the initial state to the proper stable stationary one.

twisted pair demonstrate its stability, after shedding transient radiation at the initial stage of the evolution.

The twisted-twisted soliton pair shown in Figure 2 features good agreement between the VA and numerical solutions, which are indistinguishable at some sites. This fact, along with results for the fundamental-fundamental soliton pairs displayed in Ref. [23] suggests that the VA produces accurate predictions when both components have the same symmetry.

2.5 Transient dynamics

We have found a variety of evolution routes for unstable stationary states, as shown in Fig. 3. First, in Fig. 3(a) we observe unstable solutions which decay into radiation at large times, with the core region gradually spreading out over the entire spatial domain (though some fluctuations persist for large times). In contrast to the instability resulting in the decay of the original stationary state, there are other unstable solutions which persist with a finite amplitude in a finite core region, exhibiting spatiotemporal chaos within it (see Fig. 3(b)). Spatiotemporal chaos has been observed in other lattice NLS systems, often under the action of temporal forcing [74] or nonlocality [75]. Chaos has also been observed in DNLS systems which involve nonlinear coupling between adjacent sites [76]. Our model does not include any of these ingredients, with the only change from the standard DNLS equation being the unstaggered-staggered structure, which gives rise to the increased complexity in comparison with the standard DNLS lattices.

Further, we have also found solutions featuring unstable dynamics in one component and apparently stable evolution in the other, as shown in Figs. 3(c,d). In particular, this demonstrates *amplitude death* (vanishing) of one component and persistence of the other. The amplitude death in lattice dynamical systems modelling many coupled oscillators [77, 78, 79], whereas the amplitude death in coupled continuum complex Ginzburg-Landau systems has been demonstrated in Ref. [80] for saturable kinetics and in Ref. [81] for more general yet monotone kinetics. On the other hand, it was shown in Ref. [81] that coupled continuum NLS systems do not admit amplitude death. To the best of our knowledge, the amplitude death regimes has not been found in standard DNLS systems, again highlighting the rich variety of dynamics possible in unstaggered-staggered lattices.

In Figs. 3(e,f) we plot solution pairs which maintain their overall envelope yet exhibit periodic

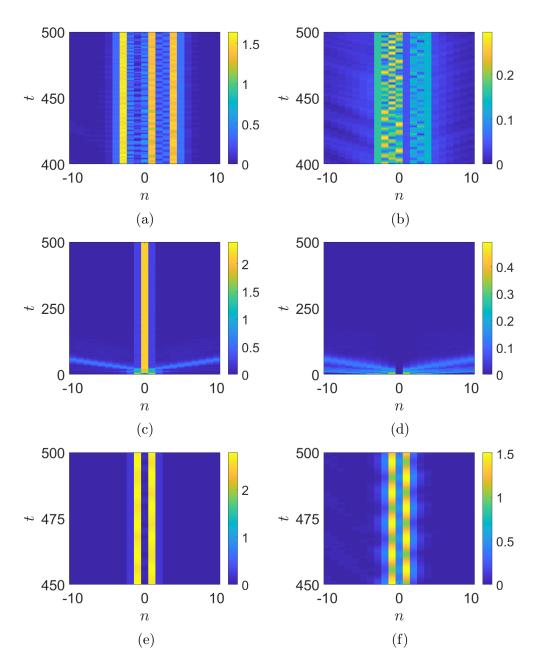


Figure 3: The simulated evolution of unstable numerical found stationary states, showing different routes of the instability development and eventual wave breakup. We plot $|\phi_n(t)|$ in (a,c,e), and $|\psi_n(t)|$ in (b,d,f), for parameters (a,b) $\beta = -5$, $\mu = 5$, $\lambda = -0.5$, M = 1 in (a,b), $\beta = -0.5$, $\mu = 2.3$, $\lambda = -2$, M = 1 in (cd), and $\beta = -0.5$, $\mu = 2.3$, $\lambda = -5$, M = 1 in (e,f). Panels (a,b) exhibit apparent spatiotemporal chaos, confined to a narrow band. In (c,d) we show a case where one of the components becomes unstable, decaying into radiation, whereas the other component persists. After transient emission of radiation, the solution in (e,f) displays spatiotemporal dynamics akin to a breather.

amplification and attenuation, thus appearing to be stable breathers. These oscillations are larger in one component, although are present in both. Breathers in other DNLS systems were reported in several works [82, 83, 84, 85].

3 Two-dimensional unstaggered-staggered lattices

In the higher-dimensional case, both theoretical [31] and experimental [29, 30] work demonstrate that a variety of dynamics are possible for the DNLS. Here, we extend the consideration of unstaggered-staggered modes to 2D lattices. As in the 1D case, we first present the dynamical system and its Lagrangian in subsection 3.1. We then derive the VA in subsection 3.2, and present representative stable stationary two-component solitons in subsection 3.3.

3.1 The 2D model and framework

The natural 2D generalization of the system of coupled DNLS equations (1) is

$$i\frac{d}{dt}\phi_{m,n} = -\frac{1}{2}\left(\phi_{m+1,n} + \phi_{m-1,n} + \phi_{m,n+1} + \phi_{m,n-1} - 4\phi_{m,n}\right) - \left(|\phi_{m,n}|^2 + \beta |\psi_{m,n}|^2\right)\phi_{m,n}, \quad (27a)$$

$$i\frac{d}{dt}\psi_{m,n} = -\frac{1}{2M}\left(\psi_{m+1,n} + \psi_{m-1,n} + \psi_{m,n+1} + \psi_{m,n-1} - 4\psi_{m,n}\right) - \left(|\psi_{m,n}|^2 + \beta |\phi_{m,n}|^2\right)\psi_{m,n}. \quad (27b)$$

Solutions with unstaggered $\phi_{m,n}$ and staggered $\psi_{m,n}$ components and two chemical potentials, λ and μ , are sought for as

$$\phi_{m,n}(t) = e^{-i\lambda t} u_{m,n}, \quad \psi_{m,n}(t) = e^{-i\mu t} (-1)^{m+n} v_{m,n}, \tag{28}$$

where real discrete fields $u_{m,n}$ and $v_{m,n}$ satisfy stationary equations,

$$(\lambda - 2) u_{m,n} + \frac{1}{2} \left(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} \right) + \left(|u_{m,n}|^2 + \beta |v_{m,n}|^2 \right) u_{m,n} = 0, \quad (29a)$$

$$\left(\mu - \frac{2}{M} \right) v_{m,n} - \frac{1}{2M} \left(v_{m+1,n} + v_{m-1,n} + v_{m,n+1} + v_{m,n-1} \right) + \left(|v_{m,n}|^2 + \beta |u_{m,n}|^2 \right) v_{m,n} = 0, \quad (29b)$$

which can be derived from the corresponding 2D Lagrangian:

$$L = \frac{1}{2} \sum_{m,n=-\infty}^{+\infty} \left[-\frac{1}{2} \left(|u_{m+1,n} - u_{m,n}|^2 + |u_{m,n+1} - u_{m,n}|^2 \right) + \frac{1}{2M} \left(|v_{m+1,n} - v_{m,n}|^2 + |v_{m,n+1} - v_{m,n}|^2 \right) + \lambda |u_{m,n}|^2 + \left(\mu - \frac{4}{M} \right) |v_{m,n}|^2 + \frac{1}{2} |u_{m,n}|^4 + \frac{1}{2} |v_{m,n}|^4 + \beta |u_{m,n}|^2 |v_{m,n}|^2 \right].$$
(30)

In particular, note that (29a) is the first variation of the Lagrangian (30) with respect to $u_{m,n}$, while (29b) is the first variation of the Lagrangian (30) with respect to $v_{m,n}$.

3.2 The variational approximation

In the 2D lattice framework, our first objective is to construct two-component solitons with the fundamental onsite-centered structure in both components. To apply the VA in this context, a practically tractable ansatz may be taken as the 2D generalization of the exponential one which was efficient in the application to the 1D DNLS equations [38, 39, 53, 23]:

$$u_{m,n} = Ae^{-P(|m|+|n|)}, \quad v_{m,n} = Be^{-Q(|m|+|n|)},$$
(31)

Considering ansatz (31) in the framework of linearized equations at $n, m \to \pm \infty$, one finds $\lambda - 2 + 2\cosh(P) = 0$ and $\mu - \frac{2}{M} - \frac{2}{M}\cosh(Q) = 0$. Solving for P and Q in terms of λ, μ , and M, we thus obtain

$$P = \operatorname{arccosh}\left(1 - \frac{\lambda}{2}\right) = \ln\left(1 - \frac{\lambda}{2} + \sqrt{-\frac{\lambda}{2}\left(2 - \frac{\lambda}{2}\right)}\right), \qquad (32a)$$

$$Q = \operatorname{arccosh}\left(\frac{M\mu}{2} - 1\right) = \ln\left(\frac{M\mu}{2} - 1 + \sqrt{\frac{M\mu}{2}\left(\frac{M\mu}{2} - 2\right)}\right).$$
(32b)

For P and Q to be real and positive, the allowed ranges of chemical potentials μ and λ are

$$\lambda < 0, \quad \mu > \frac{4}{M}, \tag{33}$$

cf. its 1D counterpart (11).

Real amplitudes A and B of ansatz (31) are again treated as variational parameters. In the 2D setting, a similar ansatz was applied to the DNLS equation with the cubic-quintic onsite nonlinearity in Ref. [86]. VA was also used in a different context, for the rigorous proof of the existence of discrete solitons as ground states in 1D [87] and 2D [88] settings. However, compound solitons built of unstaggered and staggered components were not studied previously in any form.

It is also possible to construct 2D topological discrete solitons in which at least one component is *vortical* (hence, at least one of $u_{m,n}$ or $v_{m,n}$ is complex-valued), and this shall be the focus of the present paper. For the single-component 2D DNLS equation, vortex-soliton solutions were first constructed in Ref. [31]. The initial ansatz for the vorticity in one or both components of the onsite-centered soliton may be taken as

$$u_{m,n} = Ae^{-P(|m|+|n|)}, \quad v_{m,n} = B(m+in)e^{-Q(|m|+|n|)}, \quad (34)$$

$$u_{m,n} = A(m+in) e^{-P(|m|+|n|)}, \quad v_{m,n} = Be^{-Q(|m|+|n|)}, \quad (35)$$

$$u_{m,n} = A(m+in) e^{-P(|m|+|n|)}, \quad v_{m,n} = B(m+i\sigma n) e^{-Q(|m|+|n|)}, \quad (36)$$

where P and Q are as given by Eq. (32). Ansätze (34) and (35) represent compound solitons with the vorticity embedded in one component, while in Eq. (36) both components are assumed vortical, $\sigma = \pm 1$ being the *relative vorticity* in the two components. The amplitude shape of the stationary solutions is the same for both $\sigma = +1$ and -1. In 2D continuum models, two-component vortex solitons with opposite vorticities in the two components are called *counter-rotating* two-component vortices [89], alias states with *hidden vorticity* [90], if the total angular momentum is zero. In that case, a nontrivial issue is stability of such compounds, which may be very different from that of their co-rotating counterparts. Existing studies of modulational instability in 2D DNLS lattices also suggest that plane and solitary waves are often modulationally unstable to small-wavenumber perturbations [91, 92]. The development of instability may lead to formation of highly localized structures, such as solitons or other localized states. Akin to the 1D case, the 2D solitons of the form given by Eqs. (31), (34)-(36) are expected to have narrow envelopes, with larger values of P and Q, while broad envelopes corresponding to smaller values of P and Q are likely to be unstable.

While in 1D we showed that only fundamental-fundamental, fundamental-twisted, and fundamental-fundamental soliton pairs exist in unstaggered-staggered lattices, in the 2D case we have found no asymmetric vortical-fundamental pairs, so we only present the VA for fundamental-fundamental and vortical-vortical pairs.

3.2.1 Fundamental unstaggered - fundamental staggered pairs

Here we construct 2D fundamental solitons of the unstaggered-staggered type, based on ansatz (31). Substituting it in Lagrangian (30) and carrying out the summation, we arrive at the effective Lagrangian:

$$2L_{\text{eff}} = -2A^{2} \left(\coth P\right) \tanh(P/2) + \frac{2}{M}B^{2} \left(\coth Q\right) \tanh(Q/2) + \lambda A^{2} \coth^{2} P + \left(\mu - \frac{4}{M}\right)B^{2} \coth^{2} Q + \frac{A^{4}}{2} \coth^{2}(2Pp) + \frac{B^{4}}{2} \coth^{2}(2Q) + \beta A^{2}B^{2} \coth^{2}(P+Q), \qquad (37)$$

which gives rise to the Euler-Lagrange equations,

$$A^{2} \coth^{2}(2P)A^{2} + \beta B^{2} \coth^{2}(P+Q) = 2(\coth P) \tanh(P/2) - \lambda \coth^{2} P, \qquad (38)$$

$$\beta A^2 \coth^2(P+Q) + B^2 \coth^2(2Q) = -\frac{2}{M} (\coth Q) \tanh(Q/2) - \left(\mu - \frac{4}{M}\right) \coth^2 Q.$$
(39)

As P and Q are determined already from (32), the system (38)-(39) determines the unknown amplitude parameters A and B. It is clear that a physically relevant solution pair, with $A^2 > 0$, $B^2 > 0$ does not exist for $\beta > 0$, which is a natural consequence of the fact that we are looking for the unstaggered-staggered complex. However, physical solutions may exist for $\beta < 0$, i.e., with opposite signs of the SPM and XPM onsite terms in Eqs. (1). Further, the system of Eqs. (38) and (39), considered as a system of linear equations for A^2 and B^2 , becomes degenerate if

$$\operatorname{coth}^2(2P)\operatorname{coth}^2(2Q) = \beta^2 \operatorname{coth}^4(P+Q).$$
(40)

3.2.2 Vortical unstaggered - vortical staggered pairs

The most sophisticated compound mode is built of vortical modes in both the unstaggered (u) and staggered (v) components, as per ansatz (36). Substituting the ansatz in Lagrangian (30), we obtain the respective effective Lagrangian,

$$2L_{\text{eff}} = -A^2 \eta(P) + \frac{B^2}{M} \eta(Q) + \lambda A^2 \coth^2(P) (\sinh(P))^{-2} + \left(\mu - \frac{4}{M}\right) B^2 \coth^2(Q) (\sinh(Q))^{-2} + \frac{A^4}{2} \zeta(2P) + \frac{B^4}{2} \zeta(2Q) + \beta A^2 B^2 \zeta(P+Q), \qquad (41)$$

where we have defined

$$\eta(\alpha) \equiv \frac{(\sinh(\alpha))^{-1} \coth(\alpha)}{\cosh(\alpha) + 1} + \frac{1}{2} \tanh(\alpha/2) \coth(\alpha) (\sinh(\alpha))^{-2}, \qquad (42)$$

$$\zeta(\alpha) \equiv \frac{1}{2} \coth^2(\alpha) (\sinh(\alpha))^{-4} \left(\cosh(2\alpha) + 6\right) \,. \tag{43}$$

From here we derive the corresponding Euler-Lagrange equations,

$$A^{2}\zeta(2P) + \beta B^{2}\zeta(P+Q)B^{2} = \eta(P) - \lambda \coth^{2}(Q)(\sinh(P))^{-2}, \qquad (44)$$

$$\beta A^2 \zeta(P+Q) + B^2 \zeta(2Q) B^2 = -\frac{1}{M} \eta(Q) - \left(\mu - \frac{4}{M}\right) \coth^2(Q) (\sinh(Q))^{-2}.$$
(45)

Again, as P and Q are determined already from (32), the system (44)-(45) determines the unknown amplitude parameters A and B. Since $\zeta(\alpha) > 0$ for $\alpha > 0$, in the present case we conclude, as in the 1D case, that a solution pair with $A^2 > 0$, $B^2 > 0$ does not exist for $\beta > 0$, but relevant solutions may exist at $\beta < 0$. The system of Eqs. (44) and (45) becomes degenerate if

$$\zeta(2P)\zeta(2Q) = \beta^2 \zeta(P+Q)^2 \,. \tag{46}$$

3.3 Stable 2D solution pairs

As in the case of the 1D lattice, in the 2D case we use the VA *ansätze* as initial guesses for solving Eqs. (29), and then simulate the ensuing evolution in the framework of Eq. (27). The functions and tolerances used are the same as in the 1D setting, although we here restrict the lattice to smaller sizes, for computational reasons. As the lattice is smaller, we simulated the evolution of the solutions for shorter times than in 1D, to avoid artifacts caused by radiation reflecting from the domain's boundaries. Nevertheless, concluding if the localized solution remain stable in the respective time interval (and for longer times on increasingly large lattices), it is possible to conjecture that such solutions will remain stable indefinitely.

In Figure 4 we display an example of a fundamental-fundamental soliton pair. In the course of the evolution, staggered component ψ emits some radiation, and then remains localized with a steady absolute-value profiles. In contrast, the unstaggered component ϕ emits no radiation at all. We conclude that the VA for these solutions is qualitatively accurate, although the numerically observed soliton is more localized in the $u_{m,n}$ component than the VA predicts. For these solutions, we used a lattice of size 61 by 61 and could observe a stable soliton up to T = 100. Thus, complementing what was shown in Ref. [23] for the 1D case, we find that there exist stable fundamental-fundamental soliton pairs in the 2D unstaggered-staggered lattice.

We were unable to find any parameter values which admitted solitons of the mixed staggered forms corresponding to Eqs. (34) or (35). Parameters permitting the existence of such solutions should be quite sparse, if they exist at all. This is consistent with the above-mentioned difficulty in finding 1D stable fundamental-twisted solitons, and the non-existence of twisted-fundamental pairs. In contrast, we have found pairs of vortical solitons, many of which are stable. We produce an example of such soliton pairs in Fig. 5, which are stable for both co- and counter-rotating vortical pairs (i.e., with $\sigma = \pm 1$). Again, some radiation is emitted, in the course of the short-time relaxation, only by the staggered component, and the resulting localized solution persists after the

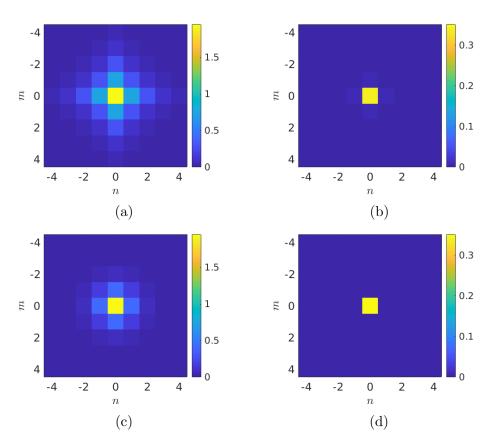


Figure 4: 2D two-component solitons of the fundamental-fundamental type for parameters $\beta = -10$, $\mu = 40$, $\lambda = -1$, M = 1. Discrete fields $|u_{m,n}|$ and $|v_{m,n}|$ are plotted in (a,c) and in (b,d), respectively, with (a,b) and (c,d) representing, respectively, VA ansatz (31) and the numerical steady- state solutions of Eq. (29).

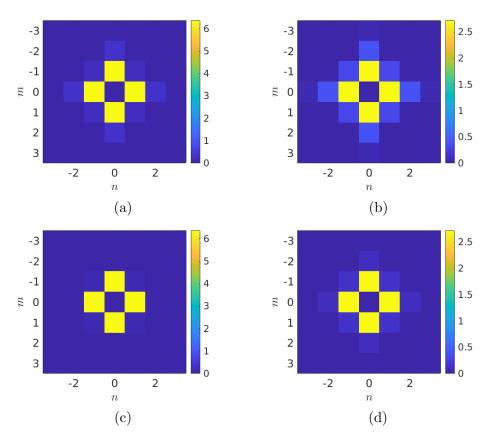


Figure 5: 2D vortical-vortical solution pairs with parameters $\beta = -0.5$, $\mu = 15$, $\lambda = -35$, M = 1, with $\sigma = 1$ used in the VA ansatz. We plot $|u_{m,n}|$ in (a,c) and $|v_{m,n}|$ in (b,d), with (a,b) corresponding to the VA ansatz (36) and (c,d) the numerical steady state solutions of (29).

completion of the relaxation. In this case too we conclude that the VA is a reasonable fit to the form of the numerically computed solutions. Here we restricted the lattice to a size of 41 by 41, and ensured the stability until T = 50. Nonetheless, we expect the solitons shown in Fig. 5 to persist indefinitely. The finding of the stable vortical-vortical soliton pairs on the unstaggered-staggered lattice adds to previously known results for discrete vortex solitons on the usual 2D DNLS lattice [31].

4 Conclusions

Extending the analysis of the recently introduced system of nonlinearly coupled DNLS equations with unstaggered and staggered components (which requires opposite signs of the SPM and XPM) nonlinearities—a situation possible in binary BEC), we have elaborated families of 1D discrete solitons with a single twisted or both twisted components, complementing the earlier work on fundamental soliton pairs on unstaggered-staggered lattices [23]. Analytical solutions for the discrete solitons are constructed by means of the VA (variational approximation). Similar to the recently studied family of fundamental solitons in this system [23], we have found that the twisted solitons produced by the VA are often stable when they are narrow, and unstable (or nonexistent in simulations) if at least one component is wide. We find that the VA is in the best agreement with numerical simulations when the solution pairs are symmetric, such as the fundamental-fundamental and twisted-twisted ones. As for asymmetric twisted-unstaggered-fundamental-staggered pairs, while stable solutions can be found numerically, they do not well agree with the VA, highlighting a limitation of the VA in that case. On the other hand, asymmetric fundamental-unstaggeredtwisted-staggered pairs are not predicted by the VA, and were not found in simulations either. therefore we conjecture that they do not exist. Through numerical simulations, we have determined the long-time evolution of initial steady states, with stable solutions maintaining their shape (sometimes after giving off a small amount of radiation, as they adjust to a true stable soliton). and unstable solutions decaying or exhibiting transient dynamics. In addition to the decay of both wave functions due to the instability, other unstable soliton initial conditions were observed to evolve into breathers or lead to the amplitude death (vanishing) of one wave function (with the other component persisting as a soliton).

Additionally, we have considered the extension of the unstaggered-staggered formulation to 2D lattices, which was not considered previously, and constructed both fundamental solitons and vortical ones, producing representative stable solutions for each case. Stable 2D two-component solitons of the asymmetric vortical-fundamental or fundamental-vortical have not been found, in agreement with the fact that, in the 1D case, twisted unstaggered-fundamental staggered pairs exist in a limited area of the parameter space, while the pairs of the symmetric types (fundamental-fundamental and twisted-twisted ones) are more common. Thus, the twisted-twisted (1D) and vortical-vortical (2D) pairs are found to be more robust than their asymmetric counterparts.

The solutions that we have found to be stable often correspond to narrow envelopes, consistent with results known form previous works. In particular, this finding is in agreement with the modulational-instability analysis for plane waves in related coupled NLS and complex Ginzburg-Landau systems in both continuum and discrete settings [93, 69, 94, 95, 96, 97], including two and three spatial dimensions [92, 80]. Indeed, perturbations with small wavenumbers often lead to modulational instability of plane waves [98, 99], which frequently results in the creation of highly localized structures, including solitons and other localized states [68]. Furthermore, the

direct modulational-instability analysis, applied to solitons in other related lattice systems, likewise suggests that modes with narrow envelopes tend to be stable [92], while wider ones fail to persist in the course of time evolution.

The 1D and 2D solutions that we have obtained here are novel in the context of unstaggeredstaggered lattices, and they may help to motivate future theoretical and experimental work in BEC and optics. Regarding theoretical extensions, there are a number of ways these results may be extended. We remark that *intersite-centered* 2D solitons and vortices may be considered too, although they are expected to be much less stable [1]. The ansatz for intersite-centered modes can be obtained from Eqs. (31) and (34)-(36) by replacing $\{m, n\} \rightarrow \{m - 1/2, n - 1/2\}$. It may also be interesting to consider solutions on periodic domains, such as a ring or torus. Work in this direction was reported in Refs. [100, 101, 102, 103]. As the soliton tails decay fairly rapidly, there may be little difference in the form of the solutions if the ring or torus is large enough, while, as they are made smaller, one may expect curvature effects to come into play. Finally, a more systematic treatment of some unsteady structures found here, such as breathers and the dynamics leading to "amplitude death", may be explored in more depth.

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