# Reconstruction of the Path Graph 

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#### Abstract

Let $P$ be a set of $n \geq 5$ points in convex position in the plane. The path graph $G(P)$ of $P$ is an abstract graph whose vertices are non-crossing spanning paths of $P$, such that two paths are adjacent if one can be obtained from the other by deleting an edge and adding another edge. We prove that the automorphism group of $G(P)$ is isomorphic to $D_{n}$, the dihedral group of order $2 n$. The heart of the proof is an algorithm that first identifies the vertices of $G(P)$ that correspond to boundary paths of $P$, where the identification is unique up to an automorphism of $K(P)$ as a geometric graph, and then identifies (uniquely) all edges of each path represented by a vertex of $G(P)$. The complexity of the algorithm is $O(N \log N)$ where $N$ is the number of vertices of $G(P)$.


## 1 Introduction

A geometric graph is a graph whose vertices are a finite set of points in general position in the plane, and whose edges are closed segments connecting distinct points. We consider the complete convex geometric graph $K(P)$, in which the vertex set is a convex set $P$ of $n$ points in the plane, and the edges are all segments connecting pairs of vertices. Without loss of generality we will henceforth assume that $P$ is the vertex set of a regular $n$-gon.

Definition 1 Let $P$ be a set of $n$ points in the plane. The path graph $G(P)$ is defined as follows. The vertices of $G(P)$ are the simple (i.e., non-crossing) spanning paths of $K(P)$. Two such vertices are adjacent in $G(P)$ if they differ in exactly two edges, i.e., if one can be obtained from the other by deleting an edge and adding another edge.

[^0]The path graph was introduced in 2001 by Rivera-Campo and UrrutiaGalicia [13 who showed that when $P$ is in convex position, $G(P)$ is Hamiltonian. Following [13, several works studied $G(P)$ in the convex case. Akl et al. 3] showed that $|V(G(P))|=n 2^{n-3}$ and that $\operatorname{diam}(G(P)) \leq 2 n-5$. Chang and $\mathrm{Wu}[6]$ determined the diameter exactly, showing that $\operatorname{diam}(G(P))=2 n-5$ for $n=3$, 4 and $\operatorname{diam}(G(P))=2 n-6$ for $n \geq 5$. Fabila-Monroy et al. [8] showed that the chromatic number of $G(P)$ is $n$. Wu et al. 15] presented algorithms for generating plane spanning paths efficiently. The general (i.e., non-convex) case is less-studied, and it is not known even whether $G(P)$ is connected for all $P$ (see $[3]$ ).

The study of $G(P)$ evolved from the study of the geometric tree graph $\mathcal{T}(P)$ which has all non-crossing spanning trees of $P$ as its vertices, and two vertices are adjacent in $G(P)$ if they differ in exactly two edges. Defined by Avis and Fukuda [4] as the geometric counterpart of the classical tree graph [7], $\mathcal{T}(P)$ was studied in quite a few works, both in the convex and in the general case (e.g., $1,2,9-12]$ ).

Some of the central results on $\mathcal{T}(P)$, such as Hamiltonicity and upper/lower bounds on the diameter (see [4, 10] ) already have counterparts for $G(P)$ (proved in $[3,6,13])$. In this paper we establish a counterpart of another result: exact determination of the automorphism group in the convex case. For $\mathcal{T}(P)$, Hernando et al. 10 showed that $\operatorname{Aut}(\mathcal{T}(P))$ is $D_{n}$, the dihedral group of rotations and reflections of a regular $n$-gon. Since $\operatorname{Aut}(K(P)) \cong D_{n}$, it follows that $D_{n}$ is isomorphic to a subgroup of $\operatorname{Aut}(G(P))$.

Our main result is that there are no other automorphisms on $G(P)$.
Theorem 2. Let $P$ be a set of $n \geq 5$ points in convex position in the plane, and let $G(P)$ be its path graph. Then $\operatorname{Aut}(G(P)) \cong D_{n}$.

The proof of Theorem 2 relies on an algorithm that allows recovering all edges of each path represented by a vertex of $G(P)$ (up to an automorphism of $K(P)$ as a geometric graph), given $G(P)$ as an abstract graph. The algorithm exploits analysis of maximal cliques in $G(P)$, following an approach pioneered by Urrutia-Galicia 14. First, we use the structure of the max-cliques to identify an ordered subset of $n$ vertices of $G(P)$ that corresponds to the boundary paths of $P$, where the identification is fixed up to an automorphism of $K(P)$. Then we show that once the ordered subset is fixed, all edges of each path can be determined uniquely by examining distances between various vertices of $G(P)$. The running time of the algorithm is $O(N \log N)$ where $N=|V(G(P))|$, which is close to optimal, since for each of the $N$ vertices of $G(P)$ we recover the $n-1=\Theta(\log N)$ edges in the path it represents. It should be noted that the determination of $\operatorname{Aut}(\mathcal{T}(P))$ in $\sqrt{10}$ is non-constructive, and no efficient algorithm is known for full recovery of $\mathcal{T}(P)$. In this sense, our result is stronger than the analogous result on $\mathcal{T}(P)$. Likewise, while the technique of Urrutia-Galicia 14 was used in several previous works, this is the first time it is used for complete recovery of $G(P)$, thus solving completely a natural graph reconstruction problem (see, e.g., 5 for a definition and survey of reconstruction problems).

The paper is organized as follows. Hereinafter, we present notations and a simple observation used throughout the paper. In Section 2 we study the structure of maximal cliques in $G(P)$. In Section 3 we prove the main theorem. We conclude the paper with a complexity analysis, in Section 4, and a few open problems.

## Notations

In this section we present notations and simple observations that will be used in the sequel.

Throughout the paper, $P$ is a set of points in convex position in the plane. The edges of $K(P)$, the complete geometric graph on $P$, are divided into two classes: $n$ boundary edges of $\operatorname{Conv}(P)$ and $\binom{n}{2}-n$ diagonals, i.e., edges internal to $\operatorname{Conv}(P)$. We denote the set of boundary edges by $\mathcal{B}(P)$, and say that $x, y \in P$ are neighboring if $(x, y) \in \mathcal{B}(P)$. An automorphism of $K(P)$ as a geometric graph is an automorphism of $K(P)$ as an abstract graph that, in addition, maps crossing edges into crossing edges and non-crossing edges into non-crossing edges.

As defined above, $G(P)$ denotes the (non-crossing) spanning path graph of $P$. For $v \in V(G(P)), P(v)$ denotes the path represented by $v$. For the sake of convenience, we sometimes use the term $P(v)$ also for the edge-set of the path represented by $v$. We stress that we usually denote this edge-set by $v$; the notation $P(v)$ is used for it only in places when the meaning is clear from the context.

The set of boundary edges of $P(v)$, that is, $P(v) \cap \mathcal{B}(P)$, is denoted by $\mathcal{B}(v)$. The set of diagonals of $P(v)$ is denoted by $\mathcal{D}(v)=P(v) \backslash \mathcal{B}(v) . P(v)$ is called a boundary path if all its edges are boundary edges. We denote the set of vertices of $G(P)$ that represent boundary paths by $\mathcal{B}$. Note that while $\mathcal{B}(v)$ denotes the boundary edges of a specific path, $\mathcal{B}$ denotes a subset of the vertices of $G(P)$.

For any graph $G$, the distance between vertices $x, y$, denoted $\operatorname{dist}(x, y)$, is the shortest length of a path in $G$ from $x$ to $y$. The distance of a vertex from a set $\mathcal{C}$ of vertices is defined as $\operatorname{dist}(x, \mathcal{C})=\min _{y \in \mathcal{C}} \operatorname{dist}(x, y)$. The degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ that emanate from $v$, and is denoted by $\operatorname{deg}_{G}(v)$. A vertex is called a leaf if its degree is 1 . An edge is called a leaf edge if one of its endpoints is a leaf. A vertex that is not a leaf is called an internal vertex.

We use the following simple observation on the structure of simple spanning paths of $P$.

Observation 3 Let $S$ be a simple spanning path of a set $P$ of points in convex position in the plane. Then:

1. Both leaf edges of $S$ are boundary edges.
2. If $S$ is not a boundary path, then its leaves cannot be neighboring vertices of the boundary.

The easy proof of the observation is omitted.

## 2 Maximal Cliques in $G(P)$

The reconstruction of the paths represented by vertices of $G(P)$ requires a fulcrum to start with. Our fulcrum is understanding of the maximal cliques in $G(P)$. We note that the approach of exploiting maximal cliques for this purpose was pioneered by Urrutia-Galicia 14 in the context of geometric tree graphs, and used recently in 12 .

Definition $4 A$ max-clique in a graph $G$ is a maximal (with respect to inclusion) clique included in G. Since a max-clique is a complete graph on its vertex set, we shall identify a max-clique with its set of vertices.

We start our discussion of max-cliques with purely combinatorial considerations that do not exploit the geometric nature of the problem. Let $u, v \in$ $V(G(P))$ be neighbors. We denote by $\bar{u}$ and $\bar{v}$ the sets of edges of $P(u)$ and $P(v)$, respectively. Clearly, $|\bar{u} \cup \bar{v}|=n,|\bar{u} \cap \bar{v}|=n-2$, and $|\bar{u} \triangle \bar{v}|=2$. Let $w$ be a common neighbor of $u$ and $v$ in $G(P)$ (if it exists). Since $|P(w)|=n-1$ and $P(w)$ differs from each of $P(v), P(u)$ in exactly two edges, there are exactly two possibilities for $\bar{w}$ :

1. $\bar{w} \cap(\bar{u} \triangle \bar{v})=\emptyset$, and then $(\bar{u} \cap \bar{v}) \subset \bar{w}$, i.e., $\bar{w}$ consists of $\bar{u} \cap \bar{v}$ plus an additional edge,
2. $(\bar{u} \triangle \bar{v}) \subset \bar{w}$, and then $\bar{w} \subset(\bar{u} \cup \bar{v})$, i.e., $\bar{w}$ consists of all edges of $\bar{u} \cup \bar{v}$ except for one edge of $\bar{u} \cap \bar{v}$.

Note that if $w$ satisfies (1), then each other common neighbor of $u, v, w$ (i.e., each other element of the max-clique that contains $u, v, w)$ also satisfies (1). Conversely, each $w, w^{\prime}$ that both satisfy (1) are neighbors. The same holds with (1) replaced by (2). Hence, we obtain:

Corollary 5 Each edge $(u, v) \in E(G(P))$ is contained in at most two maxcliques:

- An intersection max-clique

$$
I(u, v)=\{w \in V(G(P)): \bar{w}=(\bar{u} \cap \bar{v}) \cup\{e\}, \text { for some e } \notin \bar{u} \cap \bar{v}\}
$$

- $A$ union max-clique

$$
U(u, v)=\{w \in V(G(P)): \bar{w}=(\bar{u} \cup \bar{v}) \backslash\{e\}, \text { for some } e \in \bar{u} \cup \bar{v} .
$$

In addition, given three vertices in a max-clique in $G(P)$, they uniquely determine its type.

Note that by this definition, $u, v \in I(u, v)$ and $u, v \in U(u, v)$.
Remark 6 For ease of notation, we call $I(u, v)$ an "intersection max-clique" even if $I(u, v)=\{u, v\}$, i.e., it contains only two vertices. This is a slight abuse of notation, since in such a case, $I(u, v)$ may be properly contained in $U(u, v)$, and thus, not be a max-clique by the definition above. Similarly, we call $U(u, v)$ a "union max-clique" even if $|U(u, v)|=2$.


Fig. 1. These two Hamiltonian paths are neighbors in $G(P)$, and are included in an intersection-clique of size 4 , and in a union-clique of size 2 (only these two paths).

Now we present a geometric characterization of the two types of max-cliques.

Intersection max-clique. Given two neighbors $u, v \in V(G(P))$, the intersection $\bar{u} \cap \bar{v}$ can be viewed as a disjoint union of two simple paths $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$, $\left(y_{1}, y_{2}, \ldots, y_{\ell}\right)$, where $\left\{x_{i}\right\},\left\{y_{j}\right\} \subset P, 1 \leq k, \ell \leq n-1$, and $k+\ell=n$. Each element in $I(u, v)$, including $u$ and $v$, is obtained from $\bar{u} \cap \bar{v}$ by adding one of the four edges $\left(x_{k}, y_{\ell}\right),\left(x_{k}, y_{1}\right),\left(x_{1}, y_{\ell}\right),\left(x_{1}, y_{1}\right)$, such that the resulting path is non-crossing. If none of these four edges crosses edges of $\bar{u} \cap \bar{v}$, we get $|I(u, v)|=4$ (see Figure 1). If (w.l.o.g.) $\left(x_{1}, y_{1}\right)$ crosses $e \in \bar{u} \cap \bar{v}$, w.l.o.g. $e=\left(x_{j}, x_{j+1}\right)$, then $\left(x_{1}, y_{\ell}\right)$ also crosses $e$ (since all the path $\left(y_{1}, \ldots, y_{\ell}\right)$ lies on the same side of $e$ ) and then $|I(u, v)|=2$ (see Figure 2).


Fig. 2. These two Hamiltonian paths are neighbors in $G(P)$, and generate a maximal clique of size 2. I.e., the intersection-clique is identical to the union-clique, both of size 2.


Fig. 3. These two Hamiltonian paths are neighbors in $G(P)$, and are included in an intersection-clique of size 2 and in a union-clique of size $n$.

Union max-clique. Given two neighbors $u, v \in V(G(P))$, where $P(u)=$ $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, it is easy to see that $\bar{u} \cup \bar{v}$ is either of the form $\bar{u} \cup\left\{\left(x_{1}, x_{j}\right)\right\}$ where $2<j \leq n$, or of the form $\bar{u} \cup\left\{\left(x_{j}, x_{n}\right)\right\}$ where $1 \leq j<n-1$. Assume w.l.o.g. the former holds.

Each element in $U(u, v)$, including $u$ and $v$, is obtained from $\bar{u} \cup \bar{v}$ by removing an edge, such that the resulting graph is a non-crossing spanning path. We distinguish between two cases:

- If $\bar{u} \cup \bar{v}=\bar{u} \cup\left\{\left(x_{1}, x_{n}\right)\right\}$ then the edge $\left(x_{1}, x_{n}\right)$ crosses at most one edge of $\bar{u}$ (as otherwise, $P(v)$ cannot be non-crossing). If $\left(x_{1}, x_{n}\right)$ crosses $e \in \bar{u}$, then we must have $\bar{v}=\bar{u} \cup\left\{\left(x_{1}, x_{n}\right)\right\} \backslash\{e\}$ and $|U(u, v)|=2$. If $\left(x_{1}, x_{n}\right)$ does not cross any edge of $\bar{u}$ then $\bar{u} \cup \bar{v}$ is the boundary of $\operatorname{Conv}(P)$, and thus $|U(u, v)|=n$ (see Figure 3).
- If $\bar{u} \cup \bar{v}=\bar{u} \cup\left\{\left(x_{1}, x_{j}\right)\right\}$ for $j<n$ then we must have $\bar{v}=\bar{u} \cup\left\{\left(x_{1}, x_{j}\right)\right\} \backslash$ $\left\{\left(x_{j-1}, x_{j}\right)\right\}$ and then $|U(u, v)|=2$.

Summarizing the above, we have the following.
Corollary 7 Let $P$ be a set of $n \geq 5$ points in convex position in the plane, and let $G(P)$ be the path graph of $P$. Then:

- Each intersection max-clique of $G(P)$ is either of size 2 or 4.
- Among the union max-cliques, all are of size 2 except for a single max-clique of size $n$, in which each vertex represents a boundary path that contains all edges of $\mathcal{B}(P)$ except for one.


## 3 The Automorphism Group of $G(P)$

In this section we show that given $G(P)$ as an abstract graph, we can recover all edges of each path represented by a vertex of $G(P)$, up to an automorphism of
$K(P)$ as a geometric graph. This clearly implies that $\operatorname{Aut}(G(P))$ is the dihedral group of order $2 n$. (For sake of completeness, we prove this easy implication at the end of the section.)

The proof proceeds in three steps:

1. We detect all vertices of $G(P)$ that represent boundary paths. Namely, we find an ordered subset of $n$ vertices of $G(P)$ with a bijection between them and the boundary edges of $K(P)$, fixed up to an automorphism of $K(P)$ as a geometric graph.
2. We divide all vertices of $G(P)$ into levels according to their distance from the family of boundary paths, and use the identification of boundary paths to recover uniquely all boundary edges of each path represented by a vertex of $G(P)$.
3. We use the relation between vertices at adjacent levels to recover uniquely all diagonals of each path represented by a vertex of $G(P)$.

### 3.1 Identification of a "copy" of the boundary of Conv $(P)$ inside $G(P)$

As shown in Section 2, $\mathcal{B}$, the set of vertices of $G(P)$ that represent boundary paths, is a unique max-clique of size $n$ in $G(P)$. This is already a sufficient identification of $\mathcal{B}$ as a set, (i.e., without order), but for sake of obtaining an efficient algorithm for the reconstruction problem, we suggest here an alternative identification of $\mathcal{B}$ as a set, based on the fact that $\mathcal{B}$ is exactly the set of vertices of maximum degree in $G(P)$ :

Claim 8 For any $v \in \mathcal{B}$, $\operatorname{deg}_{G(P)}(v)=3 n-7$, and for any $u \in V(G(P)) \backslash \mathcal{B}$, $\operatorname{deg}_{G(P)}(u)<3 n-7$.

Proof. Let $v \in \mathcal{B}$. Any neighbor of $v$ in $G(P)$ represents a simple Hamiltonian path, obtained from $P(v)$ by deleting an edge and replacing it with another edge. If the deleted edge is a leaf edge of $P(v)$, only one neighbor of $v$ is obtained, and if the deleted edge is an internal edge of $P(v)$, then three neighbors of $v$ are obtained. Indeed, note that deletion of an internal edge transforms $P(v)$ into two boundary paths of total length $n-2$. There are four options to add an edge that will connect these paths into a single Hamiltonian path. Since $P(v)$ is a boundary path, all of them constitute simple paths. Exactly one of them is the original path $P(v)$, and so, deletion of any internal edge contributes 3 neighbors of $v$. Hence,

$$
\operatorname{deg}_{G(P)}(v)=3(n-3)+2=3 n-7
$$

On the other hand, let $u \in V(G(P)) \backslash \mathcal{B}$. By the definition of $\mathcal{B}, P(u)$ contains a diagonal $e$, and the two endpoints of $P(u)$ are located on different sides of $e$. As above, any neighbor of $u$ in $G(P)$ represents a simple Hamiltonian path, obtained from $P(u)$ by deleting an edge and replacing it with another edge. If the deleted edge is a leaf edge of $P(u)$, then after the deletion we are left with a
boundary path of length $n-2$ and an isolated vertex. The new edge replacing the removal boundary edge has to connect the isolated vertex to one of the leaves of the boundary path. However, for one of the two leaves, this edge crosses $e$ and so cannot be added. For the other leaf, we return to the original path $P(u)$. Hence, $u$ has no neighbor in $G(P)$ that is obtained by deleting a leaf edge of $P(u)$. Furthermore, by deleting an internal edge of $P(u)$, at most three neighbors of $u$ can be obtained, as above, and thus $\operatorname{deg}_{G(P)}(v)<3 n-7$.

Now, after identifying $\mathcal{B}$ as a subset of $V(G(P))$, note that each $v \in \mathcal{B}$ can be represented by the unique boundary edge of $P$ that is not contained in $P(v)$, which we denote by $e_{v}$. In order to determine (to the extent possible) what is the boundary edge $e_{v}$ that corresponds to $v$, and thus to identify a copy of the set of boundary edges of $K(P)$ in $G(P)$, we use the following observation.

Observation 9 Let $u, v \in \mathcal{B}$. The edges $e_{u}$ and $e_{v}$ share a vertex if and only if $(u, v)$ is not contained in a maximal clique of size 4 in $G(P)$.

Proof. If $e_{u} \cap e_{v}=\{x\}$ then $P(u) \cap P(v)$ is a two-component forest in which one component is a boundary path $S$ of length $n-2$ and the other component is $\{x\}$. In such a case, each element of $I(u, v)$ is obtained by adding to $P(u) \cap P(v)$ an edge that connects $x$ to an endpoint of $S$. Hence, the only elements of $I(u, v)$ are $u$ and $v$. On the other hand, from Corollary $7,|U(u, v)| \neq 4$, and therefore $(u, v)$ is not contained in any maximal clique of size 4.

If $e_{u} \cap e_{v}=\emptyset$, then $P(u) \cap P(v)$ is a two-component forest in which the components are boundary paths $S, S^{\prime}$ of length $\geq 1$, i.e., contain at least two vertices of $P$. In such a case, there are four different edges connecting an endpoint of $S$ to an endpoint of $S^{\prime}$, and hence, $|I(u, v)|=4$.

Observation 9 allows identifying a "copy" of the boundary of $\operatorname{Conv}(P)$ in $G(P)$, as follows. Define a graph whose vertex set is $\mathcal{B}$, such that $v, w \in \mathcal{B}$ are connected by an edge if and only if $e_{v}, e_{w}$ share a single vertex. Clearly, the resulting graph is a cycle of length $n$. Identify this cycle with the boundary of $\operatorname{Conv}(P)$, in such a way that each boundary edge $e$ corresponds to some $v \in \mathcal{B}$, and each $x \in P$ corresponds to a pair $\{v, w\}$ such that $e_{v} \cap e_{w}=\{x\}$. Note that the identification is fixed only up to an automorphism of $K(P)$ as a geometric graph. However, this is clearly best possible, since any automorphism of $K(P)$ induces an automorphism of $G(P)$.

### 3.2 Recovery of the boundary edges of each path

We divide the vertices of $G(P)$ into levels according to the number of diagonals they contain.

Notation 10 For $v \in V(G(P))$, the level of $v$ is $\ell(v)=|\mathcal{D}(v)|$.
The following observation shows that the levels of the vertices can be recovered from $G(P)$. This observation was made in Lemma 3.2 of [3] in order to show that the diameter of $G(P)$ is at most $2 n-5$. For sake of completeness, we also give a simple proof here.

Observation 11 For each $v \in V(G(P))$, we have $\ell(v)=\operatorname{dist}(v, \mathcal{B})$.
Proof. It is clear from the definition of $\mathcal{B}$ that $\ell(v)=0$ if and only if $v \in \mathcal{B}$, and that for any $v \in V(G(P))$ we have $\operatorname{dist}(v, \mathcal{B}) \geq \ell(v)$. The inequality $\operatorname{dist}(v, \mathcal{B}) \leq$ $\ell(v)$ will follow by induction once we show that each $v \in V(G(P))$ with $\ell(v)>0$ has a neighbor $u \in V(G(P))$ with $\ell(u)=\ell(v)-1$.

Consider a leaf $x$ of $P(v)$. Clearly, exactly one of the boundary edges of $K(P)$ that emanate from $x$ is included in $P(v)$. Denote by $(x, y)$ the boundary edge that is not included in $P(v)$. Since $\ell(v)>0, y$ cannot be a leaf of $P(v)$ (see Observation 3). Thus, $y$ is adjacent in $P(v)$ to $w, z$. Without loss of generality, the points $x, w$ lie on different sides of the edge $(y, z)$ as depicted in Figure 4 . (Otherwise, $x, z$ must lie on different sides of $(y, w)$.) In such a case, $u$, defined by $P(u)=P(v) \cup\{(x, y)\} \backslash\{(y, z)\}$, is a neighbor of $v$ in $G(P)$ that satisfies $\ell(u)=\ell(v)-1$.


Fig. 4. An illustration for the proof of Observation 11

For each $v \in V(G(P))$ of level $d$, there are exactly $d+1$ boundary edges that are not contained in $P(v)$. The following observation shows that these edges can be recovered by observing the elements of $\mathcal{B}$ whose distance from $v$ is exactly $d$. This observation follows from Lemma 5 of 6]. For sake of completeness, we give its simple proof here.

Observation 12 Let $v \in V(G(P))$ with $\ell(v)=d$. Let

$$
\mathcal{B}(P) \backslash \mathcal{B}(v)=\left\{e_{1}, e_{2}, \ldots, e_{d+1}\right\}
$$

The set $\{w \in \mathcal{B}: \operatorname{dist}(w, v)=d\}$ has exactly $d+1$ elements, which are the vertices of $\mathcal{B}$ that correspond to the edges $e_{1}, e_{2}, \ldots, e_{d+1}$.

Proof. It is clear that if $w \in \mathcal{B}$ and $\operatorname{dist}(w, v)=d$, then the only boundary edge not contained in $P(w)$ must be one of $e_{1}, e_{2}, \ldots, e_{d+1}$. On the other hand, let $w \in \mathcal{B}$ be such that $e_{i} \notin P(w)$. We claim that there exists a path of length $d$ in $G(P)$ from $v$ to $w$. By the proof of Observation 11, from each $v^{\prime} \in V(G(P))$ with $\ell\left(v^{\prime}\right)>0$ we can move to a neighbor of lower level by choosing a leaf $x$, adding a boundary edge that emanates from it, and removing another edge. Since each such $v^{\prime}$ has two leaves that are not neighboring on $\mathcal{B}(P)$ (see Observation 3), at each step there are two possible boundary edges that can be added. Hence, we can construct a path in which $e_{i}$ is not added at any step, and thus, is missing also in the path whose level is 0 . That final path must be $P(w)$.

Since the set $\{w \in \mathcal{B}: \operatorname{dist}(w, v)=d\}$ can be detected in $G(P)$, Observation 12 implies that we can recover $\mathcal{B}(v)$ for all $v \in V(G(P))$.

### 3.3 Recovery of the diagonals of each path

Our next goal is the full recovery of $P(v)$ for any path $v \in V(G(P)$ ), i.e., determination whether $(x, y) \in P(v)$ or not for each $(x, y) \in E(K(P))$. We use the following observation.

Observation 13 Let $P_{1}, P_{2}, \ldots, P_{k}$ be disjoint boundary paths, possibly including degenerate (i.e., single-vertex) paths, that cover - in the aforementioned order - all the vertices of $P$. There are at most $k$ possible ways to extend $P_{1} \cup P_{2} \cup \ldots \cup P_{k}$ into a simple spanning path $P(v)$ such that $\mathcal{B}(v)=P_{1} \cup P_{2} \cup \ldots \cup P_{k}$ by adding $k-1$ diagonals.

Proof. It is easy to see that a degenerate $P_{i}$ cannot be an endpoint of a path $P(v)$ such that $\mathcal{B}(v)=P_{1} \cup P_{2} \cup \ldots \cup P_{k}$, and that choosing an endpoint of one of the $P_{i}$ 's to be an endpoint of the path $P(v)$ determines $P(v)$ uniquely (i.e., leaves a single possibility to add the $k-1$ diagonals), see Figure 5. As there are at most $2 k$ such endpoints and each path has two endpoints, at most $k$ different paths can be constructed.

The determination of the diagonals is performed by induction on $\ell(v)$. The case $\ell(v)=0$ is already done, since the boundary edges were recovered in Section 3.2. As the case $\ell(v)=1$ is somewhat different from the other cases, we present it separately.

Let $v \in V(G(P))$ satisfy $\ell(v)=1$. In such a case, $\mathcal{B}(v)$ consists of two paths $P_{1}, P_{2}$. Clearly, neither of them is degenerate, and at least one of them - w.l.o.g., $P_{1}$ - contains at least two edges since $n \geq 5$ (see Figure 6.3. Denote the endpoints of $P_{1}$ by $a, c$ and the vertex of $P_{1}$ adjacent to $a$ by $b$. Furthermore, denote the endpoint of $P_{2}$ adjacent to $c$ by $d$, and the other endpoint of $P_{2}$ by $y$.

So far, we can recover $P_{1}$ and $P_{2}$. After they are recovered, by Observation 13 , there are only two possibilities for $P(v)$ : adding either $(c, y)$ or $(a, d)$. The following observation allows distinguishing between these two cases.

[^1]

Fig. 5. An illustration for Observation 13 where $k=4$. There are two possibilities to add 3 diagonals here (dashed).


Fig. 6. An illustration for the two cases of Observation 14

Observation 14 With the above notations, if $P(v)=P_{1} \cup P_{2} \cup\{(a, d)\}$ then there exists a neighbor $v^{\prime}$ of $v$ such that $\ell\left(v^{\prime}\right)=2$ and $(a, b) \notin P\left(v^{\prime}\right)$. If $P(v)=$ $P_{1} \cup P_{2} \cup\{(c, y)\}$ then there is no such neighbor.

Proof. If $P(v)=P_{1} \cup P_{2} \cup\{(a, d)\}$, then $v^{\prime}$ defined by $P\left(v^{\prime}\right)=P_{1} \cup P_{2} \cup$ $\{(a, d)\} \cup\{(a, c)\} \backslash\{(a, b)\}$ is the desired neighbor. If $P(v)=P_{1} \cup P_{2} \cup\{(c, y)\}$ and a neighbor $v^{\prime}$ is constructed by removing the edge $(a, b)$ (see Figure 6), the added edge must be $(a, d)$ (it must emanate from $a$ as otherwise $a$ is isolated, and the other endpoint must be $d$ as all other vertices are already of degree 2 ), and this is impossible since $(a, d)$ crosses $(c, y)$.

Observation 14 implies that if $\ell(v)=1$ then all edges of $P(v)$ can be recovered. Assume now that $\ell(v):=k-1 \geq 2$ and that for any $v$ with $\ell(v) \leq k-2$ we can recover all edges of $P(v)$. We show that all edges of $P(v)$ can be recovered.

The boundary edges of $P(v)$ can be divided into $k$ (possibly degenerate) paths $P_{1}, P_{2}, \ldots, P_{k}$ that can be recovered by the technique of Section 3.2. Once they


Fig. 7. Illustrations for Observation 15. In (a), $a$ is a leaf of $P(v)$, while in (b) $a$ is not a leaf, but $b$ is a leaf of $P(v)$.
are recovered, by Observation 13 in order to fully recover $P(v)$, it is sufficient to determine which of the endpoints of the $P_{i}$ 's is a leaf of $P(v)$. Note that as mentioned above, a degenerate $P_{i}$ cannot be a leaf of $P(v)$, and that there are at least two non-degenerate $P_{i}$ 's, as any spanning path has at least two boundary edges, and they lie in different $P_{i}$ 's unless the path is a boundary path. The leaves of $P(v)$ can be determined using the following observation.

Observation 15 Let $P_{i}$ be non-degenerate. Denote the endpoints of $P_{i}$ by $a, c$, denote the endpoint of $P_{i+1}$ adjacent to $a$ by $b$ (see Figure $7(a)$ ). Then $a$ is $a$ leaf of $P(v)$ if and only if there exists a neighbor $v^{\prime}$ of $v$ such that $\ell\left(v^{\prime}\right)=k-2$, $(a, b) \in P\left(v^{\prime}\right)$, and $c$ is a leaf of $P\left(v^{\prime}\right)$.

Proof. If $a$ is a leaf of $P(v)$, as depicted in Figure 7 (a), then $(b, c) \in P(v)$. Hence, $v^{\prime}$ defined by $P\left(v^{\prime}\right)=(P(v) \backslash\{(b, c)\}) \cup\{(a, b)\}$ is the desired neighbor.

If both $a$ and $b$ are internal vertices of $P(v)$ then there does not exist a neighbor $v^{\prime}$ with $(a, b) \in P\left(v^{\prime}\right)$, since $P(v) \cup(a, b)$ contains two vertices of degree 3 .

Finally, if $a$ is an internal vertex of $P(v)$ and $b$ is a leaf of $P(v)$, as depicted in Figure 7 (b), then $P_{i+1}$ is not degenerate. Denote its other endpoint by $d$. Then the only neighbor $v^{\prime}$ of $v$ such that $\ell\left(v^{\prime}\right)=k-2$ and $(a, b) \in P\left(v^{\prime}\right)$ satisfies $P\left(v^{\prime}\right)=(P(v) \backslash\{(a, d)\}) \cup\{(a, b)\}$. In $P\left(v^{\prime}\right), d$ is a leaf, and hence, $c$ cannot be a leaf of $P\left(v^{\prime}\right)$ as by Observation 3 this would imply that $v^{\prime}$ is a boundary path, contrary to the assumption $k \geq 3$. This completes the proof.

Combining observations 14 and 15, we can recover all edges of any $v \in$ $V(G(P))$, by induction on $\ell(v)$.

### 3.4 The automorphism group of $G(P)$ is $D_{n}$

As mentioned above, it is clear that any automorphism of $K(P)$ as a geometric graph induces an automorphism of $G(P)$. It is well-known that $\operatorname{Aut}(K(P))=$ $D_{n}$, and thus, $D_{n} \hookrightarrow \operatorname{Aut}(G(P))$ (i.e., $D_{n}$ is isomorphic to a subgroup of $\operatorname{Aut}(G(P)))$.

On the other hand, any automorphism of $G(P)$ must preserve the sizes of the max-cliques, and in particular, preserve the set $\mathcal{B}$. Moreover, it must preserve the information whether for $v, w \in \mathcal{B}$, the edges $e_{v}, e_{w}$ share a vertex (see Observation 9). Hence, it must preserve the identification of a "copy" of the boundary of $\operatorname{Conv}(P)$ in $G(P)$ presented in Section 3.1 (which is defined up to an automorphism of $K(P)$ ). Finally, it follows from the recovery process presented in Sections 3.2 and 3.3 that an automorphism of $G(P)$ is completely determined by its action on the copy of the boundary of $\operatorname{Conv}(P)$ in $G(P)$. Therefore, $\operatorname{Aut}(G(P)) \cong D_{n}$.

## 4 Complexity Analysis

The algorithmic approach presented in the previous sections allows us not only to show that $\operatorname{Aut}(G(P)) \cong D_{n}$, but also to recover the edges of all paths represented by vertices of $G(P)$ efficiently. The following theorem calculates the complexity of our algorithm.

Theorem 16. Let $G(P)$ be the path graph of a set $P$ of $n \geq 5$ points in convex position in the plane, and denote $N:=|V(G(P))|=n 2^{n-3}$ (see [3]). The edges of all paths represented by vertices of $G(P)$ can be recovered in time $O(N \log N)$.

We note that this complexity is not far from optimal, since the graph $G(P)$ contains $N$ vertices, and its recovery requires identifying the path of size $n-1 \approx$ $\log N$ that each vertex represents. In the proof of the theorem we will use an auxiliary lemma. Recall that by Claim 8 , the degree of each vertex $v$ in $G(P)$ is at most $O(n)=O(\log N)$. The lemma asserts that the average degree is much smaller - namely, bounded by a constant.

Lemma $17|E(G(P))|=O(N)$.
The proof of the lemma will be presented at the end of this section, and meanwhile we present the proof of the theorem.

Proof (of Theorem 16 ). We go over the steps of the algorithm that recovers the edges of all paths and calculate the complexity of each step.
Recovery of the boundary paths. As mentioned in Section 3.1, identifying the set $\mathcal{B}$ of all boundary paths as a set, can be done by finding the vertices of degree $3 n-7$ in $G(P)$. The complexity of this step is

$$
\sum_{v \in V(G(P))} \operatorname{deg}(v)=2|E(G(P))|=O(N)
$$

using Lemma 17 .
Detecting a "copy" of the boundary of $\operatorname{Conv}(P)$ in $G(P)$. As mentioned in Section 3.1, once the set $\mathcal{B}$ of vertices that represent the $n$ boundary paths is found, this step can be performed easily by going over all edges spanned by pairs of vertices in $\mathcal{B}$ and checking whether each such edge is contained in a max-clique of size 4 or not. By Corollaries 5 and 7, for each such pair $u, v$, it is sufficient to check whether there exists $w \in V(G(P)) \backslash \mathcal{B}$ which is a common neighbor of $u$ and $v$. Since the number of neighbors of any vertex in $G(P)$ is bounded by $O(\log N)$, the complexity of this step is less than $O\left(\log ^{4} N\right)$ operations.

## Recovering all edges of each path.

We prove that this third step can be performed in $O(N \log N)$ operations, using the following strategy. For each $v \in V(G(P))$, we store three types of information:

1. $\mathcal{B}(v)$ (i.e., the set of boundary edges of $P(v)$ ),
2. $\ell(v)$ (i.e., the level of $v$ ),
3. The endpoints of $P(v)$.

Note that by the proof of Observation 13 , items (1)-(3) yield full recovery of the edges of $\mathcal{P}(v)$.

We go over the vertices of $G(P)$ by levels, starting with level 0 , then level 1 (i.e., the neighbors of the vertices in level 0 that were not dealt with yet), then level 2, etc.

For each $v \in V(G(P))$ with $\ell(v)=i \geq 2$, items (1)-(3) for $v$ can be computed instantly given items (1)-(3) for all neighbors of $v$ at level $(i-1)$ (as described in Observation 15 and in the proof of Observation 11).

For vertices with $\ell(v)=1$, recovery of item (3) requires the knowledge of items (1)-(2) for their neighbors at levels 0,2 (as described in Observation 14). Hence, after computing items (1)-(3) for all vertices at level 0 , we compute items (1)-(2) for the vertices at level 1 , then items (1)-(2) for vertices at level 2 , then item (3) for vertices of level 1 , then item (3) for vertices at level 2 , and then all items in increasing order of levels.

The treatment of each vertex $v$ requires going over each neighbor $u$ of $v$, and (in the worst case) reading the information-type $\mathcal{B}(u)$ whose size is at most $n-1$. Eventually, each edge of $G(P)$ is considered twice, where each treatment requires $O(n)$ operations, and thus, by Lemma 17 , the total number of operations is bounded by $O(n N)=O(N \log N)$. Therefore, the total time complexity of our algorithm is $O(N \log N)$, as asserted.

Now, it only remains to prove Lemma 17
Proof (of Lemma $\sqrt{17}$ ). By symmetry, we may consider the vertices $v$ of $G(P)$ that correspond to paths $P(v)$ in which one leaf $x_{0} \in P$ is fixed, and then multiply the result by $n$. We represent any such path $P(v)=\left\langle x_{0}, x_{1}, \ldots x_{n-1}\right\rangle$ by a binary vector $\left\langle\alpha_{0}, \alpha_{1}, \ldots \alpha_{n-1}\right\rangle$ where $\alpha_{0}=0$, and $\alpha_{i}=0$ if and only if the
edge $\left(x_{i-1}, x_{i}\right)$ in $P(v)$ is a boundary edge. Note that $\alpha_{n-1}=0$. Assume that $P(v)$ contains at least two diagonals. We call $P(v)$ a path of type $t$ if

$$
t=\min _{i}\left\{\alpha_{i} \neq 0\right\}+\min _{j}\left\{\alpha_{n-j} \neq 0\right\}
$$

namely, if $P(v)$ starts with $k-1$ boundary edges and ends with $l-1$ boundary edges, for some $k, l \geq 2$ such that $k+l=t$.

We observe that a neighbor of $v$ in $P(v)$ can be obtained only by deleting one out of the first $k$ edges or the last $l$ edges of $P(v)$, and adding another edge instead. Indeed, deletion of any other edge of $P(v)$ decomposes $P(v)$ into two paths, where a leaf of the first one is $x_{0}$, a leaf of the second one is $x_{n-1}$, the diagonal $\left(x_{k-1}, x_{k}\right)$ belongs to the first path and separates $x_{0}$ from the second path, and the diagonal $\left(x_{n-l-1}, x_{n-l}\right)$ belongs to the second path and separates $x_{n-1}$ from the first path. Therefore, there does not exist any edge that can be added to the union of these two paths in order to form a simple path (except for the deleted edge).

On the other hand, for any deletion of one of the first $k$ edges or one of the last $l$ edges of a path $P(v)$ of type $t$, there exist at most 4 edges that can be added to the union of the two paths in order to obtain a Hamiltonian path. Hence, the number of neighbors of $v$ in $G(P)$ is bounded by $3(k+l)=3 t$.

In addition, for any path of type $t$ there are $t-3$ possible choices of $k, l$ as above, and thus, the number of paths of type $t$ whose endpoint is $x_{0}$ is bounded by $O\left(t \cdot 2^{n-1-t}\right)$, which implies that the total number of paths of type $t$ is bounded by $O\left(n \cdot t \cdot 2^{n-1-t}\right)$. To conclude, the number of edges of $G(P)$ of the form $\left(v, v^{\prime}\right)$ where $P(v)$ is a path of type $t$ with at least two diagonals, is bounded by $O\left(n \cdot t^{2} \cdot 2^{n-1-t}\right)$.

The number of edges of $G(P)$ of the form $\left(v, v^{\prime}\right)$ where $P(v)$ and $P\left(v^{\prime}\right)$ are paths that contain at most one diagonal, is bounded by $O\left(n^{3}\right)$ and thus is negligible.

Putting things together, the number of edges in $G(P)$ is at most

$$
\begin{aligned}
|E(G(P))| & =O\left(\sum_{t=4}^{n-1} n \cdot t^{2} \cdot 2^{n-1-t}\right)=O\left(n \cdot 2^{n-1} \cdot \sum_{t=4}^{n-1} \frac{t^{2}}{2^{t}}\right) \\
& \leq O\left(n \cdot 2^{n-1} \cdot \sum_{t=0}^{\infty} \frac{(t+2)(t+1)}{2^{t}}\right)=O\left(n \cdot 2^{n-1}\right) \\
& =O(N)
\end{aligned}
$$

where the penultimate equality follows from the well-known equality

$$
\sum_{t=0}^{\infty} \frac{(t+2)(t+1)}{2^{t}}=16
$$

that can be easily proved by differentiating twice the series $\sum_{t=0}^{\infty} x^{t}$ and substituting $x=0.5$.

## Open Problems

We conclude this paper with a few questions for further research that stem from our results.

The automorphism group of other subgraphs of $\mathcal{T}(P)$. In [11], Hernando showed that if $P$ is a set of points in convex position and $\mathcal{T}(P)$ is its geometric tree graph, then $\operatorname{Aut}(\mathcal{T}(P)) \cong D_{n}$, as we showed for $G(P)$. In view of the fact that $G(P)$ is a subgraph of $\mathcal{T}(P)$, it is reasonable to ask whether $\operatorname{Aut}\left(G^{\prime}(P)\right) \cong$ $D_{n}$ holds also for other subgraphs $G(P) \subset G^{\prime}(P) \subset \mathcal{T}(P)$. For example, does this hold for the graph of simple spanning trees with maximal degree $\leq d$ ?

Points in general position. What can be said if the points of $P$ are in general (rather than convex) position? Can we prove that $\operatorname{Aut}(G(P)) \cong \operatorname{Aut}(\mathrm{K}(\mathrm{P}))$ ?

Abstract graphs. What happens in the abstract case? That is, if $G^{\prime}(P)$ is the path graph of abstract $K(P)$, is this true that $\operatorname{Aut}\left(G^{\prime}(P)\right) \cong \operatorname{Aut}(K(P)) \cong S_{n}$ ? It was shown in 12 that this holds for the tree graph of $K(P)$.

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[^1]:    ${ }^{3}$ One can check easily that if $n=4$, then our main theorem does not hold, because of the symmetry between pairs of paths in level 1.

