

Shallow water equations in Lagrangian coordinates: symmetries, conservation laws and its preservation in difference models

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Abstract

The one-dimensional shallow water equations in Eulerian and Lagrangian coordinates are considered. It is shown the relationship between symmetries and conservation laws in Lagrangian (potential) coordinates and symmetries and conservation laws in mass Lagrangian variables. For equations in Lagrangian coordinates with a flat bottom an invariant difference scheme is constructed which possesses all the difference analogues of the conservation laws: mass, momentum, energy, the law of center of mass motion. Some exact invariant solutions are constructed for the invariant scheme, while the scheme admits reduction on subgroups as well as the original system of equations. For an arbitrary shape of bottom it is possible to construct an invariant scheme with conservation of mass and momentum or energy. Invariant conservative difference scheme for the case of a flat bottom tested numerically in comparison with other known schemes.

Keywords: shallow water, Lagrangian coordinates, Lie point symmetries, conservation law, Noether's theorem, numerical scheme

1. Introduction

The shallow water equations describe the motion of incompressible fluid in the gravitational field if the depth of the liquid layer is small enough. They are widely used in the description of processes in the atmosphere, water basins, modeling of tidal oscillations, tsunami waves and gravitational waves (see the classical papers such as [1, 2] and detailed description in, for example, [3, 4]). Even in case of the one-dimensional shallow water equations with the flat bottom one meets certain difficulties to obtain nontrivial exact solutions. Some exact solutions can be found in [3, 5, 6].

Therefore, numerical calculations and finite-difference modeling are an effective mathematical apparatus in that area. Some monographs [7, 8, 9, 10] and numerous articles, e. g., [11, 12, 13, 14, 15, 16, 17, 18], are devoted to the numerical modeling of processes described by shallow water equations.

The present paper is devoted to the construction of invariant conservative difference schemes for the shallow water equations in potential and mass Lagrangian coordinates.

Lie groups have provided efficient tools for studying ODEs and PDEs since the fundamental works of Sophus Lie [19, 20, 21]. The symmetry group of a differential equation transforms solutions into solutions while leaving the set of all solutions invariant. The symmetry group can be used to obtain new solutions from known ones and to classify equations into equivalence classes according to their symmetry groups. It can also be used to obtain exact analytic solutions that are invariant under some subgroup of the symmetry

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group, so called “group invariant solutions”. Most solutions of the nonlinear differential equations occurring in physics, mechanics and in other applications were obtained in this manner. Applications of Lie group theory to differential equations, known as group analysis, is the subject of many books and review articles [22, 23, 24, 25, 26, 27].

The group properties of the shallow water equations were studied in numerous papers (see [28, 29, 30]). Group classification and first integrals of these equations can be found in [31, 32]. It was shown (see, e. g., [32]) that the shallow water equations in Lagrangian coordinates can be obtained as Euler–Lagrange equations of Lagrangian functions of a special kind. Extended nonlinear models, such as the Green–Naghdi equations and modified shallow water equations from a group point of view were considered in [33, 34].

More recently applications of Lie groups have been extended to difference equations [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48]. The applications are quite the same as in the case of differential equations, but have certain peculiarities connected with nonlocal character of difference operators and geometrical structure of difference mesh [45].

The invariant finite-difference schemes that preserve symmetries and since the geometric properties of the original equations are of particular interest. For the shallow water equations such schemes on moving meshes were proposed in [12] (for the methods of introducing moving meshes, see [49, 50]). The difficulties of constructing energy-saving difference schemes for the shallow water equations was emphasized in [12], and there the references were made to non-invariant schemes possessing the conservation law of energy (see, for example, [51]).

The Lagrange mass coordinates¹ [52, 53] are widely used in numerical modeling of the equations of continuous medium. Notice that Lagrangian mass coordinates are related with standard Lagrangian coordinates by tangent transformations and since may not preserve the whole set of admitted Lie group of point transformations. In the papers [54, 52] completely conservative schemes for the gas dynamics equations in Lagrangian mass coordinates were constructed. One can find examples of such schemes in gas dynamics and magnetohydrodynamics in [54, 55, 56, 57, 58, 59, 60].

The paper is organized as follows. In the second section basic equations in Eulerian and Lagrangian mass coordinates and some introductory remarks are given. The third section discusses in detail the shallow water equations in Lagrangian (potential) coordinates and their connection to the Lagrangian mass coordinates. In the third section, invariant difference schemes possessing local conservation laws of energy, mass, center of mass and momentum for the shallow water equations with the flat bottom are constructed. For the case of an arbitrary bottom, difference schemes possessing local conservation laws of mass and energy (or momentum) are presented. For the flat bottom equations some invariant difference solutions and corresponding reductions of the scheme are given. In the fifth section, one of the constructed invariant schemes is numerically performed on several test problems. The same tests are performed on some know schemes, and the obtained results are compared. In Conclusion the obtained schemes and numerical results are discussed.

2. Shallow water equation for Eulerian and Lagrangian mass coordinates

System of the one-dimensional shallow water equations with an arbitrary bottom in Eulerian coordinates has the following form

$$\eta_t + ((\eta + h)u)_x = 0, \quad (1)$$

$$u_t + uu_x + \eta_x = 0. \quad (2)$$

where $u(t, x)$ is the velocity of the continuous medium particles, $\eta(t, x)$ is the height of a liquid column above the bottom at point x , and the bottom profile is described by the function $h(x)$. One can reduce the linear bottom case ($h(x) = kx$, where k is constant) to the flat bottom ($h = 0$) by the following change of variables [61]

$$\tau = kt, \quad \xi = k \left(x - \frac{kt^2}{2} \right), \quad u = \nu(\tau, \xi) + \tau, \quad \eta = w(\tau, \xi) - \xi - \frac{\tau^2}{2}$$

¹To avoid terminological confusion, further we also refer standard Lagrangian coordinates as *potential* coordinates to distinguish from the mass Lagrangian coordinates.

In case of the flat bottom, it turns out that system (1),(2) possesses especially simple form with the help of hodograph transformation

$$\begin{aligned}x &= x(\rho, u), & x_\rho &= -\frac{u_t}{\Delta}, & x_u &= \frac{\rho_t}{\Delta}, \\t &= t(\rho, u), & t_\rho &= \frac{u_x}{\Delta}, & t_u &= -\frac{\rho_x}{\Delta}, \\ \Delta &= \rho_t u_x - \rho_x u_t \neq 0\end{aligned}\tag{3}$$

that allows one to linearize [53, 2] the system into the following equations

$$\begin{aligned}x_u - ut_u + \rho t_\rho &= 0, \\x_\rho + t_u - ut_\rho &= 0.\end{aligned}$$

Now we involve the Lagrangian operator of total differentiation with respect to t

$$\frac{d}{dt} = D_t + uD_x,\tag{4}$$

which does not commute with D_x :

$$\left[\frac{d}{dt}, D_x \right] \neq 0.$$

Along with $\frac{d}{dt}$ we introduce two variables: a “density” ρ

$$\rho = h + \eta,\tag{5}$$

and a new independent (mass) coordinate s by means of *contact* transformation

$$ds = \rho dx - \rho u dt,\tag{6}$$

where ds is a total differential form, i.e.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x},\tag{7}$$

that equivalent to the equation

$$\rho_t + (\rho u)_x = 0.\tag{8}$$

Collecting all relations we get the following transform:

$$\frac{dx}{dt} = u, \quad \frac{\partial x}{\partial s} = \frac{1}{\rho}, \quad \frac{\partial s}{\partial t} = -\rho u.$$

We introduce the following operator of total differentiation with respect to s :

$$D_s = \frac{1}{\rho} D_x.\tag{9}$$

The operators $\frac{d}{dt}, D_s$ commute on the system (1),(2):

$$\left[\frac{d}{dt}, D_s \right] = \left[D_t + uD_x, \frac{1}{\rho} D_x \right] = -\frac{1}{\rho^2} (\rho_t + (\rho u)_x),\tag{10}$$

where the last equation is zero on equation (1).

So far we can change variables as following:

$$\begin{aligned} u(t, x) &= u(t, s), \quad \rho(t, s) = h(x) + \eta(t, x), \\ \frac{du}{dt} &= u_t + uu_x, \quad \frac{d\rho}{dt} = \eta_t + u(h + \eta)_x, \\ u_x &= \rho u_s, \quad \rho_x = \rho \rho_s, \quad ds = \rho dx - \rho u dt, \quad t = t. \end{aligned} \quad (11)$$

Now we can rewrite the original system in the Lagrangian coordinates (t, s, u, ρ) :

$$\frac{d}{dt} \left(\frac{1}{\rho} \right) - u_s = 0, \quad (12)$$

$$\frac{du}{dt} + \rho(\rho - h)_s = 0. \quad (13)$$

3. Shallow water equation for (potential) Lagrangian coordinates

By means of a contact transformation one can relate Lagrangian mass coordinates to potential Lagrangian coordinates which are of our main interest. Potential coordinates allow one to derive the shallow water equations as Euler-Lagrange equations of a specific Lagrangian function. What is more important, they preserve orthogonality of a difference mesh for all the symmetries of the shallow water equations that significantly simplifies the subsequent discretization procedure.

3.1. Shallow water equation with an arbitrary bottom profile

We consider a potential x , defined by

$$\frac{dx}{dt} = u; \quad \frac{\partial x}{\partial s} = \frac{1}{\rho}. \quad (14)$$

Then continuity equation (12) reads

$$x_{ts} = x_{st}, \quad (15)$$

while equation (13) will be the following (everywhere below the derivative with respect to t is a Lagrangian one):

$$x_{tt} - \frac{x_{ss}}{x_s^3} - h_x = 0. \quad (16)$$

Remark 1. Notice that equation (16) corresponds to one-dimensional gas dynamics of polytrophic gas with $\gamma = 2$. Gas dynamics equations in this case are augmented with a state equation of the form

$$p = A\rho^\gamma,$$

where p is the pressure and A is constant. For the hyperbolic shallow water equations one can put following “state equation”: [62]

$$p = \frac{1}{2}\rho^2, \quad (17)$$

which allows one to rewrite system (12), (13) in an alternative form:

$$\frac{d}{dt} \left(\frac{1}{\rho} \right) - u_s = 0, \quad \frac{du}{dt} + p_s = h_x.$$

The equation (16) admits two symmetries

$$X_1 = \frac{\partial}{\partial t} \quad \text{and} \quad X_2 = \frac{\partial}{\partial s}. \quad (18)$$

Equation (16) can be considered as Euler-Lagrange equation for the Lagrangian

$$\mathcal{L} = \frac{x_t^2}{2} - \frac{1}{2x_s} + h(x). \quad (19)$$

Now the Noether theorem can be applied. First, we consider the case where $h = h(x)$ is an *arbitrary* function. In this case one obtains the following conservation laws:

$$1. \quad X_1 = \frac{\partial}{\partial t} :$$

$$\frac{d}{dt} \left[\frac{1}{2x_s} + \frac{x_t^2}{2} - h(x) \right] + D_s \left[\frac{x_t}{2x_s^2} \right] = x_t \left\{ x_{tt} - \frac{x_{ss}}{x_s^3} - h_x \right\} = 0; \quad (20)$$

$$2. \quad X_2 = \frac{\partial}{\partial s} :$$

$$\frac{d}{dt} [x_t x_s] + D_s \left[\frac{1}{x_s} - \frac{x_t^2}{2} - h(x) \right] = x_s \left\{ x_{tt} - \frac{x_{ss}}{x_s^3} - h_x \right\} = 0. \quad (21)$$

Now we recalculate the above conservation laws of the system (12), (13) in Lagrangian mass coordinates.

$$1. \quad X_1 = \frac{\partial}{\partial t} :$$

$$\frac{d}{dt} \left[\frac{u^2 + \rho}{2} - h(x) \right] + D_s \left[\frac{\rho^2 u}{2} \right] = 0. \quad (22)$$

$$2. \quad X_2 = \frac{\partial}{\partial s} :$$

$$\frac{d}{dt} \left[\frac{u}{\rho} \right] + D_s \left[\rho - \frac{u^2}{2} - h(x) \right] = 0. \quad (23)$$

Remark 2. In verification of above conservation laws one should keep in mind:

$$dh = h'(x)dx = h' \left(udt + \frac{ds}{\rho} \right), \quad (24)$$

Let us recalculate the conservation laws for the Euler coordinate system. For the first one we have:

$$D_t^e \left[\rho \left(\frac{u^2 + \rho}{2} - h(x) \right) \right] + D_x \left[\frac{\rho^2 u}{2} + \rho u \left(\frac{u^2 + \rho}{2} - h(x) \right) \right] = 0. \quad (25)$$

or

$$\rho u (u_t + uu_x + (\rho - h)_x) + (\rho_t + (\rho u)_x) \left(\rho + \frac{u^2}{2} - h \right) = 0, \quad (26)$$

where D_t^e is the Euler total derivative with respect to t .

The second conservation law in the Euler coordinate system equals just the following

$$D_t^e(u) + D_x \left(\rho - h + \frac{u^2}{2} \right) = 0, \quad (27)$$

or

$$u_t + uu_x + \eta_x = 0.$$

3.2. The case of a linear bottom

Let us consider the linear bottom, i.e.,

$$h(x) = C_1 x + C_2,$$

where C_1 and C_2 are constant. Equation (13) becomes

$$\frac{du}{dt} + \rho(\rho - C_1 x)_s = 0,$$

or, in potential coordinates,

$$x_{tt} - \frac{x_{ss}}{x_s^3} - C_1 = 0.$$

One can cancel the constant C_1 by means of transformation

$$x = \tilde{x} + \frac{t^2}{2} C_1, \quad (28)$$

and arrive to the case of the flat bottom $h(x) = 0$

$$x_{tt} - \frac{x_{ss}}{x_s^3} = 0. \quad (29)$$

For the sake of brevity, here and further on symbol “ \sim ” is omitted.

Equation (29) admits the following symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial s}, & X_3 &= \frac{\partial}{\partial x}, & X_4 &= t \frac{\partial}{\partial x}, \\ X_5 &= 3t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x}, & X_6 &= 3s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x}, \end{aligned} \quad (30)$$

and can be considered as Euler-Lagrange equation for the Lagrangian

$$\mathcal{L} = \frac{x_t^2}{2} - \frac{1}{2x_s}, \quad (31)$$

which is invariant to the group actions of the symmetries X_1 – X_4 . Notice, that (31) is divergent-invariant to the action of the symmetry X_4 , i.e.,

$$X_4 \mathcal{L} + \mathcal{L} (D_t \xi_4^t + D_s \xi_4^s) = D_t(-x).$$

Along with the Noether theorem there exists so called direct method [63, 64] which operates with the following relation:

$$D_t(F^t) + D_s(F^s) = \Lambda F,$$

where F is the original equation and Λ is called an integrating multiplier. As far as an action of variational operator cancels any divergent expression, then we get the determining equation for the multiplier Λ :

$$\mathbf{E}(\Lambda F) \equiv 0,$$

where the variational operator \mathbf{E} has the form

$$\mathbf{E} = \frac{\partial}{\partial x} + \sum_{k=1}^{\infty} (-1)^k D_{i_1} \cdots D_{i_k} \frac{\partial}{\partial x_{i_1 \dots i_k}}.$$

Using Noether’s theorem as well as the direct method one can obtain the following conservation laws.

1. For the symmetry $X_1 = \frac{\partial}{\partial t}$ one gets the conservation law of energy (20), where $h(x) = 0$.

The corresponding integrating factor is $\Lambda_1 = u_t$.

2. For $X_2 = \frac{\partial}{\partial s}$ one obtains the conservation of mass (21), where again $h(x) = 0$, and $\Lambda_2 = u_x$ (it was shown in Section 3.1 that the conservation law (21) in Euler coordinates can be directly derived from the commutation relation $x_{ts} = x_{st}$).

3. $X_3 = \frac{\partial}{\partial x}$:

$$\frac{d}{dt}(x_t) + D_s \left(\frac{1}{2} x_s^{-2} \right) = \frac{x_{ss}}{x_s^3} - x_{tt} = 0, \quad (32)$$

with $\Lambda_3 = 1$, which is conservation of momentum.

4. $X_4 = t \frac{\partial}{\partial x}$:

$$\frac{d}{dt}(tx_t - x) + D_s \left(\frac{1}{2} tx_s^{-2} \right) = t \left\{ \frac{x_{ss}}{x_s^3} - x_{tt} \right\} = 0, \quad (33)$$

with $\Lambda_4 = t$, which means law of the center of mass motion.

In Lagrangian mass coordinates the conservation laws (32) and (33) have the following forms

$$\frac{d}{dt}(u) + D_s \left(\frac{\rho^2}{2} \right) = 0,$$

and

$$\frac{d}{dt}(tu - x) + D_s \left(\frac{1}{2} t \rho^2 \right) = 0.$$

Finally, in Euler's coordinates the conservation laws (32) and (33) become

$$D_t^e(\rho u) + D_x \left(\frac{\rho^2}{2} + \rho u^2 \right) = 0,$$

and

$$D_t^e(\rho(tu - x)) + D_x \left(\frac{1}{2} t \rho^2 + \rho u(tu - x) \right) = 0.$$

Remark 3. Generators (30) in Lagrangian mass coordinates read

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, & Y_2 &= \frac{\partial}{\partial s}, & Y_3 &= \frac{\partial}{\partial x}, & Y_4 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ Y_5 &= 3t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - 2\rho \frac{\partial}{\partial \rho}, & Y_6 &= 3s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + 2\rho \frac{\partial}{\partial \rho}. \end{aligned}$$

Now we will consider an invariant difference schemes for shallow water equations in Lagrangian coordinate system. Here we change equation (17) with constant equals 1/2 into the same equation with constant equals 1 as it was used in group classification [32] (the same result can be obtained by means of stretching $x = \alpha \tilde{x}$, where $2\alpha^3 = 1$):

$$p = \rho^2, \quad (34)$$

and then equation (29) reads

$$x_{tt} - 2 \frac{x_{ss}}{x_s^3} = 0. \quad (35)$$

4. Invariant conservative difference schemes for the shallow water equations in Lagrangian coordinates

Let us consider invariant difference schemes on the following 9-point stencil

$$(\mathbf{t}, \mathbf{s}, \mathbf{x}) = (t, \check{t}, \hat{t}; s, s_-, s_+; x, x_+, x_-, \hat{x}, \hat{x}_+, \hat{x}_-, \check{x}, \check{x}_+, \check{x}_-) \quad (36)$$

or, alternatively,

$$(\mathbf{t}, \mathbf{s}, \mathbf{x}) = (t_n, t_{n-1}, t_{n+1}; s_m, s_{m-1}, s_{m+1}; x_m^n, x_{m+1}^n, x_m^{n+1}, x_m^{n+1}, x_{m+1}^{n+1}, x_{m-1}^{n+1}, x_m^{n-1}, x_{m+1}^{n-1}, x_{m-1}^{n-1}),$$

which is depicted in Fig. 1.

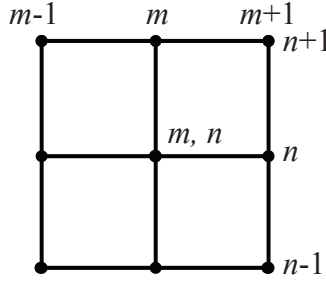


Figure 1: 9-point stencil

Here and further we will consider the simplest orthogonal regular mesh

$$h_-^s = h_+^s = h^s = \text{const}, \quad \tau_- = \tau_+ = \tau = \text{const}, \quad (37)$$

which is invariant under the whole set of the generators (30) (see criterion in [45]).

4.1. The construction of invariant difference scheme

Now we construct an invariant difference scheme for the flat bottom case (35).

One can easily state that the following difference expression admits the whole set of the generators (30):

$$\frac{\tau_-^{4/3}}{(h_-^s)^{1/3}} D_{-\tau}(x_t).$$

Hence, we are looking for an invariant scheme having the following divergent form

$$\frac{\tau_-^{4/3}}{(h_-^s)^{1/3}} \left(D_{-\tau}(x_t) + D_{-s}(\Phi) \right) = 0, \quad (38)$$

where $D_{\pm\tau}$ and $D_{\pm s}$ are the difference operators

$$D_{+\tau} = \frac{1 - S_{+\tau}}{t_{n+1} - t_n}, \quad D_{-\tau} = \frac{S_{-\tau} - 1}{t_n - t_{n-1}}, \quad D_{+s} = \frac{1 - S_{+s}}{s_{m+1} - s_m}, \quad D_{-s} = \frac{S_{-s} - 1}{s_m - s_{m-1}},$$

$S_{\pm\tau}$ and $S_{\pm s}$ are the difference shift operators that defined as follows

$$S_{\pm\tau}(f(t_n, s_m)) = f(t_n \pm \tau, s_m), \quad S_{\pm s}(f(t_n, s_m)) = f(t_n, s_m \pm h^s),$$

$x_t = \frac{D(x)}{+\tau}$ and $x_s = \frac{D(x)}{+s}$ are right difference derivatives with respect to t and s , Φ is a function, which approximates $D_s(1/x_s^2)$ to the first order by h and τ .

It is reasonable to consider Φ as a rational function

$$\Phi(\mathbf{x}) = 1/P_2(\mathbf{x}), \quad (39)$$

where $P_2 \neq 0$ — is a second-order polynomial in points $\mathbf{x} = \{x, x^+, \hat{x}, \hat{x}^+, \check{x}, \check{x}^+\}$.

Notice that being invariant to the actions of the generators X_1 – X_3 the function $\Phi(\mathbf{x})$ should depend on the differences:

$$\begin{aligned} t_{n+p+1} - t_{n+p} &= \tau_{n+p}, & s_{m+q+1} - s_{m+q} &= h_{m+q}^s, \\ x_{m+q}^{n+p+1} - x_{m+q}^{n+p}, & x_{m+q+1}^{n+p} - x_{m+q}^{n+p}, & p, q &\in \mathbb{Z}. \end{aligned}$$

Therefore, we consider the following form of the polynomial P_2 :

$$P_2(\mathbf{x}) = a_{11}x_s^2 + a_{22}\hat{x}_s^2 + a_{33}\check{x}_s^2 + 2a_{12}x_s\hat{x}_s + 2a_{13}x_s\check{x}_s + 2a_{23}\hat{x}_s\check{x}_s + b_1x_s + b_2\hat{x}_s + b_3\check{x}_s + c_1, \quad (40)$$

where a_{ij} , b_k and c_1 are some real constants, $\hat{x}_s = \frac{S(x_s)}{+\tau}$, and $\check{x}_s = \frac{S(x_s)}{-\tau}$.

Invariant scheme (38) must satisfy the infinitesimal criterions for the invariance, namely

$$X_k \left(\frac{\tau_-^{4/3}}{(h_-^s)^{1/3}} \left(\frac{D(x_t)}{-\tau} + \frac{D(\Phi)}{-s} \right) \right) \Big|_{(37)} = 0, \quad k = 1, \dots, 6.$$

After some standard algebraic simplifications one states that $b_1 = b_2 = b_3 = c_1 = 0$ and the constants a_{ij} are arbitrary. Actually, in order to approximate the original equation the following relation must hold

$$\sum_{i,j} a_{ij} = 1. \quad (41)$$

Expression $\frac{D(\Phi)}{-s}$ is divergent one, and it obviously possesses the integrating factors 1, t and any function $f(t)$. As an approximation for the integration factor x_t of the conservation law of energy (20), one can choose the following difference expression

$$\Lambda = \frac{x_t + \check{x}_t}{2}. \quad (42)$$

Following the difference analog of the direct method we demand

$$\mathcal{E} \left(\Lambda \left(\frac{D(x_t)}{-\tau} + \frac{D(\Phi)}{-s} \right) \right) \Big|_{(37)} = 0, \quad (43)$$

where

$$\mathcal{E} = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} S_{+\tau}^k S_{+s}^l \frac{\partial}{\partial x_{m-l}^{n-k}},$$

is the variational Euler operator (see [45]).

Substituting (40) and (42) into (43) and carrying out the summation of the resulting rational expressions, one gets the fraction

$$\mathcal{E} \left(\Lambda \left(\frac{D(x_t)}{-\tau} + \frac{D(\Phi)}{-s} \right) \right) \Big|_{(37)} = \frac{P_{54}}{P_{\dots}} = 0,$$

where P_{54} is a polynomial of degree 54 in terms of 20 points

$$x_{m+j}^{n+i}, \quad i = -2, \dots, 2, \quad j = -1, \dots, 2,$$

and by P_{\dots} we denote some nonzero polynomial defined on the same set of points. Then, considering the leading terms of the polynomial P_{54} , one sees that the following equation must be satisfied by the coefficients

$$a_{11}a_{22}a_{33}(a_{33} - a_{22}) = 0. \quad (44)$$

Let us choose the simplest solution $a_{11} = a_{22} = a_{33} = 0$. Substituting the solution into the polynomial P_{54} and considering the leading terms again, one gets

$$a_{13}a_{23}(a_{13} - a_{23}) = 0. \quad (45)$$

Putting $a_{13} = a_{23} = 0$, one derives the following form of the function sought

$$\Phi = \frac{1}{P_2} = (a_{23}\hat{x}_s\check{x}_s)^{-1},$$

which satisfies (43) for any value of the coefficient a_{23} . Taking (41) into account, one puts $a_{23} = 1$ and finally arrives at the following finite-difference scheme on regular orthogonal mesh:

$$\begin{aligned} F = x_{t\check{t}} + \frac{1}{h_-^s} ((\hat{x}_s\check{x}_s)^{-1} - (\hat{x}_{\bar{s}}\check{x}_{\bar{s}})^{-1}) &= 0, \\ \tau_+ = \tau_-, \quad h_+^s = h_-^s, \end{aligned} \quad (46)$$

where $x_{\bar{s}} = \frac{D}{-h}(x)$ and $x_{t\check{t}} = \frac{D}{-\tau}(x_t)$. The scheme (46) approximates equation (35) up to $O((h^s)^2 + \tau^2)$.

Remark 4. One can check that choosing another solutions of equation (44) (or equation (45)) finally leads to either the solution (46), or the trivial solution $P_2 = 0$ (which is prohibited by (39)).

Remark 5. One can rewrite (46) in the following “viscosity” form:

$$x_{t\check{t}} + \frac{1}{h_-^s} ((\hat{x}_s\check{x}_s)^{-1} - (\hat{x}_{\bar{s}}\check{x}_{\bar{s}})^{-1}) + \mu \frac{\hat{x}_{s\bar{s}} + \check{x}_{s\bar{s}}}{2} = 0, \quad (47)$$

where $|\mu| \ll (h^s)^2$ is the viscosity factor.

Remark 6. Apparently, (46) is not the only scheme that could be obtained by the procedure similar to performed above. For example, one may generalize the scheme (47), assuming

$$x_t = \frac{D}{+\tau}(w(x)), \quad x_s = \frac{D}{+s}(w(x)),$$

where w is a function. Substituting into (47) and performing series expansion, one gets

$$w'(x_{t\check{t}} - 2x_s^{-3}x_{s\check{s}}) + w''(x_t^2 - 2x_s^2) + O(\tau^2 + h^2).$$

Then, the scheme can be represented in the following generalized form

$$w(x)_{t\check{t}} + \frac{1}{h_-^s} ((w(\hat{x})_s w(\check{x})_s)^{-1} - (w(\hat{x})_{\bar{s}} w(\check{x})_{\bar{s}})^{-1}) + \mu \frac{w(\hat{x})_{s\bar{s}} + w(\check{x})_{s\bar{s}}}{2} = 0,$$

where

$$w(z) = \nu\varphi(z) + z + c,$$

and c is constant, φ is an arbitrary function, and ν is a coefficient of order $O((h^s)^2 + \tau^2)$.

4.2. Invariant representation of the scheme (46)

Consider invariants of the symmetries (30) in the space (36). There are $15 - 6 = 9$ difference invariants:

$$\begin{aligned} I_1 &= \frac{h_+^s}{h_-^s}, & I_2 &= \frac{\tau_+}{\tau_-}, \\ I_3 &= \frac{x-x_-}{\omega}, & I_4 &= \frac{x_+-x}{\omega}, & I_5 &= \frac{\check{x}+-\check{x}}{\omega}, & I_6 &= \frac{\check{x}-\check{x}_-}{\omega}, \\ I_7 &= \frac{\tau_+(x-\check{x})+\tau_-(x-\hat{x})}{\omega\tau_-}, & I_8 &= \frac{\tau_+(x-\check{x})+\tau_-(x-\hat{x}_+)}{\omega\tau_-}, \\ I_9 &= \frac{\tau_+(x-\check{x})+\tau_-(x-\hat{x}_-)}{\omega\tau_-}, \end{aligned} \quad (48)$$

where

$$\omega^3 = \tau_-^2 h_-^s, \quad h_+^s = s_+ - s, \quad h_-^s = s - s_-.$$

The scheme (46) is invariant to the actions of the whole 6-parametric group (30):

$$X_k F|_{(46)} = 0, \quad k = 1, \dots, 6.$$

Thus, the scheme (46) can be represented in terms of the invariants (48), namely,

$$\begin{aligned} \frac{(\tau_-)^{4/3}}{(h_-^s)^{1/3}} \left(x_{t\check{t}} + D_{-s}((\hat{x}_s \check{x}_s)^{-1}) \right) &= \frac{I_2 - I_6 I_7 (I_7 - I_9)}{I_2 I_6 (I_7 - I_9)} + \frac{(I_1)^2}{I_5 (I_7 - I_8)} = 0, \\ I_1 &= 1, \quad I_2 = 1. \end{aligned}$$

Remark 7. Invariant form of the “viscosity” scheme (47) is the following

$$\begin{aligned} \frac{I_2 - I_6 I_7 (I_7 - I_9)}{I_2 I_6 (I_7 - I_9)} + \frac{(I_1)^2}{I_5 (I_7 - I_9)} + \frac{\mu \alpha^2}{2} \left(\frac{I_5 + I_7 - I_9}{I_1} + I_7 - I_9 - I_6 \right) &= 0, \\ I_1 &= 1, \quad I_2 = 1, \end{aligned}$$

where $\alpha = \tau_-/h_-^s$ (on regular mesh, one can consider α as constant).

4.3. Conservation laws of the scheme (46)

It was stated in Section 4.1 that the scheme (46) has the following set of integration multipliers:

$$\Lambda_1 = 1, \quad \Lambda_2 = t, \quad \Lambda_3 = \frac{x_t + \check{x}_t}{2}, \quad (49)$$

where the last multiplier was given by the formula (42). The multipliers (49) allow one to write the following conservation laws of the scheme (46) in Lagrangian coordinates.

1. The continuity equation

$$D_{-\tau}(\hat{x}_s) - D_{-s}(x_t^+) = 0, \quad (50)$$

which is just the difference analogue of the condition (15).

2. Conservation of momentum:

$$\Lambda_1 = 1, \quad D_{-\tau}(x_t) + D_{-s}((\hat{x}_s \check{x}_s)^{-1}) = 0, \quad (51)$$

which corresponds to (32).

3. Center of mass conservation:

$$\Lambda_2 = t, \quad D_{-\tau}(tx_t - x) + D_{-s}(t(\hat{x}_s \check{x}_s)^{-1}) = 0, \quad (52)$$

which corresponds to (33).

4. Conservation of energy:

$$\Lambda_3 = \frac{x_t + \check{x}_t}{2}, \quad \frac{1}{2} D_{-\tau}(x_t^2 + x_s^{-1} + \hat{x}_s^{-1}) + \frac{1}{2} D_{-s}((x_t^+ + \check{x}_t^+)(\hat{x}_s \check{x}_s)^{-1}) = 0, \quad (53)$$

which corresponds to (20).

Remark 8. Conservation laws for the scheme (47) can be written as follows.

1. Conservation of momentum:

$$D_{-\tau}(x_t) + D_{-s}\left((\hat{x}_s \check{x}_s)^{-1} + \mu \frac{\hat{x}_s + \check{x}_s}{2}\right) = 0. \quad (54)$$

2. Center of mass conservation:

$$D_{-\tau}(tx_t - x) + D_{-s}\left(t(\hat{x}_s \check{x}_s)^{-1} + t\mu \frac{\hat{x}_s + \check{x}_s}{2}\right) = 0. \quad (55)$$

3. Conservation of energy:

$$\begin{aligned} & \frac{1}{2} D_{-\tau}(x_t^2 + x_s^{-1} + \hat{x}_s^{-1} + \mu\tau (xx_+ + \hat{x}\hat{x}_+ - 2(x^2 + \hat{x}^2))) \\ & + \frac{1}{2} D_{-s}((x_t^+ + \check{x}_t^+)(\hat{x}_s \check{x}_s)^{-1} + \mu h^s(\hat{x}\check{x}_+ - \check{x}\hat{x}_+)) = 0. \end{aligned}$$

4.4. Schemes and their conservation laws in Lagrangian mass coordinates

In the present section, through difference transformations of type (14), we construct two invariant conservative difference schemes in Lagrangian mass coordinates, which correspond to the scheme (46). Using naive difference transformations of type (14), one can derive a scheme on three time layers, which is the first scheme presented here. The second scheme is defined on two time layers, and some more sophisticated transformations are required in that case.

4.4.1. Three-level scheme

Here we use the following difference analog of the transformation (14)

$$x_t = u, \quad x_s = \rho^{-1}. \quad (56)$$

The scheme (46) becomes

$$\begin{aligned} & \left(\frac{1}{\rho}\right)_t = u_s, \\ & D_{-\tau}(u) + D_{-s}(\hat{\rho}\check{\rho}) = 0, \\ & h_+^s = h_-^s, \quad \tau_+ = \tau_-. \end{aligned} \quad (57)$$

The corresponding difference template is shown in the Figure 2

We remind that in the case of the shallow water equations the physical meaning of variable ρ is thickness of a water column above the bottom.

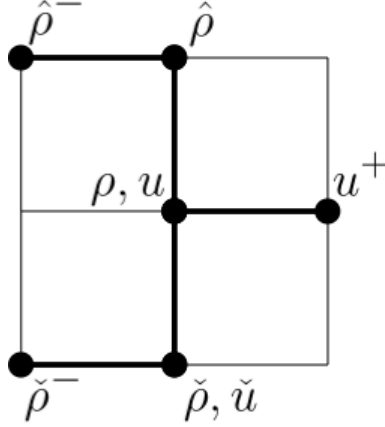


Figure 2: Three-level scheme difference template

Using the following approximation of “state” equation (34)

$$p = \rho^\gamma = \rho^2, \quad (58)$$

one can represent the scheme (57) in the form

$$\begin{aligned} \left(\frac{1}{\rho}\right)_t &= u_s, \\ D_{-\tau}(u) + D_{-s}(\sqrt{\hat{p}\check{p}}) &= 0, \\ h_+^s &= h_-^s, \quad \tau_+ = \tau_-. \end{aligned}$$

The conservation laws of the latter scheme are the following.

1. Conservation of mass:

$$\left(\frac{1}{\rho}\right)_t = u_s,$$

2. The conservation law of momentum:

$$\Lambda = 1, \quad D_{-\tau}(u) + D_{-s}(\hat{\rho}\check{\rho}) = D_{-\tau}(u) + D_{-s}(\sqrt{\hat{p}\check{p}}) = 0.$$

3. Center of mass conservation:

$$\Lambda = t, \quad D_{-\tau}(tu - x) + D_{-s}(t\hat{\rho}\check{\rho}) = D_{-\tau}(tu - x) + D_{-s}(t\sqrt{\hat{p}\check{p}}) = 0.$$

4. The conservation law of energy:

$$\Lambda = \frac{u + \check{u}}{2}, \quad \frac{1}{2}D_{-\tau}\left(u^2 + \frac{p}{\rho} + \frac{\hat{p}}{\check{\rho}}\right) + \frac{1}{2}D_{-s}\left((u^+ + \check{u}^+)\sqrt{\hat{p}\check{p}}\right) = 0.$$

4.4.2. Two-level scheme

It is of our interest to construct two-level difference schemes, as far one can perform numerical calculations much easier with the help of such a scheme. In the space of finite-differences one has a freedom to choose approximations of the relations (14) and of the “state equation” (34).

Here, instead of (56), we use the following transformation

$$\tilde{x}_s + x_s = \frac{2}{\tilde{\rho}}, \quad x_t = u, \quad (59)$$

and the following implicit approximation of “state” equation (34)

$$\frac{1}{\sqrt{\tilde{p}}} + \frac{1}{\sqrt{p}} = \frac{2}{\tilde{\rho}} \Leftrightarrow x_s = \frac{1}{\sqrt{p}}. \quad (60)$$

Then, the scheme (57) becomes

$$\begin{aligned} D_{-\tau} \left(\frac{1}{\rho} \right) - D_{-s} \left(\frac{u^+ + \tilde{u}^+}{2} \right) &= 0, \\ D_{-\tau}(u) + D_{-s}(Q) &= 0, \\ h_+^s &= h_-^s, \quad \tau_+ = \tau_-, \end{aligned} \quad (61)$$

where

$$\frac{1}{Q} = \frac{4}{\rho\tilde{\rho}} - \frac{2}{\sqrt{p}} \left(\frac{1}{\rho} + \frac{1}{\tilde{\rho}} \right) + \frac{1}{p}. \quad (62)$$

Here the scheme and the “state” equation are written in the points of *two* difference time layers. The corresponding difference template is shown on the Figure 3. The scheme (61) approximates equation (35) up to $O(h + \tau)$.

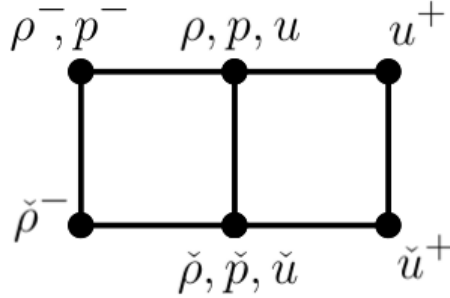


Figure 3: Two-level scheme difference template

The conservation laws of the scheme (61) are the following.

1. Conservation of mass²:

$$D_{-\tau} \left(\frac{1}{\rho} \right) - D_{-s} \left(\frac{u^+ + \tilde{u}^+}{2} \right) = 0.$$

2. The conservation law of momentum:

$$\Lambda = 1, \quad D_{-\tau}(u) + D_{-s}(Q) = 0.$$

3. Center of mass conservation:

$$\Lambda = t, \quad D_{-\tau}(tu - x) + D_{-s}(tQ) = 0.$$

²To get this conservation law one should instead of (50) get the sum of (50) and the shifted one, i.e.,

$$D_{-\tau}(\hat{x}_s + x_s) - D_{-s}(x_t^+ + \tilde{x}_t^+) = 0.$$

4. The conservation law of energy:

$$\Lambda = \frac{u + \check{u}}{2}, \quad D_{-\tau} \left(\frac{u^2}{2} + \frac{p}{2\sqrt{p} - \rho} \right) + D_{-s} \left(\frac{u^+ + \check{u}^+}{2} Q \right) = 0.$$

Remark 9. Here “viscosity” term of the scheme (47) becomes:

$$\mu \frac{\hat{x}_{s\bar{s}} + \check{x}_{s\bar{s}}}{2} = \frac{\mu}{h^s} \left(\frac{1}{\rho} + \frac{1}{\check{\rho}} - \frac{1}{\sqrt{p}} - \frac{1}{\rho^-} - \frac{1}{\check{\rho}^-} + \frac{1}{\sqrt{p^-}} \right).$$

Apparently, the physical meaning of this term is lost in Lagrangian mass coordinates.

4.5. Some invariant solutions of the scheme

In the present section we consider some invariant solutions of the scheme (46). It seems that in Lagrangian coordinates the scheme has a wider class of invariant solutions than in Eulerian representation (in particular there is no travelling wave solutions in Eulerian coordinates).

4.5.1. Example 1. Travelling wave solution

Consider travelling wave solutions

$$x = \psi(\lambda), \quad \lambda = s - \alpha t,$$

which correspond to the subalgebra $X_{1,5} = X_1 + \alpha X_5 = \partial_t + \alpha \partial_s$. Then, equation (29) becomes

$$\psi''(\alpha^2(\psi')^3 - 2) = 0.$$

Further we consider the simplest form of travelling wave solution that is

$$x = \lambda = s - \alpha t. \tag{63}$$

Constructing the scheme on the subgroup corresponding to the generator $X_{1,5}$, one should mention that the difference mesh spacing $\Delta\lambda$ along the λ -axis should be matched with the original mesh spacings h and τ [45]. This is achieved under conditions

$$\alpha = \frac{h}{\tau}, \quad \Delta\lambda = h.$$

Then, scheme (46) reduces to the following

$$\alpha^2 \psi_{\lambda\bar{\lambda}} + \left[S_{+\lambda} \left(\frac{1}{\psi_\lambda} \right) S_{-\lambda} \left(\frac{1}{\psi_\lambda} \right) \right]_{\bar{\lambda}} = 0,$$

$$\lambda_+ - \lambda = \lambda - \lambda_- = \Delta\lambda,$$

where $\psi = \psi(\lambda)$. The latter scheme possesses invariant solution (63),

$$x = \psi(\lambda) = \lambda,$$

which becomes trivial in Eulerian coordinates:

$$u = -\alpha, \quad \rho = 1.$$

4.5.2. Example 2. Nonuniform dilation in the (x, s) -space

Consider subalgebra $X_6 = 3s\partial_s + x\partial_x$ and the corresponding invariant solution of equation (29):

$$x = s^{1/3}\psi, \quad u = s^{1/2}\psi', \quad \rho = 3s^{2/3}/\psi, \quad (64)$$

where $\psi = \psi(t)$.

Equation (29) reduces to the form

$$\psi^2\psi'' + 12 = 0. \quad (65)$$

One can find the following particular exact solution of the latter equation:

$$x = s^{1/3}\psi(t) = (54st^2)^{1/3}. \quad (66)$$

The generator X_6 holds orthogonality of the mesh. Therefore, further we consider the scheme on an orthogonal mesh without requiring it to be a uniform one.

Then, the first equation of the scheme (29) possesses the form

$$\hat{\psi}\check{\psi}\psi_{t\hat{t}} + \frac{1}{s^{1/3}(s-s_-)} \left(\left(\frac{s_+ - s}{s_+^{1/3} - s^{1/3}} \right)^2 - \left(\frac{s_- - s}{s_-^{1/3} - s^{1/3}} \right)^2 \right) = 0, \quad (67)$$

and we do not set the mesh equations for now.

According to (64), one can split scheme (67) into two separate equations:

$$\hat{\psi}\check{\psi}\psi_{t\hat{t}} = K = \text{const}, \quad (68)$$

and

$$\frac{1}{s^{1/3}(s-s_-)} \left(\left(\frac{s_+ - s}{s_+^{1/3} - s^{1/3}} \right)^2 - \left(\frac{s_- - s}{s_-^{1/3} - s^{1/3}} \right)^2 \right) = -K = \text{const}. \quad (69)$$

By virtue of (65), one can set $K = \delta - 12$, where $0 < |\delta| \ll 1$ is a constant related to the mesh density.

First, consider equation (68) on a uniform lattice. One can obtain its first integral

$$\psi_t^2 + K \left(\frac{1}{\hat{\psi}} + \frac{1}{\check{\psi}} \right) = C_1 = \text{const}, \quad (70)$$

which corresponds to the integrating factor $(\hat{\psi} - \check{\psi})/(\hat{\psi}\check{\psi})$. Then, one can with the following implicit solution of the Cauchy problem

$$(\psi(t_n) - \psi_0)^2 + t_n^2 K \left(\frac{1}{\psi(t_n)} + \frac{1}{\psi_0} \right) - t_n^2 C_1 = 0, \quad (71)$$

$$t_n = n\tau, \quad n \in \mathbb{Z},$$

where $\psi_0 = \psi(0)$ is the value of the function at the initial time.

Remark 10. Notice that in some specific cases equations of type (68) can be linearized [65].

Next, consider equation (69). One can find a particular (invariant) solution of (69). Consider an invariant grid

$$\frac{s_+}{s} = \frac{s}{s_-}, \quad (72)$$

which possesses the first integral

$$s_+ = \kappa^3 s, \quad \kappa = \text{const}, \quad (73)$$

One can integrate the latter equation and get

$$s_m = \kappa^{3m} s_0, \quad m \in \mathbb{Z}, \quad (74)$$

where s_0 is an initial constant value.

It is easy to verify that equation (74) becomes a particular solution of (69) under the following condition

$$(\kappa^2 + \kappa + 1)(\kappa^2 + 1)(\kappa + 1) = (12 - \delta)\kappa^4. \quad (75)$$

The latter equation has (assuming δ is small enough) three real positive roots. Evidently, for $\delta \rightarrow 0$ one or more of the solutions κ have values close to 1.

In Lagrangian mass coordinates, functions ρ and u are added to the solution. In the simplest case ($u = x_t$, $\rho = 1/x_s$) they are

$$x = s^{1/3}\psi, \quad u = s^{1/3}\psi_t, \quad \rho = \frac{s_+ - s}{s_+^{1/3} - s^{1/3}} \frac{1}{\psi}. \quad (76)$$

On lattice equation (73) the latter solution becomes

$$x = s^{1/3}\psi, \quad u = s^{1/3}\psi_t, \quad \rho = (\kappa^2 + \kappa + 1) \frac{s^{2/3}}{\psi}. \quad (77)$$

It corresponds to the solution of the original differential equation for $\kappa \rightarrow 1$.

Particular exact solution

Now, consider the scheme on a nonuniform mesh and rewrite equation (68) in the following expanded form:

$$\frac{\psi(\hat{t})\psi(\check{t})}{t - \check{t}} \left[\frac{\psi(\hat{t}) - \psi(t)}{\hat{t} - t} - \frac{\psi(t) - \psi(\check{t})}{t - \check{t}} \right] = \delta - 12. \quad (68)$$

Substituting invariant solution

$$\hat{t} = \mu^3 t. \quad (78)$$

into (68), one gets

$$\frac{\mu^3 \psi(\mu^3 t) \psi(t/\mu^3)}{(\mu^3 - 1)^2 t^2} [\psi(\mu^3 t) - (1 + \mu^3) \psi(t) + \psi(t/\mu^3)] = \delta - 12. \quad (79)$$

Let us seek for an invariant solution of the form

$$\psi(z) = Bz^q, \quad B, q = \text{const.}$$

Analyzing the left side of equation (79), one concludes that $q = 2/3$, i. e.,

$$\psi(z) = Bz^{2/3}, \quad B = \text{const}, \quad (80)$$

which corresponds to the solution (66). Therefore, in order to obtain an exact solution, one set

$$B = 54^{1/3}.$$

Substituting into (79), one gets the following mesh constraint

$$54\mu^3(\mu + 1) = (12 - \delta)(\mu^2 + \mu + 1)^2.$$

Comparing with (75), one obtains a relation connecting the parameters that specify the grid densities with respect to t and s :

$$\frac{(\kappa^2 + \kappa + 1)(\kappa^2 + 1)(\kappa + 1)}{\kappa^4} = \frac{54\mu^3(\mu + 1)}{(\mu^2 + \mu + 1)^2}. \quad (81)$$

Hence, one gets the following particular *exact* solution:

$$\begin{aligned} x_m^n &= (54s_m t_n^2)^{1/3}, \\ t_n &= \mu^{3n} t_0, \quad s_m = \kappa^{3m} s_0, \quad m, n \in \mathbb{Z}, \\ \frac{(\kappa^2 + \kappa + 1)(\kappa^2 + 1)(\kappa + 1)}{\kappa^4} &= \frac{54\mu^3(\mu + 1)}{(\mu^2 + \mu + 1)^2}, \end{aligned} \quad (82)$$

where t_0 and s_0 are the initial data of the Cauchy problem.

Solution (77) for the functions ρ and u is

$$u_m^n = \frac{1 - \mu^2}{1 - \mu^3} \left(\frac{54 s_m}{t_n} \right)^{1/3}, \quad \rho_m^n = \frac{\kappa^2 + \kappa + 1}{54^{1/3}} \left(\frac{s_m}{t_n} \right)^{2/3}. \quad (83)$$

4.6. Schemes for an arbitrary bottom case

In this case equation (35) has the form

$$x_{tt} - 2 \frac{x_{ss}}{x_s^3} - H'(x) = 0, \quad (84)$$

where $H(x)$ is a bottom profile function.

4.6.1. Case $H'(x) \neq 0$.

In this general case, equation (84) is not a divergent one and its invariance strongly depends on the specific type of the function $H(x)$. In case $H(x)$ is arbitrary, equation (84) only admits two generators,

$$X_1 = \frac{\partial}{\partial t} \quad \text{and} \quad X_2 = \frac{\partial}{\partial s}.$$

In finite-difference case there is no obvious way to approximate the derivative $H'(x)$. One can consider separately two different representations of this derivative, namely

$$H'(x) = \frac{1}{x_t} (H'(x) x_t) = \frac{1}{x_t} D_t(H(x)) \quad (85)$$

and

$$H'(x) = \frac{1}{x_s} (H'(x) x_s) = \frac{1}{x_s} D_s(H(x)). \quad (86)$$

According to the representation (85), scheme (46) can be generalized as follows

$$\begin{aligned} x_{t\check{t}} + \frac{D}{-s} ((\hat{x}_s \check{x}_s)^{-1}) - \frac{1}{x_t + \check{x}_t} \left(\frac{D}{-\tau} H(x) + \frac{D}{+\tau} H(x) \right) &= 0, \\ \tau_+ = \tau_-, \quad h_+^s &= h_-^s, \end{aligned} \quad (87)$$

where

$$\frac{\frac{D}{-\tau} H(x) + \frac{D}{+\tau} H(x)}{x_t + \check{x}_t} = \frac{2}{x_t + \check{x}_t} \left[\frac{\frac{D}{-\tau} H(x) + \frac{D}{+\tau} H(x)}{2} \right] \sim \frac{1}{x_t'} [H_x' x_t'] \sim H'(x).$$

Scheme (87) only has the conservation law of mass (50) and the conservation law of energy:

$$\frac{D}{-\tau} (x_t^2 + x_s^{-1} + \hat{x}_s^{-1} - H(x) - H(\hat{x})) + \frac{D}{-s} ((x_t^+ + \check{x}_t^+)(\hat{x}_s \check{x}_s)^{-1}) = 0.$$

Using (59), (60), one obtains the following representation of the latter conservation law in Lagrangian mass coordinates

$$D_{-\tau} \left(\frac{u^2}{2} + \frac{p}{2\sqrt{p}-\rho} - \frac{H(x) + H(x+\tau u)}{2} \right) + D_{-s} \left(\frac{u^+ + \tilde{u}^+}{2} Q \right) = 0,$$

where Q is given by the formula (62).

Using scheme (46), one can construct new invariant schemes corresponding to the representation (86) of $H'(x)$. As an example we consider the following one

$$\begin{aligned} \frac{1}{2} \left((\hat{x}_s \check{x}_s)^{-\frac{1}{2}} + (\hat{x}_{\bar{s}} \check{x}_{\bar{s}})^{-\frac{1}{2}} \right) \left[\frac{x_s + x_{\bar{s}}}{2} x_{t\bar{t}} - D_{-s} H(x) \right] + D_{-s} ((\hat{x}_s \check{x}_s)^{-1}) &= 0, \\ \tau_+ = \tau_-, \quad h_+^s = h_-^s. \end{aligned} \quad (88)$$

Scheme (88) possesses the conservation law of mass (50) and the conservation law of momentum (which corresponds to the generator X_2)

$$D_{-\tau} \left(x_t \frac{\hat{x}_s + \hat{x}_{\bar{s}}}{2} \right) + D_{-s} \left(2(\hat{x}_s \check{x}_s)^{-\frac{1}{2}} - \frac{x_t x_t^+}{2} - H(x) \right) = 0$$

with integrating factor

$$\frac{2(\hat{x}_s \check{x}_s \hat{x}_{\bar{s}} \check{x}_{\bar{s}})^{\frac{1}{2}}}{(\hat{x}_s \check{x}_s)^{\frac{1}{2}} + (\hat{x}_{\bar{s}} \check{x}_{\bar{s}})^{\frac{1}{2}}}.$$

Using (59), (60), one obtains the following representation of the latter conservation law in Lagrangian mass coordinates

$$D_{-\tau} \left(u \left(\frac{1}{\rho} + \frac{1}{\rho^-} - \frac{1}{2\sqrt{p}} - \frac{1}{2\sqrt{p^-}} \right) \right) + D_{-s} \left(2\sqrt{Q} - \frac{uu^+}{2} - H(x) \right) = 0,$$

where Q is given by the formula (62).

4.6.2. Case $H'(x) = \text{const}$ (linear bottom).

Here the function H is $H(x) = C_1 x + C_2$, and the scheme (87) becomes

$$\begin{aligned} x_{t\bar{t}} + \frac{1}{h_-^s} ((\hat{x}_s \check{x}_s)^{-1} - (\hat{x}_{\bar{s}} \check{x}_{\bar{s}})^{-1}) - C_1 &= 0, \\ \tau_+ = \tau_-, \quad h_+^s = h_-^s. \end{aligned}$$

In case $C_1 = 0$ (a flat bottom), all the results for the latter scheme coincide with the results of Section 4.2.

In case $C_1 \neq 0$, the scheme admits the whole 6-parametric group of transformations (30). Using the difference analogue

$$x = \tilde{x} + \frac{C_1}{2} t\hat{t}$$

of transformation (28), one arrives at the flat bottom case, which was described above.

5. Numerical results

For comparison, we consider several schemes in Lagrangian mass coordinates.

- (a) Invariant scheme (61),(60), which was constructed in the previous sections.
- (b) An explicit difference scheme, which is a simple difference approximation of the shallow water equations system.
- (c) Samarskiy and Popov's completely conservative scheme for the gas dynamics equations [52], adapted for the shallow water equations.
- (d) The completely conservative Yelenin–Krylov scheme for the two-layer shallow water equations [11], adapted for the single-layer case.
- (e) A completely conservative scheme proposed in the paper [60] by V. A. Korobitsyn.

All the schemes are considered on uniform orthogonal meshes in case of the flat bottom ($h(x) = 0$) only. Also, unless specifically indicated, it is assumed that $x_t = u$. Consider schemes (b)–(e) in detail.

(b) *An explicit scheme.* Consider the following explicit scheme, which approximates the equations (12), (13) and can be written in the divergent form as follows.

$$\begin{aligned} \frac{1}{\tau} \left(\frac{1}{\hat{\rho}} - \frac{1}{\rho} \right) &= \frac{u^+ - u}{h}, \\ \frac{\hat{u} - u}{\tau} + \frac{1}{h} (\rho \hat{\rho} - \rho^- \hat{\rho}^-) &= 0. \end{aligned} \tag{89}$$

These equations are the conservation laws of mass and momentum:

$$\begin{aligned} D_{+\tau} \left(\frac{1}{\rho} \right) - D_{+s}(u) &= 0, \\ D_{+\tau}(u) + D_{+s}(\rho^- \hat{\rho}^-) &= 0. \end{aligned}$$

The center of mass conservation law is

$$D_{+\tau}(\check{t}u - x) + D_{+s}(t\rho\hat{\rho}) = 0.$$

The energy conservation law for the scheme (89) does not hold:

$$\begin{aligned} D_{+\tau} \left(\frac{u^2}{2} + \rho \right) + D_{+s}(u\rho^- \hat{\rho}^-) &= \\ = u \left(D_{+\tau}(u) + D_{+s}(\rho^- \hat{\rho}^-) \right) - \rho \hat{\rho} \left(D_{+\tau} \left(\frac{1}{\rho} \right) - D_{+s}(u) \right) + \frac{1}{2} u_t^2 \tau. \end{aligned}$$

(c) *The Samarskiy–Popov scheme.* An implicit completely conservative schemes by Samarskiy–Popov [52] for the one-dimensional gas dynamics equations can be modified for the case of the shallow water as follows.

$$\begin{aligned} u_t + \hat{p}_{\bar{s}} &= 0, & \frac{1}{\rho} &= x_s, \\ x_t &= \frac{\hat{u} + u}{2}, & p &= A\rho^2. \end{aligned} \tag{90}$$

Scheme (90) is invariant.

The original Samarskiy–Popov scheme for the one-dimensional gas dynamics equations also includes a conservation law of energy:

$$D_{+\tau} \left(\varepsilon + \frac{u_+^2}{2} \right) + D_{+s} \left(\frac{(u+\hat{u})(p+\hat{p})}{4} \right) = 0, \quad (91)$$

where the internal energy of the medium ε is determined by the equation

$$\varepsilon_t = -\hat{p} \left(\frac{1}{\rho} \right)_t, \quad (92)$$

relating the change in internal energy with the work of pressure forces.

In the case of shallow water, in contrast to the original scheme for the gas dynamics equations, the energy equation (91) is no longer a part of the scheme and it does not directly follow from (90). One can rewrite equation (91), taking into account (92), as follows:

$$\frac{\hat{u}_+ + u_+}{2} (u_t^+ + \hat{p}_s) - \hat{p} \left(\left(\frac{1}{\rho} \right)_t - \frac{1}{2} (\hat{u} + u)_s \right) - \frac{1}{4} ((u + \hat{u})p_t)_s \tau = 0. \quad (93)$$

Then, one sees that equation (91) does not hold on the solutions of system (90), and, strictly speaking, it is not a conservation law of the system. Nevertheless, we are going to use equations (91) and (92) of the original Samarskiy–Popov scheme in numerical computations in order to control the conservation law of energy on the solutions of the system.

(d) *The Yelenin–Krylov scheme.* This is a completely conservative scheme for a system of equations of the two-layer shallow water obtained in [11]. It can be reduced to the system of equations of single-layer shallow water by assuming all the parameters of the both layers to be equal. Then, after some simplifications, one arrives at the following scheme:

$$\begin{aligned} D_{+\tau} \left(\frac{1}{\rho} \right) - D_{+s} (u^{(0.5)}) &= 0, \\ D_{+\tau} (u) + D_{+s} \left(S_{-s} (P^{(0.5)} + (\rho^{-1})^{(0.5)} (P')^{(0.5)}) \right) &= 0, \end{aligned} \quad (94)$$

where

$$f^{(0.5)} = \frac{\hat{f} + f}{2} \quad \text{and} \quad P' = \frac{\hat{P} - P}{\hat{\rho}^{-1} - \rho^{-1}},$$

the function P is given by a state equation and in the case of the single-layer shallow water it should be equal to ρ^2 in the continuous limit.

The scheme possesses the conservation law of energy

$$D_{+\tau} \left(\frac{u^2 + (u^+)^2}{4} + \frac{P}{\rho} \right) + D_{+s} \left(u^{(0.5)} \left(\frac{F + F^-}{2} \right)^{(0.5)} \right) = 0,$$

where

$$F^{(0.5)} = P^{(0.5)} - (\rho^{-1})^{(0.5)} (P')^{(0.5)}.$$

In the given form, scheme (94) is a three-level one. In order to write it on two time levels, one sets the state equation that defines the function P as follows:

$$\hat{P} - P = 2(\hat{\rho}^{-1} - \rho^{-1})\rho^3. \quad (95)$$

The latter expression corresponds to the differential

$$\frac{dG}{d\rho} = \frac{d}{d\rho}(\rho^2) = 2\rho.$$

Taking into account equation (95), one can write a modified Yelenin–Krylov scheme as follows

$$\begin{aligned}
D_{+\tau} \left(\frac{1}{\rho} \right) - D_{+s} (u^{(0.5)}) &= 0, \\
D_{+\tau} (u) + D_{-s} (P^{(0.5)} + (\rho^{-1})^{(0.5)} (2\rho^3)^{(0.5)}) &= 0, \\
\hat{P} - P &= 2(\hat{\rho}^{-1} - \rho^{-1})\rho^3, \\
D_{+\tau} \left(\frac{1}{4}(u^2 + (u^+)^2) + \frac{1}{\rho}P \right) + \\
+ \frac{1}{2} D_{+s} \left(h^s u^{(0.5)} D_{-s} (P^{(0.5)} + (\rho^{-1})^{(0.5)} (2\rho^3)^{(0.5)}) \right) &= 0.
\end{aligned} \tag{96}$$

(e) *The Korobitsyn scheme.* This scheme was proposed in the paper [60], and it is a modification of the completely conservative Samarskiy–Popov scheme [52]. For the shallow water equations, according to (34), one can rewrite the scheme in the following form

$$\begin{aligned}
u_t + p_{\bar{s}} &= 0, \\
x_t &= 0.5(u + \hat{u}), \\
\left(\frac{1}{\rho} \right)_t &= 0.5(u + \hat{u})_{\bar{s}}, \\
\hat{p} &= (0.5q(\rho_- + \rho) + (1 - q)\rho) (\rho + 0.25q\tau^2 u_t^2),
\end{aligned} \tag{97}$$

where $0 \leq q \leq 1$ is a parameter. Setting $q = 0$, one gets one of the Samarskiy–Popov schemes [66]. If $q = 1$, scheme (97) becomes “thermodynamically consistent” (see. [60]), i. e., it possesses additional conservation laws if the one-dimensional flow of polytropic gas with an adiabatic exponent $\gamma = 3$ is considered. In the shallow water case, the condition of thermodynamic consistency does not have of clear physical meaning, but further we put $q = 1$, as it was done in [60]. In this case, the scheme can be represented in an explicit form. Using the relations given in [60], one can also derive the following expression for the conservation law of energy

$$\begin{aligned}
\left(\frac{u^2 + u_{\pm}^2}{2} \right)_t + \left(\frac{\hat{u} + u}{8} \left[(\rho + \rho_-)(\rho + 0.25q\tau^2 u_t^2) \right. \right. \\
\left. \left. + (\rho_- + \rho_{--})(\rho_- + 0.25q\tau^2 (u_t^-)^2) \right] \right)_s &= 0.
\end{aligned}$$

In order to reduce the schemes’ dispersion, we use an artificial linear-quadratic viscosity [52] of the form

$$\omega = -\nu \rho u_s + 0.5(1 + \gamma) \rho \frac{\kappa h^s}{\pi} u_s^2,$$

where ν is a linear viscosity coefficient, κ is a quadratic viscosity coefficient, and γ — adiabatic exponent, which equals 2 in the case of shallow water. The use of artificial viscosity means that in all the calculations the value of pressure p near strong discontinuities of solutions is replaced by

$$q = p + \omega.$$

Further, four test problems are considered (detailed description of three of them one can find in [52]). In all the tasks the mesh space steps are given as $h^s = 0.02$ and $\tau = 0.025h^s$. Viscosity coefficients ν and κ were chosen empirically so that solutions of the schemes to be close to the known exact solutions. For the modified Yelenin–Krylov scheme (96) artificial viscosity is not used, as it turned out, in this case it markedly worsens the results. For the explicit scheme we put $\nu = 0.005$, and, for scheme (97) and for the modified Samarskiy–Popov scheme (90), we put $\nu = 0.001$. In all cases, except for the scheme (96), the coefficient

of quadratic viscosity is $\kappa = 4.5$. The total substance mass S for the first three tests is taken $S = 3$, and for the fourth test $S = 4$. The initial height of the fluid column ρ_0 in all cases is 1, and the pressure p_0 is calculated by the equations of state. Calculations for all the implicit schemes were carried out by iterative methods with the accuracy requirement $\varepsilon = 0.001$.

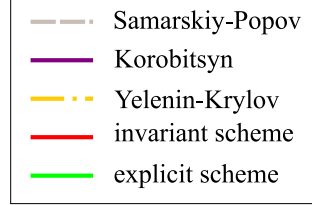


Figure 4: Legend for Fig. 5–8

Descriptions of the test problems and the numerical results are presented below. Notice that in Figures 5–8 the legend which is depicted in Figure 4 is used.

- *Test 1. Rarefaction wave.* This is the problem of a withdrawing piston, which, under the action of external forces, moves outside of the medium producing a rarefaction wave. The numerical results, as well as the values of the conservation laws on solutions, for all the schemes at the moment $t = 0.55$ are shown in the Figure 5. The height ρ of the fluid column above the flat bottom at the chosen moment of time is shown in Figure 5 a). The values of the conservation laws of energy, mass and momentum on the solution Figure 5 a) are depicted in Figures 5 b)– 5 d) appropriately.
- *Test 2. Compression wave.* Here the piston moves into the medium at a constant speed $U_0 = 0.5$, producing a shock wave. The numerical results at the moment $t = 0.6$ are given in Figure 6 by analogy with Figure 5.
- *Test 3. Unidirectional compression waves.* At the initial moment of time under the action of a piston moving at a constant speed $U_0 = 0.8$ to the right, there is a compression wave. At time $t_1 = 0.25$, the piston begins to accelerate until the moment $t_2 = 0.5$, when its speed reaches the value $U_1 = 2U_0 = 1.6$. After that the piston moves at a constant speed U_1 . The piston moves according to the following law

$$U(t) = \begin{cases} U_0, & t \leq t_1, \\ U_0 + (U_1 - U_0) \sin^2 \left(\frac{\pi}{2} \frac{t-t_1}{t_2-t_1} \right), & t_1 \leq t \leq t_2, \\ U_1, & t \geq t_2. \end{cases}$$

The numerical results at $t = 0.74$ are given in Figure 7.

- *Test 4. Collision of two opposing compression waves.* In this test, two compression waves move towards each other at speeds $U_1 = 0.5$ and $U_2 = -0.5$. The numerical results at $t = 1.0$ are given in Figure 8.

Remark 11. One can find some exact smooth solutions and generalizations of the scheme and the corresponding numerical computations in the paper [67].

Comparing the results of calculations according to five schemes, we arrive at the following conclusions.

1. All the considered schemes, with proper selection of artificial viscosity, show numerical results close to the known exact solutions.
2. The solutions of the modified Yelenin–Krylov scheme (96) change with dispersal peaks near discontinuities, while the artificial viscosity value does not significantly affect that peaks.

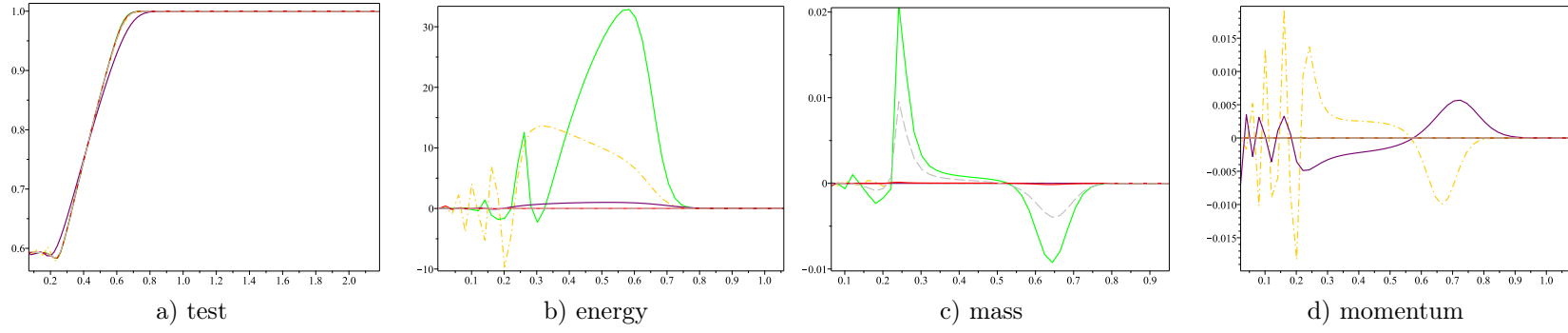


Figure 5: Test 1 numerical results

24

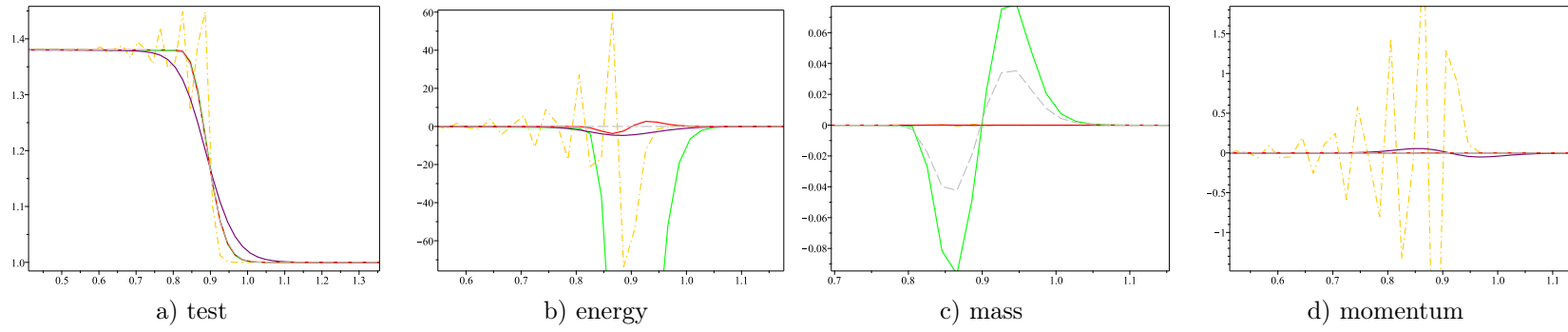


Figure 6: Test 2 numerical results

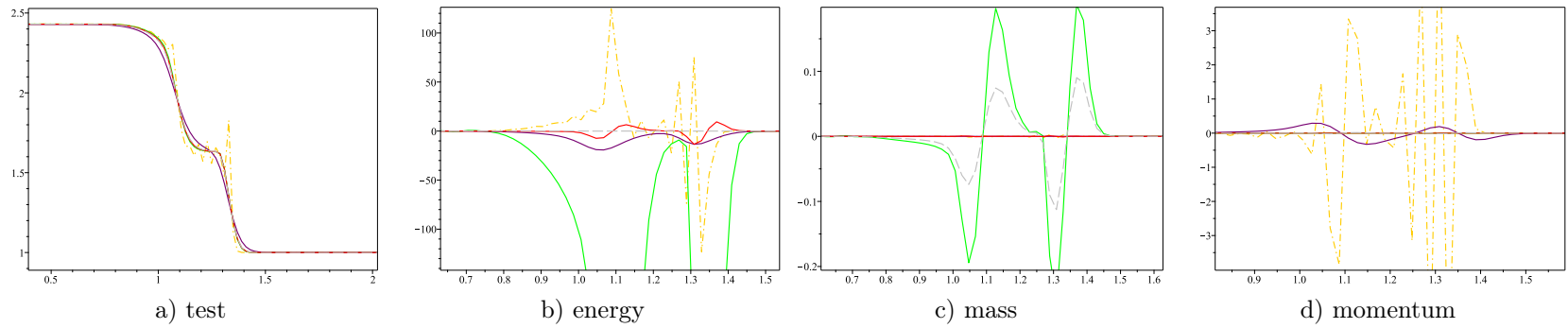


Figure 7: Test 3 numerical results

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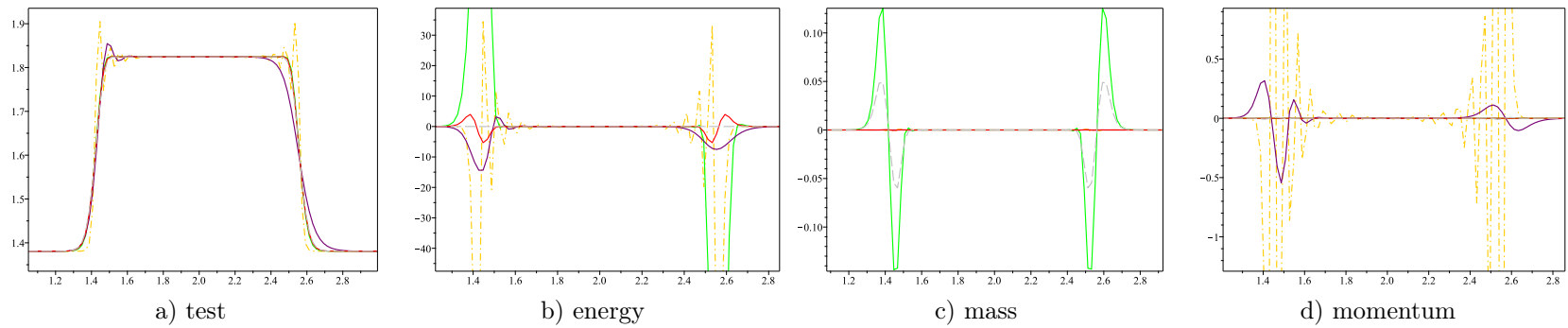


Figure 8: Test 4 numerical results

3. The modified Korobitsyn's scheme (97) turns out to be extremely sensitive to the artificial viscosity parameters, which makes big difference from scheme (96). In contrast to the implicit modified Samarskiy–Popov's scheme (90) the explicit scheme (97) shows worse results according to the conservation of energy and momentum, but it better conserves mass.
4. In all the cases, the invariant scheme (61), (60) results appear among the best results on the conservation laws on solutions.

6. Conclusion

The one-dimensional shallow water equations in Eulerian and Lagrangian coordinates were considered in the present work.

It was shown the relationship between symmetries and conservation laws in Lagrangian (potential) coordinates and symmetries and conservation laws in physical variables.

For equations in Lagrangian coordinates with a flat bottom an invariant difference scheme is constructed which possesses all the difference analogues of the differential conservation laws: mass, momentum, energy, the law of center of mass motion.

The resulting conservative scheme is three-layers scheme in time. By non-point transformations of variables such a scheme can be reduced to an invariant scheme in mass Lagrangian variables possessing the same set of conservation laws. By choosing a special form of “equation of state” (the relationship of pressure and height of a liquid column) it is possible to obtain a two-level scheme of shallow water equations in physical variables. Generalization of the scheme to the case of an arbitrary bottom faces certain difficulties, leading to that it is possible to construct either an invariant scheme with conservation of mass and momentum, or an invariant scheme with conservation of mass and energy.

For a conservative difference scheme for a flat bottom some invariant solutions are constructed. It is shown that the invariant scheme admits reduction on subgroups as well as the original system of differential equations.

Invariant conservative difference scheme for the case of a flat bottom tested numerically in comparison with other known schemes adapted for the case of shallow water in one-dimensional approximation. The numerical tests indicated a good accuracy of calculations and the validation of conservation laws on solutions with big gradients.

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