

Inserting an Edge into a Geometric Embedding^{*}

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Abstract. The algorithm to insert an edge e in linear time into a planar graph G with a minimal number of crossings on e [10], is a helpful tool for designing heuristics that minimize edge crossings in drawings of general graphs. Unfortunately, some graphs do not have a geometric embedding Γ such that $\Gamma + e$ has the same number of crossings as the embedding $G + e$. This motivates the study of the computational complexity of the following problem: Given a combinatorially embedded graph G , compute a geometric embedding Γ that has the same combinatorial embedding as G and that minimizes the crossings of $\Gamma + e$. We give polynomial-time algorithms for special cases and prove that the general problem is fixed-parameter tractable in the number of crossings. Moreover, we show how to approximate the number of crossings by a factor $(\Delta - 2)$, where Δ is the maximum vertex degree of G .

1 Introduction

Crossing minimization is an important task for the construction of readable drawings. The problem of minimizing the number of crossings in a given graph is a well-known \mathcal{NP} -complete problem [8]. A very successful heuristic for minimizing the number of crossings in a topological drawing of a graph G is to start with a spanning planar subgraph H of G and to iteratively *insert* the remaining edges into a drawing of H . The edge insertion problem for a planar graph G and two vertices $s, t \in V(G)$ asks to find a drawing $\Gamma + st$ of $G + st$ with the minimum number of crossings such that the induced drawing Γ of G is planar. The problem comes with several variants depending on whether the drawing Γ can be chosen arbitrarily or is fixed [9,10]. In the planar topological case both problems can be solved in linear time. More general problems such as inserting several edges simultaneously [2] or inserting a vertex together with all its incident edges [1] have also been studied.

All these approaches have in common that they focus on topological drawings where edges are represented as arbitrary curves between their endpoints. By contrast, we focus on geometric embeddings, i.e., planar straight-line drawings, and the corresponding rectilinear crossing number. In this scenario we are only aware of a few heuristics that compute straight-line drawings of general graphs [12,13].

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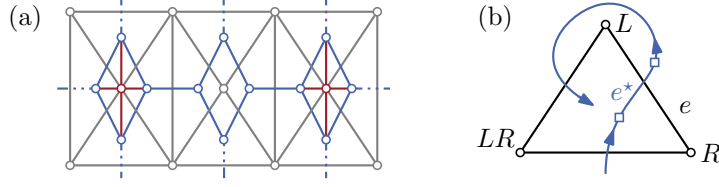


Fig. 1: (a) The extended dual (red + blue) of the primal graph (grey) and the red vertices corresponding to s and t . (b) Labeling induced by the blue path.

Clearly, if a geometric embedding Γ of the input graph G is provided as part of the input, there is no choice left; we can simply insert the straight-line segment from s to t into the drawing and count the number of crossings it produces. If, however, only the combinatorial embedding is specified, but one may still choose the outer face and choose the vertex positions so that this results in a straight-line drawing with the given combinatorial embedding, then the problem becomes interesting and non-trivial. We call this problem *geometric edge insertion*.

Contribution and Outline. We show several results on the complexity of geometric edge insertion with a fixed combinatorial embedding. Namely, we give a linear-time algorithm for the case that the maximum degree Δ of G is at most 5 (Sec. 3). For the general case, we give a $(\Delta - 2)$ -approximation that runs in linear time. Moreover, we give an efficient algorithm for testing in special cases whether there exists a way to insert the edge st so that it does not produce more crossings than when we allow to draw it as an arbitrary curve (Sec. 4). Finally, we give a randomized FPT algorithm that tests in $O(4^k n)$ time whether an edge can be inserted with at most k crossings (Sec. 5).

2 Preliminaries

Let $G = (V, E)$ be a planar graph with a given combinatorial embedding where only the choice of the outer face is free. Additionally, let s and t be two distinct vertices with $st \notin E$. Denote by $G + st$ the graph G together with the edge st . We want to insert the edge st into the embedded graph G . That is, we seek a straight-line drawing Γ of G (with the given embedding) such that st can be inserted into Γ with a minimum number of crossings. In Γ , the edge st starts at s , traverses a set of faces and ends in t . Topologically, this corresponds to a path $p(\Gamma)$ from s to t in the *extended dual* G_{st}^* of G , i.e., in the dual graph G^* plus s and t connected to all vertices of their dual faces; see Fig. 1a. The number of crossings in $\Gamma + st$ corresponds to the length of the path minus two. However, not all st -paths in G_{st}^* are of the form $p(\Gamma)$ for a straight-line drawing Γ of G .

A labeling of G is a mapping $l: V \rightarrow \{L, R\}$ that labels vertices as either left or right. Consider an edge uv of G that is crossed by a path p such that u and v are to the left and to the right of p , respectively. The edge uv is *compatible*

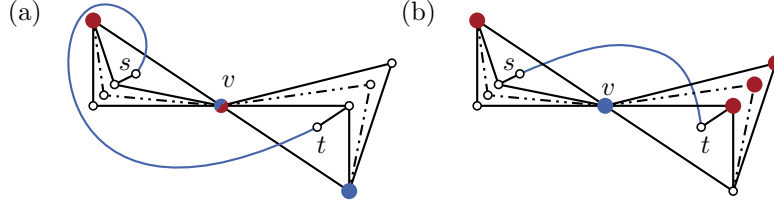


Fig. 2: Ratio between length of the shortest st path and the length of a shortest consistent st -path. The solid black edges induce a graph of maximum degree 6. Red vertices have label L , blue vertices have label R . (a) The shortest path from s to t in G_{st}^* is not consistent.

with a labeling l if $l(u) = L$ and $l(v) = R$. A path p of G_{st}^* and a labeling l of G are *compatible* if l is compatible with each edge that is crossed by p . A path p is *consistent* if there is a labeling of G that is compatible with p . Eades et al. [4] show the following result.

Proposition 1 (Eades et al. [4], Theorem 1). *An st -path in G_{st}^* is of the form $p(\Gamma)$ if and only if it is consistent, where Γ is a geometric embedding of G .*

In order to minimize the number of crossings of $\Gamma + st$, we look for a consistent st -path of minimum length in G_{st}^* . Given a path p , it is easy to check whether p is consistent. Fig. 2 shows that the ratio between the length of a shortest st -path and the length of a shortest consistent st -path can be arbitrarily large. Thus, our goal is to find short consistent st -paths.

Let $H = (V', E')$ be a directed acyclic graph. A path $p = \langle v_1, v_2, \dots, v_k \rangle$ is a *directed path* if for each $1 \leq i < k$, $v_i v_{i+1} \in E'$. It is *undirected* if for each $1 \leq i < k$, either $v_i v_{i+1} \in E'$ or $v_{i+1} v_i \in E'$. We refer to the number $|p|$ of edges of a path as the *length of p* . Two paths p and p' are *edge-disjoint* if they do not share an edge. Two paths p and p' of an embedded graph are *non-crossing* if at each common vertex v , the edges of p and p' incident to v do not alternate in the cyclic order around v in the graph induced by p and p' . We denote by $p[u, v]$ the subpath of a path p from u to v .

3 Bounded Degree

The shortest st -path of the graph in Fig. 2a is not consistent. Note that the maximum vertex degree is 6. In this section, we show that every shortest st -path in graphs of bounded degree 3 is consistent, and that in each planar graph with vertex degree at most 5, there is a shortest st -path that is consistent. Finally, we prove that there is a consistent st -path of length $(\Delta - 2)l$ in a graph with maximum vertex degree Δ and a shortest st -path of length l in G_{st}^* .

Let p be an st -path in G_{st}^* and let e^* be an edge of p . An endpoint u of the primal edge e of e^* is *left of e^** if it is locally left of p on e (Fig. 1b). A vertex v of G is *left (right)* of p if v is left (right) of an edge of p . We now consider a labeling

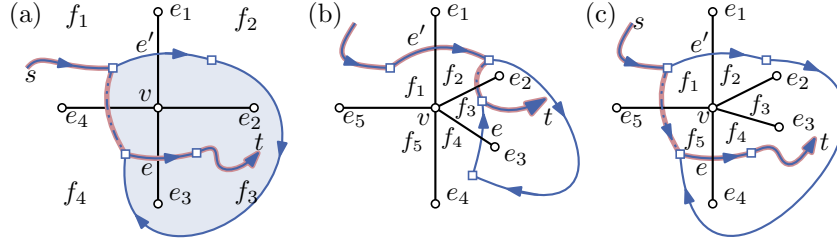


Fig. 3: Inconsistent path around (a) a degree-4 vertex and (b,c) a degree-5 vertex.

extended by two more labels LR, \perp . We define the labeling l_p induced by p as follows. Each vertex that is left and right of p gets the label LR . The remaining vertices that are either left or right of p get labels L and R , respectively. Vertices neither left nor right of p get the label \perp . Obviously, there is a labeling l of G compatible with p if and only if l_p does not use the label LR .

Theorem 2. *Let G be a planar embedded graph of degree at most 3. Then every shortest st -path in G_{st}^* is consistent.*

Theorem 3. *Let G be a planar embedded graph with maximum degree 5. Then there is a shortest st -path in G_{st}^* that is consistent.*

Proof. Let p be a shortest st -path in G_{st}^* . We call an edge e of p *good* if the vertices left and right of it do not have label LR in the labeling l_p induced by p .

If p is not consistent, then let e denote the last edge of p that is not good. Then an endpoint v of the primal edge corresponding to e has label LR . Without loss of generality, we may assume that v lies left of e . Since $l_p(v) = LR$, there is an edge e' of p that has v to its right. By the choice of e , it follows that e' lies before e on p . We now distinguish cases based on the degree of v .

If $\deg(v) \leq 3$, then we find that p enters or leaves a face twice, which contradicts the assumption that it is a shortest st -path.

If $\deg(v) = 4$, we denote the edges around v in clockwise order as e_1, \dots, e_4 such that e' crosses e_1 . Moreover, we denote the faces incident to v in clockwise order as f_1, \dots, f_4 where f_1 is the starting face of e' .

Since no face has two incoming or two outgoing edges of p , it follows that $e' = f_1f_2$ crosses e_1 and $e = f_4f_3$ crosses e_3 ; see Fig. 3a. Let p' be the path obtained from p by replacing the subpath $p[f_1, f_4]$ by the edge f_1f_4 that crosses e_4 . Since p is a shortest path, it follows that $f_2 = f_4$. By construction, it is $l_{p'}(v) = L$. Observe that $p'[f_4, t] = p[f_4, t]$ lies inside the region ρ bounded by $p[f_1, f_4]$ and a curve connecting f_1 and f_4 that crosses e_4 . The only vertex inside this region whose label changed is v . Therefore, the path $p'[f_1, t]$ consists of good edges, and we have thus increased the length of the suffix of the shortest path that consists of good edges.

Now assume that $\deg(v) = 5$. We denote the edges around v as e_1, \dots, e_5 in clockwise order such that e' crosses e_1 . We further denote the faces incident to

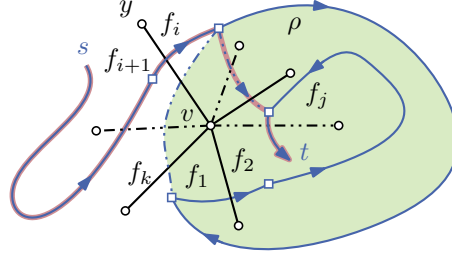


Fig. 4: Inconsistent path around a degree k vertex.

v in clockwise order as f_1, \dots, f_5 such that e' starts in f_1 . Since no face has two incoming or two outgoing edges, it follows that either e crosses e_4 from f_5 to f_4 or e crosses e_3 from f_4 to f_3 .

If e crosses e_3 , then we consider the path p' obtained from p by replacing the subpath $p[f_2, f_3]$ by the edge that crosses e_3 ; see Fig. 3b. As above, it follows that $f_2 = f_4$ and v is a cutvertex and that $p'[f_1, t]$ consists of good edges.

If e crosses e_4 , then we obtain p' by replacing $p[f_1, f_5]$ by the single edge that crosses e_5 ; see Fig. 3c. As above, we find that $f_2 = f_5$ and v is a cutvertex and that $p'[f_1, t]$ consists of good edges.

Thus, in all cases, we increase the length of the suffix of the shortest path consisting of good edges. Eventually, we thus arrive at a shortest path whose edges are all good and that hence is consistent. \square

Theorem 4. *Let $G = (V, E)$ a planar embedded graph with maximum vertex-degree Δ and let p be a shortest st -path in G_{st}^* with $s, t \in V$. Then there is a consistent path of length at most $(\Delta - 2)|p|$.*

Proof. Let p be an st -path in G_{st}^* . Assume that p is not consistent. Then there is a shortest prefix $p_2 = p[s, f_2] = p[s, f_1] \cdot f_1 f_2$ of p that is not consistent; refer to Fig. 4. Let v be a vertex incident to the primal edge of $f_1 f_2$ with $l_{p_2}(v) = LR$. Without loss of generality let f_1, f_2, \dots, f_k be the faces around v in counterclockwise order, i.e., v lies left of $f_1 f_2$.

Since p_2 is not consistent, there is a second edge of p_2 that crosses a primal edge incident to v . Let e be the last edge of $p[s, f_1]$ that crosses a primal edge incident to v . Since p_2 is the shortest inconsistent prefix of p , v lies right of e , i.e., $e = f_{i+1} f_i$ for some i with $2 < i \leq k - 1$. Moreover, let f_j be the first vertex in clockwise order from f_i that lies on the path $p[f_2, t]$. Note that such a vertex f_j exists, since at the latest f_2 satisfies the condition.

Let q be the path $f_i f_{i-1} \dots f_j$. We obtain a path p' from p by replacing $p[f_i, f_j]$ by q , i.e., $p' = p[s, f_i] \cdot q \cdot p[f_j, t]$. Note that, since f_j is the first vertex in clockwise order on $p[f_2, t]$, p' is a simple path. Since q does not contain the edges $f_k f_1$ and $f_1 f_2$, and $p[f_i, f_j]$ contains at least one edge, the path p' has length at most $|p| + (k - 2) - 1$. We claim that the prefix $p'_j = p'[s, f_j]$ is consistent.

Then, since $p'[f_j, t]$ is a subpath of $p[f_2, t]$ and $p'[s, f_j]$ is consistent, it follows that we have decreased the maximum length of a suffix of the path whose removal

results in an inconsistent path. Since this suffix has initially length at most $|p|$, we inductively find a consistent st -path of length at most $(\Delta - 2)|p|$.

It remains to prove that $p'[s, f_j]$ is consistent. Since $p[s, f_2]$ is the shortest inconsistent prefix of p , the prefix $p[s, f_1]$ is consistent. Therefore, v is right of $p[s, f_i] = p'[s, f_i]$. By construction, v is right of q . Thus, we have $l_{p'_j}(v) = R$. The only vertices w of G with $l_{p'_j}(w) = LR$ can be neighbors of v , as otherwise $p[s, f_1]$ would not be consistent.

Consider the region ρ enclosed by the path $p[f_i, f_1]$ and f_1, f_k, \dots, f_i that contains v ; refer to Fig. 4. The prefix $p[s, f_1] = p'[s, f_1]$ lies outside of ρ and the path q lies entirely in ρ . Moreover, in case that vw is crossed by $p'[s, f_i]$, w lies outside of ρ . On the other hand, if q crosses an edge vw , then w lies inside ρ . Thus, in both cases we immediately get that $l_{p'_j}(w) = L$. Therefore, the prefix $p'[s, f_j]$ is consistent. \square

4 Consistent Shortest st -paths

In Section 3 we showed that every shortest st -path in the extended dual G_{st}^* of a graph G with vertex degree at most 3 is consistent. For every graph of maximum degree 5, there is a shortest st -path G_{st}^* that is consistent. On the other hand, Fig. 2 shows that, starting from degree 6, there are graphs whose shortest st -paths are not consistent. In this section we investigate the problem of deciding whether G_{st}^* contains a consistent shortest st -path. As a consequence of Proposition 1 this problem is in \mathcal{NP} .

In Lemma 5 we show that finding a consistent st -path p in G_{st}^* is closely related to finding two edge-disjoint paths in G . Especially, we are interested in two edge-disjoint paths where the length of one is minimized. Eilam-Tzoreff [5] proved that this problem is in general \mathcal{NP} -complete. In planar graphs the sum of the length of two vertex-disjoint paths can be minimized efficiently [11]. In general directed graphs the problem is \mathcal{NP} -hard [7]. Finding two edge-disjoint paths in acyclic directed graphs is \mathcal{NP} -complete [6].

The closest relative to our problem is certainly the work of Eilam-Tzoreff. In fact their result can be modified to show that it is \mathcal{NP} -hard to decide whether a graph contains two edge-disjoint st -paths such that one of them is a shortest path. We study this problem in the planar setting with the additional restriction that s and t lie on a common face of the subgraph G_{sp} of G_{st}^* that contains all shortest paths from s to t .

Lemma 5. *An st -path p in G_{st}^* is consistent if and only if there is an st -path p' in G_{st}^* that is edge-disjoint from p and that does not cross p .*

Proof. The paths p and p' define a set of regions in the plane. Since p and p' are non-crossing, each region is bounded by one maximal subpath of p and one maximal subpath of p' (Fig. 5). We label each region ρ with either L or R , depending on whether ρ lies left or right of the unique maximal subpath of p on its boundary. We define a labeling l of G by giving each vertex v the label of the region ρ that contains it. We claim that l is compatible with p .

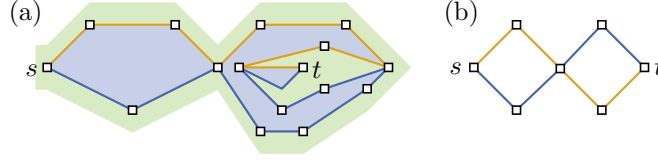


Fig. 5: (a) The green regions are right of p (blue) and the blue left of p . (b) The outer region that is not bounded by maximal subpaths of p and p' .

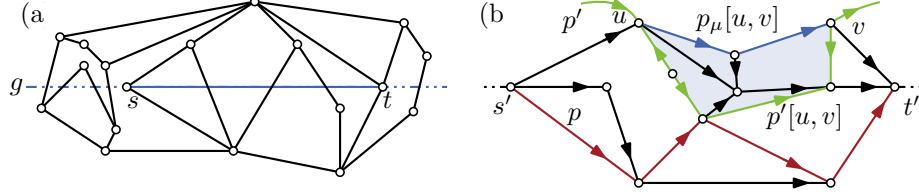


Fig. 6: (a) The line g through the segment st induces a path in G_{st}^* . (b) Modification of the undirected path p' edge-disjoint from p .

Since p and p' are edge-disjoint, every primal edge connects vertices of the same or adjacent regions. Moreover, by construction, vertices of adjacent regions have different labels. Thus all vertices left of p have label L and all vertices right of p have label R . That is l is compatible with p , i.e., p is consistent.

Conversely, assume that p is consistent. By Proposition 1 there is a straight-line drawing of G such that the segment st intersects the same edges as p and in the same order (Fig. 6a). Let g be the line that contains the segment st . Each edge of G intersects g at most once. Thus, the complement of st in g defines a path from s to t in G_{st}^* that is edge-disjoint from p and does not cross p . \square

Thus, we now consider the problem of finding a consistent shortest st -path as an edge-disjoint path problem in G_{st}^* . Our proof strategy consists of three steps. Step 1) We first show that the problem is equivalent to finding two edge-disjoint paths p and q in a directed graph \vec{G}_{st} such that p is directed and q is undirected. Step 2) We modify \vec{G}_{st} such that p is a path in a specific subgraph G_{sp} and q lies in the subgraph $\overline{G_{sp}}$. These two graphs may share an edge set \hat{E} such that each edge in \hat{E} can be an edge of p or of q . Moreover, we find pairs of edges e and e' in \hat{E} such that the path p in G_{sp} (the path q in $\overline{G_{sp}}$) contains either e or e' . Step 3) Finally, we use these properties to reduce our problem to 2-SAT.

We begin with Step 1. A directed graph $\vec{G}_{st} = (V' \cup \{s, t\}, E')$ is *st-friendly* if G_{st}^* contains a consistent shortest st -path if and only if \vec{G}_{st} contains a directed st -path p and an undirected st -path p' that is edge-disjoint from p and does not cross p . We obtain an st -friendly graph $\vec{G}_{st} = (\vec{V}, \vec{E})$ from G_{st}^* as follows. Denote by G_{sp} the directed acyclic graph that contains all shortest paths from s to t in $G_{st}^* = (V, E)$. If an edge $uv \in E$ is an edge of G_{sp} , we add it to \vec{G}_{st} .

For all remaining edges uv , we add a subdivision vertex x to \vec{G}_{st} and add the directed edges xu, xv to \vec{G}_{st} in this direction. We claim that \vec{G}_{st} is st -friendly.

Let p be a consistent shortest st -path in G_{st}^* . By Lemma 5 there is a path p' in G_{st}^* that is edge-disjoint from p and does not cross p . By construction p corresponds to a directed path in G_{sp} and p corresponds to an undirected path in \vec{G}_{st} . Conversely, due to the directions of the edges xv, xu , every directed st -path q in \vec{G}_{st} is a directed path in G_{sp} , and therefore it is a shortest st -path in G_{st}^* . If there is an undirected path q' that is edge-disjoint from q and does not cross q , we obtain a path p' from q' by contracting edges incident to split vertices x . Hence, \vec{G}_{st} is st -friendly.

We consider the following special case, where s and t lie on a common face o of the subgraph G_{sp} of \vec{G}_{st} . Without loss of generality, let o be the outer face of G_{sp} and let t lie on the outer face of \vec{G}_{st} . We denote by p_μ and p_λ the upper and lower st -path of G_{sp} on the boundary of o . A vertex v of G_{sp} is an *interior vertex* if v does not lie on o . An edge uv of G_{sp} is an *interior edge* if u and v are interior vertices. An edge e of G_{sp} is a *chord* if both its endpoints lie on o but e is not an edge on the boundary of o .

Lemma 6. *For a directed st -path p and an undirected st -path p' , that are edge-disjoint and non-crossing, there is an undirected st -path p'' that is edge-disjoint from p , does not cross p , and that does not use interior vertices of G_{sp} .*

Proof. Since p and p' are non-crossing, there are two distinct vertices u, v on p_λ or on p_μ , say p_μ , such that the inner vertices of $p'[u, v]$ lie in the interior of G_{sp} ; refer to Fig. 6b. Moreover, since p' and p are non-crossing, the region enclosed by $p'[u, v]$ and $p_\mu[u, v]$ does not contain a vertex of p in its interior. Therefore, we obtain p'' by iteratively replacing pieces in the form of $p'[u, v]$ by $p_\mu[u, v]$. \square

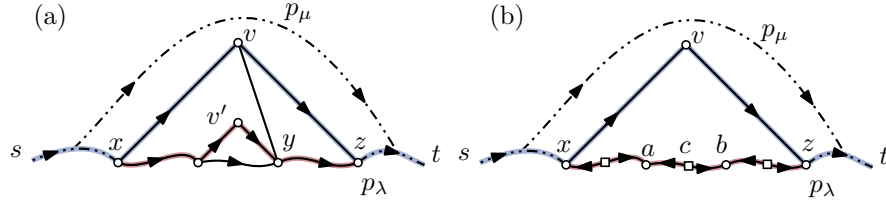


Fig. 7: (a) The red directed path can be circumvented with the blue directed path via vertex v . (b) The red path consists of avoidable edges.

This finishes Step 1, and we continue with Step 2. In the following, we iteratively simplify the structure of G_{sp} while preserving st -friendliness of \vec{G}_{st} . Due to Lemma 6, the graph G_{sp}/e , obtained from contracting an edge e of G_{sp} , is st -friendly, if e is an interior edge. This may generate a separating triangle xyz . Let v be a vertex in the interior of xyz and let p be a directed st -path

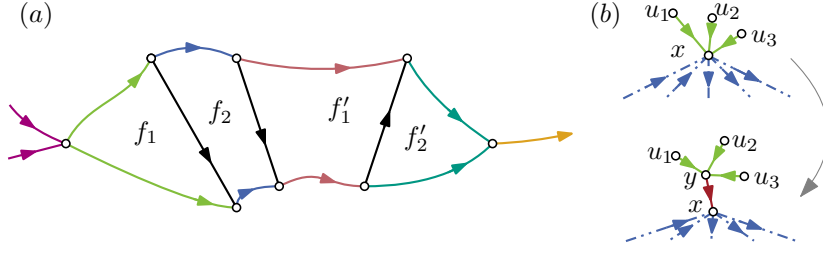


Fig. 8: (a) Interior partners decoded by color of 2-edge connected component of G_{sp} . (b) Split a vertex x on the boundary of H_{st}^* .

that contains v . Then, p contains at least two vertices of x, y, z . Hence, p can be rerouted using an edge of xyz . Thus, the graph after removing all vertices in the interior of xyz is st -friendly. After contracting all interior edges of G_{sp} , each neighbor of an interior vertex of G_{sp} lies either on p_λ or on p_μ . The remaining edges are edges on $p_\lambda \cup p_\mu$ and chords.

Consider three vertices x, y, z that lie in this order on p_λ (p_μ) and two interior vertices v and v' , with $xv, v'y, vz \in \vec{E}$; refer to Fig. 7a. Note that v and v' can coincide. Then, every directed st -path p that contains y also contains x and z . Hence, p can be rerouted through the edges xv, vz and as a consequence of Lemma 6, the graph $G_{sp} - v'y$ is st -friendly. Analogously, if G_{sp} contains the edge yv' , $G_{sp} - yv'$ remains st -friendly. We call such edges *circumventable*.

We refer to edges of a subpath $p_\lambda[x, z]$ ($p_\mu[x, z]$) as *avoidable* if there exists an interior vertex v with $xv, vz \in \vec{E}$ (Fig. 7b). If there exists a directed path p that uses an avoidable edge ab it can be rerouted by replacing the corresponding path $p_\lambda[x, z]$ with the edges xv, vz . Thus, we can split the edge ab with a vertex c and we direct the resulting edges from c towards a and b , respectively, and remove the edge ab from \vec{G}_{st} . Finally, we iteratively contract edges incident to vertices with in- and out-degree 1, and we iteratively remove vertices of degree at most 1, except for s and t . Since all interior edges of G_{sp} are contracted, circumventable interior edges are removed and avoidable edges are replaced, each 2-edge connected component of G_{sp} is an outerplanar graph whose weak dual (excluding the outer face) is a path; compare Fig. 8a. Each face f of G_{sp} , with $f \neq o$, contains at least one edge e_λ of p_λ and one edge e_μ on p_μ . Moreover, every directed st -path contains either e_λ or e_μ . We refer to the edge sets $E_{f,\lambda} = E(f) \cap E(p_\lambda)$ and $E_{f,\mu} = E(f) \cap E(p_\mu)$ as *interior partners*.

Property 7. Choosing a directed st -path in G_{sp} is equivalent to choosing for each face f of G_{sp} one of the interior partners $E_{f,\mu}$ or $E_{f,\lambda}$ such that the following condition holds. Let f_1, f_2 be two adjacent faces that are separated by a chord e that ends at p_λ (p_μ) such that f_1 is right of e (left of e), then the choice of $E_{f_2,\mu}$ ($E_{f_2,\lambda}$) implies the choice of $E_{f_1,\mu}$ ($E_{f_1,\lambda}$).

In the following, we modify the *exterior* of \vec{G}_{st} , i.e., $\overline{G_{sp}} = \vec{G}_{st} - E(G_{sp})$, with the aim to obtain an analog property for the choice of the undirected path.

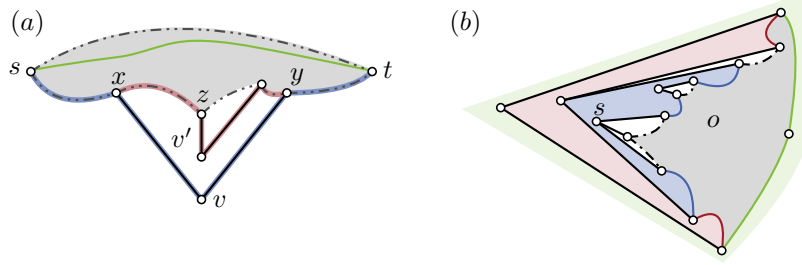


Fig. 9: (a) If the undirected path contains z , it can be rerouted to use vertex v .
 (b) The color coding of the faces indicate the exterior partners.

We refer to edges of $\overline{G_{sp}}$ as *exterior edges*. A vertex in $V(\overline{G_{sp}}) \setminus V(G_{sp})$ is an *exterior vertex*.

Since the undirected path is not allowed to cross the directed path, we split each cut vertex x into an upper copy x_μ and a lower copy x_λ . We reconnect edges of p_λ and p_μ incident to x to x_λ and x_μ , respectively. Exterior edges incident to x that are embedded to the right of p_λ are reconnected to x_λ . Likewise, edges embedded to the left of p_μ are reconnected to x_μ . Note that this operation duplicates bridges of G_{sp} . Thus, we forbid the undirected path to traverse these duplicates. Observe that after this operation the outer face o of G_{sp} is bounded by a simple cycle.

Let x be a vertex on o that is incident to an exterior edge. In this case, we insert a vertex y to \vec{G}_{st} and we remove each exterior edge ux from \vec{G}_{st} and insert as a replacement edges yx and yu ; see Fig. 8b. We refer to the edge yx as a *barrier*. Since the barrier yx is directed from y to x , the modification preserves the st -friendliness of \vec{G}_{st} . We now exhaustively contract exterior edges that are not barrier edges, and remove vertices in the interior of separating triangles.

Recall that s and t lie on a common face o of the subgraph G_{sp} of \vec{G}_{st} and t lies on the outer face of G_{sp} . Let v be an exterior vertex such that its neighbor x comes before its neighbor y on p_i , $i = \lambda, \mu$, refer to Fig. 9a. Let z be a vertex between x and y on p_i that is connected to a vertex v' such that the edge $v'z$ (zv') lies in the interior of the region bounded by yvx and $p_i[x, y]$. Consider a directed st -path p in G_{sp} and an undirected st -path p' in \vec{G}_{st} that is edge-disjoint from p , that does not cross p and that contains v' . Due to Lemma 6 we can assume, that p' does not contain an interior vertex of G_{sp} . Thus, it contains x and y . We obtain a new path p'' by replacing the subpath $p'[x, y]$ by vx, vy . Since vx, vy are exterior edges, p'' and p are edge-disjoint and non-crossing. Thus, the graph $\vec{G}_{st} - v'z$ ($\vec{G}_{st} - zv'$) is st -friendly. After removing all such edges, for any two neighbors x and y of an exterior vertex v , the paths $o[x, y]$ and $o[y, x]$ each contains either s and t . Hence, the region bounded by yvx and $o[x, y]$ contains a second exterior vertex v' if and only if $o[x, y]$ contains either s or t .

Hence, the dual of $\overline{G_{sp}}$, with the dual vertex of o removed, is a caterpillar C , refer to Fig. 9b. In case that s or t is incident to an exterior vertex v , we can

assume that the undirected path p' contains the edge sv (vt). Thus, for simplicity, we now assume that neither s nor t is connected to an exterior vertex. Let a and b be the vertices in C whose primal faces are incident to s and t , respectively. Then every undirected st -path in $\overline{G_{sp}}$ from s to t traverses the primal faces of the simple path q from a to b in C . Let f be a primal face of a vertex on q . Since we inserted the barrier edges to \vec{G}_{st} , every face contains at least one edge e_λ of p_λ and one edge e_μ of p_μ . Therefore, every undirected st -path in $\overline{G_{sp}}$ either contains e_λ or e_μ . We refer to the sets $E_{f,\lambda} = E(f) \cap E(p_\lambda)$ and $E_{f,\mu} = E(f) \cap E(p_\mu)$ as *exterior partners*.

Property 8. Choosing an undirected st -path in $\overline{G_{sp}}$ is equivalent to choosing for each face $f \neq o$ of $\overline{G_{sp}}$ one of the exterior partners $E_{f,\lambda}$ or $E_{f,\mu}$.

This finishes Step 2, and we proceed to Step 3. The problem of finding a directed st -path p and an undirected st -path p' in \vec{G}_{st} reduces to a 2-SAT instance as follows. For each exterior and interior partner we introduce variables x_f and x_g , respectively, where f and g correspond to the faces of the partners. If x_f is true, p' contains the edge of $E_{f,\lambda}$, otherwise it contains $E_{f,\mu}$. The conditions on the choice of p in Property 7 can be formulated as implications. Let $E_{f,\mu}$ and $E_{f,\lambda}$ be exterior partners and let $E_{g,\mu}$ and $E_{g,\lambda}$ be interior partners. In case that $E_{f,\lambda} \cap E_{g,\lambda} \neq \emptyset$, either p can contain edges of $E_{g,\lambda}$ or p' can contain edges of $E_{f,\lambda}$ but not both. Thus, x_f and x_g are not allowed to be true at the same time, i.e., $x_f = \overline{x_g}$. Hence, we have the following Theorem.

Theorem 9. *If s and t lie on a common face of G_{sp} , it is decidable in polynomial time whether \vec{G}_{st} has a directed st -path and an undirected st -path that are edge-disjoint and non-crossing.*

Corollary 10. *If s and t lie on a common face of G_{sp} , it is decidable in polynomial time whether G_{st}^* contains a consistent shortest st -path.*

5 Parametrized Complexity of Short Consistent st -Paths

In this section we show that edge insertion can be solved in FPT time with respect to the minimum number of crossings of a straight-line drawing of $G + st$ where G is drawn without crossings and has the specified embedding. Let l be an arbitrary labeling of G . Observe that l defines a directed subgraph of G_{st}^* by removing each edge whose dual edge has endpoints with the same label and by directing all other edges e such that the endpoint of its primal edge left of e has label L and its other endpoint has label R . We denote this graph by $G_{st}^*(l)$. Obviously, a shortest st -path in $G_{st}^*(l)$ is compatible with l , and thus a corresponding drawing exists. Clearly, given the labeling l a shortest st -path in $G_{st}^*(l)$ can be computed in linear time by a BFS.

Now assume that the length of a shortest consistent path in G_{st}^* is k . We propose a randomized FPT algorithm with running time $O(4^k n)$ for finding a shortest consistent path in G_{st}^* , based on the color-coding technique [3].

The algorithm works as follows. First, we pick a random labeling of G by labeling each vertex independently with L or R with probability $1/2$. We then compute a shortest path in $G_{st}^*(l)$. We repeat this process 4^k times and report the shortest path found in all iterations.

Clearly the running time is $O(4^k n)$. Moreover, each reported path is consistent, and therefore the algorithm outputs only consistent paths. It remains to show that the algorithm finds a path of length k with constant probability.

Consider a single iteration of the procedure. If the random labeling l is compatible with p , then the algorithm finds a path of length k . Therefore the probability that our algorithm finds a consistent path of length k is at least as high as the probability that p is compatible with the random labeling l . Let $V_L, V_R \subseteq V$ denote the vertices of V that are left and right of p , respectively. Clearly it is $|V_L|, |V_R| \leq k$. A random labeling l is consistent with p if it labels all vertices in V_L with L and all vertices in V_R with R . Since vertices are labeled independently with probability $1/2$, it follows that $\Pr[p \text{ is consistent with } l] = (1/2)^{|V_L|} \cdot (1/2)^{|V_R|} \geq (1/2)^{2k} = (1/4)^k$.

Therefore, the probability that no path of length k is found in 4^k iterations is at most $(1 - (1/4)^k)^{4^k}$, which is monotonically increasing and tends to $1/e \approx 0.368$. Thus the algorithm succeeds with a probability of $1 - 1/e \approx 0.632$. The success probability can be increased arbitrarily to $1 - \delta$, $\delta > 0$ by repeating the algorithm $\log(1/\delta)$ times. The probability that each iteration fails is then bounded from above by $(1/e)^{\log 1/\delta} = 1/e^{\log 1/\delta} = \delta$. E.g., to reach a success probability of 99%, it suffices to do $\log 100 \leq 5$ repetitions. The algorithm can be derandomized with standard techniques [3].

Theorem 11. *There is a randomized algorithm \mathcal{A} that computes a consistent path of length k if one exists with a success probability of $1 - \delta$. The running time of \mathcal{A} is $O(\log(\delta^{-1})4^k n)$.*

6 Conclusion

We have shown that the problem of finding a short consistent st -paths in G_{st}^* is tractable in special cases and fixed-parameter tractable in general. Whether G_{st}^* has a short consistent st -path is equivalent to the question of whether G_{st}^* has two edge-disjoint and non-crossing st -paths, where the length of one path is minimized. Surprisingly, this is related to yet another purely graph theoretic problem: does a directed graph G have two edge-disjoint paths where one is directed and the other is only undirected? By the result of Eilam-Tzoreff [5] the former problem is in general \mathcal{NP} -hard. For planar graphs the computational complexity of these problems remains an intriguing open question.

In this paper, we only considered planar graphs with a fixed combinatorial embedding. Allowing for arbitrary embeddings opens new perspectives on the problem and is interesting future work.

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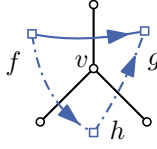


Fig. 10: Inconsistent path around a degree-3 vertex.

A Proof of Theorem 2

Theorem 2. *Let G be a planar embedded graph of degree at most 3. Then every shortest st -path in G_{st}^* is consistent.*

Proof. Let p be a shortest path in G_{st}^* . Assume that p is not consistent. Then there is a vertex v that left and right of p . Let fg be the first edge of p that crosses a primal edge incident to v . If the degree of v is at most 2, then p contains either a loop or a double edge, contradicting the assumption that p is a shortest path. Therefore, assume that the degree of v is 3. Without loss of generality, let f, g and h be the faces around v in clockwise order (Fig. 10b). Since v is left and right of p , p contains either the edge fh or hg . Thus, p contains either f or g twice. This contradicts the assumption that p is a shortest path. \square