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Convexity minimizes pseudo-triangulations

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Abstract

The number of minimum pseudo-triangulations is minimized for point sets in convex position.

Keywords: Pseudo-triangulation; Lower bound; Edge flip

1. Introduction

A *pseudo-triangle* is a planar polygon with exactly three convex vertices, called *corners*. Three reflex chains of edges join the corners. Let S be a set of n points in general position in the plane. A *pseudo-triangulation* for S is a partition of the convex hull of S into pseudo-triangles whose vertex set is S. A pseudo-triangulation is called *minimum* if it consists of exactly $n-2$ pseudo-triangles (and $2n-3$ edges), the minimum possible. Each vertex of a minimum pseudo-triangulation is *pointed*, that is,

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its incident edges span a convex angle. In fact, minimum pseudo-triangulations can be characterized as maximal planar straight-line graphs where each vertex is pointed [18]. Therefore, they have been alternatively called pointed pseudo-triangulations.

Pseudo-triangulations have received considerable attention in computational geometry due to their applications to visibility [13,14], ray shooting [8], kinetic collision detection [1,11,12], rigidity [18], and guarding [17]. Several of their interesting geometric and combinatorial properties have been discovered recently [2,9,10,16]. Still, little is known about the number of pseudo-triangulations a general point set S allows. (Assuming general position of S is necessary to avoid trivial situations.) In [15], the number of minimum pseudo-triangulations is determined for sets of points with exactly one interior point. Also, a (coarse) upper bound on the number of minimum pseudo-triangulations for sets with i interior points is given, namely 3^i times the number of triangulations. An asymptotic lower bound for the maximal number of pseudo-triangulations is derived in [6].

For standard triangulations it is not known which sets of points have the fewest or the most triangulations.[4] In contrast, we show that sets of points in convex position minimize the number of minimum pseudo-triangulations. This adds to the common belief that minimum pseudo-triangulations are more tractable in many respects. In the next section, we illustrate that a lower bound of $\Omega(3^n)$ on their number is easy to obtain for every point set, by using an inductive argument. Section 3 refines our construction and presents the main result. We close with several remarks and also include the description of an interesting parity property.

2. Incremental prelude

Let us split the given point set S into two sets H and I, containing its extreme and interior points, respectively. The set H, as being in convex position, admits exactly C_{h-2} triangulations, where $h = |H|$ and C_h denotes the hth Catalan number. Since for sets in convex position the only pseudo-triangulations are triangulations, these constitute also all possible pseudo-triangulations for H.

Since C_{h-2} is in $\Omega(3^h)$ we are done if $I = \emptyset$. Otherwise we pick any of the triangulations for H and complete it to a minimum pseudo-triangulation of S by successively adding the $n - h$ points of I in an arbitrary but fixed insertion order. The generic step inserts some point $p \in I$ into some pseudo-triangle Δ_p. There are exactly three ways to complete the interior of Δ_p to a valid minimum pseudo-triangulation: choose 2 of the 3 corners of Δ_p and connect them to p by geodesics (inner tangents) in Δ_p. This splits Δ_p into two new pseudo-triangles. Exactly one pseudo-triangle has p as a corner and will be called a *base pseudo-triangle* for p (see Fig. 2). Observe that p as well as all vertices of Δ_p stay pointed after the insertion.

Any two pseudo-triangulations we construct in this way are different, since we started with pairwise different triangulations and applied only changes restricted to already existing pseudo-triangles. Therefore this method yields $C_{h-2} \cdot 3^{n-h}$ minimum pseudo-triangulations for S.

[4] See http://www.igi.TUgraz.at/oaich/triangulations/counting/counting.html for examples of point sets that are currently conjectured to minimize the number of triangulations.

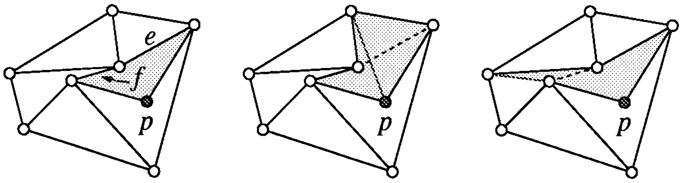

Fig. 1. Edges of Δ_p are flipped.

3. Refined incremental approach

We will show that a factor of 4 (rather than 3) can be gained per insertion step. To this end, the $n - h$ points in I are inserted in some *directional order*, and a carefully chosen edge flip is applied after inserting any point $p \in I$ to raise its degree from 2 to 3. This will lead to 4 distinct minimum pseudo-triangulations.

An *edge flip* replaces a given edge e by a unique edge e'. Edge e' lies on the geodesic between the two corners opposite to e in the two pseudo-triangles adjacent to e.

Recall that Δ_p is the pseudo-triangle into which point p is inserted. Unlike flips in triangulations, a flip of an edge of Δ_p does not need to increase the degree of p. An edge which still does will be termed an *augmenting edge* for p. See Fig. 1, where e is an augmenting edge for p while f is not.

The uniqueness of the insertion order is mandatory and can always be guaranteed by the general position assumption on S. In the following discussion we will assume that the points of the set I are inserted in increasing x-order.

Lemma 1. *For each $p \in I$ at least one of its three base pseudo-triangles contains an augmenting edge for p. An exception occurs if $h = 3$ and p is the first inserted point.*

Proof. See Fig. 2. The rightmost corner c of Δ_p must be an extreme point of S, because all points of I inserted so far lie to the left of p. Let a and b the two other corners of Δ_p. Consider the three geodesics from p to a, b and c. One of them is just the line segment pc, because there are no points of I to the right of p. Denote with t_a and t_b the points of tangency of the two other geodesics. These vertices split the boundary of Δ_p into three (not necessarily reflex) chains of edges $E(c, t_a)$, $E(t_a, t_b)$ and $E(t_b, c)$. Each chain defines a base pseudo-triangle with corner p. An edge e of Δ_p is called *visible* (from p) if there exists some point x interior to e such that the line segment xp does not cross the boundary of Δ_p. We argue that augmenting edges must be visible: consider, e.g., the chain $E(c, t_a)$. If a non-visible edge e of $E(c, t_a)$ is flipped then the new geodesic either runs via t_a or ends up at c (rather than at p). In both cases, e is no augmenting edge. Note that visible edges need not be augmenting, though, because they might be convex hull edges and hence non-flippable.

Assume now that Δ_p contains no augmenting edge. We are going to show that no edge of Δ_p is flippable then. This implies that $\Delta_p = abc$ is the convex hull of S, which is just the exceptional case stated in the lemma.

Consider the chain $E(c, t_a)$ first. When flipping a (non-augmenting) edge of $E(c, t_a)$, the new geodesic either ends at the extreme point c or runs via t_a. Moreover, only a single edge e of $E(c, t_a)$ is visible, because the edge incident to c is augmenting, otherwise. For the same reason, the vertex t_a does not

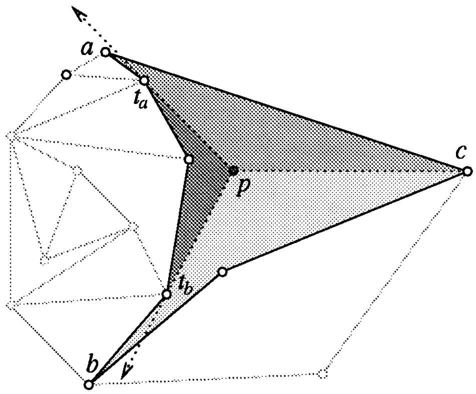

Fig. 2. Point p and its base pseudo-triangles.

lie between c and a. Let us argue next that flipping an edge of the second chain $E(t_a, t_b)$ cannot give a geodesic via t_a.

Case (1) $E(c, t_a) \neq e$.

We have $t_a \neq a$. Let E be the subchain of $E(c, t_a)$ from a to t_a. Then E and c lie on different sides of the ray pt_a. This outrules the geodesics via t_a in question.

Case (2) $E(c, t_a) = e$.

If e is a convex hull edge, we have $t_a = a$ and $e = ca$, such that t_a is an extreme point. If e is an internal edge, flipping e already yields a geodesic that runs via $t_a = a$, because e is not augmenting. In both cases, no geodesic via t_a in question is possible.

In conclusion, the vertex t_a behaves for $E(t_a, t_b)$ as does the vertex c for $E(c, t_a)$, i.e., flipping an edge of the respective chain yields a geodesic that either ends at the first endpoint of this chain or runs via its second endpoint. Clearly, the same is now true for the third chain $E(t_b, c)$. On the other hand, if we reverse the cyclic order of our argumentation (and start with the chain $E(t_b, c)$), then we get that flipping an edge of a chain yields a geodesic that either ends at the *second* endpoint of this chain or runs via its *first* endpoint, opposite as before. Since the geodesic has to fulfill both requirements, this gives a contradiction. We conclude that none of these chains contains a flippable edge. $\square$

Note that there can be more than one augmenting edge for p. We only proved the existence of one, which is best possible in the worst case.

By Lemma 1, each generic insertion step creates 4 distinct minimum pseudo-triangulations, 3 containing a base pseudo-triangle each, and one coming from the applied flip. To see that they are also distinct from all other minimum pseudo-triangulations, which do arise when starting with different triangulations for H, consider the set $\mathcal{T}$ of all generated pseudo-triangulations. We define a graph $\mathcal{G}$ for $\mathcal{T}$ by connecting each pair $T, T' \in \mathcal{T}$ by a directed edge, provided that T is obtained from T' by inserting a single point of I.

Lemma 2. *The graph $\mathcal{G}$ is a forest of trees whose roots are the C_{h-2} triangulations of H.*

Proof. Take any member $T \in \mathcal{T}$. It is sufficient to show that T has a unique predecessor T' in $\mathcal{G}$, provided that T uses points of I as vertices. Consider the rightmost point $p \in I$ in T. By construction, p is of degree 2 or 3. In the former case, p and its incident edges are simply removed from T to obtain T'. In the latter case, we first flip away a certain edge of p, and then remove p and its 2 remaining edges. The applied flip has to exactly reverse the flip of the augmenting edge whose existence is guaranteed in Lemma 1. The crucial observation is that such a flip has to restore the corresponding base pseudo-triangle for p, and thus never alters the convex angle spanned by p's edges. So we can tell from T which flip to perform: the one that relieves p from its middle edge. □

Lemma 1 combines with Lemma 2 as follows.

Theorem 1. *Let $\mathcal{N}_{n,h}$ denote the least number of minimum pseudo-triangulations attained by a set of $n \geqslant 4$ points with h extreme points.*

$$\mathcal{N}_{n,h} \geqslant \begin{cases} 3 \cdot 4^{n-4}, & h = 3, \\ C_{h-2} \cdot 4^{n-h}, & h \geqslant 4. \end{cases}$$

This bound implies a proof of a common though until now unsettled conjecture:

Corollary 1. *A set S of n points minimizes the number of minimum pseudo-triangulations if and only if S is in convex position.*

Proof. The Catalan numbers, which count the minimum pseudo-triangulations for any set of h points in convex position, increase by a factor of C_{h-1}/C_{h-2} which is strictly less than 4. Since $\mathcal{N}_{4,3} = 3$ and $C_2 = 2$, we get

$$\mathcal{N}_{n,h} > C_{n-2}, \quad n > h \geqslant 3$$

by applying to $\mathcal{N}_{n,h}$ the factor of 4 stated in Theorem 1. □

Corollary 2. *A set S of n points minimizes the number of pseudo-triangulations* (*including non-minimum ones*) *if and only if S is in convex position.*

Proof. Every pseudo-triangulation of S is minimum in the convex case, whereas sets of n points of a different structure do admit non-minimum ones as well. So the assertion follows from Corollary 1. □

4. Discussion and extensions

Remark 1. It is instructive to see why Lemma 1 fails if no restrictions are imposed on the insertion order. Without a directional insertion order, cyclically twisted geodesics can occur. In Fig. 3, none of the base pseudo-triangles for p contains an augmenting edge. Surprisingly though, this pseudo-triangulation still can be obtained by using our construction.

Remark 2. Our construction clearly does not produce all possible minimum pseudo-triangulations of a point set S. The first gap occurs for $\mathcal{N}_{5,3} = 13$, where a 13th existing pseudo-triangulation is not counted (see Fig. 4). It can be constructed, though, by using all augmenting edges of the base pseudo-triangles of each inserted point. Note that pseudo-triangulations where, for instance, the rightmost inner vertex has degree 4 or more cannot be obtained.

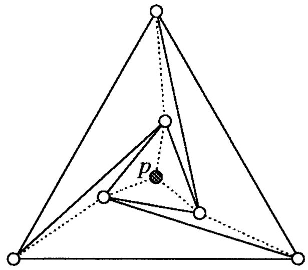

Fig. 3. Twisted geodesics.

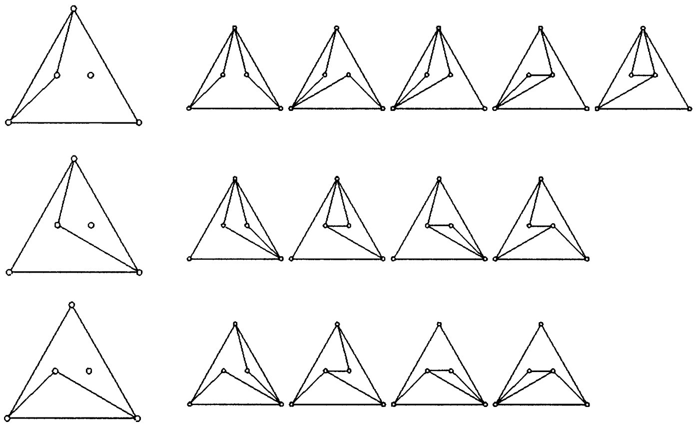

Fig. 4. Only one of the two rightmost pseudo-triangulations in the first row is counted.

We also fail to count certain pseudo-triangulations since we limit insertion to *interior* points. Lifting this restriction leads to the *Henneberg construction* from combinatorial rigidity; see e.g. [18]. This construction starts with an arbitrary triangle spanned by S, and yields all possible minimum pseudo-triangulations of S—though including duplicates. Let us revisit a proof for the existence of a corresponding insertion order.

Proof. Let $|S| = n$. If $n = 3$ we are done. Otherwise, recall that each minimum pseudo-triangulation of S has exactly $2n - 3$ edges. Thus the average degree of a point is strictly less than 4, such that a point $p \in S$ of degree 2 or 3 has to exist. Treat p as in the proof of Lemma 2, which leads to a minimum pseudo-triangulation of $S \setminus \{p\}$. The assertion now follows by induction. □

Remark 3. The point set data base of [3] has been used by [7] to obtain the values of $\mathcal{N}_{n,h}$ in Table 1. We exploited these values to slightly improve Theorem 1. For instance, for $n \geqslant 7$, we get the uniform bound

$$\mathcal{N}_{n,h} \geqslant C_{h-2} \cdot 4^{n-h}.$$

Moreover, for point sets with triangular convex hulls, our bound sharpens to

$$\mathcal{N}_{n,3} > \frac{1}{13} \cdot 4^n$$

for $n \geqslant 10$. It is conjectured that—as a counterpart to Corollary 1—the number of minimum pseudo-

Table 1
Values of $\mathcal{N}_{n,h}$ for small n and h

n/h	3	4	5	6	7	8	9	10
3	1	–	–	–	–	–	–	–
4	3	2	–	–	–	–	–	–
5	13	8	5	–	–	–	–	–
6	63	38	23	14	–	–	–	–
7	353	196	117	70	42	–	–	–
8	2095	1066	631	374	222	132	–	–
9	12881	6494	3541	2086	1230	726	429	–
10	83167	40762	20455	11998	7042	4136	2431	1430

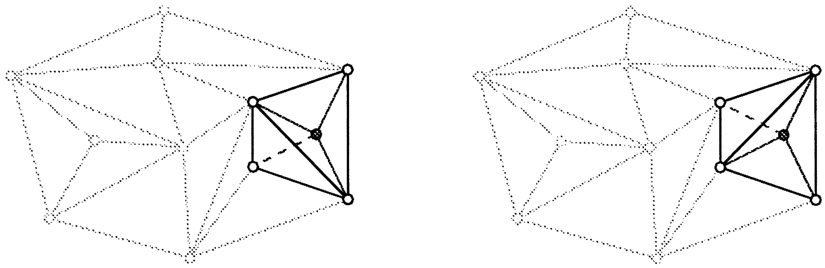

Fig. 5. Different triangulations for $S \setminus \{p\}$ yield the same triangulation for S when flipping towards the dashed edge.

triangulations is *maximized* by such sets. We believe that in fact $\mathcal{N}_{n,h}$ decreases monotonically with increasing h.

Remark 4. A straightforward approach to extending the lower bound construction to pseudo-triangulations which are *not minimum* will fail, as Lemma 2 critically relies on the pointedness of the pseudo-triangulation vertices. The construction would have been interesting for standard triangulations as well; a simple $\Omega(2^n)$ bound would have resulted. Fig. 5 illustrates the problem in this case. Beating the threshold 2^n for triangulations is by no means trivial; the currently best bound [5] is $\Omega(2.33^n)$.

Remark 5. Corollary 1 implies the existence of point sets which asymptotically have more minimum pseudo-triangulations than triangulations: the ‘double-circle’ point set (see footnote 1 in Section 1) admits only $\Theta(2^{\sqrt{12}n-\log n}) = o(C_n)$ triangulations, whereas this set has $\Omega(C_n)$ minimum pseudo-triangulations, by Corollary 1.

4.1. Parity property

The edge flipping operation for pseudo-triangulations enables us to prove a parity property for the number of minimum pseudo-triangulations, which may be useful for checking the correctness of counting algorithms. A similar property is known for the number of crossings in complete geometric graphs, see e.g. [4]. No such observations exist for triangulations.

Lemma 3. *If the number h of extreme points of a set S is even then so is the number of minimum pseudo-triangulations of S.*

Proof. Let $\mathcal{V}$ be the set of all possible minimum pseudo-triangulations for S. Consider the flip graph $\mathcal{F}$ on $\mathcal{V}$, which connects two members of $\mathcal{V}$ if and only if they can be transformed into each other by applying a single edge flip.

Any pseudo-triangulation $T \in \mathcal{V}$ has the same number of edges, namely $2n - 3$ for $n = |S|$. Thus T has exactly $i = 2n - 3 - h$ edges which are interior to the convex hull of S. These edges are known to be just the flippable edges of T, and so the degree of each vertex T in $\mathcal{F}$ is i. Since h is even, i must be odd. Consider the sum of vertex degrees in $\mathcal{F}$, which is $|\mathcal{V}| \cdot i$. Since for any graph this sum is even (as being two times the number of edges), we conclude that $|\mathcal{V}|$ is even, too. □

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