Largest triangles in a polygon*

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Abstract

We study the problem of finding maximum-area triangles that can be inscribed in a polygon in the plane. We consider eight versions of the problem: we use either convex polygons or simple polygons as the container; we require the triangles to have either one corner with a fixed angle or all three corners with fixed angles; we either allow reorienting the triangle or require its orientation to be fixed. We present exact algorithms for all versions of the problem. In the case with reorientations for convex polygons with n vertices, we also present $(1-\varepsilon)$-approximation algorithms.

1 Introduction

We study the problem of finding maximum-area triangles that are inscribed in a polygon in the plane. When the shape of the triangle is fully prescribed, this problem is related to *the polygon containment problem*, which for given two polygons P and Q, asks for the largest copy of Q that can be contained in P using rotations, translations, and scaling. The problem is related to the problem of *inscribing polygons* if the shape is partially prescribed. In inscribing polygons, we are given a polygon P and seek to find a best polygon with some specified number of vertices that can be inscribed in P with respect to some measures.

Problems of this flavor have a rich history and are partly motivated by the attempt to reduce the complexity of various geometric problems, including the shape recognition and matching problems, arising in various applications in pattern recognition, computer vision and computational geometry [7, 13, 21]. Chapter 30.5 in the Handbook of Discrete and Computational Geometry [20] provides a survey on the related works.

There has been a fair amount of work on inscribing a maximum-area convex k-gon in a polygon. A maximum-area convex k-gon inscribed in a convex n-gon can be computed in $O(kn + n\log n)$ time [2, 15]. The best algorithm for computing a maximum-area convex polygon inside a simple n-gon takes $O(n^7)$ time and $O(n^5)$ space [9]. Hall-Holt et al. [18] gave an $O(n\log n)$-time $O(1)$-approximation algorithm for finding a maximum-area convex polygon inscribed in a simple n-gon. Melissaratos et al. [17] gave an algorithm for finding a maximum-area triangle inscribed in a simple n-gon in $O(n^4)$ time. When the maximum-area triangle is restricted to have all its corners on the polygon boundary, and it takes $O(n^3)$ time.

*This research was supported by the Institute of Information & communications Technology Planning & Evaluation(IITP) grant funded by the Korea government(MSIT) (No. 2017-0-00905, Software Star Lab (Optimal Data Structure and Algorithmic Applications in Dynamic Geometric Environment))

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	Convex polygons		Simple polygons	
Fixed angles	All	One	All	One
Axis-aligned	$O(\log n)$ [16] (homothet)	$O(\log n)$	$O(n \log n)$ (homothet)	$O(n^2 \log n)$
Reorientations	$O(n^2)$ $O(\varepsilon^{-\frac{1}{2}} \log n + \varepsilon^{-1})$	$O(n^3)$ $O(\varepsilon^{-\frac{1}{2}} \log n + \varepsilon^{-1} \log \varepsilon^{-\frac{1}{2}})$	$O(n^2 \log n)$	$O(n^4)$

Table 1: Time complexities of the algorithms. All algorithms use $O(n)$ space.

For finding a maximum-area copy of a given polygon Q that can be inscribed in a polygon P, there are results known for cases of convex, orthogonal, and simple polygons, possibly with holes. A maximum-area copy of a convex k-gon that can be inscribed in a convex n-gon can be computed in $O(n + k \log k)$ time [19, 14] under translation and scaling, and in $O(nk^2 \log k)$ time [1] under translation, scaling, and rotation. For a maximum-area homothet[1] of a given triangle inscribed in a convex polygon P with n vertices, Kirkpatrick and Snoeyink gave an $O(\log n)$-time algorithm to find one [16], given the vertices are stored in order along the boundary in an array or balanced binary search tree. The maximum-area equilateral triangles of arbitrary orientation inscribed in a simple n-gon can be computed in $O(n^3)$ time [11].

There also have been works on finding a maximum-area partially prescribed shape that can be inscribed in a polygon. Amenta showed that a maximum-area axis-aligned rectangle inscribed in a convex n-gon can be found in linear time by phrasing it as a convex programming problem [6]. When the vertices are already stored in order along the boundary in an array or balanced binary search tree, the running time was improved to $O(\log^2 n)$ [12], and then to $O(\log n)$ [4]. Cabello et al. [8] considered the maximum-area and maximum-perimeter rectangle of arbitrary orientation inscribed in a convex n-gon, and presented an $O(n^3)$-time algorithm. Very recently, Choi et al. [10] gave $O(n^3 \log n)$-time algorithm for finding maximum-area rectangles of arbitrary orientation inscribed in a simple n-gon, possibly with holes. However, little is known for the case of partially prescribed triangles inscribed in convex and simple polygons, except a PTAS result by Hall-Holt et al. [18] for finding a maximum-area *fat*[2] triangle that can be inscribed in a simple n-gon.

1.1 Our results

We study the problem of finding maximum-area triangles that can be inscribed in a polygon in the plane. We consider eight versions of the problem: we use either convex polygons or simple polygons as the container; we require the triangles to have either one corner with a fixed interior angle or all three corners with fixed interior angles; we either allow reorienting the triangle or require its orientation to be fixed. We study all versions of the problem in this paper and present efficient algorithms for them. Table 1 summarizes our results.

We assume that the vertices of the input polygon are stored in order along its boundary in an array or a balanced binary search tree. We say a triangle is *axis-aligned* if one of its sides is parallel to the x-axis, and we call the side the *base* of the triangle. We say a triangle has one fixed angle if one of the two interior angles at corners incident to the base of the triangle is fixed.

For a convex polygon P with n vertices, a maximum-area homothet of a given triangle that can be inscribed in P can be computed in $O(\log n)$ time [16]. For axis-aligned triangles with one fixed angle, we present an algorithm that computes a maximum-area such triangle that can be

[1]Two shape are homothetic if one can be obtained from the other by scaling and translation.
[2]A triangle is δ-fat if all three of its angles are at least some specific constant δ.

inscribed in P in $O(\log n)$ time using $O(n)$ space.

When reorientations are allowed, we present an algorithm that computes a maximum-area triangle with fixed interior angles that can be inscribed in P in $O(n^2)$ time using $O(n)$ space. We also present an $(1-\varepsilon)$-approximation algorithm that takes $O(\varepsilon^{-\frac{1}{2}}\log n + \varepsilon^{-1})$ time. For triangles with one fixed angle, we present an algorithm to compute a maximum-area triangle that can be inscribed in P in $O(n^3)$ time using $O(n)$ space. We also present an $(1-\varepsilon)$-approximation algorithm that takes $O(\varepsilon^{-\frac{1}{2}}\log n + \varepsilon^{-1}\log\varepsilon^{-\frac{1}{2}})$ time.

For a simple polygon P with n vertices, we present an algorithm that computes a maximum-area homothet of a given triangle that can be inscribed in P in $O(n\log n)$ time using $O(n)$ space. We also present an algorithm to compute a maximum-area axis-aligned triangle with one fixed angle that can be inscribed in P in $O(n^2\log n)$ time using $O(n)$ space.

When reorientations are allowed, we present an algorithm to compute a maximum-area triangle with fixed interior angles that can be inscribed in P in $O(n^2\log n)$ time using $O(n)$ space. For triangles with one fixed angle, we present an algorithm to compute a maximum-area triangle that can be inscribed in P in $O(n^4)$ time using $O(n)$ space.

Whenever we say a *largest triangle*, it refers to a maximum-area triangle inscribed in P. We denote the triangle with three corners p, q, r (counterclockwise order) by $\triangle pqr$, where pq is base. For two fixed angles $\alpha, \beta > 0$, we call the triangle with $\angle rpq = \alpha$ *an α-triangle* and the triangle with $\angle rpq = \alpha$ and $\angle pqr = \beta$ *an (α,β)-triangle*. Let $\mathsf{area}(T)$ denote the area of a triangle T.

2 Largest Triangles in a Convex Polygon

Consider a convex polygon P with n vertices in the plane. We show for fixed angles α and β how to find largest (α,β)-triangles and largest α-triangles, aligned to the x-axis or of arbitrary orientation, that can be inscribed in P.

2.1 Largest (α,β)-triangles

Since all interior angles of an (α,β)-triangle are fixed, this problem is to find a largest copy of a given triangle that can be inscribed in P using rotations, translations, and scaling. When the orientation of triangles is fixed, the problem reduces to finding a largest *homothet* of a given triangle that can be inscribed in P. A *homothet* of a figure is a scaled and translated copy of the figure.

A largest homothet of a given triangle that can be inscribed in P can be computed in $O(\log n)$ time [16]. Thus, we focus on the case in which arbitrary orientations are allowed. This problem is similar to finding a largest equilateral triangle in a convex polygon [11]. A largest (α,β)-triangle in a convex polygon P must have at least one corner lying on a vertex of P by the same argument for largest equilateral triangles in Theorem 1 of [11].

Consider an (α,β)-triangle $\triangle t_0t_1t_2$. Let $\Phi_v(s,\vartheta)$ denote the affine transformation that scales s and rotates ϑ in counterclockwise direction around a vertex v of P. Let

$$\varphi_{0,v} = \Phi_v(\frac{\sin\beta}{\sin(\alpha+\beta)}, \alpha),\ \varphi_{1,v} = \Phi_v(\frac{\sin(\alpha+\beta)}{\sin\alpha}, \beta),\ \varphi_{2,v} = \Phi_v(\frac{\sin\alpha}{\sin\beta}, \pi-\alpha-\beta).$$

For t_i lying on a vertex v of P, we observe that $t_{i+1}, t_{i+2} \in P$ if and only if $t_{i+2} \in P \cap \varphi_{i,v}(P)$ with indices under modulo 3. See Figure 1(a) for an illustration. Thus, for a fixed vertex v of P, we can compute a largest triangle $\triangle t_0t_1t_2$ with t_0 at v in $O(n)$ time by finding the longest segment vt_{i+2} contained in $P \cap \varphi_{i,v}(P)$. By repeating this for every vertex v of P such that corner t_i lies on v for $i = 0, 1, 2$, a largest (α,β)-triangle can be computed in $O(n^2)$ time.

Theorem 1. *Given a convex polygon P with n vertices in the plane and two angles α, β, we can compute a maximum-area (α, β)-triangle of arbitrary orientations that can be inscribed in P in $O(n^2)$ time.*

We give an example of a convex polygon with $\Omega(n^2)$ combinatorially distinct candidates of an optimal triangle with $\alpha = \beta = 60°$. Let P be a convex polygon with $3n$ vertices such that $2n$ vertices of P are placed uniformly on a circular arc of interior angle $120°$, and the remaining n vertices are placed densely in the neighborhood of the center of the arc as shown Figure 1(b). If v is one of the n vertices near the center, $P \cap \varphi_{0,v}(P)$ has $2n-1$ vertices along by the arc, and thus there are $\Theta(n)$ candidates of t_2 for each such vertex v to consider for the longest vt_2 in $P \cap \varphi_{0,v}(P)$. This gives $\Theta(n^2)$ candidates for the longest vt_2 of similar lengths, and thus $\Theta(n^2)$ combinatorially distinct (α, β)-triangles with side vt_2. Any algorithm iterating over all such triangles takes $\Omega(n^2)$ time.

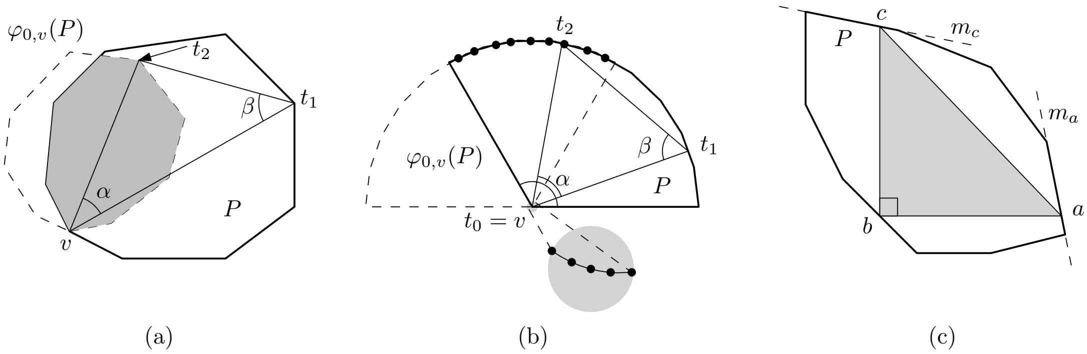


Figure 1: (a) A largest (α, β)-triangle $\triangle t_0 t_1 t_2$ with t_0 lying on a vertex v of P is determined by the longest segment vt_2 contained in $P \cap \varphi_{0,v}(P)$. (b) An example of $\Theta(n^2)$ combinatorially distinct (α, β)-triangles with side vt_2 to consider for an optimal triangle. (c) The largest axis-aligned right triangle with $m_a < 0$ and $m_c < 0$.

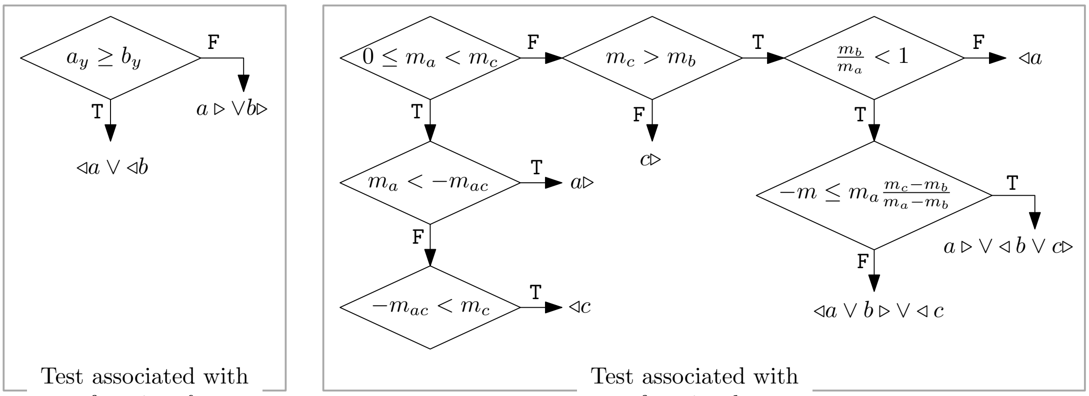
2.2 Largest α-triangles

This problem is to find a largest triangle with one corner angle fixed to a constant α that can be inscribed in a convex polygon P. We consider α-triangles that are either axis-aligned or of arbitrary orientations.

2.2.1 Largest axis-aligned α-triangles

We start with an algorithm to compute a largest axis-aligned α-triangle for $\alpha = 90°$. Alt et al. [5] presented an algorithm of computing a largest axis-aligned rectangle that can be inscribed in a convex polygon. We follow their approach with some modification. If two corners of the triangle are on the polygon boundary, the algorithm by Alt et al., works to compute a largest axis-aligned right triangle that can be inscribed in P.

So in the following, we focus on the case that a largest axis-aligned right triangle has all its corners on the boundary of P. Consider a largest axis-aligned right triangle $\triangle bac$ with all three corners on the boundary of P. See Figure 1(c) for an illustration. Let m_a and m_c denote the slopes of the polygon edges where a and c lie, respectively, and let m_{ac} denote the slope of ac. Then, either (1) $m_a < 0$ or $m_c < 0$, or (2) $m_a > 0$ and $m_c > 0$. Observe that no rectangle containing a largest axis-aligned right triangle belonging to case (1) is contained in P, and thus the algorithm by Alt et al. fails to find a largest axis-aligned right triangle belonging to the case.



Test associated with function f.

Test associated with function h.

Figure 2: Modified tests associated with functions f and h.

We compute a largest axis-aligned right triangle in $O(\log n)$ time using the tentative prune-search algorithm [5] by replacing the tests associated with functions f and h by the ones in Figure 2. Using the tests, we can determine a half of the candidate triples of polygon edges in which a largest axis-aligned right triangle cannot have their corners, and continue to find a largest axis-aligned right triangle on the remaining half of the candidate triples of polygon edges.

The algorithm to compute a largest axis-aligned right triangle can compute the axis-aligned α-triangle using linear transformation $L_\alpha = \begin{pmatrix} 1 & \cot\alpha \\ 0 & 1 \end{pmatrix}^{-1}$ for α. Then, $L_\alpha(T)$ is an axis-aligned right triangle for an α-triangle T. Observe that T is a largest axis-aligned α-triangle inscribed in P if and only if $L_\alpha(T)$ is a largest axis-aligned right triangle inscribed in $L_\alpha(P)$, a convex polygon. However, it takes $O(n)$ time for computing entire description of $L_\alpha(P)$. To reduce the time complexity, we compute $L_\alpha(p)$ only when we need the slope of the polygon edge containing p or the right side of an α-triangle with corner at p on the polygon boundary. Since there are $O(\log n)$ decision steps in the algorithm and each decision step uses only a constant number of points on the polygon boundary, we have following theorem.

Theorem 2. *Given a convex polygon P of n vertices in the plane and an angle α, we can find a maximum-area axis-aligned α-triangle that can be inscribed in P in $O(\log n)$ time.*

2.2.2 Largest α-triangles of arbitrary orientations

We can compute a largest α-triangle of arbitrary orientations by simply iterating over all triples of edges of P. For each triple of edges of P, we can find a largest α-triangle T with corners on the edges of the triple in $O(1)$ time.

Theorem 3. *Given a convex polygon P of n vertices in the plane and an angle α, we can find a maximum-area α-triangle of arbitrary orientation that can be inscribed in P in $O(n^3)$ time using $O(n)$ space.*

One may wonder if the running time can be improved. Cabello et al. showed a construction of a convex polygon with n vertices that has $\Theta(n^3)$ combinatorially distinct rectangles that can be inscribed in the polygon. By using a similar construction, we can show that there are $\Theta(n^3)$ combinatorially distinct α-triangles that can be inscribed in a convex polygon with n vertices. Thus, any algorithm iterating over all those combinatorially distinct triangles takes $\Omega(n^3)$ time.

2.3 FPTAS in arbitrary orientations

Let T_{opt} be a largest (α, β)-triangle that can be inscribed in P. We can compute an (α, β)-triangle inscribed in P whose area is at least $(1-\varepsilon)$ times $\mathsf{area}(T_{\text{opt}})$ in $O(\varepsilon^{-\frac{1}{2}} \log n + \varepsilon^{-1})$ time using ε-kernel [8]. For any $\varepsilon \in (0,1)$, an *ε-kernel* for a convex polygon P is a convex polygon P_ε such that for all unit vectors u in the plane, $(1-\varepsilon)w(P,u) \le w(P_\varepsilon, u)$, where $w(P,u)$ is the length of the orthogonal projection of P onto any line parallel to u.

Theorem 4. *Given a convex polygon P with n vertices in the plane, two angles α, β, and $\varepsilon > 0$, we can find an (α, β)-triangle that can be inscribed in P and whose area is at least $(1-\varepsilon)$ times the area of a maximum-area (α, β)-triangle inscribed in P in $O(\varepsilon^{-\frac{1}{2}} \log n + \varepsilon^{-1})$ time.*

Proof. By Lemma 1 in [8], an ε-kernel P_ε of P has $O(\varepsilon^{-\frac{1}{2}})$ vertices and it can be computed in $O(\varepsilon^{-\frac{1}{2}} \log n)$ time. A largest (α, β)-triangle inscribed in $P_{\frac{\varepsilon}{32}}$ has area at least $(1-\varepsilon)$ times the area of the largest (α, β)-triangle inscribed in P by Lemma 8 in [8], and it can be computed in $O(\varepsilon^{-1})$ time. $\square$

2.3.1 Largest α-triangles of arbitrary orientations

Let T_{opt} denote a largest α-triangle that can be inscribed in P. We can compute an α-triangle inscribed in P whose area is at least $(1-\varepsilon)$ times $\mathsf{area}(T_{\text{opt}})$ in $O(\varepsilon^{-\frac{1}{2}} \log n + \varepsilon^{-\frac{3}{2}})$ time using the algorithm by Cabello et al. [8]. We can improve the time complexity further to $O(\varepsilon^{-\frac{1}{2}} \log n + \varepsilon^{-1} \log \varepsilon^{-\frac{1}{2}})$ using the approximation method by Ahn et al. [3].

We use d to denote the *diameter* of a convex polygon P which is the maximum distance between any two points in P, and w to denote the *width* of P which is the minimum distance between two parallel lines enclosing P. Let $c_1 = \min\{\frac{1}{16}, \frac{\cot(\alpha/2)}{4}, \frac{|\tan\alpha|}{4}\}$ and $c_2 = 2\min\{\frac{1-\cos\alpha}{\alpha}, \frac{1+\cos\alpha}{\pi-\alpha}\}$ be the constants defined by α.

Lemma 5. *$\mathsf{area}(T_{\text{opt}}) \ge c_1 dw$.*

Proof. Let pq be a diameter of P, and let R be a rectangle circumscribed to P with two sides parallel to pq such that P touches all four sides of R. Let w' be the side of R orthogonal to pq. See Figure 3(a).

Consider two interior-disjoint triangles with pq as the base and total height w' that are inscribed in P. Let $\triangle pqr$ be one of the triangles whose height is at least $\frac{w'}{2}$. Without loss of generality, assume that the bisecting line of pq intersects the boundary of qr at s and let $\angle pqs = \gamma$.

If $\alpha < \gamma$ or $\alpha > \pi - 2\gamma$, then either $\triangle qtp$ (with $\mathsf{area}(\triangle qtp) = \frac{\tan\alpha}{4} d^2$) or $\triangle tpq$ (with $\mathsf{area}(\triangle tpq) = \frac{\cot(\alpha/2)}{4} d^2$) is an α-triangle, where t is the point on the bisecting line of pq achieving $\angle pqt = \alpha$ or $\angle qtp = \alpha$. See Figure 3(b). If $\gamma \le \alpha \le \pi - 2\gamma$, $\triangle tqs$ (with $\mathsf{area}(\triangle tqs) \ge \frac{1}{16} dw$) is an α-triangle inscribed in P, where t is the point on pq achieving $\angle stq = \alpha$ or $\angle qst = \alpha$, while satisfying $\angle stq \le \frac{\pi-\gamma}{2}$. See Figure 3(c). Since $w \le d$, the lemma holds. $\square$

Let $\overrightarrow{u}$ be one of the directions of the lines defining the width of P. Let ϑ be the angle from $\overrightarrow{u}$ to the ray from p bisecting $\angle rpq$ for a largest α-triangle $\triangle pqr$ inscribed in P in counterclockwise direction, where $-\pi \le \vartheta \le \pi$. Then we have the following technical lemma.

Lemma 6. $\min\{\left||\vartheta| - \frac{\alpha}{2}\right|, \left|\pi - \frac{\alpha}{2} - |\vartheta|\right|\} \le \frac{w}{c_1 c_2 d}$.

Proof. Clearly, T_{opt} is contained in the strip of w. Then $\mathsf{area}(\triangle pqr) \le \frac{1}{2} \frac{w}{|\sin(|\vartheta| - (\alpha/2))|} \cdot \frac{w}{|\sin(|\vartheta| + (\alpha/2))|} = \frac{w^2}{|\cos\alpha - \cos|2\vartheta||}$. See Figure 4.

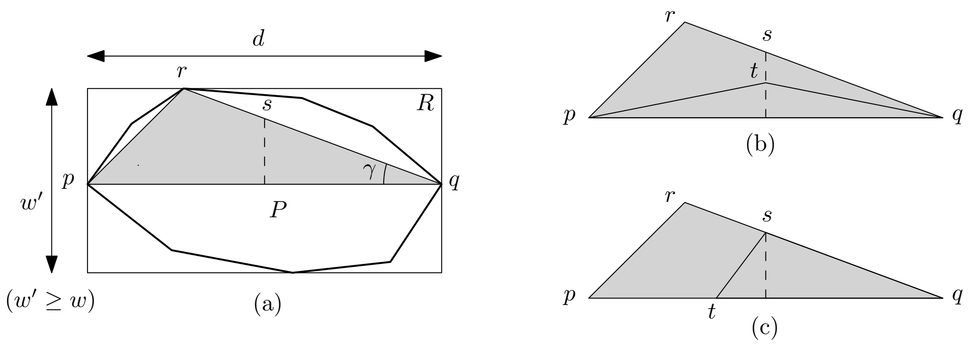


Figure 3: (a) Proof of Lemma 5. (b) $\alpha < \gamma$ or $\alpha > \pi - 2\gamma$. (c) $\gamma \le \alpha \le \pi - 2\gamma$.

For $\theta \in [0, \frac{\pi}{2}]$, $|\cos\alpha - \cos 2\theta| \ge 2\min\{\frac{1-\cos\alpha}{\alpha}, \frac{1+\cos\alpha}{\pi-\alpha}\}|\theta - \frac{\alpha}{2}|$. Observe also that the graph of $|\cos\alpha - \cos 2\theta|$ is symmetric with respect to $\theta = \frac{\pi}{2}$. Therefore, $c_1 dw \le \mathsf{area}(T_{\mathrm{opt}}) \le \frac{w^2}{|\cos\alpha - \cos|2\vartheta||}$, and $d_\alpha \min\{\big||\vartheta| - \frac{\alpha}{2}\big|, \big|\pi - \frac{\alpha}{2} - |\vartheta|\big|\} \le |\cos\alpha - \cos|2\vartheta|| \le \frac{w}{c_1 d}$. $\square$

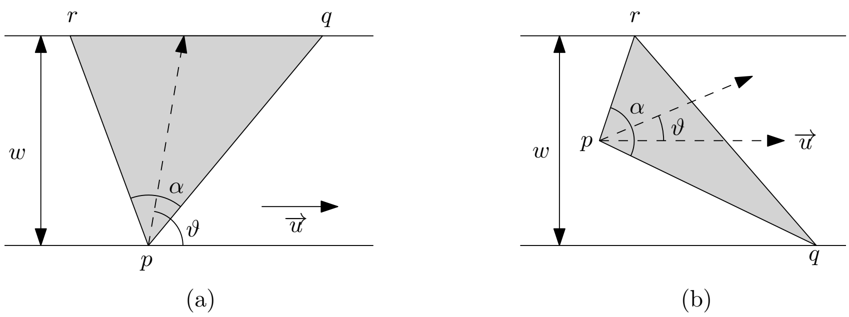


Figure 4: $\mathsf{area}(T_{\mathrm{opt}}) \le \mathsf{area}(\triangle pqr)$ if (a) $\frac{\alpha}{2} < |\vartheta| < \pi - \frac{\alpha}{2}$, or (b) $|\vartheta| < \frac{\alpha}{2}$ or $\pi - \frac{\alpha}{2} < |\vartheta|$.

Let $T_{\mathrm{opt}}(\theta)(-\pi \le \theta \le \pi)$ denote a largest α-triangle $\triangle pqr$ such that the angle from $\overrightarrow{u}$ to the ray from p bisecting $\angle rpq$ in counterclockwise direction is θ.

Lemma 7. *Given $\varepsilon > 0$, $\mathsf{area}(T_{\mathrm{opt}}(\vartheta + \delta)) > (1-\varepsilon)\mathsf{area}(T_{\mathrm{opt}}(\vartheta))$ for $|\delta| \le \min\{\frac{\alpha}{2}, \frac{\pi-\alpha}{2}, \frac{c_1 w}{2d}\varepsilon\}$.*

Proof. Let $\triangle pqr$ be $T_{\mathrm{opt}}(\vartheta)$. Without loss of generality, assume $|pq| \le |pr|$. Let p' and q' be points on pq and qr, respectively, and let $\angle p'rp = \angle q'pq = |\delta|$. Also, let t be the intersection point of pq' and rp'. See Figure 5 for an illustration.

If $|\delta| \le \min\{\frac{\alpha}{2}, \frac{\pi-\alpha}{2}\}$, then

$$\begin{aligned}
\mathsf{area}(T_{\mathrm{opt}}(\vartheta)) - \mathsf{area}(T_{\mathrm{opt}}(\vartheta+\delta)) &\le \mathsf{area}(\triangle pqr) - \mathsf{area}(\triangle tq'r) \\
&\le \mathsf{area}(\triangle rpp') + \mathsf{area}(\triangle pqq') \\
&\le \frac{1}{2}\Big(\frac{\sin\alpha}{\sin(\alpha + |\delta|)}|pr|^2 + |pq|^2\Big)\sin|\delta| \\
&\le (|pq|^2 + |pr|^2)\sin|\delta| \le 2\max(|pq|^2, |pr|^2)\sin|\delta|.
\end{aligned}$$

By Lemma 5, $1 - \frac{\mathsf{area}(T_{\mathrm{opt}}(\vartheta+\delta))}{\mathsf{area}(T_{\mathrm{opt}}(\vartheta))} \le \frac{2\max(|pq|^2,|pr|^2)}{\mathsf{area}(T_{\mathrm{opt}}(\vartheta))}\sin|\delta| \le \frac{2d^2}{c_1 wd}\sin|\delta| \le \frac{2d}{c_1 w}|\delta| \le \varepsilon$, and the lemma follows. $\square$

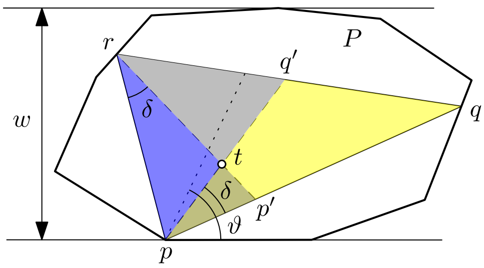


Figure 5: $\mathsf{area}(\triangle pqr) - \mathsf{area}(\triangle tq'r) \leq \mathsf{area}(\triangle rpp') + \mathsf{area}(\triangle pqq')$ and $\mathsf{area}(T(\vartheta+\delta)) \geq \mathsf{area}(\triangle tq'r)$ in proof of Lemma 7.

Lemma 8. *Given a convex polygon P with n vertices in the plane, an angle α, and $\varepsilon > 0$, we can find an α-triangle that can be inscribed in P and whose area is at least $(1-\varepsilon)\mathsf{area}(T_{\mathrm{opt}})$ in $O(\varepsilon^{-1}\log n)$ time.*

Proof. We sample all orientations θ in $-\pi \leq \theta \leq \pi$ at interval $\min\{\frac{\alpha}{2}, \frac{\pi-\alpha}{2}\frac{c_1 w}{2d}\varepsilon\}$ that satisfy $\min\{\left||\theta| - \frac{\alpha}{2}\right|, \left|\pi - \frac{\alpha}{2} - |\theta|\right|\} \leq \frac{w}{c_1 c_2 d}$.

For each sampled orientation θ, $L_{\alpha,\theta}(T_{\mathrm{opt}}(\theta))$ is a largest axis-aligned right triangle in $L_{\alpha,\theta}(P)$ for the linear transformation $L_{\alpha,\theta} = \begin{pmatrix} 1 & \cot\alpha \\ 0 & 1 \end{pmatrix}^{-1} R\left(\frac{\alpha}{2} - \theta\right)$, where $R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$. So, we can compute $T_{\mathrm{opt}}(\theta)$ in $O(\log n)$ time using the same technique used for the axis-aligned α-triangles.

We can obtain an orientation θ such that $\mathsf{area}(T_{\mathrm{opt}}(\theta)) \geq (1-\varepsilon)\mathsf{area}(T_{\mathrm{opt}})$ for at least one of the sampled orientations by Lemmas 6 and 7. The running time of the algorithm is $O(\varepsilon^{-1}\log n)$. ☐

After applying the inner approximation using an *ε-kernel*, we can obtain the following theorem.

Theorem 9. *Given a convex polygon P with n vertices in the plane, an angle α, and $\varepsilon > 0$, we can find an α-triangle that can be inscribed in P and whose area is at least $(1-\varepsilon)$ times the area of a maximum-area α-triangle inscribed in P in $O(\varepsilon^{-\frac{1}{2}}\log n + \varepsilon^{-1}\log\varepsilon^{-\frac{1}{2}})$ time.*

Proof. By Lemma 1 in [8], an ε-kernel P_ε of P has $O(\varepsilon^{-\frac{1}{2}})$ vertices and it can be computed in $O(\varepsilon^{-\frac{1}{2}}\log n)$ time. A largest α-triangle in $P_{\frac{\varepsilon}{64}}$ has area at least $(1-\frac{\varepsilon}{2})\mathsf{area}(T_{\mathrm{opt}})$ by Lemma 8 in [8]. Then, an $(1-\frac{\varepsilon}{2})$-approximation to the largest α-triangle in $P_{\frac{\varepsilon}{64}}$ is an $(1-\varepsilon)$ -approximation of the largest α-triangle in P. We can compute an $(1-\frac{\varepsilon}{2})$-approximation to the largest α-triangle inscribed in $P_{\frac{\varepsilon}{64}}$ in $O(\varepsilon^{-1}\log\varepsilon^{-\frac{1}{2}})$ time by Lemma 8. ☐

3 Largest (α,β)-triangles in a Simple Polygon

In this section, we show how to find a largest (α,β)-triangle that can be inscribed in a simple polygon P with n vertices in the plane. Without loss of generality, we assume no three vertices of P are collinear.

A triangle T inscribed in P may touch some boundary elements (vertices and edges) of P. We call an edge of P that a corner of T touches *a corner contact* of T, and a vertex of P that a side of T touches in its interior *a side contact* of T. We call the set of all corner and side contacts of T *the contact set* of T. We say a triangle T *satisfies* a contact set C if C is the contact set of T.

We use $\eta_\theta(p)$ to denote the ray emanating from p that makes angle θ from the positive x-axis in counterclockwise direction. The inclination of line (or segment) is the angle that the line makes from the positive x-axis in counterclockwise direction.

3.1 Largest axis-aligned (α, β)-triangles

Finding a largest axis-aligned (α, β)-triangle is equivalent to finding a largest homothet inscribed in P. For an axis-aligned (α, β)-triangle T inscribed in P, we use $r(T)$ to denote the left endpoint of the base of T, and call it *the anchor* of T. For an axis-aligned (α, β)-triangle T inscribed in P and satisfying a contact set C, we say T is *maximal* if there is no axis-aligned (α, β)-triangle of larger area inscribed in P and satisfying a contact set C' with $C \subseteq C'$. For a point r in the interior of P, consider the largest axis-aligned (α, β)-triangle, denoted by $T(r)$, with r at its anchor. For ease of description, we say C is the contact set of r and r satisfies C, for the contact set C of $T(r)$. We use $C(r)$ to denote the contact set of r. We also say r is *maximal* if $T(r)$ is maximal.

To compute a largest axis-aligned (α, β)-triangle that can be inscribed in P, we consider all maximal axis-aligned (α, β)-triangles and find a largest triangle among them. To find all maximal (α, β)-triangles, we construct a subdivision of P by (α, β) such that a maximal (α, β)-triangle T has $r(T)$ at a vertex of the subdivision.

3.1.1 Subdivision of P by angles (α, β)

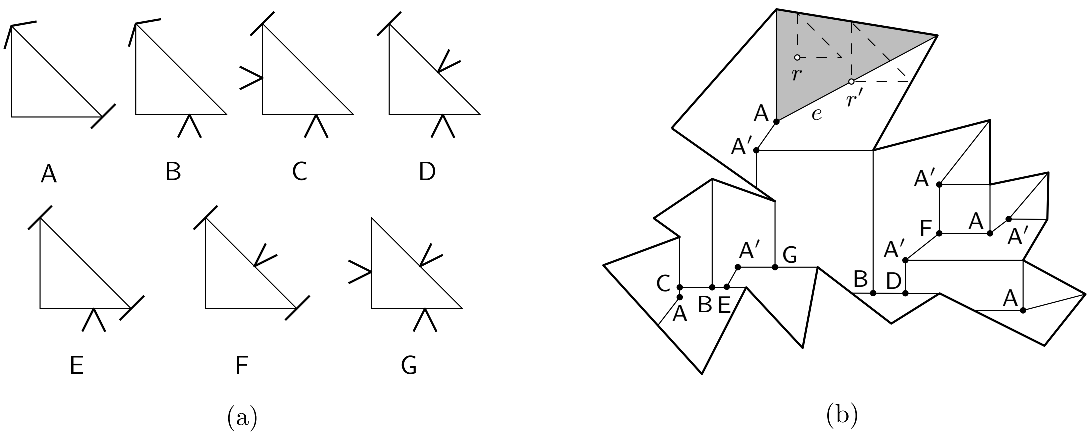


Figure 6: (a) Contact sets of type 3. Symmetric cases are omitted. (b) Subdivision of a polygon for $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{4}$. Each vertex of the subdivision is labeled by its corresponding contact set in (a). A' is the symmetric case of A. Point r has a contact set of type 1 and it is in a cell of the subdivision. Point r' has a contact set of type 4 and it is in an edge of the subdivision.

For a point r in the interior of P, consider the contact set C of r, which may consist of polygon edges that a corner of $T(r)$ touches and polygon vertices that a side of $T(r)$ touches in its interior. We classify the contact set C of r into four types as follows.

1. C consists of exactly one edge of P.
2. C consists of one or two reflex vertices of P that the side of $T(r)$ opposite to r touches in its interior.
3. C belongs to one of the configurations shown in Figure 6(a) or their symmetric configurations with respect to the anchor of $T(r)$. A superset of C also belongs to this type.

4. Other than types 1, 2, and 3.

Observe that each interior point r of P has a contact set, which belongs to one of the four types defined above. For a contact set C, consider the set $R(C)$ of the points r in the interior of P such that C remains to be the contact set of $T(r)$ under translations and scaling. Then the classification of contact sets above induces a subdivision of P into cells, edges, and vertices. A vertex of the subdivision has degree 1 (endpoint of a subdivision edge on the boundary of P) or larger.

Lemma 10. *Let S be the subdivision of P by (α, β), and let C be the contact set of a point in the interior of P. Then $R(C)$ is a cell of S if C is of type 1 or 2, a vertex of S if C is of type 3, and an edge of S if C is of type 4.*

Proof. Observe that $R(C)$ is a cell of S if C is of type 1 or 2, a vertex of S if C is of type 3. A contact set C of type 4 is (1) a proper subset of a configuration in 6(a), (2) the set consists of a contact set C' belongs in (1) and additional side contacts on the side which contains a side contact in C', and (3) the set consists of corner contacts on one corner c, except the anchor, and side contacts on both sides incident to c. Since type 1 and 2 contains all the contact set which contains exactly one element, C contains at least two elements. Then $R(C)$ for C belongs in (1) is line segment. The additional side contacts on the side which contains a side contact does not restrict $R(C)$. Thus, $R(C)$ for C belongs in (2) is also line segment. It is obvious that $R(C)$ for C belongs in (3) is line segment. Therefore, $R(C)$ is an edge of S if C is of type 4. □

See Figure 6(b) that illustrates the subdivision of P for $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{4}$ into cells, edges and vertices. Any point r in a cell has the same contact set of type 1 or 2. (The gray cell has a contact set of type 1.) Any point on an edge of the subdivision has the same contact set of type 4. (The edge labeled with e corresponds to a contact set of type 4.) Each vertex of the subdivision has a contact set of type 3 and is labeled by its corresponding contact set in Figure 6(a).

Observe that an axis-aligned (α, β)-triangle is not maximal if its anchor lies in a cell or edge of S. Thus, we have the following lemma.

Lemma 11. *Every maximal axis-aligned (α, β)-triangle has its anchor at a vertex of the subdivision of P.*

Now we explain how to construct the subdivision S for P. We use a plane sweep algorithm with a sweep line L which has inclination $\pi - \beta$ and moves downwards. The status of L is the set of rays and edges of P intersecting it, which is maintained in a balanced binary search tree $\mathcal{T}$ along L. While L moves downwards, the status in $\mathcal{T}$ changes when L meets particular points. We call each such particular point *an event point* of the algorithm. To find and handle these event points, we construct a priority queue $\mathcal{Q}$ as the event queue which stores the vertices of P in the beginning of algorithm as event points. As L moves downwards from a position above P, some event points are newly found and inserted to $\mathcal{Q}$ and some event points are removed from $\mathcal{Q}$.

The invariant we maintain is that at any time during the plane sweep, the subdivision above the sweep line L has been computed correctly. Consider the moment at which L reaches a vertex v of P. If v is convex vertex, we add a ray $\eta_\gamma(v)$ to S and update $\mathcal{T}$ and $\mathcal{Q}$ if it is contained in P locally around v. Since every point in $\eta_\gamma(v)$ near v has the same contact set of type 4 consisting of the two edges, γ is determined uniquely by the two edges. If v is a reflex vertex, we add at most two rays, $\eta_{\pi+\alpha}(v)$ and $\eta_\pi(v)$ from v, to S and update $\mathcal{T}$ and $\mathcal{Q}$ accordingly if the ray is contained in P locally around v and every point in the ray near v has the same contact set of type 4. Consider the moment at which L reaches the intersection of a ray with the boundary of P. Then the ray simply stops there. Consider now the moment at which L reaches the intersection point x of two rays η_1 and η_2. Then the two rays stop at x. We add one ray η emanating from x to S and update $\mathcal{T}$ and $\mathcal{Q}$ accordingly. Observe that the contact set of points on η near x is of type 4 consisting of contact elements of the points in η_1 and η_2. Thus, the orientation of η is uniquely determined in $O(1)$ time. If η_1 or η_2 emanates from a reflex vertex of P, η makes counterclockwise angle $\pi + \alpha$ or π from the positive x-axis. Imagine we move a point p from x along η. Then the contact set of p may change at some point $q \in \eta$ to another contact set consisting of contact elements of the points in η_1 and η_2. We call such a point q *a bend point* of η. Again a bend point of a ray can be found in $O(1)$ time. We add q to $\mathcal{Q}$ as an event point. Finally, consider the moment at which L reaches a bend point q of a ray η which emanates from the intersection of two rays. Then η stops at q. We add a new ray η' emanating from q to S and update $\mathcal{T}$ and $\mathcal{Q}$ accordingly. The orientation of η' is uniquely determined by the contact elements of the points in η in $O(1)$ time. Observe that η' makes a counterclockwise angle other than $\pi + \alpha$ and π from the positive x-axis.

At each of these event points, we update $\mathcal{T}$ and $\mathcal{Q}$ as follows. At an event point where a ray η is added to S, we insert η to $\mathcal{T}$, compute the event points at which η intersects with its neighboring rays along L and with the boundary of P, and add the event points to $\mathcal{Q}$. At an event where a ray η stops, we remove it from $\mathcal{T}$ and remove the events induced by η from $\mathcal{Q}$. We also compute the event at which the two neighboring rays of η along L intersect and add them to $\mathcal{Q}$.

After we have treated the last event, we have computed the subdivision of P.

Lemma 12. *We can construct the subdivision S of P in $O(n \log n)$ time using $O(n)$ space.*

Proof. First we show that the number of event points in the plane sweep algorithm is $O(n)$. A polygon vertex induces at most two rays and generates at most three events at the vertex and two points where the rays intersect the boundary of P for the first time. Thus, there are $O(n)$ rays induced by polygon vertices and they generate $O(n)$ event points. At an event, either (a) two neighboring rays merge into one at their intersection point or (b) a ray making counterclockwise angle $\pi + \alpha$ or π with the positive x-axis has at most one bend point, and every ray emanating from a bend point makes a counterclockwise angle other than $\pi + \alpha, \pi$ from the positive x-axis. An event of case (a) generates $O(1)$ new event points and the number of rays decreases by 1. Thus, the total number of event points of case (a) is $O(n)$. An event point of case (b) generates $O(1)$ new events, but only once for a ray making counterclockwise angle $\pi + \alpha$ or π with the positive x-axis. Again, the total number of event points of case (b) is $O(n)$.

For each event in the plane sweep algorithm, we stop at most two rays and add at most two rays to S in $O(1)$ time. Then we update $\mathcal{T}$ and $\mathcal{Q}$ accordingly in $O(\log n)$ time since there are $O(n)$ elements in $\mathcal{T}$ and $O(n)$ events in $\mathcal{Q}$. Thus, we can handle an event in $O(\log n)$ time, and we can construct S in $O(n \log n)$ time. The data structures S, $\mathcal{T}$ and $\mathcal{Q}$ all use $O(n)$ space. □

3.1.2 Computing a largest axis-aligned (α, β)-triangle

By Lemma 11, it suffices to check all vertices of S to find all maximal axis-aligned (α, β)-triangles. For each vertex w of S, the triangle $T(w)$ is a maximal axis-aligned (α, β)-triangle satisfying $C(w)$ by definition. We can compute the area of $T(w)$ in $O(1)$ time by storing $C(w)$ to w when it is added into S. Then, we can find a largest axis-aligned (α, β) triangle by choosing a largest one among all maximal axis-aligned (α, β)-triangles. By Lemmas 11 and 12, we have following theorem.

Theorem 13. *Given a simple polygon P with n vertices in the plane and two angles α, β, we can find a maximum-area (α, β)-triangle that can be inscribed in P in $O(n \log n)$ time using $O(n)$ space.*

3.2 Largest (α, β)-triangles of arbitrary orientations

We describe how to find a largest (α, β)-triangle of arbitrary orientations that can be inscribed in a simple polygon P with n vertices. We use C_θ to denote the coordinate axes obtained by rotating the standard xy-Cartesian coordinate system by θ degree counterclockwise around the origin. We say a triangle T with base b is θ-aligned if b is parallel to the x-axis in C_θ.

We use $\mathsf{S}(\theta)$ to denote the subdivision of P in C_θ. We construct the subdivision $\mathsf{S}(\theta)$ of P at C_0 using the algorithm in Section 3.1, and maintain it while rotating the standard xy-Cartesian coordinate axes from angle 0 to 2π. During the rotation, we maintain the combinatorial structure of $\mathsf{S}(\theta)$ (not the embedded structure $\mathsf{S}(\theta)$) and update the combinatorial structure for each change so that the changes of $\mathsf{S}(\theta)$ are handled efficiently. We abuse the notation $\mathsf{S}(\theta)$ to refer the combinatorial structure of $\mathsf{S}(\theta)$ if understood in the context. For each vertex of $\mathsf{S}(\theta)$, we store the function which returns the actual coordinate of the vertex in the embedded structure $\mathsf{S}(\theta)$. Thus, an edge of $\mathsf{S}(\theta)$ is determined by the functions stored at its two endpoints. For each edge of $\mathsf{S}(\theta)$, we store the contact set of the points in the edge.

We say a contact set C is *feasible* at an angle θ_0 if there exists a θ_0-aligned (α, β)-triangle inscribed in P and satisfying $C' \supseteq C$. For a contact set C, consider all angles at which C is feasible. Then these angles form connected components in $[0, 2\pi)$ which are disjoint intervals. We call each such interval a feasible interval of C.

For a fixed angle θ_0, consider a θ_0-aligned (α, β)-triangle satisfying a contact set C. Let $I = [\theta_1, \theta_2]$ be a feasible interval of C containing θ_0. For $\theta_0 \in I$, we say a θ_0-aligned (α, β)-triangle T satisfying a contact set $C_1 \supseteq C$ is *maximal* for I if there is no $\theta' \in I$ such that a θ'-aligned (α, β)-triangle satisfying a contact set $C_2 \supseteq C$ has larger area.

3.2.1 Maintaining subdivision under rotations

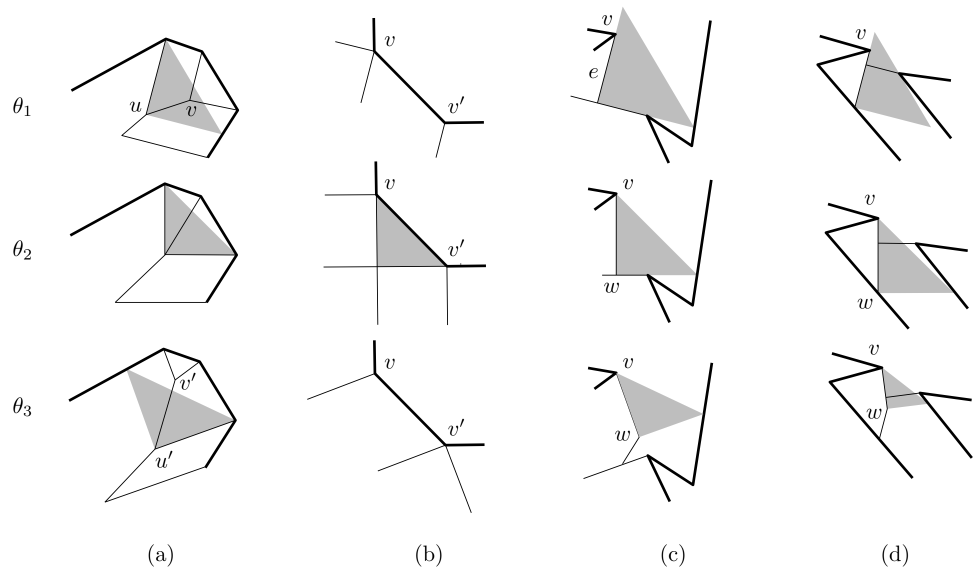


Figure 7: Changes of the subdivision $\mathsf{S}(\theta)$ during the rotation with θ at $\theta_1, \theta_2, \theta_3$ with $\theta_1 < \theta_2 < \theta_3$. (a) An edge event at which edge uv of $\mathsf{S}(\theta_1)$ becomes a vertex of $\mathsf{S}(\theta_2)$ and then it splits into two with edge $u'v'$ in between them in $\mathsf{S}(\theta_3)$. (b) A vertex event at which an edge of $\mathsf{S}(\theta)$ incident to a reflex vertex v' suddenly appears in $\mathsf{S}(\theta_2)$ and an edge of incident to a reflex vertex v suddenly disappears from $\mathsf{S}(\theta_3)$. (c) An align event at which a subdivision edge e splits into two edges with vertex w in between. (d) A boundary event at which w meets the boundary of P in $\mathsf{S}(\theta_2)$.

The combinatorial structure of $\mathsf{S}(\theta)$ changes during the rotation. Each change is of one of the following types:

- Edge event: an edge of $\mathsf{S}(\theta)$ degenerates to a vertex of $\mathsf{S}(\theta)$. Right after the event, the vertex splits into two with an edge connecting them in $\mathsf{S}(\theta)$. See Figure 7(a).
- Vertex event: an edge of $\mathsf{S}(\theta)$ incident to a reflex vertex v suddenly appears or disappears on $\mathsf{S}(\theta)$. This event may occur only when an edge e of P incident to v has inclination 0, α, or $\pi - \beta$. See Figure 7(b).
- Align event: an edge e of $\mathsf{S}(\theta)$ with inclination 0 or α splits into two edges with a vertex w of $\mathsf{S}(\theta)$ in between or two such edges merge into one. This event may occur only for vertex w of $\mathsf{S}(\theta)$ such that the maximal θ-aligned (α, β)-triangle $T(w) = \triangle wpq$ (or its symmetric one) has a reflex vertex of P on q, a reflex vertex of P on its base wp, and an edge on p, or $T(w)$ has a vertex of P on p and q, with one of them being reflex. See Figure 7(c).
- Boundary event: a vertex of $\mathsf{S}(\theta)$ with degree 2 or larger meets the boundary of P or a vertex of $\mathsf{S}(\theta)$ with degree 1 meets a vertex of P on the boundary of P. See Figure 7(d).

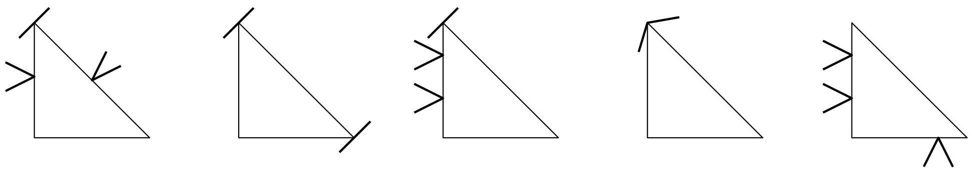


Figure 8: Classification of contact sets of (α, β)-triangles with anchor at the vertex at which an edge, align, boundary event occurs.

Recall that the set $C(p)$ for a point p on an edge of $\mathsf{S}(\theta)$ has at least two elements and $C(v)$ for a vertex v of $\mathsf{S}(\theta)$ has at least three elements. We classify all the contact sets of the vertices of $\mathsf{S}(\theta)$ at which an event (except a vertex event) occurs. For a vertex w of $\mathsf{S}(\theta)$, $C(w)$ belongs to one of the following types. See Figure 8 for an illustration.

- Type X_1: $C(w)$ contains a corner contact at a corner c and side contacts on both sides incident to c.
- Type X_2: $C(w)$ contains no side contact on a side s and a corner contact on each corner incident to s.
- Type X_3: $C(w)$ contains two side contacts on a side s and a corner contact on a corner incident to s.
- Type X_4: $C(w)$ contains two corner contacts on a corner, i.e. a corner is on a vertex of P, except the case that it contains a corner contact on each corner.
- Type X_5: $C(w)$ contains two side contacts on a side, one side contact on another side, and no corner contact on the corner shared by the sides.

Since there are no two vertices of $\mathsf{S}(\theta)$ such that the contact sets of them are same, a vertex where an event (except a vertex event) occurs has contact set containing more than three elements. Observe that an (α, β)-triangle T is not maximal if there is a side s of T such that no contacts are on both s and corners incident to s. Thus, for a vertex w of $\mathsf{S}(\theta)$ at which an event (except a vertex event) occurs, $T(w)$ has no side s such that no contacts on both s and corners incident to s. Observe that $C(w)$ belongs to a type defined above. Thus, the number of events other than vertex events is at most the number of maximal (α, β)-triangles which have a contact set of one of the types above. We need the following two technical lemmas to bound the number of (α, β)-triangles satisfying types X_1 and X_2.

Lemma 14. *Let $F = \{f_i \mid 1 \leq i \leq n\}$ be a finite family of real value functions such that every f_i is of single variable and continuous, any two functions f_i and $f_{i'}$ intersect in their graphs at most once, every function f_i has domain D_i of size d. If there is a constant c such that $|\bigcup D_i| = cd$, then the complexity of the lower envelope of F is $O(n)$.*

Proof. Let $D_F = \bigcup D_i$. Since there is a constant c, we can construct a finite partition A of D_F such that each element of A has size d, except one element of size smaller than or equal to d. Then every D_i intersects at most two elements of A. Let l_j be the left endpoint of $t_j \in A$ and assume $l_j < l_{j'}$ for all $j < j'$. Let $L(l_j)$ and $R(l_j)$ be the sets of functions f'_i which are functions f_i restricted to $D_i \cap [l_{j-1}, l_j]$ and $D_i \cap [l_j, l_{j+1}]$, respectively. Since any two functions $f'_i, f'_k \in L(l_j)$ (or two in $R(l_j)$) intersect in their graphs at most once and their domains have the same start point or end point, the sequence of the lower envelope of $L(l_j)$ (or $R(l_j)$) is a Davenport-Schinzel sequence of order 2. Then the lower envelope of set $L(l_j)$ (and of set $R(l_j)$) has complexity $O(k)$, where $k = |L(l_j)|$ (or $k = |R(l_j)|$). Since $|\bigcup L(l_j)| = |\bigcup R(l_j)| = O(n)$, the lower envelope of $\bigcup L(l_j)$ and $\bigcup R(l_j)$ has complexity $O(n)$.

Now, consider a new partition of D_F obtained by slicing it at every point at which the lower envelopes of $\bigcup L(l_j)$ and $\bigcup R(l_j)$ change combinatorially. Since there are $O(n)$ such points and the lower envelope of F restricted to a component of the new partition has constant complexity, the complexity of the lower envelope of F is $O(n)$. □

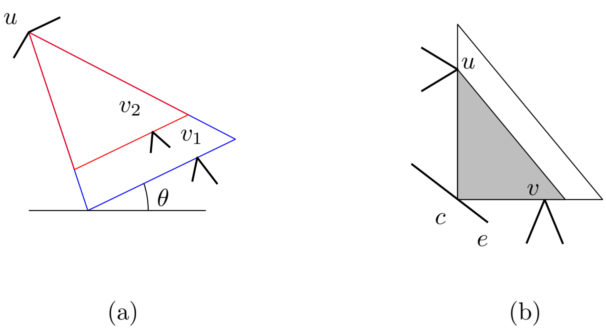


Figure 9: (a) For a fixed angle θ, $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$ for any two θ-aligned triangles T_1 and T_2 with u at a corner c and a vertex v_1 and v_2 on the side opposite to c, respectively. (b) An (α, β)-triangle T satisfying a contact set $C = \{u, v, e\}$ of type X_1 contains an (α, β)-triangle T' which shares a corner c with T lying on e, has a corner at u, and has v on the side opposite to c.

Lemma 15. *For a fixed vertex u of P, there are $O(n)$ maximal (α, β)-triangles T with a corner at u such that T has a corner on the interior of an edge of P and has a vertex of P on the side opposite to u.*

Proof. For a fixed u, consider a maximal (α, β)-triangle T with a corner at u such that T has a corner on the interior of an edge e of P and has a vertex v of P on the side opposite to u. Let θ_T denote the angle of rotation such that the base of T is parallel to the x-axis of C_{θ_T}.

To count such triangles, let $\mathcal{T}_v$ denote all nontrivial (α, β)-triangles with a corner at u, with another vertex of P on the side opposite to u. Observe that for a fixed angle θ, $T_1 \subseteq T_2$ or

$T_2 \subseteq T_1$ for any two θ-aligned triangles $T_1, T_2 \in \mathcal{T}_v$. Moreover, no θ-aligned triangle of $\mathcal{T}_v$, except the smallest one, is inscribed in P. See Figure 9(a). Let $F_v(\theta)$ be the function that returns the area of the smallest θ-aligned triangle of $\mathcal{T}_v$ at angle θ. Let $\mathcal{T}_e$ denote all nontrivial (α, β)-triangles with a corner at u, and another corner on the interior of an edge of P which can be inscribed in P. Observe that for a fixed angle θ, there is at most one θ-aligned triangle in $\mathcal{T}_e$ as it is required to satisfy the constraint to be inscribed in P. Let $F_e(\theta)$ be the function that returns the area of the θ-aligned triangle of $\mathcal{T}_e$ at angle θ. Then T occurs at the angle of an intersection of the graphs of F_v and F_e.

Now we count the intersections of the graphs of F_v and F_e. For any vertex v of P, the domain of $\mathsf{area}(T_v(\theta))$ has size $\pi - \alpha - \beta$, where $T_v(\theta)$ is the θ-aligned (α, β)-triangle with u at a corner and v on its side opposite to u. For any pair of vertices v, v' of P, $(\mathsf{area}(T_{v_1}(\theta))$, $\mathsf{area}(T_{v_2}(\theta))$ have at most one intersection. The union of all domains of $\mathsf{area}(T_v(\theta))$'s is $[0, 2\pi)$. Thus, by Lemma 14, the complexity of F_v is $O(n)$.

For any two edges e_1, e_2 of P, the portions of the graph of F_e corresponding to $\mathsf{area}(T_{e_1}(\theta))$ and $\mathsf{area}(T_{e_2}(\theta))$ are disjoint since $\eta_{\pi+\alpha}(u)$ hits only one edge at an angle θ, where $T_e(\theta)$ denotes the θ-aligned (α, β)-triangle with u at a corner and e on its anchor.

For any pair of a vertex v and an edge e of P, the graphs of $\mathsf{area}(T_v(\theta))$ and $\mathsf{area}(T_e(\theta))$ intersect at most twice since the trajectory of point c such that $\angle acv = \beta$ forms a circular arc, and a circular arc intersects a line segment at most twice. Thus, the graphs of F_v and F_e intersect $O(n)$ times for a fixed vertex u. $\square$

In the following lemma, we bound the number of (α, β)-triangles satisfying one of the types X_i for $i = 1, \ldots, 5$.

Lemma 16. *There are $O(n^2)$ events that occur to $\mathcal{S}(\theta)$ during the rotation.*

Proof. Observe that there are $O(n)$ align events. Consider an (α, β)-triangle T satisfying a contact set C belonging to type X_1. Then T contains an (α, β)-triangle T' which shares a corner c with T lying on an edge, has a corner at a vertex $u \in C$, and has a vertex of P on the side opposite to c. See the gray triangle in Figure 9(b). Thus, we find all such triangles T' for every vertex u. By Lemma 15, there are $O(n)$ such triangles for a vertex u of P, and in total $O(n^2)$ (α, β)-triangles satisfying the contact sets of type X_1.

Consider an (α, β)-triangle T satisfying a contact set C belonging to type X_1. Then there is at least one vertex v of P such that the boundary of P contains v. It is obvious that the number of (α, β)-triangles with corner at v and satisfies a contact set of type X_2. Thus, assume v is side contact of T. Let e_1 and e_2 be two edges that C contains and x be the vertex or edge of P which is contained in C and distinct to v, e_1, and e_2. We can consider two θ-aligned (α, β)-triangles $T_1(\theta)$ and $T_2(\theta)$ which satisfies the contact set consists of $\{v, x, e_1\}$ and $\{v, x, e_2\}$ respectively. Similar to Lemma 15, we consider the number of intersection of the graphs of $\mathsf{area}(T_1(\theta))$ and $\mathsf{area}(T_2(\theta))$. Then we can prove that the number of such intersection is $O(n^2)$. Thus, the number of (α, β)-triangles satisfying a contact set of X_4 is $O(n^2)$.

Observe that there are constant number contact set C of type X_3 if two vertices of P which lies on a same side is given. And each C has constant number of feasible orientations. The number of contact sets of type X_4, except the case of contact set contain two side contacts v_1 and v_2 on the side opposite to the corner which is on a vertex u of P, is also $O(n^2)$ because of the same reason. The number of contact set of excepted case can be considered in a similar way to Lemma 15, by considering two triangles T_{v_1} and T_{v_2} such that both T_{v_1} and T_{v_2} has a corner at u and v_i is on the side opposite to u of T_{v_i}. Thus, the number of (α, β)-triangles satisfying a contact set of X_4 is $O(n^2)$.

An (α, β)-triangle T satisfying a contact set of type X_5 contains an $(\alpha, \beta$-triangle T' which has a corner at v, where v is a side contact of T similar to type X_1. Since the contact set of T' is of type X_4, the number of contact set of X_5 is $O(n^2)$.

Therefore, the number of events that occurs to $\mathsf{S}(\theta)$ during the rotation is also $O(n^2)$. □

To capture the combinatorial changes and maintain $\mathsf{S}(\theta)$ during the rotation, we construct and maintain the following data structures: (1) An event queue $\mathcal{Q}$ which is a priority queue that stores events indexed by their angles. (2) A planar graph representing the combinatorial structure of $\mathsf{S}(\theta)$. (3) For each edge e of P, a balanced binary search trees $\mathcal{T}(e)$. The tree store degree-1 vertices of $\mathsf{S}(\theta)$ in order along e.

In the initialization, we construct $\mathsf{S}(0)$, and then $\mathcal{T}(e)$ for each edge of P. Then we initialize $\mathcal{Q}$ with the events defined by the vertices of $\mathsf{S}(0)$ and the vertex events and boundary events defined by the polygon vertices. For each vertex v of $\mathsf{S}(0)$, we compute the angle at which v and a neighboring vertex of v meet (edge event), an edge incident to v splits into two (align event), or v meets a polygon vertex (boundary event). These angles can be computed in $O(1)$ time for each v using the contacts corresponding to v. Then the size of $\mathcal{Q}$ is $O(n)$ which can be constructed in $O(n \log n)$ time.

We update each data structure whenever an event occurs. Note that each event changes a constant number of elements of $\mathsf{S}(\theta)$, creates a constant number of events to $\mathcal{Q}$, and removes a constant number of events from $\mathcal{Q}$. Thus we can update the subdivision in $O(1)$ time, and the tree $\mathcal{T}(e)$ in $O(\log n)$ time for edge e where a boundary event occurs.

Lemma 17. *We can construct the subdivision of P and maintain it in $O(n^2 \log n)$ time using $O(n)$ space during rotation.*

Proof. By Lemma 12, we can construct $\mathsf{S}(0)$ in $O(n \log n)$ time using $O(n)$ space. We can construct all the data structures for maintaining $\mathsf{S}(\theta)$ in $O(n \log n)$ time using $O(n)$ space. By Lemma 16, there are $O(n^2)$ events during the rotation. Each event, except vertex events, can be handled in $O(\log n)$ time. For each vertex event, we reconstruct $\mathsf{S}(\theta)$, $\mathcal{Q}$ and $\mathcal{T}(e)$ for each edge e of P. Since there are $O(n)$ vertex events and it takes $O(n \log n)$ time for the reconstruction, it takes $O(n^2 \log n)$ time to handle all vertex events. The space complexity remains to be $O(n)$ since the complexity of the data structures is $O(n)$. □

3.2.2 Computing a largest (α, β)-triangle

Whenever a vertex w appears on the subdivision or $C(w)$ changes at θ_0 by an event occurring at angle θ_0, we store θ_0 at w. Whenever $C(w)$ changes or w disappears from the subdivision at angle θ_1, we compute the largest θ-aligned (α, β)-triangle satisfying $C(w)$ for $\theta \in [\theta_0, \theta_1]$, where θ_0 is the angle closest from θ_1 at which w appears or $C(w)$ changes with $\theta_0 < \theta_1$. We do this on every vertex w of the subdivision, and then return the largest triangle among the triangles on the vertices. We can compute the largest one among all θ-aligned (α, β)-triangles satisfying $C(w)$ for $\theta \in [\theta_0, \theta_1]$ in $O(1)$ time using the area function of the θ-aligned (α, β)-triangle satisfying $C(w)$. Thus, from Lemma 17, we have the following theorem.

Theorem 18. *Given a simple polygon P with n vertices in the plane and two angles α, β, we can find a maximum-area (α, β)-triangle that can be inscribed in P in $O(n^2 \log n)$ time using $O(n)$ space.*

4 Largest α-triangles in a Simple Polygon

In this section, we compute a largest α-triangle that can be inscribed in a simple polygon P with n vertices in the plane. Without loss of generality, we assume no three vertices of P are collinear.

4.1 Largest axis-aligned α-triangles

We consider a largest axis-aligned α-triangle that can be inscribed in P. We use $\mathsf{S}(\theta)$ to denote the subdivision of P defined for two angles α and θ. We say a contact set C is *feasible* at an angle θ_0 if there exists an axis-aligned (α, θ_0)-triangle inscribed in P satisfying $C' \supseteq C$. For a feasible interval I of a contact set C and $\theta_0 \in I$, we say an axis-aligned (α, θ_0)-triangle T satisfying a contact set $C_1 \supseteq C$ is *maximal* in I if there is no $\theta' \in I$ such that an axis-aligned (α, θ')-triangle satisfying a contact set $C_2 \supseteq C$ has larger area. The point at which $\eta_\theta(p)$ meets the boundary of a simple polygon $Q \subseteq P$ for the first time other than its source point is called the *foot of* $\eta_\theta(p)$ *on* Q and denoted by $\delta_\theta(Q, p)$. In an α-triangle, we say the side opposite to the anchor is *the diagonal* of the triangle. For a point $p \in P$, we define the *visibility region* of p as $\mathsf{Vis}(p) = \{x \in P \mid px \subset P\}$. For a ray η and an angle θ, we define the θ*-visibility region* $\mathsf{Vis}_\theta(\eta)$ of η as the set of points x in the segment $y\delta_\theta(P, y)$ contained in P for every $y \in \eta$.

First, we find the largest axis-aligned $(\alpha, \frac{\pi-\alpha}{2})$-triangle T using the algorithm in Section 3.1. Let d be the diameter of P. Since every side of a triangle inscribed in P has length less than or equal to d, any (α, θ)-triangle that can be inscribed in P has area less than or equal to $\frac{d^2 \sin\alpha \sin\theta}{2\sin(\alpha+\theta)}$. Thus it suffices to consider (α, θ)-triangles for $\theta > \theta_T$ to find a largest axis-aligned α-triangle, where θ_T satisfies $\frac{d^2 \sin\alpha \sin\theta_T}{2\sin(\alpha+\theta_T)} = \mathsf{area}(T)$. To find a largest axis-aligned α-triangle, we choose an angle $\theta_0 \le \theta_T$, construct $\mathsf{S}(\theta_0)$ and maintain it while increasing θ from θ_0 to $\pi - \alpha$. Note that we can compute T in $O(n \log n)$ time by Theorem 13 and we can find θ_0 in $O(1)$ time. The combinatorial structure of $\mathsf{S}(\theta)$ changes while increasing θ. We use the definitions for the combinatorial changes of $\mathsf{S}(\theta)$ in Section 3.2: edge, vertex, align and boundary events. See Figure 10. Note that a vertex event occurs at $\theta = \pi - \gamma$, where γ is the inclination of an edge of P.

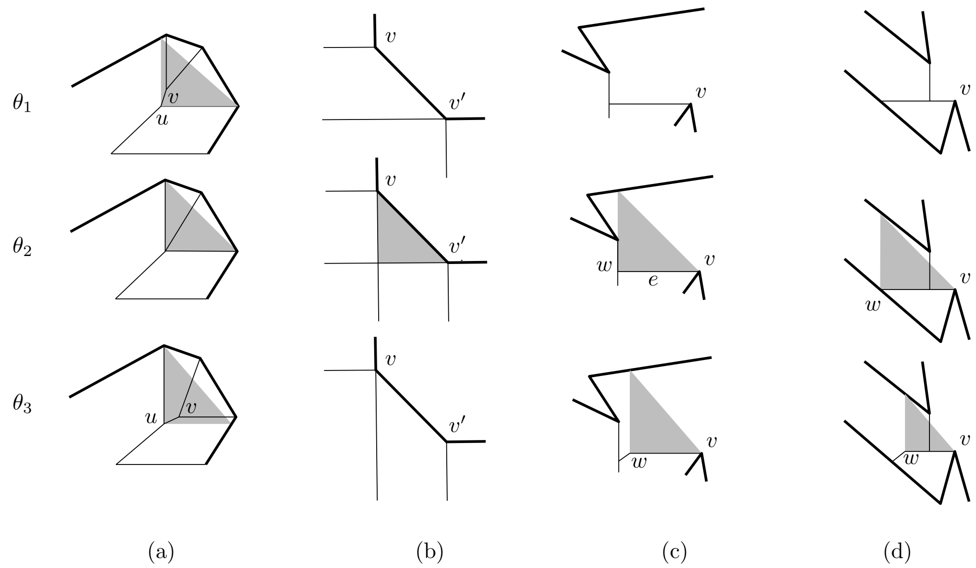


Figure 10: Changes of the subdivision $\mathsf{S}(\theta)$ while increasing θ at $\theta_1, \theta_2, \theta_3$ with $\theta_1 < \theta_2 < \theta_3$. (a) An edge event. (b) A vertex event. (c) An align event. (d) A boundary event.

4.1.1 The number of edge and vertex events

We count all α-triangles satisfying a contact set of a vertex at which an event occurs. Figure 11 shows a classification of all contact sets of vertices of $\mathsf{S}(\theta)$ at which an event (except vertex events)

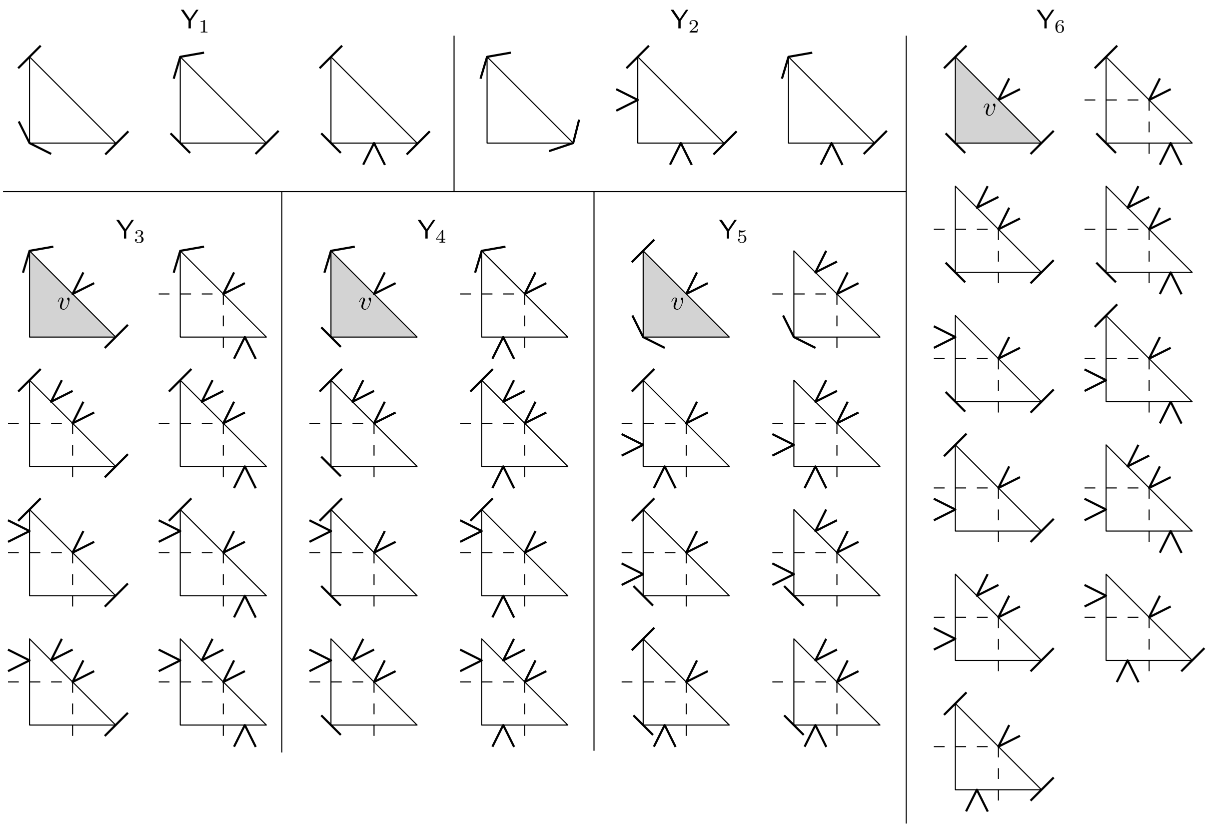


Figure 11: Classification of contact sets. Symmetric cases are omitted.

occurs. No contact set of types Y_1 or Y_2 contains a diagonal contact, while contact sets of other types contain a diagonal contact. We first show that the number of α-triangles satisfying contacts sets of types Y_1 and Y_2 is $O(n^2)$. Then we show that the number of α-triangles satisfying contact sets of other types is also $O(n^2)$.

Lemma 19. *There are $O(n^2)$ axis-aligned α-triangles satisfying a contact set of type Y_1 or Y_2.*

Proof. Any contact set of type Y_1 or Y_2 contains vertices of P. For a vertex v of P, there is at most one axis-aligned α-triangle which satisfies a contact set of type Y_1 containing v. For any two polygon vertices v_1 and v_2, there is at most one axis-aligned α-triangle which satisfies a contact set of type Y_2 containing v_1 and v_2. Thus, there are $O(n^2)$ axis-aligned α-triangles satisfying a contact set of type Y_1 or Y_2. $\square$

To count all axis-aligned α-triangles satisfying a contact set of a type Y_i for $i = 3, \ldots, 6$, we consider such triangles whose diagonal contains a reflex vertex of P. For a reflex vertex v of P, let $P'(v) = \big(\mathsf{Vis}(v) \cap \mathsf{Vis}_\alpha(\eta_\pi(v))\big) \cup \big(\mathsf{Vis}(v) \cap \mathsf{Vis}_0(\eta_{\pi+\alpha}(v))\big) \cup \big(\mathsf{Vis}_{\pi+\alpha}(\eta_\pi(v)) \cap \mathsf{Vis}_\pi(\eta_{\pi+\alpha}(v))\big)$. See Figure 12(a). Then, every axis-aligned α-triangle with v on its diagonal is inscribed in $P'(v)$. The gray triangles and their contacts in Figure 11 show axis-aligned α-triangles T satisfying a contact set of a type Y_i for $i = 3, \ldots, 6$ with their diagonals containing v and their contact sets with respect to $P'(v)$.

Lemma 20. *For a reflex vertex v of P, the number of axis-aligned α-triangles satisfying a contact set of type Y_i for $i = 3, \ldots, 6$ and containing v on their diagonals is $O(n)$.*

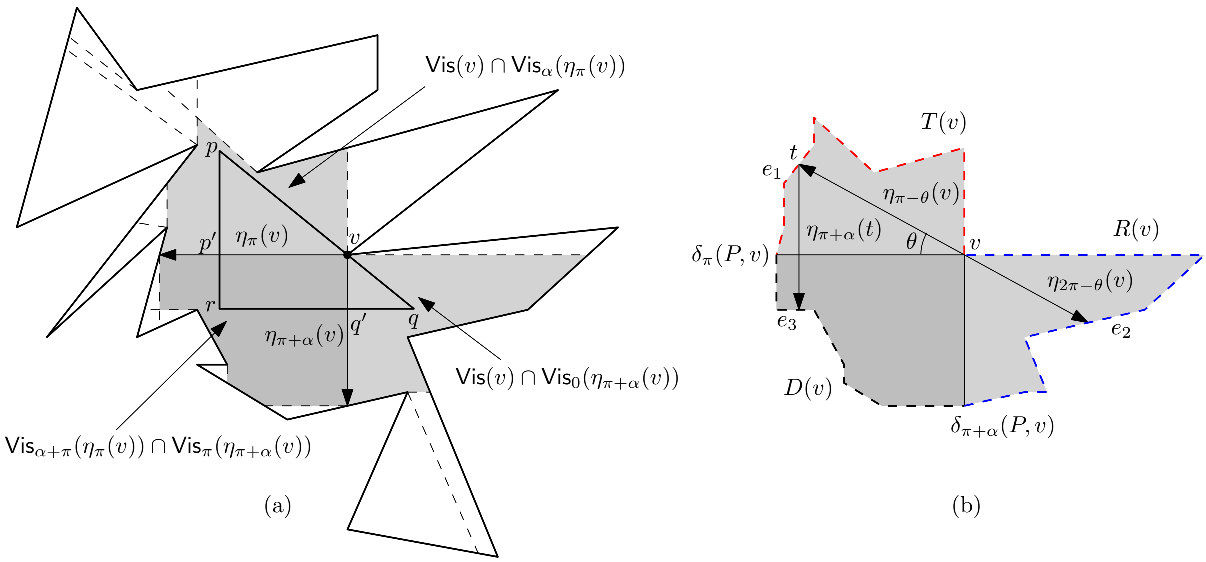


Figure 12: (a) Preprocessing for computing α-triangles satisfying a contact set of type Y_i for $i = 3,\ldots,6$. (b) A quadruplet (v, e_1, e_2, e_3) that can be a contact set of type Y_6. Let $t = \delta_{\pi-\theta}(P'(v), v)$.

Proof. Let $T(v), R(v)$, and $D(v)$ be the polygonal chains of $P'(v)$ from $\delta_\pi(P, v)$ to v, from v to $\delta_{\pi+\alpha}(P, v)$, and from $\delta_{\pi+\alpha}(P, v)$ to $\delta_\pi(P, v)$ in clockwise, respectively. See Figure 12(b).

For a vertex r of $T(v)$, there is at most one axis-aligned α-triangle which satisfies a contact set of type Y_3, contains v on its diagonal, and contains a corner at r. Thus, there are $O(n)$ axis-aligned α-triangles satisfying a contact set of type Y_3.

By the same reason, there are a constant number of α-triangles satisfying a contact set of type Y_4 for each vertex of $T(v)$, and a constant number of α-triangles satisfying a contact set of type Y_5 for each vertex of $D(v)$.

Now we count the number of quadruplets of elements including v that can form a contact set of type Y_6. This quadruplet changes only if $\eta_{\pi-\theta}(v)$, $\eta_{2\pi-\theta}(v)$, $\eta_{\pi+\alpha}(\delta_{\pi-\theta}(P'(v), v))$ meets another vertex of $P'(v)$. Since $\eta_{\pi-\theta}(v)$ and $\eta_{2\pi-\theta}(v)$ rotate clockwise around v, and $\eta_{\pi+\alpha}(\delta_{\pi-\theta}(P'(v), v))$ moves rightwards while increasing θ, each of the rays meets a vertex of $P'(v)$ at most once. See Figure 12(b). Thus, there are $O(n)$ contact sets of type Y_6 containing v as a diagonal contact. For a contact set $C = \{v, e_1, e_2, e_3\}$ of type Y_6 containing v as a diagonal contact, we can find the number of axis-aligned α-triangles satisfying C in a way similar to the proof on type X_2 of Lemma 16. If there is an axis-aligned (α, θ_T)-triangle T satisfying C, then there are two axis-aligned (α, θ)-triangles $T'(\theta)$ and $T''(\theta)$ such that $T'(\theta)$ satisfies $\{v, e_1, e_2\}$ and $T''(\theta)$ satisfies $\{v, e_2, e_3\}$ and $\mathsf{area}(T) = \mathsf{area}(T'(\theta_T)) = \mathsf{area}(T''(\theta_T))$. Since each area function consists of a constant number of trigonometric functions with period 2π, there are $O(1)$ distinct angles at which $\{v, e_1, e_2, e_3\}$ is feasible.

Therefore, the number of axis-aligned α-triangles satisfying a contact set of each type in Figure 11, except types Y_1 and Y_2, and containing v on their diagonals is $O(n)$ for each v. □

By Lemmas 19 and 20, we can conclude with the following lemma.

Lemma 21. *There are $O(n^2)$ events, except vertex events, that occur to $\mathsf{S}(\theta)$ while increasing θ from θ_0 to $\pi - \alpha$.*

4.1.2 Maintaining the subdivision while increasing θ

To capture the combinatorial changes and maintain $\mathsf{S}(\theta)$ while increasing θ, we maintain the same data structures defined in Section 3.2.1, but with different equations for computing angles at which an event occurs. By following the same initialization and update steps in Section 3.2.1, we can construct and maintain subdivision $\mathsf{S}(\theta)$ while increasing θ. By Lemmas 12 and 21, we have following lemma.

Lemma 22. *We can construct the subdivision $\mathsf{S}(\theta)$ of P and maintain it in $O(n^2 \log n)$ time using $O(n)$ space while increasing θ from θ_0 to $\pi - \alpha$.*

4.1.3 Computing the largest axis-aligned α-triangles

If an event occurs at a vertex w of the subdivision and an angle θ, $C(w)$ changes. We find all feasible intervals of contact sets while maintaining the subdivision. We can compute the maximal axis-aligned α-triangle satisfying a contact set in $O(1)$ time if a feasible interval is given. Thus, we have the following theorem by Lemma 22.

Theorem 23. *Given a simple polygon P with n vertices in the plane and an angle α, we can compute a maximum-area axis-aligned α-triangle that can be inscribed in P in $O(n^2 \log n)$ time using $O(n)$ space.*

4.2 Largest α-triangles of arbitrary orientations

To find a largest α-triangle of arbitrary orientations, we follow the approach by Melissaratos et al. [17] in computing a largest triangle with no restrictions in a simple polygon, with some modification. Their algorithm considers all triangles but we consider α-triangles only. They divide the cases by the number of corners of the triangle lying on the boundary of P. They denote by m-case the case that m corners of a triangle lie on the boundary of P, for $m = 0, 1, 2, 3$.

Consider a contact set C consisting of at most three elements. Then any α-triangle satisfying C can be enlarged into another α-triangle while satisfying C. So, the contact set of a largest α-triangle consists of at least four elements. Also, for triangles of the 0-case, their contact sets consists of five elements and the side opposite to the fixed angle corner contains two side contacts. See Figure 13(i). This can be proved in a way similar to Lemma 6.3 in [17].

Figure 13 illustrates the classification of contact sets of the largest α-triangles for each case.

For cases (i) and (ii) of Figure 13, we fix two reflex vertices C and D lying on the same side and find the points G, H where the line containing CD intersects the boundary of P with $GH \subset P$. See Figure 14(a) and (b). Then by walking on the shortest-path maps of G and H along the boundary of P in a way similar to [17], we can compute a largest α-triangle for each case using $O(n^4)$ time.

For cases (iv), (vii), (x) of Figure 13, we find the largest α-triangle satisfying a contact set C without any restriction on the boundary of P. If the largest α-triangle satisfying C is not inscribed in P, then the maximum area is achieved at the boundary angles of the feasible intervals of C, which are handled for other cases.

A contact set C of the remaining cases contains at most one side contact for each side of the α-triangle satisfying C. For each of these cases, we find a largest α-triangle that can be inscribed in P using the method in [17]. We decompose the problem into $O(n^4)$ simple optimization problems and add a constraint such that one interior angle of triangles must be α to each optimization problem. Since the original optimization problem can be solved in constant time, our problem can also be solved in $O(1)$ time.

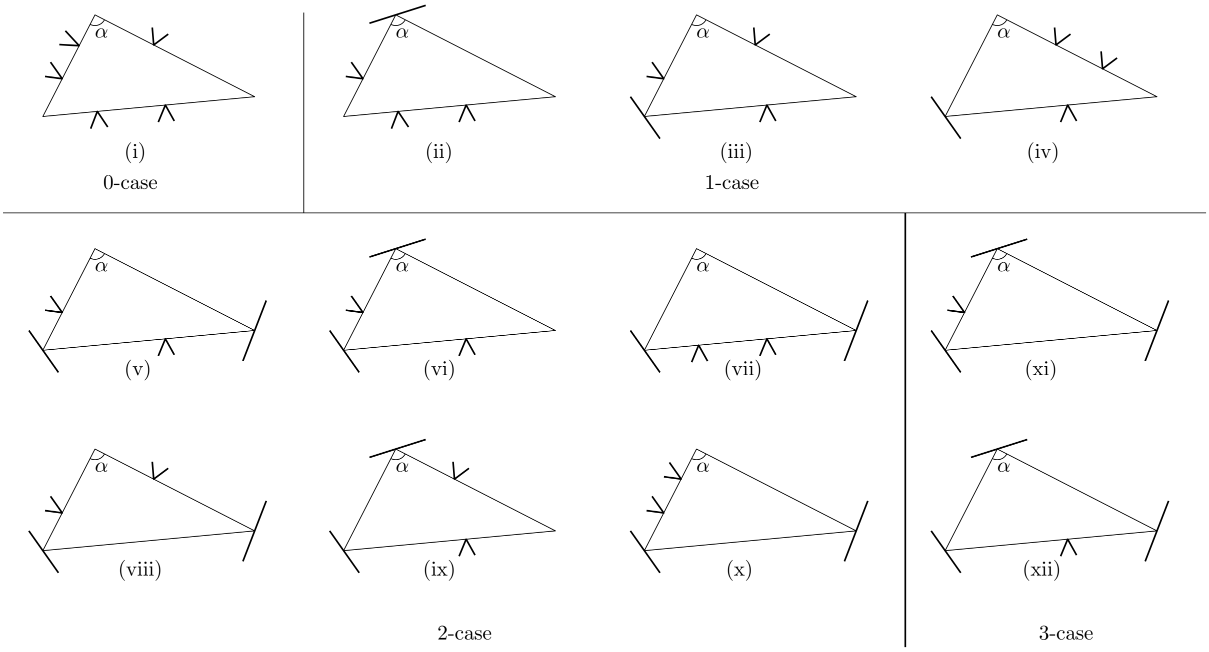


Figure 13: Classification of the largest α-triangles

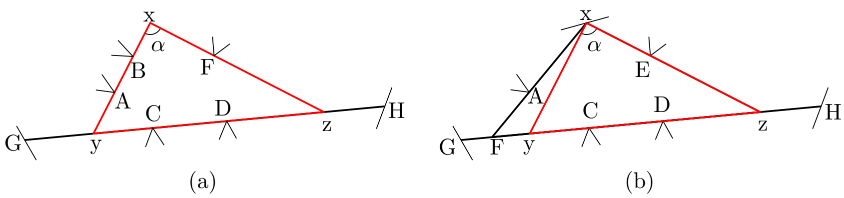


Figure 14: (a) Configuration (i) in Figure 13. (b) Configuration (ii) in Figure 13.

Theorem 24. *Given a simple polygon P with n vertices in the plane and an angle α, we can compute a maximum-area α-triangle inscribed in P in $O(n^4)$ time using $O(n)$ space.*

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