Tiling space and slabs with acute tetrahedra

David Eppstein¹ John M. Sullivan² Alper Üngör³

Dept. of Information and Computer Science, Univ. of California at Irvine, eppstein@ics.uci.edu
Dept. of Mathematics, Univ. of Illinois at Urbana-Champaign, jms@math.uiuc.edu
Dept. of Computer Science, Duke Univ., ungor@cs.duke.edu

Abstract

We show it is possible to tile three-dimensional space using only tetrahedra with acute dihedral angles. We present several constructions to achieve this, including one in which all dihedral angles are less than 77.08° , and another which tiles a slab in space.

1 Problem definition

Triangulations of two and three-dimensional domains find numerious applications in scientific computing, computer graphics, solid modeling and medical imaging. Most of these applications impose a quality constraint on the elements of the triangulation. Among the most popular quality criteria for elements [4] are the aspect ratio (circumradius over inradius), the minimum dihedral angle, and the radius-edge ratio (circumradius over shortest edge). However, many other quality criteria have been considered, including maximum dihedral angle. Bern *et al.* for instance, studied nonobtuse triangulations [3, 5], where domains are meshed with simplices having no obtuse angles. In this paper, we consider a slightly stronger quality constraint: all the dihedral angles in the mesh are forced to be acute (strictly less than 90°). Although acuteness seems only slightly stronger than nonobtuseness, this problem turns out to be considerably harder than the nonobtuse triangulation problem, as we observe below in Section 3.

Definition. An angle is *acute* if it is strictly less than a right angle $(\frac{\pi}{2} = 90^{\circ})$. A simplex is *acute* if all its (interior) dihedral angles are acute. A triangulation is *acute* if all of its simplices are acute.

Problem 1. Given a domain Ω , compute an acute triangulation of Ω .

There has been extensive work on the two-dimensional version of this problem, for the special cases where the domain Ω is a triangle, square, quadrilateral, or a finite point set [5, 10, 24, 31, 32, 34]. We review those results in Section 3. In three-dimensional space, however, almost nothing has been known about acute triangulations before now. To the best of our knowledge, even the following relaxed form of the problem, where the input domain is the entire space, had not been addressed in the literature.

Problem 2. Is it possible to tile three-dimensional Euclidean space using acute tetrahedra?

We present an affirmative answer to this question, by several different constructions. The twodimensional analog of this problem has a trivial positive answer: congruent copies of any single triangle will tile the plane. However, this idea does not extend to 3D, as the regular tetrahedron (for instance) cannot tile space. All tetrahedra known to tile space have right angles, as further discussed in Section 3.

We started this research on acute triangulations because of a method developed for space-time meshing which required an acute base mesh. This and our other motivations are discussed in Section 2. Section 3 surveys previous research in acute triangulations. Section 4 investigates what acuteness means for a three-dimensional simplex and gives a comparison of acute and Delaunay triangulations. Constructions tiling three-dimensional space, and hence solving Problem 2, are given in Section 5. The paper concludes in Section 6 with a quality assessment of these constructions.

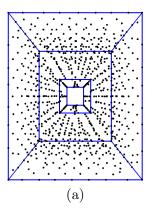
2 Motivation

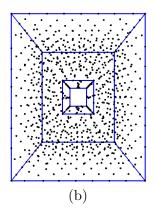
We were originally motivated to study acute triangulations by the space-time meshing algorithm of Üngör and Sheffer [42]. This tent-pitcher algorithm was designed to discretize space-time domains into meshes that obey a certain cone constraint, which requires all faces in the mesh to have smaller slopes than the cones that define the domain of influence imposed by the numerical (engineering) problem. (For instance, we might require simply that all faces make at most a 45° angle with the horizontal.) Because there is then a well-defined direction of information flow across element boundaries, such meshes enable the use of very efficient element-by-element methods (including space-time discontinuous Galerkin methods) to solve a wide variety of numerical problems, for instance in elastodynamics. The tent-pitching algorithm starts with a space mesh of the two-or three-dimensional input domain and constructs the space-time mesh using an advancing front approach. The algorithm is known to generate a valid space-time mesh if the initial space mesh is an acute triangulation [42], but may fail if there is an obtuse angle or even a right angle.

Later, Erickson *et al.* [16] proposed an improved version of the tent-pitching algorithm. By removing the acute angle requirement, the new space-time algorithm works over arbitrary spatial domains. However, there is a loss of efficiency (more elements are required) whenever there is a nonacute angle.

Thus the study of Problem 1 is motivated by current space-time meshing algorithms. But even a solution to Problem 2 is useful, since it leads to a better understanding of the acute triangulation problem for more general input domains, and it also finds some direct applications in mesh generation.

Spacial tilings of high quality have been used for designing meshing algorithms: Fuchs [23], Field and Smith [17, 19] and Naylor [35] built meshes by overlaying standard tilings onto the given polyhedral domain. They used tilings known at the time, such as Sommerville constructions





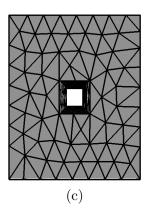


Figure 1: An almost regular triangulation of a cube with a hole (A. Fuchs); (a) the point set of a body-centered cubic (BCC) lattice overlayed with the domain; (b) the adjusted point set; (c) the conforming Delaunay triangulation.

(Figure 5) and subdivided cubes (Figure 7), which we discuss in Section 3.2. Their approach has three steps, illustrated in Figure 1:

- (a) Overlay the chosen tiling with the given domain. The main challenge in this step is finding the right scaling, location and orientation for the tiling so that it matches the domain boundary as closely as possible.
- (b) Adjust the points to get a better fit. For this purpose, one of the standard smoothing techniques [9, 18, 22] can be used. Alternatively, Fuchs [23] suggested minimizing a function which penalizes configurations that produce irregular vertices.
- (c) Construct the mesh by computing the conforming Delaunay triangulation of the adjusted point set and the domain boundary.

Fuchs [23] reports good performance of his experiments when he used the second Sommerville construction (Figure 5(b)) as the space tiling. (This tiling is the Delaunay triangulation of the body-centered cubic lattice.) The dihedral angles of his mesh in Figure 1(c) range between 7.6° and 168.2°. However, most of the angles (here and also in his meshes of similar geometric domains) cluster around 60° and 90°, which are exactly the dihedral angles of the BCC tetrahedron in the input tiling. Some of the constructions we propose in Section 5 are considerably better in terms of dihedral angles and also other quality measures. Our new constructions can find immediate use to improve the results of this previous research [23, 17, 19, 35] on tiling-based meshing.

Bossavit suggests [7, 8] that acute triangulations may also be useful in computational electromagnetics.

3 Background

3.1 Acute and nonobtuse triangulations

There has been considerable research [1, 3, 4, 6, 14] on the nonobtuse triangulation problem, which imposes a slightly weaker constraint than the acute triangulation problem. Angles in a nonobtuse triangulation are less than or equal to 90°. Bern *et al.* [3] showed that any *d*-dimensional point set can be triangulated with $O(n^{\lceil d/2 \rceil})$ simplices, none of which has any obtuse dihedral angles. However, they also proved that no similar result is possible if all angles are required to be at most $90^{\circ} - \epsilon$. This indicates that the acute triangulation problem is much more challenging than nonobtuse triangulation. To appreciate this difference, consider the two problems for a square domain in two dimensions. A single diagonal cuts a square into two nonobtuse triangles, as in Figure 2(a). Finding an acute triangulation, however, can be a challenging recreational math problem.

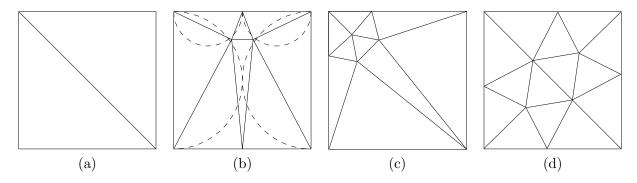


Figure 2: (a) nonobtuse triangulation of a square (b) a square meshed with eight acute triangles (c) a square meshed with ten acute triangles (d) triangulation where maximum angle is 72°

Lindgren [31] showed that at least eight triangles, as in Figure 2(b), are needed. Later, Cassidy and Lord [10] showed that for any $n \ge 10$ (but not for n = 9) there is an acute triangulation with exactly n triangles. Figure 2(c) shows the solution with ten triangles. We can use the maximum angle in a triangulation as a quality measure. The triangulations in Figure 2(b,c) can be realized with maximum angles about 85° and 80.3°, respectively. Eppstein [15] improved this angle to 72° using fourteen acute triangles, as shown in Figure 2(d). Using Euler's formula, Eppstein also showed that any acute triangulation of a square must have an interior vertex of valence five, implying that 72° is the best possible. It is unknown whether there is a triangulation achieving this with fewer than fourteen triangles.

The acute triangulation problem has been studied for other simple polygons as well. Gardner [24] asked the question for triangles. Manheimer proved that seven acute triangles are necessary and sufficient to subdivide a nonobtuse triangle [34]. Recently, Maehara [32] showed that an arbitrary quadrilateral can be tiled by 10 (but perhaps not by any fewer) acute triangles. Gerver [25]

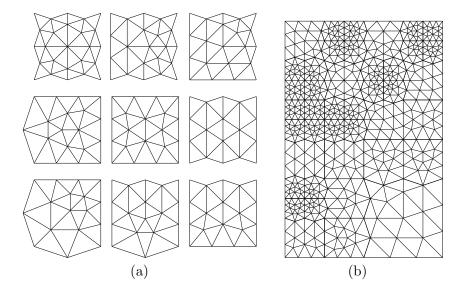


Figure 3: Acute triangulation gadgets and their use on a point set (M. Bern and D. Eppstein [5])

considered the problem of finding triangulations with a stricter upper bound (between 72° and 60°) on their angles, and gave necessary conditions for a polygonal domain to have such a triangulation. If we restrict ourselves to two-dimensional point sets, a solution to the acute triangulation problem is given by Bern *et al.* [5]. Their approach starts with a quadtree, and replaces the squares by tiles with protrusions and indentations. Figure 3 shows sample tiles together with an acute triangulation resulting from their algorithm.

Krotov and Krizek [29] studied refinement methods to subdivide a nonobtuse tetrahedral partition into another finer one. Unfortunately, they called the resulting triangulations acute type instead of nonobtuse even though 90° dihedral angles were ubiquitous in them. Another related work is by Hangan, Itoh and Zamfirescu [27, 28] who studied acute surface triangulations of certain special shapes such as a cube, sphere and icosahedron.

3.2 Acute and nonobtuse tilings

Aristotle claimed that regular tetrahedra could meet five-to-an-edge to tile space, and this claim was repeated over the centuries (see [36]). This of course is false, because the dihedral angle of a regular tetrahedron is not 72° but $\arccos \frac{1}{3} \approx 70.53^{\circ}$. Figure 4 shows the small gap left when five tetrahedra are placed around an edge.

There are, however, tetrahedral shapes which can tile space. Sommerville [38] found four such tetrahedra, shown in Figure 5. Four decades later, Davies [12] and Baumgartner [2] independently rediscovered three of the Sommerville tetrahedra; Baumgartner also found a new example. Goldberg [26] surveyed the list of all known space-tiling tetrahedra, and found three infinite families, including the one shown in Figure 6(a).

The construction of this family is based on a tiling of the plane by equilateral triangles of sidelength e. The infinite prism over each triangle is filled with tetrahedra whose sides are 3a, b, b, b, c, c, where $b^2 = a^2 + e^2$ and $c^2 = 4a^2 + e^2$, as shown in Figure 6. Since the ratio a/e is arbitrary, there is a continuous family of tetrahedral space-fillers of this type. Goldberg's two other families can be derived simply by cutting these tetrahedra into two conguent pieces, either by the plane through C, D and the midpoint of \overline{AB} , or by the plane through A, B and the midpoint of \overline{CD} . Whether the list is complete or not is still an open problem [13, 36]. None of the known space-tiling tetrahedra is acute (although several are nonobtuse). In fact, the tilings all contain edges of valence four. (In Goldberg's family, these are the edges of length c.)

Since it seems likely that there is no tiling of space by congruent acute tetrahedra, we will now consider tilings with several shapes of tetrahedra. There are now many more ways to fill space, for instance by subdividing the cube into five or six tetrahedra as in Figure 7. These tilings also, of course, have 90° dihedral angles, and so are nonobtuse but not acute.

There are many results (like minmax and maxmin angle results) known about optimality of Delaunay triangulations in the plane. But these do not extend to three dimensions, and little is yet known about optimum triangulations in space. Thus it is not surprising that the constructions of acute triangulations do not easily extend from two to three dimensions. It is perhaps remarkable that acute triangulations of space can be constructed.

4 Acute Tetrahedra

An acute tetrahedron does not necessarily have high quality in terms of either aspect ratio or radius-edge ratio. Low-quality tetrahedra have been classified into nine types [11], and three of these (the spire, splinter and wedge) can have all their dihedral angles acute. However, fortuitously, the tetrahedra in our constructions are mostly quite close to regular, and are high-quality for use in mesh generation and numerical simulations.

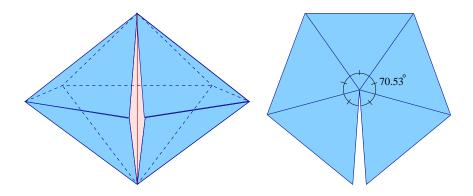


Figure 4: The regular tetrahedron does not tile space.

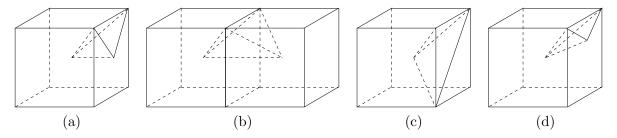


Figure 5: Sommerville tetrahedra. The first tetrahedron is half of the third. The fourth tetrahedron is one fourth of the second.

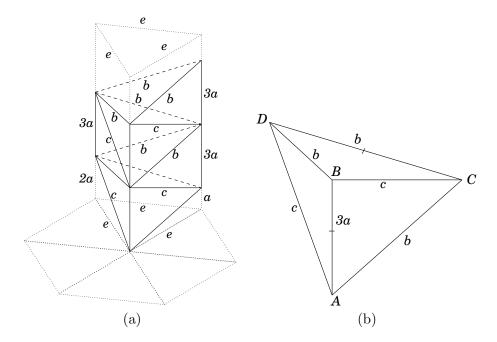


Figure 6: Family of space tilings

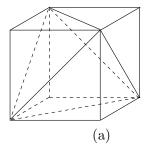
4.1 Acuteness test

By definition, a tetrahedron is acute if each of its six dihedral angles is less than 90°.

Lemma 1. Consider an edge ab of a tetrahedron abcd, and let Π denote projection to a plane normal to ab. The dihedral angle along ab is acute if and only if $\Pi(a) = \Pi(b)$ lies strictly outside the circle with diameter $\Pi(c)\Pi(d)$.

Proof. The dihedral angle along ab is by definition the angle $\angle\Pi(c)\Pi(a)\Pi(d)$; the lemma follows from standard plane geometry (Thales' theorem, see Figure 8(c)).

This lemma can be applied to each of the edges of a tetrahedron. We now examine some alternate criteria for acuteness.



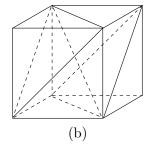


Figure 7: Cube subdivided into (a) 5 or (b) 6 tetrahedra.

Lemma 2. A tetrahedron is acute if and only if the orthogonal projection of each vertex onto the plane of the opposite facet lies strictly inside that facet. An acute tetrahedron has acute facets, but this is not a sufficient condition.

Proof. Suppose the projection of a vertex d is not inside the opposite triangle $\triangle abc$. Then, the dihedral angle along any edge that separates the region it projects to from $\triangle abc$ must be nonacute, as in Figure 8(a). (If the projection is on a triangle edge, then the corresponding dihedral angle is exactly 90°.) Conversely, if the dihedral angle along edge ab is nonacute, then d projects outside $\triangle abc$, as in Figure 8(b).

We prove the second statement in contrapositive form, while noting that nonacute sliver tetrahedra can have acute face angles. Suppose tetrahedron abcd has a nonacute face angle $\angle bac$; we will show the tetrahedron is nonacute. If the projection of d onto $\triangle abc$ is not in the interior we are done by the first part of the lemma. Otherwise, we claim the dihedral angle along ad is larger than $\angle bac$ and thus is nonacute. To check the claim, remember the spherical dual law of cosines (see [41]):

$$\cos d' = -\cos b' \cos c' + \sin b' \sin c' \cos \angle bac$$

where b', c' and d' are the dihedral angles along the edges ab, ac and ad, respectively. (See Figure 8(d).) Assuming b', $c' < \frac{\pi}{2}$, this gives $\cos d' < \cos \angle bac$ as desired.

4.2 Acuteness of Delaunay Triangulations

Given a set of vertices, the Delaunay triangulation is optimal in many ways. However, a Delaunay triangulation in any dimension can have obtuse angles. In this section, we investigate the converse, whether an acute triangulation is necessarily the Delaunay triangulation for its vertices. The answer is positive in the plane, but negative in three-space.

Lemma 3. Any acute two-dimensional triangulation \mathcal{T} is Delaunay.

Proof. Since \mathcal{T} is acute, the diametral circle of each edge is empty of other vertices. By definition, this means the edge is in the Gabriel graph of the vertex set, which is a subgraph of the Delaunay

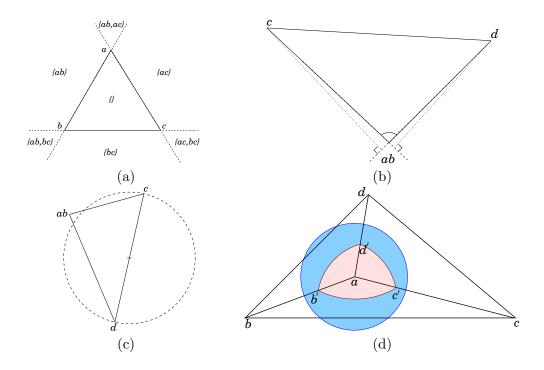


Figure 8: Acuteness tests: (a) if vertex d projects outside $\triangle abc$, the label shows which edges have obtuse dihedral angles; (b) if the dihedral angle along ab is obtuse, both c and d project outside their opposite triangles; (c) Thales' theorem says that the angle at ab is obtuse exactly when it lies outside the circle with diameter cd; (d) if the vertex d projects inside the triangle abc then the face angle $\triangle bac$ is smaller than the dihedral angle on ad.

triangulation. But since the edges of \mathcal{T} form a triangulation, it must be the entire Delaunay triangulation. See also Figure 9.

Corollary. If an acute triangulation of a two-dimensional vertex set exists then it is unique. \Box

Lemma 4. There is an acute triangulation \mathcal{T} in three dimensions which is not Delaunay.

Proof. Consider a "cube corner" tetrahedron, and glue it to a copy of itself across the equilateral face. Then move the two corner vertices away from each other a tiny amount to make the tetrahedra acute. In coordinates, take a suitable small $\epsilon > 0$, and let $a = (-\epsilon, -\epsilon, -\epsilon)$, b = (1, 0, 0), c = (0, 1, 0), d = (0, 0, 1), and $e = (2/3 + \epsilon, 2/3 + \epsilon, 2/3 + \epsilon)$. The two tetrahedra abcd and bcde are acute, but the Delaunay triangulation of these five points consists of three tetrahedra: abce, acde, and abde, as in Figure 10, because e is inside the circumsphere of abcd. Note that the three Delaunay tetrahedra are obtuse, having 120° dihedral angles along edge ae. The acute triangulation we started with is obtained by performing a 3-to-2 flip on the Delaunay triangulation.

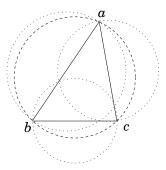


Figure 9: Acute triangles in the plane are Delaunay. An alternate proof uses the fact that the circumcircle is contained in the union of the three diametral circles around the edges.

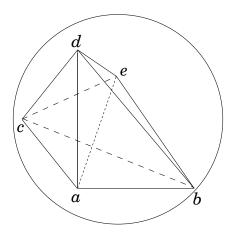


Figure 10: An example of an acute triangulation in space which is not Delaunay.

5 Constructions for Acute Tilings

5.1 TCP triangulations

Our first set of acute triangulations basically come from the crystallography literature. Chemists studying alloys of two transition metals have often found that since the two types of atoms are similar (but slightly different) in size, the Delaunay triangulation of their positions is built of nearly regular tetrahedra. These TCP (tetrahedrally close packed) structures were first described by Frank and Kasper [20, 21] and have been studied extensively by the Shoemakers [37] among others.

A combinatorial definition of the TCP class was given by Sullivan [39]: A triangulation is called TCP if every edge has valence 5 or 6, and no triangle has two 6-valent edges. This definition includes all the chemically known TCP structures, but also allows some new structures [40] not yet seen in nature.

It is not hard to check that the definition allows exactly four types of vertex star in a TCP triangulation. Dually, the voronoi cell around any vertex has one of the four combinatorial types

shown in Figure 11: these are the polyhedra with pentagonal and hexagonal faces but no adjacent hexagons. (It is interesting that these dual structures are seen in some other crystal structures: in some zeolites, silicon dioxide outlines the voronoi edges, while in clathrates, water cages along the voronoi skeleton trap large gas molecules.)

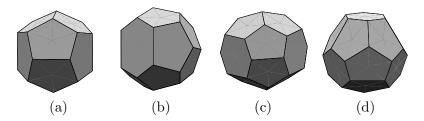


Figure 11: Foam cells with pentagonal and hexagonal faces

All known TCP structures can be viewed as convex combinations of the three basic ones (called A15, Z and C15) shown in Figure 12. There are many ways to understand these structures [39]. To construct A15, we can start with a BCC lattice. Its Delaunay triangulation is one of the Sommerville tilings discussed above; since the edges have even valence the tetrahedra can be colored alternately black and white. If we take the BCC lattice together with the circumcenters of all black tetrahedra, we have the vertices of A15: their Delaunay tetrahedra are all now nearly regular. Similarly, the C15 structure arises from the diamond lattice by adding selected circumcenters, and the Z structure can be obtained similarly starting with hexagonal prisms.

The C15 structure (also known as the cubic Friauf–Laves phase) is shown in Figure 13, where the red spheres are centered on a diamond lattice (FCC together with a certain translate) and the blue spheres are at selected circumcenters.

In any triangulation of space, the average dihedral angle multiplied by the average edge valence is exactly 360°. If a tiling could be made of regular tetrahedra, the average edge valence would thus be $n_0 := 360^{\circ}/\arccos(\frac{1}{3}) \approx 5.1043$. But by symmetry, the regular tetrahedron is a critical point for average dihedral angle, so any tiling made of nearly regular tetrahedra should have average valence

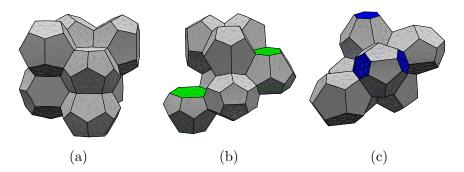


Figure 12: The voronoi cells for the three basic TCP structures, A15, Z, and C15.

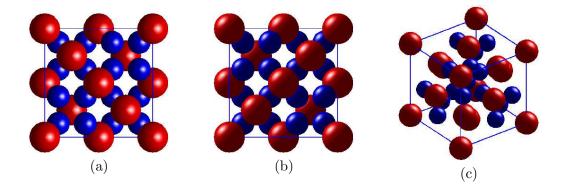


Figure 13: The vertices of the C15 triangulation are at the centers of these balls.

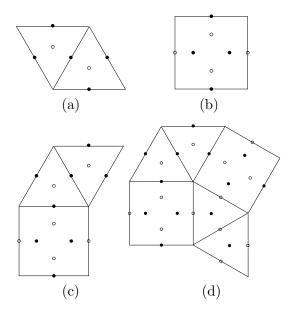


Figure 14: Four simple periodic square/triangle tilings of the plane which lead to the TCP structures named (a) Z; (b) A15; (c) σ ; and (d) H.

quite close to n_0 . Indeed, all known TCP structures have average valence between $5\frac{1}{10}$ and $\leq 5\frac{1}{9}$, the values for C15 and A15.

Sullivan [40] has formalized a construction suggested by Frank and Kaspar for mixing the basic TCP structures. Start with any tiling of the plane by copies of an equilateral triangle and a square, like one of the four shown in Figure 14. Suppose the side length of the square and triangle is 4. Mark black and white dots on the tilings as shown. (The dots are at edge midpoints, at triangle centers, and at distance 1 from the sides of the squares.) Then the vertices of the corresponding TCP structure are at heights 4k - 1 above the black dots, at heights 4k + 1 above the white dots, and at heights 4k and 4k + 2 above the vertices of the square/triangle tiling. (Here k ranges over

all integers.)

Again, in each case the TCP triangulation is simply the Delaunay triangulation of this periodic point set. See Figure 16. The triangulations constructed in this way are all combinations of the A15 and Z structures. A variant of this construction [40] builds combinations of the Z and C15 structures, again starting from an arbitrary square/triangle tiling.

Especially in the mixed structures like σ and H, the particular geometry we have described here may differ slightly from that found in the actual crystals with the same combinatorics. Presumably, these slight adjustments do not affect the shapes of the tetrahedra very much. The quality figures we present below are measured using the exact geometry we have just described.

Sullivan's original interest in these structures was for the mathematical study of foam geometry. The Kelvin problem asks for the most efficient partition of space into unit-volume cells, that is, for the partition with least surface area. Lord Kelvin's suggested solution was a slightly relaxed form of the BCC voronoi cells (truncated octahedra). But in 1994, Weaire and Phelan [43] discovered that a relaxed form of the voronoi cells for the TCP structure A15 is more efficient than the Kelvin foam [30].

It is perhaps not surprising that TCP structures are related to foams: Plateau's rules for singularities in soap films minimizing their surface area imply that a foam is combinatorially dual to some triangulation, preferably one with nearly regular tetrahedra. It is an interesting question whether any triangulation meeting the combinatorial definition of TCP can be built with tetrahedra close to regular, but certainly for the known TCP structures this seems always to be the case. Thus our acute triangulations arising from this construction have high quality by almost all measures.

5.2 Icosahedral Construction of the Z Structure

An alternate construction for the TCP Z structure is inspired by the work of Field [17]. His tilings involved right-angled tetrahedra, but by selectively adjusting the point set, we obtain a tiling with only acute tetrahedra. A regular icosahedron can be subdivided into 20 acute (and nearly regular) tetrahedra simply by coning to the center point. We place icosahedra in a hexagonal lattice in the plane, each in the same orientation, touching edge to edge, as in Figure 15.

This layer then gets repeated vertically, with each icosahedron sharing a horizontal face with the ones just above and below it. Our point set is then the vertices and centers of all the icosahedra. Its Delaunay triangulation, shown in Figure 16(d), is combinatorially the TCP Z structure, but with slightly different geometry than that constructed before. The horizontal faces (seen head-on as equilateral triangles in Figure 15) are shared by two icosahedra. Each other face separates an icosahedron from one of the four types of Delaunay tetrahedra that fill the gaps. There are two type of gaps. The deeper gaps are defined by the points a, b, c, d, e, f, g, and g' (opposite to g) and are filled with two type of tetrahedra, e.g., bdfg and abfg. The shallower gaps are defined by the points d, f, h, i, j, k, e, and e' (opposite of e) and are filled with two type of tetrahedra, e.g., ee'df and ee'fh.

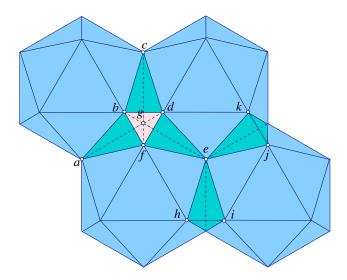


Figure 15: Icosahedron construction.

5.3 An Acute Triangulation of a Single Slab

The acute triangulations we have described so far, though periodic, do not have any planar boundaries within them. Here we describe an acute triangulation of a slab (which can of course be repeated to fill all of space). We view this as partial progress towards the problem of triangulating an arbitrary domain, although it seems much harder to find an acute triangulation of a cube or even an infinite square prism. We triangulate the bottom half of the slab in the following eight steps. Let h be the height of the slab and $\gamma = h/14.2$.

- 1. Start with a grid of equilateral triangles of side length 6γ on the base plane, as in Figure 17(a).
- 2. Place a near-regular tetrahedron (with height 4γ) over each triangle, as in Figure 17(b).
- 3. Add a tetrahedron in the gap between each pair of adjacent tetrahedra, as in Figure 17(c). The resulting surface has deep hexagonal dimples at the original vertices in the base plane.
- 4. Add six tetrahedra in each dimple, each with one vertex on the starting plane, two vertices at height 4γ , and one new vertex at height 4.6γ over the starting vertex, as in Figure 17(d). Now we have a surface with shallow hexagonal bumps.
- 5. Place a vertex at height at 7.1γ over the midpoint of the edge between each pair of adjacent bumps. Each such vertex and edge form a vertical triangle; let this separate two new tetrahedra whose fourth vertices are the two nearby bump vertices, as in Figure 17(e). The surface is now covered by tall diamond-shaped bumps.
- 6. Place a tetrahedron between each adjacent pair of bumps, as in Figure 17(f). We now have an alternating grid of medium-depth six-sided holes (over each of the shallow hexagonal bumps)

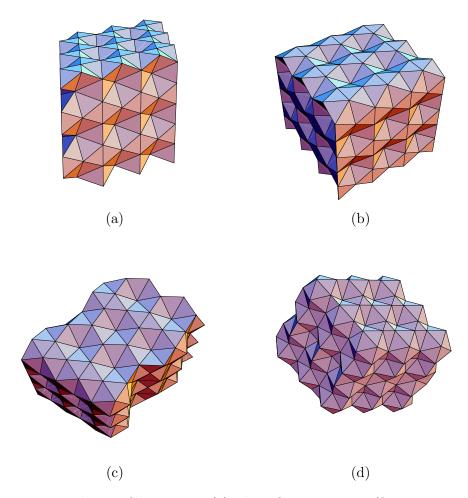


Figure 16: Acute triangulations filling space. (a) The TCP structure Z (from a triangle tiling). (b) The TCP structure A15 (from a square tiling). (c) The TCP structure σ , a mixture of A15 and Z. (d) Icosahedron construction of Figure 15.

and deep tetrahedral holes (over the points where three of the shallow hexagonal bumps meet).

- 7. Fill each tetrahedral hole, to form a surface alternating between six-sided holes and flat triangles, as in Figure 17(g).
- 8. Place six tetrahedra into each medium-height hexagonal hole to turn it into a medium-height hexagonal bump, as in Figure 17(h). In order to make the bumps equal to the holes, the height of the new vertices is chosen as $[2(7.1-4.6)+4.6]\gamma = 9.6\gamma$.

To complete the triangulation of the slab, we now repeat the first seven steps in reverse order.

Any of the constructions given in this section serves to prove our main result:

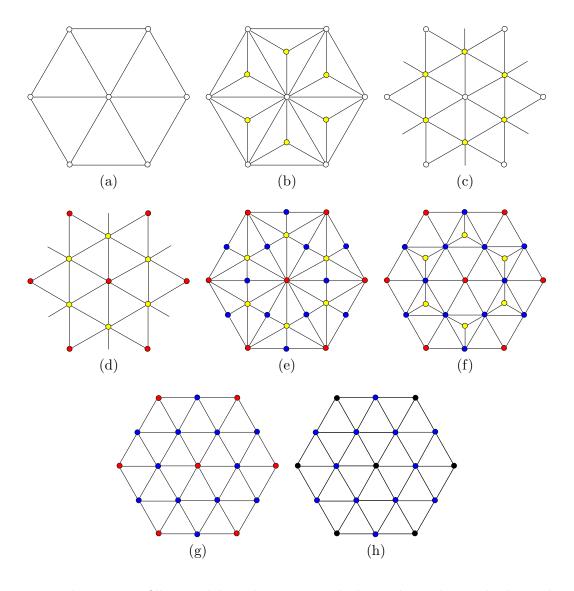


Figure 17: Eight steps in filling a slab with acute tetrahedra. The nodes in the base plane are colored white; successive layers above that plane are then colored yellow, red, blue and black, in order.

Theorem 5. It is possible to tile three-dimensional Euclidean space with acute tetrahedra. \Box

6 Evaluation of the constructions

Our constructions use tetrahedra of quite good quality (as summarized in Table 6) and so they are quite suitable for mesh generation.

There are still some challenges left to make use of these tilings in real-life meshing techniques. A strategy is required to fit the tilings into a planar projection of the spatial domain. Malkevitch studies this problem in [33]. He describes the conditions for a polygon to be tiled by squares and

	radius-edge		smallest		largest	
	ratio		dihedral angle		dihedral angle	
Construction	min	max	min	max	min	max
TCP Z from triangle tiling	.651	.737	53.13	67.37	73.89	77.07
TCP A15 from square tiling	.645	.707	53.13	67.79	73.39	78.46
TCP σ	.645	.737	53.13	67.79	73.39	78.46
TCP C15	.612	.711	60	70.52	70.52	74.20
TCP Z from icosahedra	.629	1.000	41.81	69.09	71.99	83.62
Slab	.636	.938	46.83	67.88	74.39	87.70
Sommerville I	1.118	1.118	45	45	90	90
Sommerville II	.645	.645	60	60	90	90
Sommerville III	.866	.866	45	45	120	120
Sommerville IV	1.581	1.581	30	30	131.81	131.81
Cube V	.612	.866	54.73	70.53	70.53	90
Cube VI	.866	.866	45	45	90	90
Regular tetrahedron	.612	.612	70.53	70.53	70.53	70.53
Cube corner	.866	.866	54.73	54.73	90	90

Table 1: The quality of the tetrahedra in our constructions (and of the regular tetrahedron) can be measured in terms of the radius-to-shortest-edge ratio and the maximum dihedral angles.

equilateral triangles. Also, even though one of our constructions fits between two parallel planes in a slab, all of them have dimples (cavities on the surface) in most directrions, making them not very suitable for meshing domains with flat surfaces.

Open problems related to this work include the following

- 1. Given a point set in 3D that has an acute triangulation, how can we compute it?
- 2. Are these constructions the best possible? For instance, which tiling of space with tetrahedra minimizes the maximum dihedral angle?
- 3. Is it possible to tile the space with congruent copies of some single acute tetrahedron?
- 4. Is it possible to subdivide a cube (or even an acute tetrahedron) into acute tetrahedra?
- 5. Is it possible to extend the planar acute triangulation algorithm of [5] into three dimensions, using constructions similar to ours?

Acknowledgments

Many thanks to Jeff Erickson for discussions and asking how to subdivide a cube into acute tetrahedra, Marshall Bern, Olaf Delgado, Herbert Edelsbrunner, and Sariel Har-Peled for discussions, and Damrong Guoy for his help on visualization.

This work was partially supported by NSF ITR grant DMR-0121695. David Eppstein was supported by NSF grant CCR-9912338. John Sullivan was also partially supported by NSF grant DMS-0071520. Alper Üngör was also partially supported by a UIUC Computational Science and Engineering Fellowship.

References

- [1] B. S. Baker, E. Grosse, and C. S. Rafferty. Nonobtuse triangulation of polygons. *Disc. & Comp. Geometry* 3:147–168, 1988.
- [2] L. Baumgartner. Zerlegung des vierdimensioanlen raumes in kongruente fünfzelle. *Math.-Phys.* 15:76–86, 1968.
- [3] M. Bern, L. P. Chew, D. Eppstein, and J. Ruppert. Dihedral bounds for mesh generation in high dimensions. *Proc. 6th ACM-SIAM Symp. Discrete Algorithms*, 89–196, 1995.
- [4] M. Bern and D. Eppstein. Polynomial-size nonobtuse triangulation of polygons. *Int. J. Comp. Geometry and Applications* 2:241–255, 1992.
- [5] M. Bern, D. Eppstein, and J. Gilbert. Provably good mesh generation. J. Comp. System Sciences 48:384–409, 1994.
- [6] M. Bern, S. Mitchell, and J. Ruppert. Linear-size nonobtuse triangulation of polygons. Proc. 10th ACM Symp. Comp. Geometry, 221–230, 1994.
- [7] A. Bossavit. Generalized finite differences in computational electromagnetics. *Progress in Electromagnetics Research* 32:45–64, 2001. (http://ceta-mac1.mit.edu/pier/pier32/pier32.html).
- [8] A. Bossavit. The sommerville mesh in yee-like schemes. *Proc. Scientific Computing in Electrical Engineering (to appear)*, 2002.
- [9] S. A. Canann, M. B. Stephenson, and T. Blacker. Optismoothing: An optimization-driven approach to mesh smoothing. *Finite Elements in Analysis and Design* 13:185–190, 1993.
- [10] C. Cassidy and G. Lord. A square acutely triangulated. J. Rec. Math. 13(4):263–268, 1980.
- [11] S.-W. Cheng, T. K. Dey, H. Edelsbrunner, M. A. Facello, and S.-H. Teng. Sliver exudation. *Proc.* 15th ACM Symp. Comp. Geometry, 1–13, 1999.
- [12] H. L. Davies. Packings of spherical triangles and tetrahedra. *Proc. Colloquium on Convexity*, 42–51, 1965.
- [13] H. Edelsbrunner. Geometry and topology for mesh generation. Cambridge University Press, 2001.

- [14] D. Eppstein. Faster circle with applications nonobtuse packing to Tech. Rep. 94-33, Univ. California, 1994. triangulation. of Irvine, (http://www.ics.uci.edu/~eppstein/pubs/Epp-TR-94-33.pdf).
- [15] D. Eppstein. Acute square triangulation, 1997. $\langle \text{http://www.ics.uci.edu/}^{\sim} \text{eppstein/junkyard/acute-square/} \rangle$.
- [16] J. Erickson, D. Guoy, J. Sullivan, and A. Üngör. Building space-time meshes over arbitrary spatial domains. *Proc.* 11th Int. Meshing Roundtable, 391–402, 2002.
- [17] D. A. Field. Implementing Watson's algorithm in three dimensions. *Proc. 2nd ACM Symp. Comp. Geometry*, 246–259, 1986.
- [18] D. A. Field. Laplacian smoothing and Delaunay triangulations. Comm. in Applied Numer. Analysis 4:709–712, 1988.
- [19] D. A. Field and W. D. Smith. Graded tetrahedral finite element meshes. Int. J. Numer. Meth. Eng. 31:413–425, 1991.
- [20] F. C. Frank and J. S. Kasper. Complex alloy structures regarded as sphere packings. I. Definitions and basic principles. Acta Crystall. 11:184–190, 1958.
- [21] F. C. Frank and J. S. Kasper. Complex alloy structures regarded as sphere packings. II. Analysis and classification of representative structures. *Acta Crystall.* 12:483–499, 1959.
- [22] L. Freitag, M. Jones, and Ρ. Plassmann. An efficient parallel algorithm 4thMeshingRoundtable, mesh smoothing. Proc.Int.47-58, 1995. (http://www.andrew.cmu.edu/user/sowen/abstracts/Fr162.html).
- [23] A. Fuchs. Automatic grid generation with almost regular Delautetrahedra. Proc.7thInt.Meshing Roundtable, 133-1471998. nav (http://www.andrew.cmu.edu/user/sowen/abstracts/Fu432.html).
- [24] M. Gardner. Mathematical games. Scientific American 202:177–178, 1960.
- [25] J. L. Gerver. The dissection of a polygon into nearly equilateral triangles. *Geometriae Dedicata* 16:93–106, 1984.
- [26] M. Goldberg. Three infinite families of tetrahedral space-fillers. J. Comb. Theory 16:348–354, 1974.
- [27] T. Hangan, J. Itoh, and T. Zamfirescu. Acute triangulations. Bulletin Math. de la Soc. des Sci. Math. de Roumanie 43:279–286, 2000.
- [28] J.-I. Itoh. Acute triangulations of sphere and icosahedron. *Josai Mathematical Monographs* 3:53–62, 2001.

- [29] S. Korotov and M. Krizek. Acute type refinements of tetrahedral partitions of polyhedral domains. SIAM J. Numer. Anal. 39(2):724–733, 2001.
- [30] R. Kusner and J. M. Sullivan. Comparing the Weaire-Phelan equal-volume foam to Kelvin's foam. Forma 11(3):233–242, 1996.
- [31] H. Lindgren. Geometric dissections. Van Nostrand, Princeton, N. J., 1964.
- [32] H. Maehara. Acute triangulations ofpolygons. Proc.JapanConf. Disc.Comp.Geometry, 237 - 243, 2000. Springer-Verlag, LNCS 2098. (http://link.springer.de/link/service/series/0558/bibs/2098/20980237.htm).
- [33] J. Malkevitch. Tiling convex polygons with equilateral triangles and squares. *Discrete Geometry and Convexity*, 299–303, 1985. New York Academy of Sciences.
- [34] W. Manheimer. Solution to problem E1406: Dissecting an obtuse triangle into acute triangles. American Mathematical Monthly 67, 1960.
- [35] D. J. Naylor. Filling space with tetrahedra. Int. J. Numer. Meth. Engng. 44:1383–1395, 1999.
- [36] M. Senechal. Which tetrahedra fill space? Mathematics Magazine 54:227–243, 1981.
- [37] D. Shoemaker and C. Shoemaker. Concerning the relative numbers of atomic coordination types in tetrahedrally close packed metal structures. *Acta Crystall.* 42:3–11, 1986.
- [38] D. M. Y. Sommerville. Space-filling tetrahedra in Euclidean space. *Proc. Edinburgh Mathematical Society*, 49–57, 1923. vol. 41.
- [39] J. M. Sullivan. The geometry of bubbles and foams. Foams and Emulsions (NATO ASI volume E 354) 379–402. Kluwer, 1999.
- [40] J. M. Sullivan. New tetrahedrally close-packed structures. Foams, Emulsions and their Applications, 111–119, 2000. Verlag MIT. Proceedings of Eurofoam 2000, Delft.
- [41] W. Thurston. *Three-Dimensional Geometry and Topology, Volume 1.* Princeton University Press, Princeton, NJ, 1997.
- [42] A. Üngör and A. Sheffer. Pitching tents in space-time: Mesh generation for discontinuous Galerkin method. J. of Foundations of Computer Science 13(2):201–221, 2002.
- [43] D. Weaire and R. Phelan. A counter-example to Kelvin's conjecture on minimal surfaces. *Phil. Mag. Lett.* 69(2):107–110, 1994.