State space description of national economies: the V4 countries

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Abstract

We present a new approach to description of national economies. For this we use the state space viewpoint, which is used mostly in the theory of dynamical systems and in the control theory. Gross domestic product, inflation, and unemployment rates are taken as state variables. We demonstrate that for the considered period of time the phase trajectory of each of the V4 countries (Slovak Republic, Czech Republic, Hungary, and Poland) lies approximately in one plane, so that the economic development of each country can be assocated with a corresponding plane in the state space. The suggested approach opens a way to a new set of economic indicators (for example, normal vectors of national economies, various plane slopes, 2D angles between the planes corresponding to different economies, etc.).

The tool used for computations is orthogonal regression (alias orthogonal distance regression, alias total least squares method), and we also give general arguments for using orthogonal regression instead of the classical regression based on the least squares method.

A MATLAB routine for fitting 3D data to lines and planes in 3D is provided.

Keywords: orthogonal regression, orthogonal distance regression, total least squares method, state space description, phase trajectory, national economy.

1 Introduction

In this article we consider two different versions of the least squares method – the classical one and the so called orthogonal regression. We incline to using the orthogonal regression, and give several arguments in favour of this. The main reason is the fact that orthogonal regression is a suitable tool for fitting lines and surfaces in multidimentional space, while even the use of classical regression in 2D is not so natural.

The article is organized as follows.

First, we briefly recall the history of the classical least squares method to remind that the justification of its applications is unclear from its very beginning.

Second, we take a critical look at the traditional interpretation of the classical least squares method and conclude that it is absolutely artificial and meaningless. Changing a viewpoint from "squares" to "circles", we find a natural geometric interpretation for quantities to be minimized, and this change leads to the method of orthogonal regression. After that, we provide several general arguments in favour of using orthogonal regression. We provide an easy example – "The Flight of a Bumblebee" – in order to illustrate how the orthogonal regression method can help in describing dependences in multidimentional space.

Then we apply the orthogonal regression method to describing the national ecomonies of the countries of the V4 group in the state space of three variables – gross domestic product, inflation, and unemployment. The development of the national economies under study is described by their trajectories in the chosen state space. We discovered that for each particular country the trajectory of its national economy lies approximately in one plane, so the normal vector of this plane and one of its points (for example, the one coinciding with the data centroid) can be associated with a particular economy.

In the conclusion we discuss some new possibilities that our approach opens in the field of description of complex dynamic economic systems such as national economies.

2 The history of the least squares method

One can hardly find another topic, which is so famous and which has such misterious and unclear history as the least squares method.

The controversal circumstances around the the appearance of the method of least squares were investigated by many authors. According to Celmins [1], it might be summarized as follows.

In 1805 Legendre published "Nouvelles methodes pour la determination des cometes", in which he introduced the method of least squares and gave it this famous name.

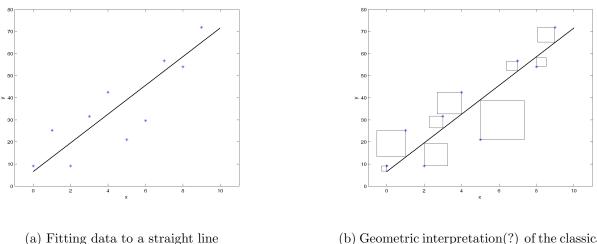
In 1809, Gauss published the book "Theoria motus corporum coelestium in sectionibus conicis solem ambientium" [2, 3], where he discussed the method of least squares and, mentioning Legendre's work, stated that he himself had used the method since 1795.

Legendre felt offended by Gauss's statement. In a letter to Gauss about his new book Legendre wrote that claims of priority should not be made without proof by previous publications. Gauss did not have such a publication, but he stated that he was convinced that the idea of least squares method is so simple that many people must have used the method even before him.

Later Gauss tried to prove his claim but had only little success. His own computational notes were – as he said – lost. His colleagues apparently did not remember discussions with him or did not want to be involved in the controversy. As Celmins mentions [1], only the astronomer Olbers included in a paper in 1816 a footnote asserting that Gauss had shown him the method of least squares in 1802, and Bessel published a similar note in a report in 1832.

Later Gauss gave up the search but did not retract his claim. In 1820 Legendre published a supplement to his 1805 memoir with an appendix where he publicly attacked Gauss's claims of priority. The controversy continued, and in 1831 H. C. Schumacher (cf. Celmins [1]) wrote to Gauss about a publication published in 1799 that contained data and adjustment results by Gauss. Schumacher suggested repeating the calculations and thereby demonstrating that the method of least squares was indeed used by Gauss in 1799. Gauss's answer was that he was well aware of the data but would not permit a recalculation, since such attempts would only suggest that he could not be trusted.

After describing the above circumstances, Celmins tried to repeat Gauss's calculations. Celmins's main conclusion was that Gausss results are not consistent with any obvious and reasonable adjustment of observational errors nor with a least-square adjustment in the parameter space.



) Fitting data to a straight line

(b) Geometric interpretation(?) of the classical LSM

Figure 1: Fitting by the classical least squares method (LSM)

3 Classical LSM

The well known least squares method is a mathematical procedure for finding the best-fitting curve to a given set of points by minimizing the sum of the squares of the offsets ("the residuals") of the points from the curve.

The classical least squares fitting consists in minimizing the sum of the squares of the vertical deviations of a set of data points

$$E = \sum_{i} [y_i - f(x_i, \alpha_1, \alpha_2, \dots, \alpha_n)]^2$$
(1)

from a chosen function f.

For a simple illustration, let us recall the classical linear regression problem, in which we have to fit the set of data by a straight line in 2D plane. This situation is shown in Fig. 1(a).

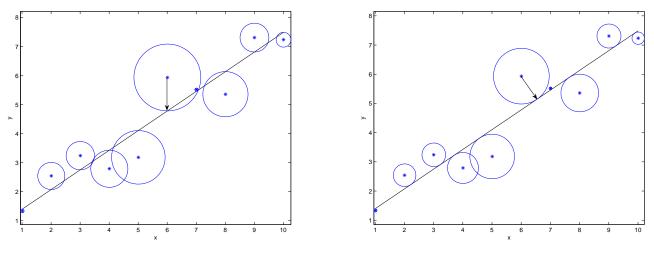
One may use the edge of a transparent ruler or a tightly stretched black string to get a line which seems to fit the data points. Mathematically, what we have to do is to determine the parameters k and b of the equation of a straight line:

$$y = kx + b \tag{2}$$

where x – the variable considered as independent; y – the variable considered as dependent on x; k, b – constants, often called parameters, to be determined so that the line fits the data optimally (in some sense...).

To find a way to calculate the parameters, let us return to the simplest case, equation (2) and Fig. 1(a), and let us agree to choose k and b so as to minimize the sum of the squares of the errors. Fig. 1(b) shows these squares graphically. We twist and push the line through these points until the sum of the areas of the squares is the smallest. The natural question is: "Why the squares? Why not just get the smallest sum of distances of the data points from the line?" The only real explanation is that it is easy to compute k and b to minimize the sum of squares off offsets of y (vertical offsets), but it is quite difficult to minimize (using analytic derivations) the sum of distances of the data points from the line, and it is really this ease that is responsible for the generally accepted preference for the squares of offsets of y.

In fact, we must always keep in mind that the least squares approach is basically a "last resource" tool that is used for obtaining at least some mathematical model for a process under



(a) The case of classical least squares fitting

(b) The case of orthogonal distance fitting

Figure 2: "Least circles" viewpoint

study, when obtaining a better model by analytical derivations is impossible or extremely time and/or effort consuming. As such, it is based on several presumptions, which are based only on some intuition or prejustice; among them we must mention postulating the role of variables ("independent/dependent") and the type of dependance between them (linear, polynomial, exponential, logarithmic, etc.)

4 The method of least circles?!

Looking at the geometric "interpretation" of the least squares method shown in Fig. 1(b), which so often appears in numerous textbooks and lectures, we can conclude that it is absolutely artificial and does not contain any sign of mathematical beauty. The picture shown in Fig. 1(b) again provokes the question: "Why the squares?"

To change a viewpoint, let us note that the criterion (1) can be painlessly replaced with

$$E = \beta \sum_{i} [y_i - f(x_i, \alpha_1, \alpha_2, \dots, \alpha_n)]^2$$
(3)

Indeed, multiplication by a non-zero number β does not affect the point of minimum. Only the minimum value of the criterion function (*E*) will be multiplied by β – but this value itself is not the subject of interest, since we look for the values of $\alpha_1, \alpha_2, \dots, \alpha_n$.

Taking $\beta = \pi$, we obtain

$$E = \sum_{i} \pi [y_i - f(x_i, \alpha_1, \alpha_2, \dots, \alpha_n)]^2$$
(4)

Geometrically, the formula (4) means the sum of areas of the circles shown in Fig. 2(a). The radii of the circles in Fig. 2(a) are the vertical offsets of y_i from the fitting line. Each of those circles has two points of intersection with the line. It is clear, that one cannot consider this picture as elegant. Changing the radii slightly, one can preserve n pairs of intersection of the circles and the line, so one can draw an infinite number of pictures lloking similar to Fig. 2(a). But Fig. 2(a) is just a reformulation of the standard geometric "illustration" of the least squares method (recall Fig. 1(b)). Instead of the "least squares method" we now deal with the "least circles method". But the circles shown in Fig. 2(a) are clearly not the best.

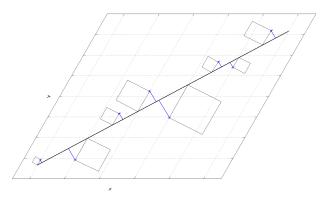


Figure 3: The case of non-carthesian coordinates.

However, the circles shown in Fig. 2(b) are really optimal: the fitting line is a tangent line to all circles. The radii of the circles in Fig. 2(b) are equal to minimal distances between the points (x_i, y_i) and the fitting line, and this guarantees the unique picture.

The criterion to minimize in this case is

$$E = \sum_{i} \pi \left[d\left((x_i, y_i), f(x, \alpha_1, \alpha_2, \dots, \alpha_n) \right) \right]^2,$$
(5)

which is up to a constant multiplier π the formula known under the name of orthogonal regression or total least squares [5, 4, 6, 10]. Here $d((x_i, y_i), f)$ denotes the distance between the point (x_i, y_i) and the fitting line f.

5 Arguments for orthogonal regression approach

There are numerous works and significant activities devoted to the theory of the total least squares method [12]. To support its wider applications, we would like to list the following arguments in favour of the orthogonal distance (or total least squares) fitting.

1. The shortest (orthogonal) distance is the most natural viewpoint on any fitting.

The section 4 and the comparison of Fig. 2(a) and Fig. 2(b) give a clear visual evidence for this.

2. The sum of orthogonal distances is invariant with respect to the choice of the system of coordinates.

This is obvious, since the distance is independent of the choice of the coordinate system. In addition, let us imagine that we replace rectangular coordinates with non-rectangular. For example, in Fig. 3 an affine system with the angle of 60 degrees between the axes is shown. In this picture the squares of Δy_i have even less meaning than in Fig. 1(b). However, the orthogonal distances, which are also shown in Fig. 3 still have solid interpretation and can be used for fitting.

3. There are no conjugate regression lines, which appear after swapping x and y, because in the case of orthogonal regression the fitting y = f(x) gives exactly the same line as the fitting $x = f^{-1}(y)$.

Suppose one wants to find the dependance between the height (x) and the weight (y) of people. The dependance is presumed to be linear (a straight line) described by equation (2). After determining k and b, this relationship can be used for estimating the weight of a person of a given height.

Table 1: Data for the Nievergelt's example.

Х	1	3	4	5	7
у	4	2	6	8	5

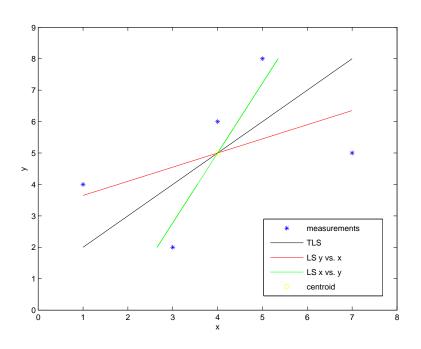


Figure 4: Nievergelt's example: classical regression versus orthogonal regression.

However, the vewpoint can be inverted: for a given weight, estimate the height of a person. If one already has the equation (2), then the solution should normally be

$$x = \frac{1}{k}y - \frac{b}{k}.$$
(6)

However, the classical regression approach leads to a different result expressed by a conjugate regression line.

In Fig. 4 a solution to the Nievergelt's easily understandable example [10] is shown. To make the situation more obvious, we added one to the ordinates in the Nievergelt's example, so we used the data shown in Table 1. The classical least squares regression y versus x gives the regression line y = 0.45x + 3.2. After swapping x and y, classical least squares regression gives the conjugate regression line x = 0.45y + 1.75. However, in the case of the orthogonal distance regression we obtain the same line y = x + 1 (x = y - 1) independently on the order of x and y.

It is worth noting that all three lines run through the centroid, and that the orthogonal distance regression line is located between the "scissors" formed by the conjugate regression lines obtained by the classical least squares regression.

4. There are no problems with causality (normally, determination of what is an independent variable and what is a dependent variable is simply unclear or even impossible; this is always postulated).

In all textbooks on statistics discussing the classical regression analysis it is always underlined that that the choice of what is the "independent" variable and what if the "dependent" variable is extremely important. However, in many cases it is not so easy to make a decision what is what, and justification of such a decision is based only on some subjective judgement or on a prejustice. We already mentioned the relationship between the weight and the height of people. It is obvious that any of these characteristics can be taken as an independent one.

5. Implementation of the orthogonal fitting does not depend on the number of dimensions.

To realize this, just recall the square of the distance between the two points $P(x_1, x_2, \ldots, x_n)$ and $Q(y_1, y_2, \ldots, y_n)$ in *n*-dimensional orthogonal coordinate system is

$$[d(P,Q)]^{2} = \sum_{i=1}^{n} (x_{i} - y_{i})^{2}$$
(7)

6 State space description of national economies

In everyday professional and non-professional communication one can frequently hear the word "state of economy". This expression can be given an exact meaning by adopting the tools that are available in the theory of dynamical systems and in automatic control. Namely, we will use the technique called state space description and phase trajectories.

Suppose we have a dynamical process (that is, the process deveoping in time), which is characterized by three quantities x(t), y(t), z(t) (of course, number or dimensions can be different, not necessarily 3). Chosing x, y, and z as the three coordinates, we can assign a point (x(t), y(t), z(t)) to each value of t – that is, we assign a point in a 3D space to a state of the considered process at time t. The variables x, y, z are called the state variables. The line formed by the points (x(t), y(t), z(t)) when t takes on the values from a given interval (usually [0, T] for some finite T, or $[0, \infty)$) is called the phase trajectory of the process.

As state variables for this study, we selected those which are standard: gross domestic product (GDP), inflation, and unemployment. In the subsequent sections, we demonstrate the advantages of the state space description of national economies and possible ways for further developments and scientific investigations.

7 Example: "The Flight of the Bumblebee" – fitting 3D data by a straight line in 3D

Before continuing with real data for the countries of the V4 group, let us consider a simple general example. Suppose a bumblebee is flying from point A to point B. In an ideal case, the trajectory of its flight would be a straight line. However, for the reasons which are mostly unknown to us, we observe the deviations which we (due to lack of knowledge, or due to lack of a full and precise model of the bumblebee's motion) consider as random.

Having a set of positions of the bumblebee (depicted as discrete points in Fig. 5), we can try to make a conclusion about the main direction of its flight. The simplest model is a straight line. The only way to determine the parameters of the equations of a fitting straight line in 3D space is using the method of orthogonal regression (total least squares). The result is shown in Fig. 5 as a straight line.

Of course, the general trajectory of a bumblebee can be more complex than a straight line. Although in the case of orthogonal regression we do not need to postulate the roles of variables, we still need to postulate the structure of the model – we have to decide which line is more "promising" for modeling the bumblebee's flight – it could be a straight line, or a conic section, or some spiral, or something else.

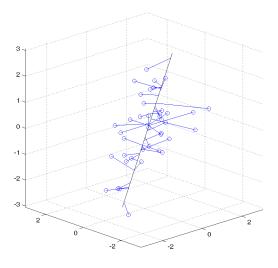


Figure 5: The Flight of the Bumblebee. (Inspired by a figure from [9]).

The "bumblebee flight" problem can be considered also as a problem of determining the direction of the flight of an unknown aircraft, or determining the state of economy at some time instant in the future based on the observations in the past.

8 The case of the Central European countries of the V4 group

The traditional approach to visualization of the change of GDP, inflation, and unemployment uses 2D plots – see Fig. 6(a), Fig. 6(b), and Fig. 6(c). Although such visualization provides partial information about each particular aspect of a particular economy, and even allows comparisons of a given particular indicator for several economies, it does not provide any information about the economy as a whole.

This is why we plot the data for each country of the V4 group – Slovak Republic (SK), Czech Republic (CZ), Hungary (HU), and Poland (PL) – in the state space with three coordinates: GDP, inflation, and unemployment. We use the data for the period from 1994 to 2000 from the MESA 10 report [7].

One can see that, in conrast with the traditional 2D plots, the phase trajectories shown in Figs. 7(a)-7(d) nicely show how the national economies developed in time.

The most interesting observation regarding these phase trajectories is that for each particular country its phase trajectory lies approximately in one plane. This observation indicates that we can associated such planes in state space with particular economies, and that the global properties of each particular national economy as a whole are described by the associated plane (or, in other word, by its normal vector and the data centroid, which is a point belonging to the plane).

The values of normal vectors and centroids of the planes describing national economies of the V4 countries are listed in Table 3. As it is well known from the analytic geometry, each plane is uniquely described by these data. In the last column the values of the total error of approximation are given.

It is worth mentioning that the fitting of the state space data points to planes was done using total least squares (TLS) method, implemented in the form of the principal component analysis [13], although in many cases the SVD decomposition approach is used [4, 10]. The

	1994	1995	1996	1997	1998	1999	2000
CZ	3.2	2.9	3.5	5.2	7.5	9.4	8.7
HU	11.2	10.5	9.2	7.7	7	6.5	6.5
PL	16.0	14.9	13.5	10.5	10.4	13.0	13.5
SK	13.7	13.1	11.3	11.8	12.5	16.2	18.5

Table 2: Economic indicators of the V4 countries, 1994–2000

(a) Unemployment in the V4 countries – in percents

	1994	1995	1996	1997	1998	1999	2000
CZ	2.2	5.9	4.8	-0.1	-2.2	-0.2	2.5
HU	2.9	1.5	1.3	4.4	5.1	4.5	5.6
PL	5.2	7	6.0	6.8	4.8	4.1	5.0
SK	4.8	6.7	6.2	6.2	4.1	1.9	2.0

(b) GDP of the V4 countries - change in percents

	1994	1995	1996	1997	1998	1999	2000
CZ	10	9.1	8.8	8.5	10.7	2.1	4.1
HU	18.8	28.2	23.6	18.3	14.3	10	9.3
PL	33.2	28	19.9	14.8	11.6	7.3	9.9
SK	13.4	9.9	5.8	6.1	6.7	10.6	11.5

(c) Inflation in the V4 countries – in percents

description of our function for MATLAB is given in the Appendix.

9 Conclusions

In this article we presented a new approach to description of national economies. This approach is based on the state space viewpoint, which is used mostly in the theory of dynamical systems and in the control theory. Gross domestic product, inflation, and unemployment rates were taken as state variables. We demonstrated that for the considered period of time the phase trajectory of each of the V4 countries (Slovak Republic, Czech Republic, Hungary, and Poland) lies approximately in one plane, so that the economic development of each country can be assocated with a corresponding plane in the state space.

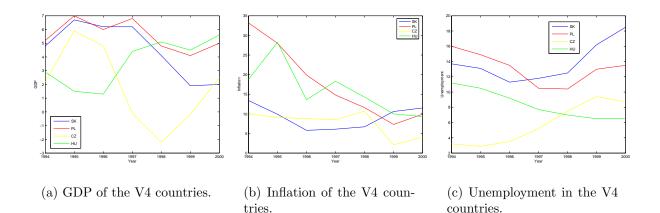


Figure 6: Classical 2D visualization of economic indicators of the V4 countries.

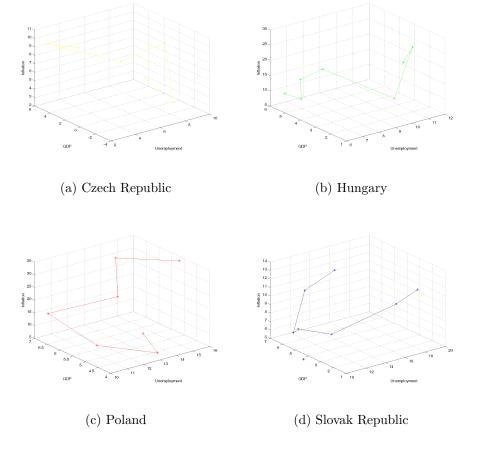


Figure 7: Phase trajectories of the national economies of the V4 countries in state space.

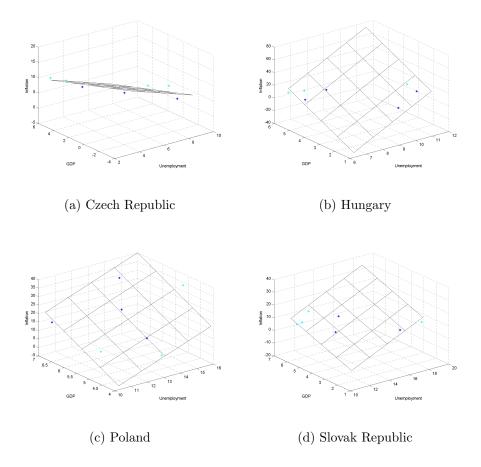


Figure 8: Planes of the national economies of the V4 countries in state space.

Error Country Normal vector Centroid SK(0.6704, 0.7195, -0.1811)(13.8714, 4.5571, 9.1429)4.2633PL(-0.4083, -0.9059, 0.1123)(13.1143, 5.5571, 17.8143)4.3106CZ(0.7632, 0.4525, 0.4612)(5.7714, 1.8429, 7.6143)4.6111 HU (0.7362, 0.6745, -0.0545)(8.3714, 3.6143, 16.0714)3.7431

Table 3: Normal vectors, centroids, and errors for the planes corresponding to the countries of the V4 group.

The suggested approach opens a way to a new set of economic indicators. Among possible indicators of economies we can mention, for example, normal vectors of national economies, various plane slopes, 2D angles between the planes corresponding to different economies, etc.

The tool used for computations is orthogonal regression (alias orthogonal distance regression, alias total least squares method), and we also gave general arguments for using orthogonal regression instead of the classical regression based on the least squares method.

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A Brief description of the MATLAB routine

We created a MATLAB routine called fit_3D_data for performing computations used in our article. Since our routine can be used in many other investigations utilizing the orthogonal regression (total least squares) method, we published it at MATLAB File Exchange. The URI for download is:

```
http://www.mathworks.com/matlabcentral/fileexchange/
loadFile.do?objectId=12395&objectType=file
```

The description of fit_3D_data and its parameters is given below.

```
function [Err,N,P] = fit_3D_data(XData, YData, ZData,
                                 geometry, visualization, sod)
%
% [Err, N, P] = fit_3D_data(XData, YData, ZData,
%
                            geometry, visualization, sod)
%
% Orthogonal Linear Regression in 3D-space
% by using Principal Components Analysis
%
% This is a wrapper function to some pieces of the code from
% the Statistics Toolbox demo titled "Fitting an Orthogonal
% Regression Using Principal Components Analysis"
% (http://www.mathworks.com/products/statistics/
  demos.html?file=/products/demos/shipping/stats/orthoregdemo.html),
%
% which is Copyright by the MathWorks, Inc.
%
```

```
% Input parameters:
% - XData: input data block -- x: axis
% - YData: input data block -- y: axis
% - ZData: input data block -- z: axis
% - geometry: type of approximation ('line', 'plane')
% - visualization: figure ('on', 'off') -- default is 'on'
% - sod: show orthogonal distances ('on', 'off') -- default is 'on'
%
% Return parameters:
% - Err: error of approximation - sum of orthogonal distances
\%\, - N: normal vector for plane, direction vector for line
% - P: point on plane or line in 3D space
%
% Example:
%
\% >> XD = [4.8 \ 6.7 \ 6.2 \ 6.2 \ 4.1 \ 1.9 \ 2.0]';
% >> YD = [13.4 9.9 5.8 6.1 6.7 10.6 11.5]';
% >> ZD = [13.7 13.1 11.3 11.8 12.5 16.2 18.5]';
% >> fit_3D_data(XD,YD,ZD,'line','on','on');
% >> fit_3D_data(XD,YD,ZD,'plane','on','off');
%
% Note: Written for Matlab 7.0 (R14) with Statistics Toolbox
%
% We sincerely thank Peter Perkins, the author of the demo,
% and John D'Errico for their comments.
%
% Ivo Petras, Igor Podlubny, May 2006
% (ivo.petras@tuke.sk, igor.podlubny@tuke.sk)
```