# Strongdeco: Expansion of analytical, strongly correlated quantum states into a many-body basis 

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#### Abstract

We provide a Mathematica code for decomposing strongly correlated quantum states described by a first-quantized, analytical wave function into manybody Fock states. Within them, the single-particle occupations refer to the subset of Fock-Darwin functions with no nodes. Such states, commonly appearing in two-dimensional systems subjected to gauge fields, were first discussed in the context of quantum Hall physics and are nowadays very relevant in the field of ultracold quantum gases. As important examples, we explicitly apply our decomposition scheme to the prominent Laughlin and Pfaffian states. This allows for easily calculating the overlap between arbitrary states with these highly correlated test states, and thus provides a useful tool to classify correlated quantum systems. Furthermore, we can directly read off the angular momentum distribution of a state from its decomposition. Finally we make use of our code to calculate the normalization factors for Laughlin's famous quasi-particle/quasi-hole excitations, from which we gain insight into the intriguing fractional behavior of these excitations.


Keywords: Fractional quantum Hall effect, Strongly correlated systems, Manybody quatum states, geometric gauge fields, Ultra-cold atoms

## Program Summary

Title of program: Strongdeco
Catalogue identifier:
Program summary URL: http://cpc.cs.qub.ac.uk/summaries
Program available from: CPC Program Library, Queen's University of Belfast, N. Ireland

Operating systems: Linux, Windows, Mac
Programming language used: Mathematica
Number of bytes in distributed program, including test code and documentation:
Distribution format: .nb
Nature of Problem: Analysis of strongly correlated quantum states
Method of Solution: The program makes use of the tools developed in mathematica to deal with multivariate polynomials to decompose analytical strongly correlated states of bosons and fermions into a standard many-body basis. Operations with polynomials, determinants and permanents are the basic tools.

## 1. Introduction

In recent years there has been a growing interest in the study of strongly correlated quantum states and their possible realization in two-dimensional (2D) systems of ultracold atoms [1, 2]. Such states, which were first postulated when studying the dynamics of electrons subjected to strong magnetic fields, can also be produced in systems of neutral atoms subjected to so-called artificial gauge fields. One of the first examples was obtained by rotating a 2 D atomic cloud such that the centrifugal force on the atoms mimics the Lorentz force which a charged particle would experience in the presence of a constant magnetic field perpendicular to the system [3, 4, 5, 6]. The main drawback of this approach is that large rotations are needed in order to observe strongly correlated states such as the Laughlin [7], while it is difficult to stabilize in this fast rotating regime. For this reason, the Laughlin state has not yet been engineered in the pioneering experiments [8, 9]. Recently, Roncaglia et al. have proposed an alternative experiment to avoid the instability difficulty by using a Mexican-hat $\operatorname{trap}$ [10].

Other very promising proposals to overcome this problem come from quantum optics and consider the coupling of the atoms to one or several laser fields. These make the atoms experience a Berry phase 11, 12, 13, 14, 15], which, due to the mathematical equivalence between geometric phases and external gauge fields, can then be interpreted as if it were due to the presence of an external gauge field. The experimental realization of such an artificial gauge field has already been achieved [16].

An important motivation to study these systems is the possibility of producing strongly correlated quantum states and quasi-particle/quasi-hole excitations which are neither described by fermionic nor by bosonic commutation laws. The latter are expected to have strong impact in the context of anyonic quantum computation [17]. One thus needs to quantify the strongly correlated states produced by different proposals and the properties of their quasi-particle/quasi-hole excitations.

A common way to study such systems theoretically is by means of exact diagonalization of small-sized systems [3, 6]. The usual methodology is to employ Fock-Darwin (FD) wave functions, which describe single particles with fixed angular momentum in a fixed Landau level, and which are the eigenfunctions of a 2D non-interacting system with a perpendicular magnetic field in the symmetric gauge. The many-body basis for bosons (fermions) is then built up by symmetric (antisymmetric) combinations of the FD states. While this basis is practical for definite calculations, many relevant states in the literature have been found by proposing a first-quantized, analytic wave function. Here the Laughlin [7], the Pfaffian, also called Moore-Read [18], or the Laughlin quasi-particle states [19] are the most prominent examples. Translating the first-quantized wave functions into the language of second quantization, however, turns out to be a hard task 20, 21].

In this paper, we present a computer code which achieves this goal for arbitrary states described by an analytic function, and thus provides practitioners of this field with a simple and yet powerful tool to quantify the degree of correlation by examining its expansion into an independent particle motion basis. The code is written in Mathematica [22], which is a computer language specially suited for symbolic evaluation.

We begin by presenting the first-quantized expression of the most important strongly correlated states in Sect. 2. In Sect. 3, we briefly describe and construct the many-body basis into which we then decompose the states in Sect. 4. Finally, in Sect. 5, we consider two applications which can be tackled making use of the described decomposition scheme. The most relevant routines contained in Strongdeco.nb are explained within the text, a brief description of all routines is given in the Appendix.

## 2. Analytical strongly correlated states

Strongly correlated states in 2D systems exposed to a gauge field are usually studied in the regime where all particles occupy the lowest Landau level. The Hilbert space of an $N$-body system in this regime can be represented by wave functions of the form

$$
\begin{equation*}
\Psi\left(z_{1}, \ldots, z_{N}\right)=\mathcal{N} f\left(z_{1}, \ldots, z_{N}\right) \mathrm{e}^{-\sum\left|z_{i}\right|^{2} / 2 \lambda_{\perp}^{2}} \tag{1}
\end{equation*}
$$

where $z_{i}=\left(x_{i}+i y_{i}\right) / \lambda_{\perp}, \mathcal{N}$ is a normalization constant, and $f$ is a polynomial in its arguments $z_{i}$. The typical length scale of the system is given by $\lambda_{\perp}$. The most famous wave function of this form is the Laughlin function [7]:

$$
\begin{equation*}
\Psi_{\mathcal{L}}\left(z_{1}, \ldots, z_{N}\right)=\mathcal{N}_{\mathcal{L}} f_{m}\left(z_{i}, \ldots, z_{N}\right) \mathrm{e}^{-\sum\left|z_{i}\right|^{2} / 2 \lambda_{\perp}^{2}} \tag{2}
\end{equation*}
$$

with $\mathcal{N}_{\mathcal{L}}$ a normalization constant, and

$$
\begin{equation*}
f_{m}\left(z_{i}, \ldots, z_{N}\right) \equiv \prod_{i<j}\left(z_{i}-z_{j}\right)^{m} \tag{3}
\end{equation*}
$$

where $m$ is an integer directly related to the filling factor $\nu=1 / m$ of the lowest Landau level. Originally intended to describe electrons, this wave function had to be antisymmetric, restricting $m$ to odd numbers. However, as shown for instance in Ref. [3], also the ground state of a two-dimensional system of rotating bosons with contact interaction is, for certain values of the angular rotation, described by the Laughlin state, if $m$ is taken as an even integer [23, 24]. One important property of the Laughlin wave function is that $f_{m}$ is a homogeneous polynomial. Its degree determines the well-defined total angular momentum of the system, given by $L=\frac{1}{2} m N(N-1)$.

Besides the Laughlin state, other states of the form given by Eq. (1) show up as the ground state of a rotating Bose gas, if we vary the rotation frequency [3], 19, 25]. For a broad range of rotation frequencies, for instance, a large overlap
is found with the so-called Pfaffian state, which has $L=N(N-2) / 2$ for even $N$, and $L=(N-1)^{2} / 2$ for odd $N$. It explicitly reads,

$$
\begin{equation*}
\Psi_{\mathcal{P}}=\mathcal{N}_{p} \operatorname{Pf}([z]) \prod_{i<j}\left(z_{i}-z_{j}\right) \tag{4}
\end{equation*}
$$

with $\mathcal{N}_{p}$ a normalization coefficient and,

$$
\begin{equation*}
\operatorname{Pf}([z])=\mathcal{A}\left[\frac{1}{\left(z_{1}-z_{2}\right)} \frac{1}{\left(z_{3}-z_{4}\right)} \cdots \frac{1}{\left(z_{N-1}-z_{N}\right)}\right] \mathrm{e}^{-\sum\left|z_{i}\right|^{2} / 2 \lambda_{\perp}^{2}} \tag{5}
\end{equation*}
$$

where $\mathcal{A}$ is an antisymmetrizer of the product. As explained in Ref. [3], the Pfaffian state can also be computed as,

$$
\begin{equation*}
\Psi_{\mathcal{P}}=\mathcal{S} \prod_{i<j \in \sigma_{1}}\left(z_{i}-z_{j}\right)^{2} \prod_{k<l \in \sigma_{2}}\left(z_{k}-z_{l}\right)^{2} \mathrm{e}^{-\sum\left|z_{i}\right|^{2} / 2 \lambda_{\perp}^{2}} \tag{6}
\end{equation*}
$$

where $\sigma_{1}$ and $\sigma_{2}$ are two subsets containing $N / 2$ particles each if $N$ is even, and $(N+1) / 2$ and $(N-1) / 2$ if $N$ is odd. $\mathcal{S}$ symmetrizes the expression.

Another relevant state is the Laughlin quasi-particle state,

$$
\begin{equation*}
\Psi_{\mathcal{L} q p}\left(\xi, \xi^{*}\right)=\mathcal{N}_{q p}\left(\xi, \xi^{*}\right) \mathrm{e}^{-\sum\left|z_{i}\right|^{2} / 2 \lambda_{\perp}^{2}} \prod_{i \leq N}\left(\partial_{z_{i}}-\xi\right) f_{m} \tag{7}
\end{equation*}
$$

where $\xi$ represents the position of the quasiparticle. If we pin the quasi-particle to the origin, the Laughlin quasi-particle state has a definite angular momentum $L=\frac{1}{2} m N(N-1)-N$, and, as shown in Ref. [19] for $m=2$ and $N=4$, its overlap with the ground state of rotating ultracold atoms is fairly large in certain regions of the rotation.

The analog of Laughlin's quasi-particle state is the quasi-hole state,

$$
\begin{equation*}
\Psi_{\mathcal{L} q h}\left(\xi, \xi^{*}\right)=\mathcal{N}_{q h}\left(\xi, \xi^{*}\right) \mathrm{e}^{-\sum\left|z_{i}\right|^{2} / 2 \lambda_{\perp}^{2}} \prod_{i \leq N}\left(z_{i}-\xi\right) f_{m} \tag{8}
\end{equation*}
$$

with an increased angular momentum $\frac{1}{2} m N(N-1) \leq L \leq \frac{1}{2} m N(N-1)+N$. An interesting property of quasiparticles and quasiholes is their anyonic nature 26] and fractional charge. Note that in the case of an electroneutral system, one may define the analog of a charge by looking at the Berry phase a particle acquires when moving in the presence of the artificial gauge field.

## 3. The many-body basis

After defining in the previous section the general structure for all wave functions of interest, we now construct the many-body basis into which we wish to decompose these states. A convenient choice to span the Hilbert space of the many-body system are the eigenfunctions of the non-interacting problem, i.e. the FD wave functions,

$$
\begin{equation*}
\phi_{\ell}(z)=\frac{z^{\ell}}{\sqrt{\pi \ell!}} \frac{1}{\lambda_{\perp}^{\ell+1}} e^{-|z|^{2} /\left(2 \lambda_{\perp}^{2}\right)}, \quad \ell=0, \ldots, \infty \tag{9}
\end{equation*}
$$

where we have restricted ourselves to the lowest Landau level. These states satisfy,

$$
\begin{equation*}
\int_{-\infty}^{\infty} d x \int_{-\infty}^{\infty} d y \phi_{\ell}^{*}(z) \phi_{\ell^{\prime}}(z)=\delta_{\ell, \ell^{\prime}} \tag{10}
\end{equation*}
$$

The many-body basis can be generated by considering products of the FD functions, which in the case of bosons have to be combined in a symmetric way, while antisymmetric combinations must be constructed in the case of fermions. Here, we will concentrate on the bosonic case, but with only slight modifications which are explicitly shown in the code file, fermionic systems can be treated in the same way. For the bosonic system we write the many-body state as,

$$
\begin{equation*}
\left\{\ell_{1}, \ell_{2}, \ldots, \ell_{N}\right\} \equiv \mathcal{S}\left[\phi_{\ell_{1}}\left(z_{1}\right) \phi_{\ell_{2}}\left(z_{2}\right) \ldots \phi_{\ell_{N}}\left(z_{N}\right)\right] \tag{11}
\end{equation*}
$$

where $\mathcal{S}$ symmetrizes over the $N$ particles. These states are called permanents, which are the bosonic analog of the Slater determinants, with the difference that all terms have a positive sign. Without loss of generality we may assume that $\ell_{1} \leq \ell_{2} \leq \cdots \leq \ell_{N}$. The orthonormality of the permanents then reads

$$
\begin{equation*}
\left\{\ell_{1}, \ell_{2}, \ldots, \ell_{N}\right\} \cdot\left\{\ell_{1}^{\prime}, \ell_{2}^{\prime}, \ldots, \ell_{N}^{\prime}\right\}=\delta_{\ell_{1}, \ell_{1}^{\prime}} \delta_{\ell_{2}, \ell_{2}^{\prime}} \cdots \delta_{\ell_{N}, \ell_{N}^{\prime}} \tag{12}
\end{equation*}
$$

For simplicity we will from now on set the scale factor $\lambda_{\perp}=1$, and suppress the exponential term which is common to all $N$-body states, and, as an overall Gaussian, fixes the center of mass to the origin. We can then simplify the problem to dealing with permanents of the form,

$$
\begin{equation*}
\mathcal{S}\left[z_{1}^{\ell_{1}} z_{2}^{\ell_{2}} \ldots z_{N}^{\ell_{N}}\right] \tag{13}
\end{equation*}
$$

From Eq. (12) follows that, for a given $N$, all states of a fixed total angular momentum $L=\sum_{i=1}^{N} \ell_{i}$ form a subspace which is orthogonal to the subspace with total angular momentum $L^{\prime} \neq L$. We can therefore perform the decomposition independently in each subspace, and thus restrict ourselves to a subspace with fixed $L$. Its basis (up to normalization factors and the overall exponential term) can be constructed through the command,

```
ConjS[na_, L_] := Module[{poty, dimy},
    poty = Pots[na, L];
    dimy = Dimensions[poty][[1]];
    Table[Perm[na, poty[[i]]], {i, 1, dimy}]]
```

which makes use of the function Perm [27], that builds the appropriate permanent, and of $\operatorname{Pots}[\mathbf{N}, \mathbf{L}]$, which constructs the set of indexes $\ell_{1}, \ldots \ell_{N}$ for a given $N$ and $L$, represented by na and $\mathbf{L}$ in the code,

```
cc[0] = 0;
tab[n_, l_] :=
Table[{cc[i], cc[i - 1],
If[i == 1, l, (l - Sum[cc[j], {j, 0, i - 1}])/2]},
{i, 1, n - 1}];
Pots[na_, L_] := If[na == 2, Table[{i, L - i}, {i, 0, L/2}],
Module[{pat},
Clear[pat];
pat[na] = Join[Table[
cc[i], {i, 1, na - 1}], {L - Sum[cc[i], {i, 1, na - 1}]}];
pat[a_] := Table[pat[a + 1], Evaluate[tab[na, L][[a]]]];
Flatten[pat[1], na - 2]]]
```

For instance, for $N=4$ and $L=2$ we have ${ }^{1}$,
Pots $[4,2]=\{\{0,0,0,2\},\{0,0,1,1\}\}$
and correspondingly,

```
ConjS[4, 2]=
{6 z[1]^2+6 z[2]^2+6 z[3]^2+6 z[4]^2,
4 z[1] z[2]+4 z[1] z[3]+4 z[2] z[3]
+4 z[1] z[4]+4 z[2] z[4]+4 z[3] z[4]}
```

As can be seen in this example, due to multiple occupation of the same singleparticle state, some of the permutations contributing to the symmetrized wavefunction are described by the same monomials which thus have prefactors given by the factorial of the number of permutations. These factors need to be taken into account to correctly normalize the many-body states, and can be obtained through, nami[ $\mathbf{N}, \mathbf{L}]$, which gives a table with the same ordering as Pots or ConjS, for our previous example, nami $[4,2]=\{6,4\}$, as could be inferred from the obtained expressions.

```
nami[na_, L_] := Module[{potty, pp, inde, ta},
    potty = Pots[na, L];
    pp = Dimensions[potty][[1]];
    inde = Table[Complement[potty[[i]]], {i, 1, pp}];
    ta = Table[Table[Count[potty[[i]], inde[[i, j]]],
{j, 1, Dimensions[inde[[i]]][[1]]}], {i, 1, pp}];
    Table[ Product[ta[[i, j]]!,
{j, 1, Dimensions[ta[[i]]][[1]]}] , {i, 1, pp}]]
```

Once these factors are known it is easy to build the normalization coefficient by looking into the prefactors in the Fock-Darwin states Eq. (9), $1 / \sqrt{\pi \ell!}$. The function $\operatorname{tip}[\mathbf{N}, \mathbf{L}]$ gives the normalization coefficients. Their explicit coding is,

[^0]```
tip[na_, L_] := Module[{potty, nimy},
potty = Pots[na, L];
nimy = nami[na, L];
Table[Sqrt[nimy[[i]]]Sqrt[Product[Pi Gamma[potty[[i, jj]]+1],
{jj, 1, na}] ], {i, 1, Dimensions[nimy][[1]]}]]
```


## 4. Decomposition of the states

All states described in Sect. 2] are, up to the common exponential factor, polynomials in the $z$ variables. To write down the states in terms of the manybody ones, we can suppress the exponential and work out the decomposition of the polynomial in terms of the permanents. While the Laughlin and the Pfaffian state have a definite total angular momentum, for the quasi-hole and quasi-particle states this is only true if we fix the position of the quasi-particle to the origin $\xi=0$. Otherwise, we must first sort the polynomial by the different contributions with a definite order in $z$, and can then proceed, for each contribution separately, in the way described here, where we assume an analytical state, $\Psi\left(z_{1}, z_{2}, \ldots, z_{N}\right)$ with fixed $N$ and $L$. We are looking for an expansion of the form,

$$
\begin{equation*}
\Psi\left(z_{1}, z_{2}, \ldots, z_{N}\right)=\sum_{j=1}^{n_{D}} C_{j}\left\{\ell_{1, j}, \ell_{2, j}, \ldots, \ell_{N, j}\right\} \tag{14}
\end{equation*}
$$

where $n_{D}$ is the total size of the many-body basis, which can be computed as $n_{D}=\operatorname{Dimensions}[\operatorname{PotsN}[N, L]][[1]]$. To get a feeling of how this grows with $N$ and $L$ the dimension of these spaces for the $L$ corresponding to the Laughlin wave functions are, $n_{D}=7,34,192,1206,8033,55974$ for $N=3,4,5,6,7$ and 8 , respectively.

To decompose a polynomial into these states, we have to find the monomials which correspond to a given Fock state and read out their coefficients. Since we know that the polynomial is symmetric (antisymmetric) under exchange of two coordinates, it is sufficient to find only one monomial contributing to a given Fock state, as all the others must have the same coefficient (up to a sign in the antisymmetric case). This can be achieved by taking derivatives:

$$
\begin{equation*}
\left.\partial_{z_{1}}^{\ell_{1, j}} \cdots \partial_{z_{N}}^{\ell_{N, j}} \Psi\left(z_{1}, z_{2}, \ldots, z_{N}\right)\right|_{z_{1}=0, \cdots, z_{N}=0}=c_{j} \tag{15}
\end{equation*}
$$

where $c_{j}$ is not yet the coefficient $C_{j}$ in Eq. (14), but is directly related to it through the normalization procedure described in the previous section. Hereby, we have to take into account that an additional factor $\prod_{i=1}^{N} \ell_{i, j}$ ! occurs through the derivatives. Thus we obtain

$$
\begin{equation*}
C_{j}=c_{j}\left(P \prod_{i=1}^{N} \ell_{i, j}!\pi\right)^{-1 / 2} \equiv \alpha_{j} c_{j} \tag{16}
\end{equation*}
$$

where the $P$ is the factorial of permutations leading to the same expression, obtained by nami $[\mathbf{N}, \mathbf{L}]$. We thus see that $\alpha_{j}$ equals the inverse of the $j$ th component of $\operatorname{tip}[\mathbf{N}, \mathbf{L}]$. The decomposition of, for instance, the Laughlin wavefunction can therefore be obtained by the following piece of code:

```
DDecoLaug[na_,nu_] :=
Module[{Lmin, Lmax, state, base, dim, factors, d, prf, outp},
    Lmin = na (na - 1);
    Lmax = na (na - 1);
    state = Laughlin[na,nu];
    base = Flatten[Table[Pots[na, i], {i, Lmin, Lmax}], 1];
    dim = Dimensions[base][[1]];
    factors = Flatten[Table[tip[na, i], {i, Lmin, Lmax}], 1];
    d[0] = state;
    prf = Table[
            For[i = 1, i < na + 1, i++,
            d[na] = 0;
            d[i] = D[d[i - 1], {z[i], base[[j, i]]}];
            d[i] = d[i] /. z[i] -> 0;
            If[d[i] == 0, Break[]]
            ];
        d[na]/factors[[j]],
        {j, 1, dim}];
    outp = prf/Sqrt[prf.prf]]
```

Here, Laughlin [ $\mathbf{N}, \mathbf{n u}$ ] describes the Laughlin wavefunction for $N$ particles at filling $\nu$. For even $1 / \nu$, this is a symmetric function describing bosons, while odd values yield an antisymmetric function for fermionic systems. In principle, we can use the code for both the symmetric and the antisymmetric case. In the latter, however, it is convenient to exclude states with multiple occupied singleparticle levels from the basis, as they obviously make no contribution. This can be done by replacing Pots[ $\mathbf{N}, \mathbf{L}]$ by its fermionic analogue $\operatorname{PotsF}[\mathbf{N}, \mathbf{L}]$ defined in the code file. Consequently, we will also have to replace $\mathbf{t i p}[\mathbf{N}, \mathbf{L}]$ by $\operatorname{tipF}[\mathbf{N}, \mathrm{L}]$.

An alternative way to achieve the decomposition is by means of a particular, built-in Mathematica function, PolynomialReduce. This function provides the decomposition of a given multivariate polynomial in terms of a set of polynomials. The code for decomposing the bosonic Laughlin state then reads

```
LaugDeco[na_,nu_]:=Module[{state, base, symb, laur, prf, outp},
    state = Laughlin[na,nu];
    base = ConjS[na, na (na - 1)];
    symb = Table[z[i], {i, 1, na}];
    laur = PolynomialReduce[state, base, symb];
    If[laur[[2]] != 0, Print["Problem in reduction"]];
    prf = laur[[1]] tip[na, na (na - 1)];
    outp = prf/Sqrt[prf.prf];
    outp]
```

For most states that we have considered, the decomposition by means of deriva-
tives is faster. However, making use of PolynomialReduce turns out to be quicker for the fermionic Laughlin state as well as for quasiparticle excitations.

In figure 1, a snapshot of the code for the decomposition of the Laughlin state is provided for $N=3, N=4$ and $N=5$. The code has been tested for $N \leq 7$ on a laptop running on linux with 1 Gb of RAM memory. A listing of the different commands defined in Strongdeco.nb is provided in the Appendix. The notebook is provided with some examples built-in inside.


Figure 1: Snapshot of the code where the decomposition of the Laughlin state with $\nu=1 / 2$ is obtained for $N=3, N=4$ and $N=5$.

## 5. Applications

Finally, we make use of the presented decomposition scheme, and consider as examples three problems which might be tackled with the given code.

### 5.1. Wave-function overlaps

The decomposition achieved by our code turns out to be very useful if the state, e.g. the eigenstates of a certain problem or the evolved state at a given time, of a system is known in the many-body basis. This is the case if a system is studied via exact diagonalization [3, 6, 13, 14]. It is then customary to ask

| $\ell=$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m=2$ | $\frac{257}{553}$ | $\frac{264}{553}$ | $\frac{303}{553}$ | $\frac{446}{553}$ | $\frac{447}{553}$ | $\frac{330}{553}$ | $\frac{165}{553}$ | 0 | 0 | 0 |
| $m=3$ | $\frac{185}{706}$ | $\frac{185}{706}$ | $\frac{209}{706}$ | $\frac{321}{706}$ | $\frac{417}{706}$ | $\frac{465}{706}$ | $\frac{455}{706}$ | $\frac{339}{706}$ | $\frac{186}{706}$ | $\frac{62}{706}$ |

Table 1: Angular-momentum distribution for Laughlin states of $N=4$ particles for bosons ( $m=2$ ) and fermions $(m=3)$.
whether, and to which degree, the obtained state resembles one of the wellknown strongly correlated states described in Sect. 2. To answer this question one has to calculate the overlap between both states, which can either be done by straightforwardly expressing the many-body state in first quantization and then evaluating the overlap integrals. For reasonable system sizes of $N \geq 3$, the latter is a very lengthy task. The second possibility consists of finding an expression of the analytic wave function in terms of the many-body basis, which then reduces the overlap calculation to a simple scalar multiplication of two vectors. In this case, the first step is the non-trivial one, but it is directly achieved by the code we have presented here.

### 5.2. Angular-momentum distribution

As a second application, one can consider a system which is known to have eigenstates of the form given by Eq. (1). Many of the properties of such states are better computed by first transforming it into the independent motion basis. A clear example is the calculation of the angular momentum distribution of the state, from which one also gains insight into its one-body density matrix and other correlation functions. For the fermionic Laughlin state this problem has been considered in Ref. [21], where exact results are obtained by calculating the density and then extracting the angular momentum distribution. This method, however, fails for systems larger than $N=3$, for which MonteCarlo methods have been applied. By means of our code, we are able to reproduce these results by decomposing the Laughlin state into a basis from which the angular momentum of each particle can be directly read off. It is straightforward to go beyond the analytical results of Ref. [21]. As an example, we give in Table 1 the angular momentum distributions for the fermionic Laughlin state at $m=3$ and the bosonic Laughlin state at $m=2$ of a system with $N=4$ particles.

### 5.3. Fractional charge of excitations

Another useful application is the calculation of the normalization factor for a state of the form (11). As explained in Ref. [28], the normalization factors $\mathcal{N}_{q h}\left(\xi, \xi^{*}\right)$ and $\mathcal{N}_{q p}\left(\xi, \xi^{*}\right)$ of the quasi-particle in Eq. (7) and the quasi-hole state in Eq. (8), contain information about the Berry phase $\mathbf{a} \cdot \mathrm{d} \mathbf{x}$ which these
excitations acquire during an adiabatic movement:

$$
\begin{equation*}
a_{\xi} \equiv \frac{i}{2} \partial_{\xi} \ln \mathcal{N}\left(\xi, \xi^{*}\right), \tag{17}
\end{equation*}
$$

with $\partial_{\xi} \equiv \frac{1}{2}\left(\partial_{x}-i \partial_{y}\right)$ and $a_{\xi} \equiv \frac{1}{2}\left(a_{x}-i a_{y}\right)$. With this expression one may consider the phase picked up by the quasi-particle when it is moved adiabatically around a closed loop, which is given by $I_{a} \equiv \oint \mathrm{~d} \mathbf{x} \cdot \mathbf{a}$. Comparing this with the phase a moving particle acquires in the system, $I_{A} \equiv \oint \mathrm{~d} \mathbf{x} \cdot \mathbf{A}$, where $\mathbf{A}$ is the external gauge potential, one is able to deduce the fractional charge and fractional statistics of the quasi-particles. If both loop integrals are equal, i.e. $\eta \equiv I_{a} / I_{A}=1$, the quasi-particle behaves like a normal particle, while the mismatch by a fractional factor, i.e. $\eta=1 / p$ with $p$ an integer, allows to interpret the quasi-particle as a "fractional"particle.

The difficulty of this analysis lies in the calculation of the normalization factors. For the Laughlin state one usually circumvents this by applying the Plasma analogy [7, 26] to determine the normalization factor of the corresponding quasi-holes and quasi-particles, avoiding the explicit calculation. It is found that $p=m$, i.e. the fractional behavior of the excitation follows from the fractional filling of the lowest Landau level. The plasma analogy, however, is not applicable to all the relevant states, which exhibit such anyonic excitations, but are different from the Laughlin. Calculating the normalization factors by direct integration is much too complicated, even for systems of only a few particles. However, by transforming the corresponding quasi-hole or quasi-particle state into the many-body basis by means of our code, the normalization factors are obtained by simply taking the scalar product of the decomposition vector.

As an example, we calculate $\eta_{q h}$ and $\eta_{q p}$ for quasi-holes and quasi-particles in small Laughlin systems. First, we notice that the polynomial in Eqs. (7) and (8) contains terms of different order in $z$. It is therefore necessary to separate the $N$ contributions with fixed angular momentum, and then apply the decomposition explained in Sect. 4 to each of them. Then we can write the normalization factor as a polynomial in $\xi$ and $\xi^{*}$. For instance, a the wavefunction of a quasi-hole in the $m=2$ Laughlin state of $N=5$ bosons, is found to have the following normalization factor:

$$
\begin{equation*}
C \propto 1+0.477|\xi|^{2}+0.117|\xi|^{4}+0.0211|\xi|^{6}+0.00334|\xi|^{8}+0.000668|\xi|^{10} \tag{18}
\end{equation*}
$$

which, even in high orders of $\xi$, agrees reasonably well with the prediction by the plasma analogy, according to which $C \propto \exp \left(\frac{1}{m}|\xi|^{2}\right)$.

However, from Table 2 showing the results for different numbers of particles, we find that, especially for the quasi-particle excitation, significant deviations occur for very small systems $(N=4)$. For $N=6$, the mismatch is almost cured, and the agreement with the plasma analogy prediction is better than $3 \%$ for both quasi-holes and quasi-particles.

| $N$ | $\eta_{q h}$ | $\eta_{q p}$ |
| :---: | :---: | :---: |
| 4 | 0.473 | 0.354 |
| 5 | 0.477 | 0.438 |
| 6 | 0.487 | 0.501 |

Table 2: Fractional charge of quasi-holes and quasi-particles in a bosonic Laughlin system at filling $\nu=1 / 2$ for different number of particles $N$.

## 6. Conclusion

We have presented a code which, by means of the symbolic package Mathematica, analytically decomposes relevant analytical, strongly-correlated manybody states into the many-body basis built up by single-particle angular momentum eigenstates. This basis is commonly used to describe 2 D quantum systems subjected to gauge fields, and thus the described decomposition is a useful tool for calculating the overlap of different states with famous test states like the Laughlin state, the Pfaffian state and many others. It also allows for studying the angular-momentum distribution of strongly correlated states, which can be related to the ground-state density of the system, or normalization constants of the states. The latter has been shown to provide insights into the fractional character of quasiparticle excitations.

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## Appendix A. List of routines in Strongdeco.nb

We provide a brief description of the routines,

| Command Name | Brief Description |
| :---: | :---: |
| Many-body states |  |
| $\operatorname{Pots}[\mathbf{N}, \mathrm{L}]$ | Many-body states for $N$ bosons of fixed total $L$ |
| ConjS[N,L] | Permanents for $N$ bosons of fixed total $L$ |
| PotsF[ $\mathrm{N}, \mathrm{L}$ ] | Many-body states for $N$ fermions of fixed total $L$ |
| ConjSF[N,L] | Slaters for $N$ fermions of fixed total $L$ |
| Sample states |  |
| Laughlin [ $\mathbf{N}, \nu$ ] | Laughlin state of $N$ atoms and filling $\nu$. |
| Pfaffian [ $\mathrm{N}, \nu$ ] | Pfaffian state of $N$ atoms using a Laughlin of filling $\nu$ |
| Conje[N] | Pfaffian state for $N$ atoms, filling 1/2 |
| deltaL2[ $\mathrm{N}, \mathrm{I}]$ | Generalized Laughlin state [13] |
| deltaL4[ $\mathrm{N}, \mathrm{I}]$ | Generalized Laughlin state [13] |
| $\mathbf{q h}[\mathbf{N}, \nu, \xi]$ | Laughlin quasi-hole state at position $\xi$ |
| quah $[\mathbf{N}, \nu$ ] | Laughlin quasi-hole state at the center |
| $\mathbf{q p}[\mathbf{N}, \nu, \xi]$ | Laughlin quasi-particle state at position $\xi$ |
| quap[ N$]$ | Laughlin quasi-particle state at the center ( $\nu=1 / 2)$ |
| deltaQP2[ $\mathrm{N}, \mathrm{i}]$ | Generazlied Laughlin quasi-particle |
| edge[N] | Laughlin ( $\nu=1 / 2$ ) edge excitation |
| deltaP2[ $\mathrm{N}, \mathrm{i}]$ | Generalized Pfaffian |
| deltaP $4[\mathrm{~N}, \mathrm{i}]$ | Generalized Pfaffian |
| Normalizations |  |
| nami[ $\mathrm{N}, \mathrm{L}$ ] | Repeated terms, bosons |
| $\operatorname{tip}[\mathrm{N}, \mathrm{L}]$ | Normalization coefficient, bosons |
| $\operatorname{tipF}[\mathbf{N}, \mathrm{L}]$ | Normalization coefficient, fermions |
| Decomposition of states |  |
| Bosons |  |
| LaugDeco[ $\mathbf{N}, \nu$ ] | Laughlin decomposition |
| DDecoLaug [ $\mathrm{N}, \nu$ ] | Laughlin decomposition using Eq. (15) |
| EdgeDeco[N] | Laughlin edge state |
| Laug2Deco[N] | Generalized Laughlin |
| Laug4Deco[N] | Generalized Laughlin |
| QuasiHDeco[ N$]$ | Quasihole |
| DDecoQuah[N] | Quasihole using Eq. (15) |
| NormQh[N] | Quasihole norm |
| QuasiPDeco[N] | Laughlin quasiparticle |
| DDecoQuap[N] | Laughlin quasiparticle using Eq. (15) |
| QuasiP2Deco[N] | Generalized Laughlin quasiparticle |
| PfaffDeco[N] | Pfaffian |
| ConjeDeco[N] | Pfaffian using Eq. (6) |
| DDecoConje[N] | Pfaffian using Eq. (6) and Eq. (15) |
| Pfaff2Deco[N] | Generalized Pfaffian |
| Fermions |  |
| LaugDecoF [ $\mathbf{N}, \nu$ ] | Laughlin |
| DDecoLaugF[ $\mathbf{N}, \nu]$ | Laughlin using Eq. (15) |
| QuasiHDecoF[ $\mathbf{N}, \nu]$ | Laughlin quasi hole |

Table A.3: List of commands provided in Strongdeco.nb.

## References

[1] M. Lewenstein et al., Adv. Phys. 56, 243 (2007).
[2] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[3] N.K. Wilkin and J.M.F. Gunn, Phys. Rev. Lett. 84, 6 (2000).
[4] A. L. Fetter, Rev. Mod. Phys. 81, 647 (2009).
[5] N. Cooper, AiP 57, 539 (2008).
[6] D. Dagnino, N. Barberán, M. Lewenstein, and J. Dalibard, Nature Phys. 5, 431 (2009); D. Dagnino, N. Barberán, and M. Lewenstein, Phys. Rev. A 80, 053611 (2009).
[7] R.B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[8] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, Phys. Rev. Lett. 92, 050403 (2004).
[9] V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendoff, and E. A. Cornell, Phys. Rev. Lett. 92, 040404 (2004).
[10] M. Roncaglia, M. Rizzi, and J. Dalibard, Scientific Reports 1, 43 (2011).
[11] G. Juzeliūnas and P. Öhberg, Phys. Rev. Lett. 93, 033602 (2004).
[12] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, in press Reviews of Modern Physics, arXiv:1008.5378 (2010).
[13] B. Juliá-Díaz, D. Dagnino, K. J. Günter, T. Graß, N. Barberán, M. Lewenstein, and J. Dalibard, Phys. Rev. A 84, 053605 (2011).
[14] B. Juliá-Díaz, T. Graß, N. Barberán, and M. Lewenstein, arXiv:1111.0798 (2011).
[15] B. G. Levi, Phys. Today 64, 13 (2011).
[16] Y.-J. Lin, R. L. Compton, K. Jimenez-Garcia, J. V. Porto, and I. B. Spielman, Nature 462, 628 (2009); Y.-J. Lin, R. L. Compton, A. R. Perry, W. D. Phillips, J. V. Porto, and I. B. Spielman, Phys. Rev. Lett. 102, 130401 (2009).
[17] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[18] G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
[19] M. Popp, B. Paredes, and J. I. Cirac, Phys. Rev. A 70, 053612 (2004).
[20] K. Takano and A. Isihara, Phys. Rev. B 34 (R), 1399 (1986).
[21] S. Mitra and A. H. MacDonald, Phys. Rev. B 48, 2005 (1993).
[22] See http://www.wolfram.com/mathematica/.
[23] N. R. Cooper, N. K. Wilkin, and, J. M. F. Gunn, Phys. Rev. Lett. 87, 120405 (2001).
[24] N. Regnault and T. Jolicoeur, Phys. Rev. Lett. 91, 030402 (2003).
[25] M. Roncaglia, M. Rizzi, and J. I. Cirac, Phys. Rev. Lett. 104, 096803 (2010).
[26] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. 53, 722 (1984).
[27] E. W. Weisstein, "Permanent." From MathWorld-A Wolfram Web Resource. http://mathworld.wolfram.com/Permanent.html
[28] X.-G. Wen, Quantum Field Theory of Many-Body Systems, Oxford University Press (2004).


[^0]:    ${ }^{1}$ Note that the state $\{1,1,0,0\}$ is equivalent to $\{0,0,1,1\}$ due to the symmetrization of the states.

