Signed Sequential Rank CUSUMs

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Abstract

CUSUMs based on the signed sequential ranks of observations are developed for detecting location and scale changes in symmetric distributions. The CUSUMs are distribution free and fully self-starting: given a specified in-control median and nominal in-control average run length, no parametric specification of the underlying distribution is required in order to find the correct control limits. If the underlying distribution is normal with unknown variance, a CUSUM based on the Van der Waerden signed rank score produces out-of-control average run lengths that are commensurate with those produced by the standard CUSUM for a normal distribution with known variance. For heavier tailed distributions, use of a CUSUM based on the Wilcoxon signed rank score is indicated. The methodology is illustrated by application to real data from an industrial environment.

Keywords: CUSUM, distribution-free, self starting, signed sequential ranks, symmetric distributions

1 Introduction

CUSUM procedures were developed to signal the onset of a persistent change away from a specified product quality characteristic, such as a mean or median. In the application treated in Section 7 of the paper, the raw data consist of a series of matched pairs $(V_{1i}, V_{2i}), i \geq 1$, the result from two treatments applied to the same sample of material. The process is deemed to be in control as long as the treatment effect, defined as the mean or median of $X_i = V_{1i} - V_{2i}$, is zero. Otherwise the process is out of control. Page (1954) developed the first CUSUM for detecting a shift in the mean of a normal distribution with known variance and in-control mean zero. However, the normality assumption is often in doubt and it is known that a standard normal CUSUM performs poorly when the true underlying distribution has substantially heavier tails than a normal distribution (Hawkins and Olwell, 1997, Section 3.7.1). It is therefore surprising that extension of the methodology to heavier tailed, symmetric distributions has received almost no attention in the literature. Furthermore, implementation of a normal distribution CUSUM requires a known standard deviation, σ , in the underlying distribution. Misestimation of σ from Phase I data and subsequent use of the estimate in the Phase II CUSUM can result in an in-control average run length substantially different from the nominal value - see, for instance, Hawkins and Olwell (1997, pages 159-161) and Keefe, et al. (2015). This paper proposes CUSUM schemes that largely overcome these problems.

A natural approach towards extending a normal distribution CUSUM to other symmetric distributions is to replace the observed data by rank-based equivalents, which leads to distribution-free procedures. By "distribution free" is meant that the in-control properties of the CUSUM do not depend on the functional form of the underlying distribution. This paper develops distribution-free CUSUMs for single observations, the SSR (signed sequential rank) CUSUMs. The CUSUM is based on the series of signed sequential ranks $s_i r_i^+$, $i \ge 1$, of the observations, where $s_i = sign(X_i)$ and r_i^+ is the rank of $|X_i|$ in the sequence $|X_1|, \ldots, |X_i|$, that is, the number among $|X_1|, \ldots, |X_i|$ less than or equal to $|X_i|$. When the process is in control, the r_i^+ form a series of independent random variables with r_i^+ uniformly distributed on the integers $1, \ldots, i$ - see Barndorff-Nielsen (1963, Theorem 1.1). Since the s_i are mutually independent and independent of the $|X_i|$ series, $s_i r_i^+$, $i \ge 1$ is a sequence of independent random variables with $s_i r_i^+$ uniformly distributed on $\pm 1, \ldots, \pm i$, no matter what the common distribution of the X_i is. The independence, distribution freeness and naturally sequential nature of the signed sequential ranks makes them ideally suited to the construction of CUSUMs for time ordered data. Control limits guaranteeing any specified

in-control average run length (ARL) can be determined once and for all for a range of reference constants and nominal in-control average run lengths. The SSR CUSUM thus overcomes the estimation problem that besets the standard normal CUSUM. Furthermore, its validity is not dependent upon the existence of any moments in the underlying distribution.

Among other existing distribution-free CUSUMs for singly arriving observations are the sequential rank CUSUM of McDonald (1990) and the changepoint CUSUM of Hawkins and Deng (2010). These are based on unsigned (sequential or ordinary) ranks of the X-data. Since unsigned ranks remain unchanged when the data are transformed monotonically, these CUSUMs cannot incorporate specific information about the in-control mean or symmetry of the in-control distribution. As a result, they are not competitive with SSR CUSUMs when such information is available. In fact, they are tailor made for situations in which the in-control value of the characteristic in question, as well as the underlying distribution, is unknown and where the current state of the process, whatever it may be, is declared to be the in-control state. SSR CUSUMs are not applicable in such instances.

The paper is structured as follows. The SSR CUSUMs are defined in Section 2. In Section 3, various out-of-control properties of the CUSUMs are examined using theoretical results coupled with Monte Carlo simulations. It is shown that the quantitative out-of-control behaviour of an SSR CUSUM, which requires no knowledge of any parameters in the underlying distribution, can be inferred from the behaviour of a standard normal distribution CUSUM. In Section 4 it is shown that a distribution-free CUSUM based on the Van der Waerden rank score behaves like a standard normal CUSUM, hence provides a simple solution to the unknown variance problem that besets the latter. Section 5 demonstrates that a CUSUM based on Wilcoxon signed rank scores can serve usefully in an omnibus role, especially when the underlying distribution has heavier tails than the normal distribution. In Section 6, a sequential rank CUSUM to detect a change in the dispersion of a symmetric distribution is developed. Implementation of the proposed methodology is illustrated in Section 7 by application to a set of data from a coal mining operation. Section 8 gives a summary of the main results and conclusions. The supplementary material to the paper includes tables of control limits guaranteeing a specified in-control average run length for three specific distribution free CUSUMs.

2 Signed sequential rank CUSUMs

The following generic version of CUSUM methodology will be used. Let ξ_i be a function of X_1, \ldots, X_i for which $E[\xi_i] = 0$ when the process is in control (the data come from a symmetric distribution with zero median) and $E[\xi_i] > 0$ when it is out-of-control. A one-sided (upper) CUSUM then consists in computing recursively the sequence D_i^+ , $i \ge 1$, where

$$D_0^+ = 0; \ D_i^+ = max \left[0, D_{i-1}^+ + \xi_i - \zeta^+ \right], \ i \ge 1$$
(1)

and where $\zeta^+ > 0$, the reference value, is a positive constant. The CUSUM signals a change as soon as D_i^+ exceeds a control limit $h^+ > 0$, the interpretation being that a change from an in-control to an out-of-control situation has possibly occurred somewhere along the observed sequence X_1, \ldots, X_i . The run length, N, is the index at which a signal first occurs. Because the barrier at 0 forces the CUSUM to be non-negative, the CUSUM will eventually signal regardless of whether a change has taken place (a valid signal) or not (a false signal). To compensate for this unit type I error, the control limit $h^+ > 0$ is chosen to ensure that the average in-control ARL (IC ARL) equals a pre-specified value, denoted by ARL_0 .

To control for downward shifts, a second sequence

$$D_0^- = 0; \ D_i^- = \min\left[0, D_{i-1}^- + \xi_i + \zeta^-\right], \ i \ge 1$$

with control limit $h^- < 0$ is computed. The CUSUM signals a shift as soon as either $D_i^+ > h^+$ or $D_i^- < h^-$. If the median of a symmetric distribution is being monitored, one has $\zeta^+ = \zeta^- = \zeta$ and $h^+ = h^- = h$. It is customary to exhibit the pairs (i, D_i^+) and (i, D_i^-) in a single (x, y) plot together with horizontal lines at the control limits $y = h^+$ and $y = h^-$ - see Figure 1 in Section 7. A "normal CUSUM" is the special case in which $\xi_i = X_i$ and the X_i have a normal (0, 1) distribution. This CUSUM has been studied extensively and its properties are well known - see Hawkins and Olwell (1997, Chapter 3).

Let the score function J(u), -1 < u < 1 be odd and square-integrable on the interval (-1, 1) with $\int_0^1 J^2(u) du = 1$ and set

$$v_i^2 = \frac{1}{i} \sum_{j=1}^i J^2\left(\frac{j}{i+1}\right).$$

Then, under the in-control regime, the signed sequential rank statistics

$$\xi_i = J\left(\frac{s_i r_i^+}{i+1}\right) / \nu_i = s_i J\left(\frac{r_i^+}{i+1}\right) / \nu_i, \tag{2}$$

 $i \geq 1$, are independently distributed with zero means and unit variances. The proposed SSR (signed sequential rank) CUSUM consists in using ξ_i in the one-sided procedure (1) or in its two-sided version. If the median shifts from zero to a non-zero value, or if the distribution becomes asymmetric, the expected value of ξ_i ceases to be zero. Consequently, the CUSUM should be effective in detecting a shift away from zero as well as detecting the onset of substantial asymmetry in the underlying distribution.

A Wilcoxon SSR CUSUM, abbreviated to "W-CUSUM", is based upon the W-score

$$J_W(u) = \sqrt{3}u. \tag{3}$$

Here, $v_i^2 = (2i+1)/(2(i+1))$, whence

$$\xi_i = \sqrt{\frac{6}{(2i+1)(i+1)}} s_i r_i^+$$

is used in (1). The Wilcoxon score is well suited to practical implementation because of its simple form. Another popular score is $\Phi^{-1}((1+u)/2)$, the Van der Waerden score. The corresponding CUSUM will be referred to as the "VdW-CUSUM".

The "distribution-free when in control" character of SSR CUSUMs allows fairly precise estimation by Monte Carlo simulation of the IC ARL for any given score function J, control limit h and reference constant ζ . Tables S1 and S2 in the supplementary material give control limits for a matrix of (ζ, ARL_0) pairs for use with the W- and VdW-CUSUMs.

Regardless of the reference value actually used, the existence of Phase I data is not a prerequisite for initiating an SSR CUSUM. Given a reference constant ζ , any specified ARL_0 is guaranteed upon use of the appropriate h. Thus, the SSR CUSUM is fully self-starting in the sense defined by Hawkins and Olwell (1997) and the between-practitioner variation, as defined in Saleh, et al. (2016), is zero. The effects of using a "wrong" ζ will become evident only in the out-of-control ARL properties of the CUSUM. These effects will be discussed in the sections that follow.

3 Out-of-control properties

Denote by τ the point in time (the changepoint) at which the underlying process shifts from an in-control to an out-of-control state. The efficacy of a CUSUM can be judged by the out-of-control average run length (OOC ARL)

$$E[N-\tau|N \ge \tau],\tag{4}$$

the expected time-to-signal after onset of an out-of-control state, conditional upon no signal occurring prior to its onset. Some general insights into the OOC ARL behaviour of SSR CUSUMs can be gained by restricting attention to nominally "small" shifts and "large" changepoints τ . This criterion is in line with the primary objective of CUSUM methodology, which is to detect quickly relatively small persistent shifts. Furthermore, the shift δ in the median is expressed in units of an (unknown) underlying scale parameter, σ , which is typically a measure of dispersion in the underlying distribution. The fact that the ranks of any set of data are scale invariant actually necessitates such a specification. Define

$$\theta_0 = E[f_0(Y)J'(2F_0(Y) - 1)], \tag{5}$$

where f_0 and F_0 denote the pdf and cdf of $Y = X/\sigma$, and notice that θ_0 is functionally independent of σ . Then, for $i \ge 1$,

$$E[\xi_{\tau+i}] \approx \theta_0 \delta \neq 0, \tag{6}$$

implying that the CUSUM will show a sustained upward ($\delta > 0$) or downward ($\delta < 0$) drift after the changepoint, resulting in a finite OOC ARL.

It follows that the larger θ_0 is, the better. Table 1 shows the values of θ_0 for the *W*- and *VdW*- CUSUMs in Student *t*-distributions with ν degrees of freedom, standardized to unit standard deviation for $\nu \geq 3$ and to unit inter-quartile range for $\nu = 2$ and $\nu = 1$. The *t*-distributions are chosen as benchmarks because they exhibit a range of tail thicknesses that would mimic most cases occurring in practice. Inspection of Table 1 reveals that the *W*-CUSUM should be the preferred one among the two, *except* when the distribution is normal ($\nu = \infty$).

Table 1 Values of θ_0 for the *W*- and *VdW*-CUSUMs in Student t_{ν} -distributions.

	ν					
	∞	4	3	2	1	
W-CUSUM	0.98	1.18	1.37	1.18	1.10	
VdW-CUSUM	1.00	1.12	1.29	1.06	0.93	

Because the distribution of partial sums of the ξ_i tend to normality, there is an expectation that the SSR CUSUMs will share some of the good properties of a normal CUSUM. Indeed, Proposition 1 in the Appendix suggests the following heuristic: If δ is "small" and a shift of size $\delta\sigma$ occurs at a "large" $n = \tau$, then an SSR CUSUM with reference value ζ and control limit h behaves approximately like a normal CUSUM with the same ζ and h when a shift of size $\delta\theta_0$ occurs at $n = \tau$.

(7)

Some implications of this heuristic will now be explored.

3.1 Specification of a reference value

The optimal choice of reference constant to detect a target shift of size δ_1 in a standard normal distribution CUSUM is $\delta_1/2$ - see, for instance, Bagshaw and Johnson (1974, Section 2). Thus, the heuristic (7) suggests $\zeta = \theta_0 \delta_1/2$ as an appropriate reference value in an SSR CUSUM. The variation of θ_0 values seen in Table 1 is not substantial so that default values $\theta_0 = 1$ or $\theta_0 = 1.3$ seem appropriate, depending on the anticipated tail thickness of the underlying distribution.

An estimate of θ_0 is useful when designing a CUSUM - see Section 3.2. Such an estimate can be made if some Phase I data V_1, \ldots, V_m are available. For the Wilcoxon score, for instance, (5) reduces to $\theta_0 = \sqrt{12}E[f(Y)]$, which can be estimated by

$$\hat{\theta}_0 = \sqrt{12} \frac{\hat{f}_0(V_1/\hat{\sigma}) + \dots + \hat{f}_0(V_m/\hat{\sigma})}{m}$$
 (8)

where $\hat{\sigma}$ is a location invariant and scale equivariant estimator of σ (such as a sample standard deviation or inter-quartile range) and \hat{f}_0 is an estimator of the density f_0 based upon the observations $Y_i = V_i/\hat{\sigma}$, $1 \leq i \leq m$. The suggested reference value for use in Phase II is then $\hat{\zeta} = \hat{\theta}_0 \delta_1/2$. Use of the appropriate control limit \hat{h} (read from Table S1 or Table S2, for instance) then guarantees a Phase II IC ARL equal to the nominal value. There is again no practitioner-to-practitioner IC ARL variation.

3.2 Out of control ARL

The heuristic (7) suggests that approximations to the OOC ARL of the SSR CUSUM can be found by pretending that the underlying distribution is normal. Such approximations are useful in CUSUM design. The following example illustrates this numerically. Let ζ , h and τ be given. Denote by $\mathcal{W}(\delta)$ and $\mathcal{N}(\delta)$ respectively the ARL of a one sided W-CUSUM and a normal

CUSUM at a persistent mean shift $\delta > 0$ which starts at $n = \tau$. The two CUSUMs use the same ζ and h. An implication of the heuristic (7) is that

$$\mathcal{W}(\delta) \approx \mathcal{N}(\theta_0 \delta) \tag{9}$$

when δ is "small". To gauge the extent to which this approximation is useful, data were generated from two underlying distributions, a standard normal distribution and a heavier tailed t_3 distribution, both standardized to unit variance. Various mean shifts δ were induced at $\tau = 50$. For the normal distribution the (ζ, h) pairs (0.1, 12.01) and (0.25, 7.25) were used and for the t_3 distribution the pairs (0.15, 9.86) and (0.35, 5.66), based on larger ζ values to allow for tail heaviness, were used. The *h* values, taken from Table S2 in the supplementary material, guarantee a *W*-CUSUM $ARL_0 = 500$ in all four cases.

 $\mathcal{W}(\delta)$ was estimated from 10,000 Monte Carlo trials in each of the two distributions (normal and t_3), the estimates serving as nominal "true" values of $\mathcal{W}(\delta)$. If an analytic formula or software for determining the exact value $\mathcal{N}(\theta_0 \delta)$ were available, the quality of the approximation (9) could now be assessed directly. However, except for $\tau = 0$ and $\tau \to \infty$, these are not available for arbitrarily specified τ . In their absence a "Monte Carlo formula" can be used. This entails estimating $\mathcal{N}(\theta_0 \delta)$ for $\theta_0 = 0.98$ and $\theta_0 = 1.37$ at each of the shifts δ by 10,000 (or more) Monte Carlo trials using normal random numbers only.

The first column in Table 2 shows the first three in a series of shifts $\delta = 0.125 : 0.125 : 1.5$ that were induced at $\tau = 50$. The columns headed d_{ζ} show the differences $\mathcal{W}(\delta) - \mathcal{N}(\theta_0 \delta)$ between the true and predicted OOC ARLs rounded to the nearest integer, the subscript on d indicating the reference constant. The third entry in each column shows the maximal difference over all $\delta > 0.25$. (An unabridged version, Table S2.1, is in the supplementary material.) Clearly, the normal approximation is excellent at all shift sizes that would typically be considered to be of practical relevance and would certainly be useful for the purpose of CUSUM design. In the design phase, given an estimate $\hat{\theta}_0$ of θ_0 , the "Monte Carlo formula" with various values of δ, ζ, h and τ as inputs will yield corresponding outputs $\mathcal{N}(\hat{\theta}_0 \delta)$. These outputs are estimates of the unknown $W(\delta)$ and can be used to gauge the likely Phase II behaviour of the CUSUM under various specifications of the input parameters.

Table 2
W-CUSUM ARL approximations in normal and t_3
distributions. $ARL_0 = 500$; changepoint $\tau = 50$.

	nor	mal	t_3		
δ	$d_{0.10}$	$d_{0.25}$	$d_{0.15}$	$d_{0.35}$	
0.125	-1	0	-1	5	
0.25	1	0	1	3	
>0.25	1	1	2	2	

The same simulations were also run at $\tau = 0$, an instance in which the condition in the heuristic that τ must be large is violated. As expected, there were consistent differences between the (estimated) true values and the values predicted by the heuristic. In particular, the heuristic *underestimated* substantially the true OOC ARL values. (Table S2.2 in the supplementary material gives a full set of results.) This is not surprising because accumulation of a non-negligible number of ξ_i in (1) is required to effect approximate normality. However, the results in Table 2 suggest that $\tau = 50$ observations is already sufficient for this purpose even if the underlying distribution, such as a t_3 , has considerably heavier tails than a normal distribution.

3.3 Behaviour under asymmetry

Since an SSR CUSUM is constructed on an assumption of symmetry in the in-control distribution, it should have an ability to detect asymmetry. This aspect of SSR CUSUM behaviour was assessed in a small simulation study. The in-control distribution was a standard normal distribution. From $\tau = 51$ onwards, data were generated from skew-normal distributions (Azzalini, 2005) with zero mean, unit variance and skewness parameters $\lambda = 1$ (lightly skewed), $\lambda = 3$ (moderately skewed) and $\lambda = 5$ (heavily skewed). A two-sided W-CUSUM with $ARL_0 = 500$ was run and the OOC ARL at each value of λ was estimated from 10,000 simulated data sets. Table 3 shows the estimates.

Table 3 OOC ARL of *W*-CUSUM in skew-normal distributions.

	ζ					
	0.05	0.15	0.25			
$\lambda = 1$	388	421	464			
$\lambda = 3$	113	119	149			
$\lambda = 5$	84	82	101			

The very large out-of-control ARLs at $\lambda = 1$ indicate that the CUSUM is unable to detect efficiently such a small degree of asymmetry, thus implying some robustness in that respect. On the other hand, the results at $\lambda = 3$ and $\lambda = 5$ indicate an ability to detect substantial degrees of asymmetry. Consequently, a signal from the CUSUM is *not necessarily* an indication that the mean or median has changed. The subsequent data analysis should include an assessment of the possibility that the signal resulted from the onset of asymmetry.

4 An efficient self-starting CUSUM for a normal distribution

Suppose the data come from a normal distribution with unknown standard deviation σ . A naive approach consists in estimating σ from Phase I data and pretending in Phase II that the estimate, $\hat{\sigma}$, is error free. It is well known that such an approach is defective because the Phase II IC ARL could differ vastly from the nominal value - see Keefe, et al. (2015), where further references can also be found. Saleh, et al. (2016) propose to ameliorate the effect by estimating appropriate control limits for use in Phase II via bootstrapping from Phase I data. If such a method is used, control limits must be generated afresh whenever the CUSUM is applied to a new data series. A "once and for all" table, such as Table S1 or Table S2, is out of the question.

A result of Chernoff and Savage (1958, Theorem 3) states that if X has finite variance and J(u) is the VdW-score $\Phi^{-1}((1 + u)/2)$, then θ_0 in (5) satisfies $\theta_0 \geq 1$, the minimum value $\theta_0 = 1$ being attained only if X has a normal distribution. This fact, in conjunction with the heuristic (7), suggests that the VdW-CUSUM offers a fully self-starting procedure that requires no bootstrapping or parameter estimation of any kind. Since no Phase I data are required and the CUSUM is guaranteed to achieve the specified ARL_0 , there is no between-practitioner variation. The VdW-CUSUM is asymptotically efficient: asymptotically in the sense that ARL_0 should be "large" and the OOC target "small"; and efficient in the sense that under these conditions the OOC ARLs should be equal to those of a normal CUSUM with the same IC ARL and the same OOC target. The only further restriction is that τ must be "large".

To form some idea of what "large" and "small" would mean in the present context, ARLs of the normal- and VdW-CUSUMs were estimated by Monte Carlo simulation at $ARL_0 = 500$ with typical target OOC shifts $\delta_1 = 0.5$ and $\delta_1 = 1.0$. Shifts ranging from $\delta = 0.25$ to $\delta = 1.50$ were induced at $\tau = 0$, 50 and at 100. Denote by $\mathcal{V}(\delta)$ and $N(\delta)$ the respective ARLs of the VdW- and normal CUSUMs. Table 4.1 shows the differences

$$d_{\delta} = V(\delta) - N(\delta), \tag{10}$$

rounded up to the nearest integer, at the various shifts δ . The boldface entries are those where the shift is greater than or equal to the target.

The only instance in which the relevant differences could be called substantial is at $\tau = 0$, a setting that violates the "large τ " requirement. Results at $ARL_0 = 1,000$ (Table S4.1 in the supplementary material) follow the same pattern: a substantial difference at $\tau = 0$ but a difference of only 1 at $\tau = 50$ and $\tau = 100$. Overall, the results suggest that $\tau \geq 50$ and $\delta_1 \leq 1$ meet the respective descriptions "large" and "small" whenever $ARL_0 \geq 500$.

	a_{δ} from (10) at $ARL_0 = 500$.								
	$\tau = 0$		$\tau = 50$		$\tau =$	100			
δ	$\delta_1 = 0.5$	$\delta_1 = 1.0$	$\delta_1 = 0.5$	$\delta_1 = 1.0$	$\delta_1 = 0.5$	$\delta_1 = 1.0$			
0.25	8	24	2	10	2	9			
0.4	7	20	1	7	0	4			
0.5	7	18	1	4	1	2			
0.75	7	13	1	2	1	1			
1.0	7	11	1	1	1	1			
1.25	7	11	1	1	1	1			
1.5	8	11	1	1	1	1			

Table 4.1 d_{δ} from (10) at $ARL_0 = 500$.

It is interesting to see what transpires when δ_1 is apparently "not small", say $\delta_1 = 2$. Then $\zeta = 1$ is an optimal choice and the control limit h = 2.2ensures $ARL_0 = 500$ in the VdW-CUSUM. The appropriate control limit for the normal CUSUM is h = 2.323. Table 4.2 shows the results for $\tau = 50$ and $\tau = 100$.

Table 4.2 d_{δ} from (10) at $ARL_0 = 500$ and $\delta_1 = 2.0$.

		δ							
	0.25	0.50	1.00	1.50	2.00	2.50	3.00		
$\tau = 50$	27	29	6	2	2	1	1		
$\tau = 100$	22	15	2	1	1	1	1		

Again, the differences are practically negligible at the larger shifts $\delta \geq 2.0$. Overall, the Van der Waerden CUSUM, which does not require any knowledge of the unknown σ , is not in any substantive manner inferior to a normal CUSUM, which requires a known σ .

5 An omnibus self-starting CUSUM

From Table 1 it is clear that θ_0 is larger for the Wilcoxon score than for the Van der Waerden score, except when the underlying distribution is normal, the difference there being almost negligible. Thus, the W-CUSUM would be preferred in distributions with heavier than normal tails and would be almost as good as the VdW-CUSUM in a normal distribution. This accords with the conclusions in Hodges and Lehmann (1960, Section 5) regarding the relative performances of the two scores in a hypothesis testing context. The W-CUSUM can therefore be recommended as an omnibus self-starting CUSUM that will be effective in many situations. The following are some possible limitations of the W-CUSUM.

First, the CUSUM has $|\xi_i| \leq \sqrt{3}$. Thus, if the target δ_1 exceeds $2 \times \sqrt{3} = 6.92$ and the default reference constant $\zeta = \delta_1/2$ is used, the ARL at all $\delta > 0$ will be infinite because then $\xi_i - \zeta$ is always negative, whence $D_n^+ = 0$ for all n. However, this is not a substantive practical limitation because the typical range of out-of-control target shifts δ_1 in applications of the CUSUM are considerably less than 6.92.

Second, given $\zeta < 2\sqrt{3}$ and h, a W-CUSUM requires at least $[h/(\sqrt{3} - \zeta)] + 1$ observations to reach the control limit. The maximum value of this quantity over all (ζ, h) pairs in Table S1 is 5 observations. A maximum of five possible additional observations seems a small price to pay for the simplicity involved in applying the W-CUSUM and reaping the benefits of (i) its high efficiency in non-normal distributions and of (ii) its bounded score function, which inhibits transient special causes from producing signals - see Section 7 for an example.

When observations occur naturally in groups of two or more without a time ordering, no SSR CUSUM is applicable because sequential ranks are then not uniquely defined. In such a case the grouped signed rank CUSUM of Bakir and Reynolds (1979), which is also distribution-free, can be used.

6 A sequential rank CUSUM for dispersion

While the in-control properties of the SSR CUSUM do not depend upon the variability of the underlying data, its proper application does require the variability to remain unchanged. Suppose that after $\tau > 0$ observations there is a change to a distribution with density $g(x) = f(x/\sigma)/\sigma$, $\sigma \neq 1$. Then σ is the fraction by which the current, unknown, dispersion changes. To detect an increase in dispersion, one can use a CUSUM based on the scores $J^2(u)$, thus eliminating the effect of the sign of X. With the Wilcoxon score, the corresponding sequential rank statistic to be used in (1) is then

$$\xi_i = \frac{6(r_i^+)^2}{(2i+1)(i+1)} - 1, \tag{11}$$

which has zero in-control expected value. The corresponding CUSUM will be referred to as the " W^2 -CUSUM". Since the sequential ranks r_i^+ are invariant under scale changes, it is clear that a CUSUM based on them cannot detect changes from a specified value of σ . Only changes away from the current value of σ , whatever it may be, will be detectable, and this only if the change occurs after a sufficiently long time lapse $\tau > 2$. Furthermore, the effect of the pre-change value of the scale parameter becomes negligible as observations continue to accrue after a change to different value. Thus, the CUSUM will eventually return to a nominally in-control state after a change has occurred. This behaviour is similar to that of self-starting CUSUMs, and is a warning to users of the need for corrective action as soon as a change is diagnosed see Hawkins and Olwell (1997, Section 7.1).

If a change from an unspecified σ to $\sigma\Delta$, $\Delta > 0$, occurs, the analogue of (6) is,

$$E[\xi_{\tau+i}] \approx \theta_1 \log \Delta$$

where

$$\theta_1 = 12E\left[\left(2F_0(Y) - 1\right)Yf_0(Y)\right].$$

Thus, appropriate reference constants for a $100\alpha\%$ change up or down from the current dispersion level would be $\zeta^+ = \theta_1 log (1 + \alpha) / 2$ in an upper CUSUM and $\zeta^- = -\theta_1 log \alpha / 2$ in a lower CUSUM. Table 6 shows values of θ_1 in some t_{ν} distributions. It seems that $\theta_1 = 1$ could serve usefully as a default value. If an estimator is required,

$$\hat{\theta}_1 = 12 \sum_{i=1}^m \left(2\frac{i}{m+1} - 1 \right) \frac{V_{(i)}}{\hat{\sigma}} \hat{f}\left(\frac{V_{(i)}}{\hat{\sigma}}\right)$$

will do, where $V_{(1)} < \cdots < V_{(m)}$ denote the order statistics of the Phase I data and where \hat{f} and $\hat{\sigma}$ are as in (8). Table S6 in the supplementary material has control limits that cover all θ_1 values in Table 5 and all $0.5 < \alpha < 1$.

Table 5 Values of θ_1 for the W^2 -CUSUM in t_{ν} -distributions.

	ν							
	∞	4	3	2	1			
θ_1	1.10	0.94	0.89	0.8	1.10			

7 Application

The data consist of successive pairs of determinations (V_{1i}, V_{2i}) , $1 \le i \le 240$ of the ash content of coal, reported as a percentage per unit mass, from two nominally identical laboratories. The measurements V_{1i} and V_{2i} were made on two identical coal samples extracted from a single batch of coal. If the true value of the ash content is T_i , then the determinations by the two laboratories may be represented as

$$V_{1i} = T_i + \varepsilon_{1i}, \ V_{2i} = T_i + \varepsilon_{2i}$$

where ε_{1i} and ε_{2i} represent the respective laboratory measurement errors. These errors may be taken to be statistically independent since the laboratories operate independently of one another. Given that the laboratories are operating to ISO or ASTM specifications, the errors should also be identically distributed with zero means and common, albeit not precisely known, standard deviation σ . The T_i reflect the characteristics of various seams from which the coal is extracted and are typically neither independently nor identically distributed. Nevertheless, the differences

$$X_i = V_{1i} - V_{2i} = \varepsilon_{1i} - \varepsilon_{2i},$$

which are the focus of interest here, do not depend upon the T_i and are independently and symmetrically distributed around zero. A non-zero mean or asymmetry in the distribution, or a change in the variance of X, indicates a deviation from specifications in one or both of the laboratories. This would typically lead to an audit of the analysis procedures used in the laboratories to isolate the cause of the deviation.

To monitor the mean and standard deviation of X, four CUSUMs with $ARL_0 = 2,000$ were run concurrently: a two-sided W-CUSUM - see (3) - for the mean and a two-sided W^2 -CUSUM - see (11) - for the standard deviation. The overall IC ARL would then be approximately 500. In the present instance, no formal Phase I data were available. However, based on the operating specifications for ash analysis, the measurement error ε in a laboratory should have a standard deviation of about 0.45 (% ash per unit mass of coal), implying that the standard deviation, σ , of X should be between 0.6 and 0.7. The target mean change size was specified as $\delta_1 = 0.25$. To accommodate heavier than normal tails, $\theta_0 = 1.18$ from Table 1 was used to arrive at a reference value $\zeta = 1.18 \times 0.25/2 \approx 0.15$ for the CUSUM. The corresponding control limit from Table S1 is h = 14.063. To detect a 50% increase or decrease in dispersion with the dispersion CUSUM, reference values of $\zeta^+ = 0.20$ ($\approx \log 1.5/2$) and $\zeta^- = -0.35$ ($\approx \log 0.5/2$) are used. The control limits from Table S6 are $h^+ = 10.29$ and $h^- = -6.29$.

The CUSUMs are shown in Figure 1. The W-CUSUM (left-hand panel) signals an increase in the mean at observation 235. The usual CUSUM-based estimator of the changepoint after occurrence of a signal is the last index at which the CUSUM (upper or lower) was at zero, which in this case is $\hat{\tau} = 214$. The locations of both on the time axis is indicated by vertical dotted lines. The estimate of the new mean from observations 215 through 235 is 0.595 while the mean of the first 214 observations is -0.027. The change in the mean from -0.027 to 0.595 is highly statistically significant (*p*-value = 0.0001) from a bootstrap two sample t-test on 10,000 bootstrap samples). The time series plot in Figure 2 shows an apparent outlier at $X_{103} = 3.81$ which, after investigation turned out to be due to a transcription error in a spreadsheet. That this value was not detected by either of the CUSUMs points to their robustness against transient special cause effects. Figure 3 shows a Q-Q plot (left panel) and kernel density estimate (right panel) made from the data $\{X_1, \ldots, X_{214}\}$ after removal of the outlier. Both plots suggest a degree of non-normality and slight asymmetry in the underlying distribution. In view of the highly significant difference between the means, it is unlikely that asymmetry was the cause of the CUSUM signal.

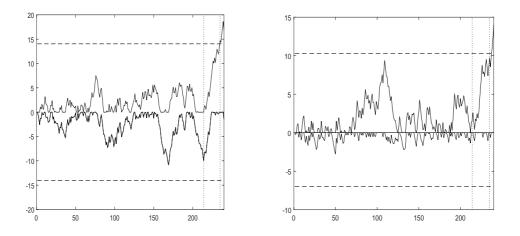


Fig. 1. W-CUSUM (left panel) and W^2 -CUSUM (right panel) CUSUMs of the ash data. Observation index n on the horizontal axis. Horizontal dashed lines indicate the control limits.

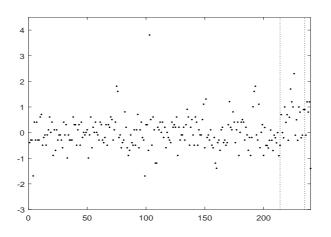


Fig. 2. Time series plot of dataset X_1, \ldots, X_{240} .

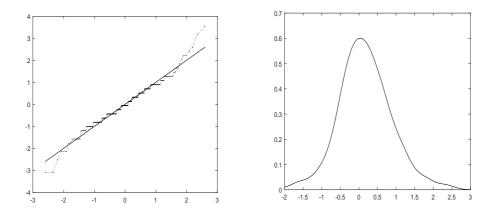


Fig. 3. Q-Q plot (left panel) and density estimate (right panel) from data X_1, \ldots, X_{214} after removal of one outlier.

The J_W^2 CUSUM (right-hand panel in Figure 1) signals at n = 238, shortly after the W-CUSUM. The standard deviation estimates from the segments $\{X_1, \ldots, X_{214}\}$ and $\{X_{215}, \ldots, X_{235}\}$ are very similar, namely $\hat{\sigma}_1 =$ 0.61 and $\hat{\sigma}_2 = 0.68$. A bootstrap F-test for equality of variances in these two segments yields a p-value of 0.66. Thus, on the available evidence, the signal from the variance chart is most likely a result of the substantial mean change. Further substantiation of this conclusion comes from a Monte Carlo simulation in which data were generated from the density estimate in Figure 3, shifted to the right by an amount 0.027 to make the resulting density have mean zero. Repeated sampling from this distribution ensures a constant standard deviation. An increase of 0.622 (= 0.595+0.027) was induced in the median after $\tau = 214$ observations. The estimate of the ARL $E[N - \tau | N > \tau]$ resulting from 10,000 such trials was 22. This is of the same order of magnitude as the excess $N - \hat{\tau} = 235 - 214 = 21$ in the observed data and confirms the likely reaction by the dispersion CUSUM to the mean shift.

The impact of the choices $\zeta = 0.15$ and $\zeta = 0.20$ on the CUSUMs can be assessed if the first 50 observations, say, are treated as in-control Phase I data. These have a standard deviation of $\hat{\sigma} = 0.45$, which is somewhat less than the original estimate of 0.6. The default bandwidth for a Gaussian kernel density estimate made on these Phase I data is b = 0.22. Then, using (8), a computation gives $\hat{\theta}_0 = 1.03$ so that the suggested reference constant for a target shift of $\delta_1 = 0.5$ would be

$$\zeta = 1.03 \times 0.5/2 \approx 0.25.$$

For the W^2 -CUSUM, analogous computations give

$$\hat{\theta}_1 = 1.12,$$

which suggests $\zeta = 0.23$ as reference constant, very close to the value that was actually used. Running the CUSUMs on the Phase II observations X_{51}, X_{52}, \ldots with this new reference constant has no material effect on the results: The W-CUSUM then signals at n = 50 + 180 = 230 and the changepoint is again estimated to be at n = 50 + 164 = 214.

For b = 0.44 and b = 0.11, respectively double and one half the default bandwidth, the corresponding estimated reference values for the *W*-CUSUM are $\zeta = 0.21$ and $\zeta = 0.28$. Again, when these are used, the CUSUM results are almost identical to those found at $\zeta = 0.15$. This points to the fact that the performance of the CUSUM is not overly sensitive to misestimation of θ_0 and consequent misestimation of the "optimal" reference value $\zeta = \theta_0 \delta_1$.

8 Summary

This paper develops CUSUMs based on signed sequential ranks to detect changes away from a specified in-control median of an unknown symmetric distribution. The in-control behaviour of the CUSUMs is distribution-free while their out-of-control properties are shown to be well approximated by those of a normal distribution CUSUM. In particular, no estimates of distribution parameters are required to initiate the CUSUMs and they exhibit no between-practitioner variation. When the underlying distribution is normal with unknown variance, a CUSUM based on the Van der Waerden rank score is efficient compared to a normal distribution CUSUM. A Wilcoxontype CUSUM is fully self-starting and near-efficient for heavy tailed distributions. A CUSUM to detect changes in dispersion is also developed. The methodology is illustrated in an application to a set of data from an industrial environment.

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9 Appendix

9.1 A justification for the heuristic (7)

The following result, which is a special case of Theorem 1 in Lombard (1981), forms the basis of the heuristic and follows upon making identifications between the notation used in this paper and that used in Lombard (1981). Alternatively, the result can be obtained from Theorem 1 of Lombard and Mason (1985) upon making appropriate Taylor expansions.

Proposition 1

Let h and τ^* be positive numbers. For every t > 0, denote the integer part of $h^2 t$ by $\lfloor h^2 t \rfloor$ and set $\tau = \lfloor h^2 \tau^* \rfloor$. Suppose the independent observations X_1, \ldots, X_{τ} have common cdf F(x) while $X_{\tau+1}, X_{\tau+2}, \ldots$ have cdf $F(x - \delta \sigma/h)$. With ξ_i from (2), set

$$S_{\lfloor h^2 t \rfloor} = \sum_{i=1}^{\lfloor h^2 t \rfloor} (\xi_i - \zeta), \ t \ge 0$$
(12)

where $\zeta = \gamma/h$ and $S_0 = 0$. Then the continuous time process $S_{\lfloor h^2 t \rfloor}/h$, $t \ge 0$ converges in distribution as $h \to \infty$ to the continuous time process

$$\Theta(t) = W(t) - \gamma t + \theta \delta \sigma max\{0, t - \tau^*\}$$
(13)

where W denotes a standard Brownian motion and where

$$\theta = -\int_{-1}^{1} J(u) \frac{f'(F^{-1}(\frac{1+u}{2}))}{f(F^{-1}(\frac{1+u}{2}))} du.$$

Here, convergence in distribution is meant in the sense of weak convergence of probability measures on the space $D[0, \infty)$ - see Billingsley (1999, Section 16).

Straightforward calculation involving an integration by parts gives $\theta = \theta_0/\sigma$ for θ_0 defined in (5). Then, upon evaluating (12) and (13) at $t = n/h^2$, $1 \leq n \leq \tau$ and $t = (\tau + k)/h^2$, $k \geq 1$, and using the fact that $W(n/h^2)$ and W(n)/h, $n \geq 1$, have the same joint distributions, Proposition 1 suggests that the joint distributions of the partial sums S_n , $n \geq m$ that figure in the SSR CUSUM can be approximated by those of the sequence

$$\Theta(n) = W(n) - \zeta n + \theta_0 \delta max\{0, n - \tau\}$$

where $\tau > m$ and m is a "large" positive integer. Let $\xi_1^*, \ldots, \xi_{\tau}^*$ be i.i.d. normal(0, 1), let $\xi_{\tau+k}^*$, $k \ge 1$ be i.i.d. normal($\theta_0 \delta, 1$) and set $S_n^* = \xi_1^* + \cdots + \xi_n^* - n\zeta$. Then the sequences S_n^* , $n \ge 1$ and $\Theta(n)$, $n \ge 1$ are identically distributed. Thus, the distribution of the normal CUSUM based on ξ_n^* , $n \ge 1$ provide an approximation to the SSR CUSUM based on ξ_n^* , $n \ge 1$. This is the content of the heuristic.

10 Supplementary Material

10.1 Control limits for the W- and VdW-CUSUMs (Section 2)

The computations used to obtain the control limits in Tables S1 and S2 below were as follows. Since the partial sums of the ξ_i are approximately normally distributed, it is not difficult to imagine that the control limits h of the CUSUM will correspond closely to those of a standard normal cusum. Given a set of reference values and nominal IC ARL values ARL_0 , denote by h_1 the corresponding control limits from a standard normal CUSUM. The first step of an iterative process was to estimate the IC ARL of the SSR CUSUM on a (ζ, h_1) grid using, for instance, 10,000 independent Monte Carlo generated realizations with a uniform distribution on [-1, 1] serving as in-control distribution. Denote the estimate by $A(\zeta, h_1)$. Cubic spline interpolation from $(\zeta, A(\zeta, h_1))$ to (ζ, h) then yielded new estimates, h_2 , of the correct control limits. A further 10,000 independent Monte Carlo generated realizations using h_2 produced a new estimated IC ARL $\hat{A}(\zeta, h_2)$. This process was repeated until all the differences $|A(\zeta, h) - ARL_0|$ were less than 3. For $\zeta \leq 0.25$, no more that three iterations were required, while for $\zeta \geq 0.25$, six iterations sufficed. Finally, the control limits were all checked independently in 100,000 Monte Carlo runs. The largest difference between nominal and simulation estimated IC ARLs was 3.

		ARL_0								
ζ	100	250	500	1,000	2,000					
0.1	6.45	9.44	12.01	14.79	17.93					
0.15	5.65	7.91	9.86	11.88	14.06					
0.2	5.00	6.89	8.37	9.96	11.57					
0.25	4.46	6.02	7.25	8.52	9.84					
0.3	4.01	5.33	6.37	7.45	8.53					
0.35	3.62	4.75	5.66	6.58	7.51					
0.4	3.29	4.29	5.06	5.87	6.66					
0.45	2.99	3.89	4.56	5.24	5.96					
0.5	2.73	3.52	4.13	4.74	5.34					

Table S1 Control limits for the W - CUSUM.

	ARL_0							
ζ	100	250	500	1,000				
0.1	5.995	9.041	11.743	14.485				
0.15	5.318	7.778	9.922	12.14				
0.2	4.640	6.514	8.100	9.796				
0.25	4.186	5.816	7.208	8.607				
0.3	3.731	5.118	6.315	7.417				
0.35	3.410	4.661	5.698	6.685				
0.4	3.089	4.204	5.080	5.952				
0.45	2.829	3.863	4.665	5.458				
0.5	2.568	3.521	4.249	4.964				

Table S2 Control limits for the VdW-CUSUM. For $ARL_0 > 1000$ normal distribution control limits can be used.

10.2 ARL predicted by the heuristic (Section 3.2)

Table S2.1

	W-CUSUM ARL approximations in normal and t_3								
	distributions. $ARL_0 = 500$; changepoint $\tau = 100$.								
		normal: (ℓ	$\theta_0 = 0.9$	8)		t ₃ : $(\theta_0 =$	= 1.37)		
(ζ, h)	(0.10)	, 12.01)	(0.25)	5, 7.25)	(0.15)	5, 9.86)	(0.35)	5, 5.66)	
δ	$W(\delta)$	$N(\theta_0 \delta)$	$W(\delta)$	$N(\theta_0 \delta)$	$W(\delta)$	$N(\theta_0 \delta)$	$W(\delta)$	$N(\theta_0 \delta)$	
0.125	126	127	161	157	93	93	131	122	
0.25	57	57	70	67	38	37	48	45	
0.375	36	35	37	37	23	22	25	23	
0.5	26	25	25	24	17	16	16	15	
0.625	20	19	18	17	14	12	12	11	
0.75	17	16	14	14	11	10	10	8	
1.0	11	12	11	10	9	7	7	6	
1.25	10	9	8	7	8	6	6	5	
1.5	9	8	7	6	7	5	6	4	

Table 3.2 shows the results when $\tau = 0$, that is, when the process is out of control from the outset and the condition in the heuristic that τ be large, is not met. While it is clear in this instance that the CUSUM does have the ability to detect an initial out-of-control situation, the approximation tends to underestimate quite substantially the true OOC ARL.

Table S2.2	
W-CUSUM ARL approximations in normal and t_3	
distributions. $ARL_0 = 500$; changepoint $\tau = 0$.	
	_

		normal: $(\ell$	$\theta_0 = 0.98)$		$t_3: (\theta_0 = 1.37)$			
(ζ, h)	(0.10	, 12.01)	(0.25, 7.25)		(0.15, 9.86)		(0.35, 5.66)	
δ	$W(\delta)$	$N(\theta_0 \delta)$	$W(\delta)$	$N(\theta_0 \delta)$	$W(\delta)$	$N(\theta_0 \delta)$	$W(\delta)$	$N(\theta_0 \delta)$
0.125	146	145	171	167	107	106	138	125
0.25	72	68	78	72	48	44	57	47
0.375	46	43	46	41	32	27	33	25
0.5	35	31	32	27	25	19	23	16
0.625	29	24	25	20	21	15	19	12
0.75	25	20	21	16	19	12	16	9
1.0	21	15	16	11	16	9	13	7
1.25	18	12	14	9	15	7	12	5
1.5	17	10	13	7	14	6	11	4

10.3 VdW-CUSUM. (Section 4)

Table S4.1

a_{δ} at $ARL_0 = 1,000$											
	au = 0		$\tau = 50$		$\tau = 100$						
δ	$\delta_1 = 0.5$	$\delta_1 = 1.0$	$\delta_1 = 0.5$	$\delta_1 = 1.0$	$\delta_1 = 0.5$	$\delta_1 = 1.0$					
0.25	11	38	4	24	3	18					
0.4	9	31	3	11	2	10					
0.5	8	24	2	8	2	5					
0.75	8	15	1	3	1	2					
1.0	8	13	1	1	1	1					
1.25	9	12	1	1	1	1					
1.5	9	12	1	1	1	1					

d_{δ} at $ARL_0 = 1,000$

10.4 W^2 -CUSUM (Section 6)

Control limits for the W^2 -CUSUM.										
	ARL ₀									
ζ	100	250	500	1,000	2,000					
0.05	6.57	10.08	13.39	17.34	21.61					
0.1	5.69	8.20	10.47	12.90	15.60					
0.15	4.97	6.98	8.68	10.49	12.36					
0.2	4.40	6.08	7.45	8.87	10.29					
0.25	3.96	5.39	6.53	7.77	8.83					
0.3	3.63	4.86	5.83	6.83	7.86					
0.35	3.28	4.39	5.25	6.11	6.97					
0.4	3.02	4.02	4.76	5.52	6.31					

Table S6