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A normal approximation for the chi-square distribution

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Abstract

An accurate normal approximation for the cumulative distribution function of the chi-square distribution with n degrees of freedom is proposed. This considers a linear combination of appropriate fractional powers of chi-square. Numerical results show that the maximum absolute error associated with the new transformation is substantially lower than that found for other power transformations of a chi-square random variable for all the degrees of freedom considered ($1 \leqslant n \leqslant 1000$).

Keywords: Chi-square distribution; Maximum absolute error; Normal approximation; Power transformation

1. Introduction

Power transformations of the chi-square random variable can be employed to improve its approximate normality. Among these, the best known are the Fisher's square root transformation $(2n\chi^2)^{1/2}$ (Fisher, 1922) and the third root transformation $(\chi^2/n)^{1/3}$ (Wilson and Hilferty, 1931), where n stands for the number of degrees of freedom (d.f.). Also the fourth root transformation $(\chi^2/n)^{1/4}$ has a distribution which is close to normality for all degrees of freedom (Cressie and Hawkins, 1980; Hawkins and Wixley, 1986).

The third root transformation produces a closer approximation to normality than the square root transformation (Merrington, 1941; Goldberg and Levine, 1946; Zar, 1974).

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For 1 and 2 degrees of freedom the fourth root is superior to the third root; however, for larger numbers of degrees of freedom the third root is better.

Goria (1992) proposed a linear combination of the square and of the fourth root transformations, which are the only that, within the family of power transformations, have the property that the Pearson's kurtosis index is zero to an appropriate order. However, the results were substantially similar to those obtained using the third root transformation, with clear benefits only for high values of the degrees of freedom. In this paper a more accurate linear combination of power transformations of the chi-square random variable will be proposed, that fits well regardless of the number of degrees of freedom.

2. Power approximation

The basic idea underlying the transformation was to combine different powers of χ^2 in order to have the Pearson's kurtosis index equal to zero to an appropriate order, and the symmetry near to zero also for a small number of degrees of freedom. A preliminary search suggested that a linear combination of powers $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$ would be suitable. So, the following linear combination was considered:

$$L = a\left[\left(\frac{\chi^2}{n}\right)^{1/6} + b\left(\frac{\chi^2}{n}\right)^{1/3} + c\left(\frac{\chi^2}{n}\right)^{1/2}\right],$$

where the constants a, b, c must be determined in such a way that both the skewness and the kurtosis of L tend rapidly to those of the normal distribution. It is clear from the above equation that the constant a affects only the variance, but neither skewness nor kurtosis, so it was set equal to one. An admissible solution was found for $b = -\frac{1}{2}$ and $c = \frac{1}{3}$ (see the appendix for details); therefore the linear combination proposed as normal approximation for the chi-square distribution is the following:

$$L = \left(\frac{\chi^2}{n}\right)^{1/6} - \frac{1}{2}\left(\frac{\chi^2}{n}\right)^{1/3} + \frac{1}{3}\left(\frac{\chi^2}{n}\right)^{1/2}.$$

The resulting linear combination does have an appealing feature as it can be expressed as

$$6L = 6\left(\frac{\chi^2}{n}\right)^{1/6} - 3\left(\frac{\chi^2}{n}\right)^{1/3} + 2\left(\frac{\chi^2}{n}\right)^{1/2}.$$

The skewness and kurtosis coefficients of L are, respectively, $\frac{1}{27\sqrt{2}n^{3/2}} + \mathrm{O}(n^{-2})$ and $\mathrm{O}(n^{-2})$, while its expected value is

$$E[L] = \frac{5}{6} - \frac{1}{9n} - \frac{7}{648n^2} + \frac{25}{2187n^3} + \mathrm{O}(n^{-4}) \tag{1}$$

and the variance is

$$\mathrm{Var}[L] = \frac{1}{18n} + \frac{1}{162n^2} - \frac{37}{11664n^3} + \mathrm{O}(n^{-4}). \tag{2}$$

3. Numerical results

The numerical accuracy of the linear combination L in terms of the maximum absolute error (MAE) was investigated for n between 1 and 1000 employing the approach of Li and Martin (2002). To evaluate the performance of each approximation function, $V(x;n)$, the MAE was defined over the range of the chi-square random variable with n degrees of freedom for which the cumulative probability $F_{\chi^2}(x;n)$ lies between 0.0001 and 0.9999, in steps of 0.0001. This set is denoted by $\boldsymbol{X}$:

$$\mathrm{MAE} = \max_{x \in \mathbf{X}} |F_{\chi^2}(x;n) - V(x;n)|.$$

Each evaluation was performed twice, once using Mathematica version 4.2 (Wolfram, 1999) and the other using a FORTRAN program which employed the algorithms AS91, AS111, AS147, AS66, ACM291 (Griffiths and Hill, 1985). The same set of numerical results was obtained. These are displayed in Fig. 1 and, for selected values of n, in Table 1, along with the MAE for the approximation function, $V(x;n)$, taken as

1. the ordinary normal asymptotic approximation: $V(x;n) = \Phi[(x-n)/\sqrt{2n}]$;
2. the Fisher's square root approximation (Fisher, 1922): $V(x;n) = \Phi[x - \sqrt{2n-1}]$;
3. the third root approximation (Wilson and Hilferty, 1931): $V(x;n) = \Phi[(x-\mu)/\sigma]$ with $\mu = 1 - 2/9n$ and $\sigma^2 = 2/9n$;
4. the fourth root approximation (Hawkins and Wixley, 1986): $V(x;n) = \Phi[(x-\mu)/\sigma]$ with $\mu = 1 - 3/16n - 7/512n^2 + 231/8192n^3$ and $\sigma^2 = 1/8n + 3/128n^2 - 23/1024n^3$;
5. the linear combination of Goria (Goria, 1992): $V(x;n) = \Phi[(x-\mu)/\sigma]$ with $\mu = 5 - 1/n - 3/128n^2 + 311/2048n^3$ and $\sigma^2 = 9/2n + 1/8n^2 - 207/256n^3$;

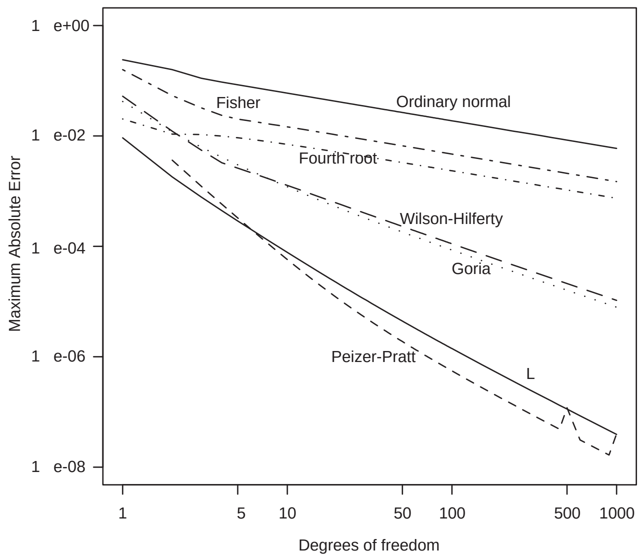

Fig. 1. Maximum absolute errors of selected approximations for the cumulative distribution function of the χ^2 random variable.

Table 1
Comparison of maximum absolute errors for seven approximations to the chi-square cumulative distribution function with n degrees of freedom

n	L	Ordinary normal	Fisher	Wilson–Hilferty	Fourth root	Goria	Peizer–Pratt
1	9.2E − 03	2.4E − 01	1.6E − 01	5.2E − 02	2.0E − 02	4.2E − 02	—
2	1.8E − 03	1.6E − 01	5.4E − 02	1.2E − 02	1.1E − 02	1.2E − 02	3.6E − 03
3	7.8E − 04	1.1E − 01	3.2E − 02	5.5E − 03	1.1E − 02	6.3E − 03	1.2E − 03
4	4.4E − 04	9.4E − 02	2.4E − 02	3.3E − 03	9.9E − 03	4.1E − 03	5.8E − 04
5	2.9E − 04	8.4E − 02	2.0E − 02	2.6E − 03	9.3E − 03	3.0E − 03	3.3E − 04
6	2.0E − 04	7.7E − 02	1.9E − 02	2.2E − 03	8.7E − 03	2.3E − 03	2.1E − 04
7	1.5E − 04	7.1E − 02	1.7E − 02	1.8E − 03	8.2E − 03	1.9E − 03	1.4E − 04
8	1.2E − 04	6.7E − 02	1.6E − 02	1.6E − 03	7.7E − 03	1.6E − 03	1.0E − 04
9	9.5E − 05	6.3E − 02	1.5E − 02	1.4E − 03	7.4E − 03	1.4E − 03	7.5E − 05
10	7.8E − 05	6.0E − 02	1.5E − 02	1.3E − 03	7.0E − 03	1.2E − 03	5.8E − 05
11	6.5E − 05	5.7E − 02	1.4E − 02	1.1E − 03	6.7E − 03	1.1E − 03	4.6E − 05
12	5.5E − 05	5.4E − 02	1.3E − 02	1.0E − 03	6.5E − 03	9.5E − 04	3.7E − 05
13	4.8E − 05	5.2E − 02	1.3E − 02	9.6E − 04	6.3E − 03	8.6E − 04	3.1E − 05
14	4.2E − 05	5.0E − 02	1.2E − 02	8.9E − 04	6.0E − 03	7.9E − 04	2.6E − 05
15	3.7E − 05	4.9E − 02	1.2E − 02	8.2E − 04	5.9E − 03	7.2E − 04	2.2E − 05
20	2.2E − 05	4.2E − 02	1.0E − 02	6.1E − 04	5.1E − 03	5.1E − 04	1.2E − 05
25	1.5E − 05	3.8E − 02	9.3E − 03	4.8E − 04	4.6E − 03	4.0E − 04	7.3E − 06
30	1.1E − 05	3.4E − 02	8.5E − 03	3.9E − 04	4.2E − 03	3.2E − 04	5.0E − 06
40	6.5E − 06	3.0E − 02	7.4E − 03	2.9E − 04	3.7E − 03	2.3E − 04	2.9E − 06
50	4.4E − 06	2.7E − 02	6.6E − 03	2.3E − 04	3.3E − 03	1.8E − 04	1.9E − 06
60	3.3E − 06	2.4E − 02	6.1E − 03	1.9E − 04	3.0E − 03	1.5E − 04	1.4E − 06
80	2.0E − 06	2.1E − 02	5.2E − 03	1.4E − 04	2.6E − 03	1.1E − 04	8.1E − 07
100	1.4E − 06	1.9E − 02	4.7E − 03	1.1E − 04	2.3E − 03	8.6E − 05	5.5E − 07
120	1.0E − 06	1.7E − 02	4.3E − 03	9.2E − 05	2.1E − 03	7.1E − 05	4.1E − 07
150	7.3E − 07	1.5E − 02	3.8E − 03	7.3E − 05	1.9E − 03	5.6E − 05	2.8E − 07
200	4.7E − 07	1.3E − 02	3.3E − 03	5.4E − 05	1.7E − 03	4.1E − 05	1.8E − 07
240	3.5E − 07	1.2E − 02	3.0E − 03	4.5E − 05	1.5E − 03	3.4E − 05	1.3E − 07
400	1.6E − 07	9.4E − 03	2.3E − 03	2.7E − 05	1.2E − 03	2.0E − 05	5.9E − 08
600	8.5E − 08	7.7E − 03	1.9E − 03	1.8E − 05	9.6E − 04	1.3E − 05	3.1E − 08
800	5.5E − 08	6.6E − 03	1.7E − 03	1.3E − 05	8.3E − 04	9.9E − 06	2.0E − 08
1000	3.9E − 08	5.9E − 03	1.5E − 03	1.0E − 05	7.4E − 04	7.9E − 06	4.0E − 08

6. the Peizer–Pratt approximation (Peizer and Pratt, 1968): $V(x;n) = \Phi[-(1/3 + 0.08/n)/(2n-2)^{1/2}]$ if $x = n-1$ and $V(x;n) = \Phi[((x-n+(2/3)-(0.08/n))/|x-(n-1)|) \times ((n-1)\log((n-1)/x) + x - (n-1))^{1/2}]$ if $x \neq n-1$;
7. the linear combination L proposed: $V(x;n) = \Phi[(x-\mu)/\sigma]$ with $\mu = 5/6 - 1/9n - 7/648n^2 + 25/2187n^3$ and $\sigma^2 = 1/18n + 1/162n^2 - 37/11664n^3$.

From Table 1 it can be seen that the accuracy of L is to four decimal places from 3 degrees of freedom onward and that its performance is considerably superior to the other power approximations considered; the greatest MAE for L, which is achieved with 1 d.f., is less than 0.01.

Figures shown in Table 1 reproduce and extend those reported in Table 3 of Ling (1978) and in Table 18.3 of Johnson et al. (1995). Compared with the Peizer–Pratt approximation, the linear combination L is more accurate for 2–6 degrees of freedom (for 1 d.f. the Peizer–Pratt approximation is not defined). Although the Peizer–Pratt approximation does not belong to the power transformation family, it was chosen since it was the best normal approximation among those evaluated in the studies by Ling (1978) and El Lozy (1982).

Appendix.

Let $Y_h \equiv (\chi^2/n)^h$ a power transformation of the chi-square random variable and $\xi \equiv \chi^2 - n$. We can write (Stuart and Ord, 1994)

$$Y_h = \left(1 + \frac{\xi}{n}\right)^h = \sum_{j=0}^{\infty} \binom{h}{j} \left(\frac{\xi}{n}\right)^j.$$

Taking expectations we find the first six moments of Y_h using the following relation between the moment of order r of Y_h and the moment of order kr of $Y_{h/k}$, where k is a constant:

$$E[(Y_h)^r] = E\left[\left(\left(\frac{\chi^2}{n}\right)^{h/k}\right)^{kr}\right] = E[(Y_{h/k})^{kr}].$$

It is now possible to calculate the moments of $L = Y_{1/6} + bY_{1/3} + cY_{1/2}$ and from the relations between moments and cumulants, the cumulants of L. We require that the Pearson's skewness and kurtosis coefficients of L tend to zero to order n^{-3}; therefore, we consider the third and the fourth cumulant

$$\begin{aligned} k_3(L) &= (3c-1)(1+2b+3c)^2 \frac{1}{108n^2} \\ &\quad + \left(\frac{b}{324} + \frac{5b^2}{108} + \frac{32b^3}{729} + \frac{13c}{432} + \frac{5bc}{27} \right. \\ &\quad\quad \left. + \frac{7b^2c}{36} + \frac{61c^2}{432} + \frac{25bc^2}{108} + \frac{c^3}{16} - \frac{55}{11664}\right)\frac{1}{n^3} + \mathrm{O}(n^{-4}), \\ k_4(L) &= (1+2b+3c)^2(2-b-4b^2-12c-15bc)\frac{1}{729n^3} + \mathrm{O}(n^{-4}). \end{aligned}$$

Putting the largest term of $k_3(L)$ and $k_4(L)$ equal to zero we find as solutions the two sets $\{b=-1, c=\frac{1}{3}\}$ and $\{b=-\frac{1}{2}, c=\frac{1}{3}\}$. However, the first set must be discarded since, in this case, the variance of L is zero.

Therefore the linear combination is

$$L = \left(\frac{\chi^2}{n}\right)^{1/6} - \frac{1}{2}\left(\frac{\chi^2}{n}\right)^{1/3} + \frac{1}{3}\left(\frac{\chi^2}{n}\right)^{1/2}$$

with mean and variance given respectively in (1) and (2).

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