

Data-driven model checking for errors-in-variables varying-coefficient models with replicate measurements

Miaomiao Wang^a Chunling Liu^b Tianfa Xie^a Zihua Sun^{c*}

^a College of Applied sciences, Beijing University of Technology, Beijing 100190, China

^b Department of Applied Mathematics, The Hong Kong Polytechnic University Kowloon, Hong Kong

^c School of Mathematic sciences, University of Chinese Academy of Sciences, Beijing, China

Abstract

In this work, we investigate the adequacy check of errors-in-variables varying-coefficient models when replicate measurements are available. Estimation using the naive method that ignores measurement errors is biased. After the calibration of the estimators of the regression coefficient functions, we construct an empirical-process-based test statistic by the attenuation of corrected residuals. The asymptotic properties of the test statistic under the null hypothesis, global and various local alternatives are established. Simulation studies and real data analyses reveal that the proposed test performs satisfactorily.

Keywords: Additive measurement error, Empirical process, Model check, Replicate measurements, Varying-coefficient models

1. Introduction

The varying-coefficient regression models are widely known tools used to characterize the association between a response and a group of dynamic covariates. The dynamic pattern is due to the fact that the regression coefficients may be functions that have unknown forms and depend on some other covariates. Hence, they still inherit the structure and interpretability of the traditional linear models, but possess the flexibility of nonparametric regression. See [1] and [2], among others.

Varying-coefficient regression modeling is widely used in a wide range of scientific areas, where dynamic features are of importance. On the other hand, highly frequently, the covariates inevitably

*Corresponding author: E-mail address: sunzh@amss.ac.cn

suffer from random measurement error. Within the errors-in-variables varying-coefficient data setting, there is a vast amount of literature on estimation procedures. However, surprisingly, there are relatively fewer works on model adequacy checking. See [3], [4], [5], [6], [7] and [8], among others. Among the existing literature, the works mainly focus on parametric model diagnostics. For example, [5] considered the goodness-of-fit test of parametric models in the context of the classical measurement error framework.

Parametric modeling often has to face the embarrassment of misspecification; distribution-free test procedures would be a robust alternative. Nevertheless, the treatments are quite different in the aforementioned two scenarios. In the situation where the measurement error distribution is completely unknown, practical needs call for more effective test procedures. To the best of our knowledge, there are few works that can address such test problems in varying coefficient models. Though the score-type test and the modification of estimation equations of [5] can be applied to many semiparametric models, they lose effect for the varying-coefficient models because of the absence of the finite-dimensional parameter. Therefore, we investigate such model testing problems by presenting empirical process (EP) based test methods. Such procedures enable effective local and omnibus goodness-of-fit tests for a errors-in-variables varying-coefficient regression model with replicate measurements.

The lack-of-fit test of regression models serves as a useful tool to avoid model misspecification. However, for research on hypothesis testing for measurement error regression models, the main difficulty lies in the impossibility of calculating the likelihood or forming residuals when extending classical testing procedures. In our methodology, we first calibrate the measurement error, which avoids the biased estimation of residuals. Then, we use the idea of traditional residual analysis and derive a powerful test statistic based on an empirical process representation. Through the combination of the local linear smoothing technique and correction of attenuation, we may provide an accurate estimate of the regression coefficient function. Our novel approach is superior to the existing methods such as deconvolutional kernel techniques or minimum distance procedures because the accumulated residuals turn into an EP for model adequacy checking. Hence, this data-driven test method enjoys the advantage of the EP to avoid the nonparametric smoothing of the unknown distribution function of model errors.

The contribution of our work is trifold. First, we fill the gap of model checking by a new residual analysis technique for the errors-in-variables varying coefficient models with replicate mea-

40 surements. Our proposed test can detect infinite moment conditions rather than only one or several moment conditions in score-type tests. Second, we establish the asymptotic properties for local and omnibus test procedures and obtain the optimality of the test in testing local alternative models. The subsequent inference is relatively challenging compared to that of most parametric models in the literature. Finally, the EP-based test has a dimension-reduction effect compared to the ap-
 45 proachesthat apply the nonparametric smoothing to the model errors, for example, the weighted integrated squared distance (WISD) test [9] and the test based on the U-statistic [10, 11].

The rest of the paper is organized as follows. In section 2, we propose our test statistic and establish the main results for an omnibus test. In section 3, we investigate several other local test procedures and establish the corresponding asymptotic theory. In section 4, we demonstrate the
 50 numerical analysis by the design of simulation studies and two real-world data analyses. Finally, we provide a brief summary. The proofs are detailed in the appendix.

2. Methodology and omnibus test

We formulate the varying-coefficient model in the form of

$$Y = \alpha(U)^\top \mathbf{X} + \varepsilon, \quad (2.1)$$

where Y is the scalar response variable, $\mathbf{X} = (X_1, \dots, X_p)^\top$ is a $p \times 1$ covariate, and $\alpha(\cdot)$ is a p -
 55 dimensional vector of unknown coefficient functions depending on a single covariate U . A common assumption is that $E(\varepsilon^2|U, \mathbf{X}) < \infty$.

In practice, some covariates are often unavailable because they suffer from measurement error. In (2.1), let $\mathbf{X} = (\mathbf{X}_1^\top, \mathbf{X}_2^\top)^\top$ with $\mathbf{X}_1 = (X_1, \dots, X_q)^\top$ and $\mathbf{X}_2 = (X_{q+1}, \dots, X_p)^\top$. We consider the case where the covariate \mathbf{X}_1 is measured with the classical additive error, and the covariate \mathbf{X}_2
 60 is exactly observed. Instead of the covariate \mathbf{X}_1 , we observe $\tilde{\mathbf{X}}_1$, which satisfies

$$\tilde{\mathbf{X}}_1 = \mathbf{X}_1 + \mathbf{e}_1, \quad (2.2)$$

where \mathbf{e}_1 is the measurement error with mean zero and the covariance matrix Σ_1 , which may be unknown. We assume that \mathbf{e}_1 is independent of $(\mathbf{X}^\top, U, \varepsilon)^\top$, and the measurement error is nondifferential in the sense that $E(Y|\mathbf{X}, \tilde{\mathbf{X}}_1, U) = E(Y|\mathbf{X}, U)$. Let $\tilde{\mathbf{X}}_2 = \mathbf{X}_2$ and $\mathbf{e} = (\mathbf{e}_1^\top \ \mathbf{0}^\top)^\top$. Then, $\tilde{\mathbf{X}} = \mathbf{X} + \mathbf{e}$. Denote the covariance matrix of \mathbf{e} by Σ_e . Then, the observed data from the
 65 population $(Y, \tilde{\mathbf{X}}, U)$ may be written as $\{(Y_i, \tilde{\mathbf{X}}_i, U_i), i = 1, \dots, n\}$.

One challenging issue in implementing varying coefficients models is that the nonlinear functional form of the continuous variables may be misidentified. Therefore, it is important to implement a suitable testing procedure before further statistical analyses. To the best of our knowledge, there is no existing work addressing the adequacy check of varying-coefficient models when the variable
70 is measured with error and distribution information is absent. We aim to fill this gap and consider the following testing problem:

$$\mathcal{H}_0 : \exists \alpha(U) \text{ s.t. } E[Y|\mathbf{X}, U] = \alpha(U)^\top \mathbf{X}, \text{ a.s.} \quad (2.3)$$

against the alternative hypothesis that \mathcal{H}_0 is not true when some components of \mathbf{X} , for example, \mathbf{X}_1 , are measured with classical error. Under the alternative hypothesis, the covariates \mathbf{X} , or U , or both need to enter the model as an unknown function, but $\alpha(U)^\top \mathbf{X}$. Therefore, we consider an
75 omnibus test.

When the data are measured with classical error, the naive method ignores the measurement error and applies directly the method appropriate for exactly observed data, yielding $\hat{\alpha}_{naive}(u)$, which is the biased estimator of the coefficient function in (2.1). The corresponding naive estimated model error may be decomposed into

$$\begin{aligned} \hat{\epsilon}_{naive}(Y_i, \tilde{\mathbf{X}}_i, U_i) &=: Y_i - \hat{\alpha}_{naive}(U_i)^\top \tilde{\mathbf{X}}_i \\ &= \{Y_i - \alpha(U_i)^\top \mathbf{X}_i\} + \{\alpha(U_i) - \hat{\alpha}_{naive}(U_i)\}^\top \mathbf{X}_i + \hat{\alpha}_{naive}(U_i)^\top (\mathbf{X}_i - \tilde{\mathbf{X}}_i), \end{aligned}$$

80 for $i = 1, 2, \dots, n$. It is readily validated that under the null hypothesis, the expectations of the first and the third summands tend asymptotically to zero, but this is not true for the second term as the sample size converges to ∞ . Therefore the expectation of the naive residuals do not converge to zero under the null hypothesis. Consequently, the naive model checking method for Model (2.1) tends to reject the null hypothesis and loses its effect. It is natural that we calibrate the estimator of
85 the coefficient function vector $\alpha(u)$ as the first step, denoted by $\hat{\alpha}_n(u)$. We provide a brief overview in the Appendix as it is standard and analog to [12].

Note that the null hypothesis (2.3) is tautological to $E[\{Y - \alpha(U)^\top \mathbf{X}\} \mathbf{1}(\mathbf{X} < \mathbf{x}, U < u)] = 0$ for $\mathbf{x} \in \mathcal{R}^p$ and $u \in \mathcal{R}$. When data are accurately observed, [13] constructed the following test statistic:

$$\mathcal{T}_n^{XZ} = \int [\widehat{CR}_{n,XZ}(\mathbf{x}, u)]^2 dF_n^{xz}(\mathbf{x}, u) \quad (2.4)$$

where $F_n^{xz}(\cdot)$ is an empirical distribution function based on $\{(\mathbf{X}_i, U_i), i = 1, \dots, n\}$. The cumulative summation process $\widehat{CR}_{n,XZ}(\mathbf{x}, u)$ is defined as

$$\widehat{CR}_{n,XZ}(\mathbf{x}, u) = n^{-1/2} \sum_{i=1}^n \{Y_i - \tilde{\alpha}_n(U_i)^\top \mathbf{X}_i\} \mathbf{1}(\mathbf{X}_i < \mathbf{x}, U_i < u),$$

90 where $\tilde{\alpha}_n(u)$ is the local constant kernel estimator of the coefficient function $\alpha(u)$ based on the sample $\{(Y_i, \mathbf{X}_i, U_i), i = 1, \dots, n\}$.

Because the predictor \mathbf{X}_1 is measured with error, as shown in the above discussion, the direct application of the method of [13] cannot control Type I error since the naive residuals do not converge to zero under the null hypothesis. A natural alternative is to use the corrected estimator 95 $\hat{\alpha}_n(u)$ of the coefficient function and replace the true variable \mathbf{X} by the observed variable $\tilde{\mathbf{X}}$. Thus, the following test statistic is constructed:

$$\mathcal{T}_n^{direct} = \int [\widehat{CR}_{n,direct}(\mathbf{x}, u)]^2 d\tilde{F}_n(\mathbf{x}, u) \quad (2.5)$$

where

$$\widehat{CR}_{n,direct}(\mathbf{x}, u) = n^{-1/2} \sum_{i=1}^n \{Y_i - \hat{\alpha}_n(U_i)^\top \tilde{\mathbf{X}}_i\} \mathbf{1}(\tilde{\mathbf{X}}_i < \mathbf{x}, U_i < u),$$

and $\tilde{F}_n(\cdot)$ is the empirical distribution function based on $\{(\tilde{\mathbf{X}}_i, U_i), i = 1, \dots, n\}$.

It is readily validated that under the null hypothesis (2.3), $E\{\widehat{CR}_{n,direct}(\mathbf{x}, u)\} = \sqrt{n}E\{\alpha(U)^\top (\mathbf{X} - \tilde{\mathbf{X}}) \mathbf{1}(\tilde{\mathbf{X}} < \mathbf{x}, U < u)\} + o(1)$ and $E\{\alpha(U)^\top (\mathbf{X} - \tilde{\mathbf{X}}) \mathbf{1}(\tilde{\mathbf{X}} < \mathbf{x}, U < u)\} \neq 0$. Thus, even under the 100 null hypothesis, \mathcal{T}_n^{direct} converges to ∞ as $n \rightarrow \infty$. Clearly, the direct application of the variable with measurement error in the indicator function causes the test method to lose effect.

Under (2.3), by the nondifferential condition and because \mathbf{e} is independent of $(\mathbf{X}^\top, U, \varepsilon)^\top$, we have $E[(Y - \alpha(U)^\top \tilde{\mathbf{X}}) \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] = 0$. Under the alternative hypothesis, $E[(Y - \alpha(U)^\top \tilde{\mathbf{X}}) \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)]$ tends to be away from zero. Therefore, it is reasonable to develop 105 a test method based on $E[(Y - \alpha(U)^\top \tilde{\mathbf{X}}) \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)]$. Another outstanding merit of $E[(Y - \alpha(U)^\top \tilde{\mathbf{X}}) \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)]$ is that it can be estimated consistently without involving the unavailable true variable \mathbf{X}_1 .

Thus we further concentrate on the shrunken estimated empirical process of $E[\{Y - \alpha(U)^\top \tilde{\mathbf{X}}\} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)]$,

$$\widehat{CR}_n(\mathbf{z}, u) = n^{-1/2} \sum_{i=1}^n \{Y_i - \hat{\alpha}_n(U_i)^\top \tilde{\mathbf{X}}_i\} \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u). \quad (2.6)$$

110 Then, we have a modified representation for \mathcal{T}_n^{direct} as our final version of the test statistic:

$$\mathcal{T}_n = \int [\widehat{CR}_n(\mathbf{z}, u)]^2 dF_n(\mathbf{z}, u), \quad (2.7)$$

where $F_n(\cdot)$ is an empirical distribution function based on $\{(\mathbf{X}_{2i}, U_i), i = 1, \dots, n\}$.

Let $Q(U) = E(\mathbf{X}\mathbf{X}^\top | U)$. Denote

$$\begin{aligned} & \mathcal{IF}_{\mathbf{z}, u}(\mathbf{X}_i, \tilde{\mathbf{X}}_i, U_i, Y_i, \varepsilon_i, \mathbf{e}_i) \\ &= \varepsilon_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) - \alpha(U_i)^\top \mathbf{e}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ & \quad + \tilde{\mathbf{X}}_i \varepsilon_i E\{Q^{-1}(U) \tilde{\mathbf{X}} \mathbf{1}(\mathbf{X}_2 < \mathbf{x}, U < u) | U = U_i\} \\ & \quad - E\{Q^{-1}(U) \tilde{\mathbf{X}}^\top \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u) | U = U_i\} \alpha(U_i) \mathbf{X}_i \mathbf{e}_i^\top \end{aligned}$$

for $i = 1, \dots, n$. Then, we have the following asymptotic convergence of the omnibus test.

Theorem 2.1. *Under the regular conditions listed in the Appendix, when the null hypothesis (2.3)*

115 *holds, we have the following results:*

(i) $E[\widehat{CR}_n(\mathbf{z}, u)]$ converges to 0 as $n \rightarrow \infty$.

(ii) *The estimated empirical process $\widehat{CR}_n(\mathbf{z}, u)$ has the following asymptotic expansion:*

$$\widehat{CR}_n(\mathbf{z}, u) = n^{-1/2} \sum_{i=1}^n \mathcal{IF}_{\mathbf{z}, u}(\mathbf{X}_i, \tilde{\mathbf{X}}_i, U_i, Y_i, \varepsilon_i, \mathbf{e}_i) + o_p(1). \quad (2.8)$$

Furthermore, we can prove that $\mathcal{T}_n \xrightarrow{L} \int \{R(\mathbf{z}, u)\}^2 dF(\mathbf{z}, u)$, where $R(\mathbf{z}, u)$ is a centered Gaussian process with the covariance function,

$$\text{Cov}\{R(\mathbf{z}_1, u_1), R(\mathbf{z}_2, u_2)\} = E\{\mathcal{IF}_{\mathbf{z}_1, u_1}(\mathbf{X}, \tilde{\mathbf{X}}, U, Y, \varepsilon, \mathbf{e}) \mathcal{IF}_{\mathbf{z}_2, u_2}(\mathbf{X}, \tilde{\mathbf{X}}, U, Y, \varepsilon, \mathbf{e})\}$$

120 and $F(\mathbf{z}, u)$ is the distribution function of (\mathbf{X}_2, U) .

Theorem 2.1 suggests that the expectation of $\widehat{CR}_n(\mathbf{z}, u)$ converges to zero when $n \rightarrow \infty$ for any $\mathbf{z} \in \mathcal{R}^{p-q}$ and $u \in \mathcal{R}$ under \mathcal{H}_0 . The test statistic \mathcal{T}_n tends to be zero under the null hypothesis but becomes larger under the alternative hypothesis. Thus, the test is one-sided, and the null hypothesis should be rejected for some large enough value of \mathcal{T}_n .

125 **Remark 1.** *The equation $E[\{Y - \alpha^\top(U) \tilde{\mathbf{X}}\} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] = 0, \forall \mathbf{z} \in \mathcal{R}^{p-q}, u \in \mathcal{R}$ is equal to $E\{Y - \alpha(U)^\top \tilde{\mathbf{X}} | \mathbf{X}_2, U\} = 0$. Moreover, it can be proven that $E\{Y - \alpha(U)^\top \tilde{\mathbf{X}} | \mathbf{X}_2, U\} = 0$ under the null hypothesis. Therefore, the proposed test \mathcal{T}_n can judge the null hypothesis and control*

Type I error. Further, because the measurement error variable \mathbf{e} is independent of $(\mathbf{X}^\top, U, \varepsilon)^\top$, the condition that $\mathbb{E}\{Y - \alpha(U)^\top \tilde{\mathbf{X}} | \mathbf{X}_2, U\} = 0$ is tantamount to $\mathbb{E}\{Y - \alpha(U)^\top \mathbf{X} | \mathbf{X}_2, U\} = 0$. Though
130 the proposed test \mathcal{T}_n cannot detect the alternative models satisfying $\mathbb{E}[Y - \alpha(U)^\top \mathbf{X} | \mathbf{X}, U] \neq 0$ a.s. but $\mathbb{E}[(Y - \alpha(U)^\top \mathbf{X}) | \mathbf{X}_2, U] = 0$ a.s., compared with the score-type test, the above properties are still desirable. The proposed test can detect infinite moment conditions. However, the score-type test only detects one or several moment conditions. For example, looking into the expression $\widehat{CR}_n(\infty, \infty)$, we obtain a score-type test, which actually tests the specific direction, $\mathbb{E}(Y - \alpha(U)^\top \tilde{\mathbf{X}}) = 0$.

135 3. Local tests and asymptotic results

In this section, we will introduce several local test procedures that are important in practice. First, we consider the alternative hypothetical models:

$$\mathcal{H}_{1n} : Y = g(\mathbf{X}, U) + \varepsilon \text{ a.s.} \quad (3.1)$$

with some arbitrary bounded measurable nonzero function $g(\mathbf{X}, U)$ which cannot take the form of $\alpha(U)^\top \mathbf{X}$ for any measurable function $\alpha(U)$.

140 **Theorem 3.2.** *Under the regular conditions listed in the Appendix, under the alternative hypothesis (3.1), if $\mathbb{E}(\eta | \mathbf{X}_2, U) \neq 0$ a.s., where $\eta = g(\mathbf{X}, U) - \alpha(U)^\top \mathbf{X} + \varepsilon$, then the test statistic \mathcal{T}_n converges to ∞ as $n \rightarrow \infty$.*

The alternative hypothesis (3.1) is equivalent to $E(\eta | \mathbf{X}, U) \neq 0$ a.s.. It is only for the scenarios when $E(\eta | \mathbf{X}, U) \neq 0$ a.s., but $E(\eta | \mathbf{X}_2, U) = 0$ a.s., that the proposed test loses some power. For
145 any other scenarios, the proposed test has asymptotic power one. This is an outstanding merit of the proposed test compared with the score-type test.

The second local test that we consider is Pitman local alternative hypothetical models:

$$\mathcal{H}_{2n} : Y = \alpha(U)^\top \mathbf{X} + n^{-1/2} \mathcal{D}(\mathbf{X}, U) + \varepsilon, \text{ a.s.} \quad (3.2)$$

with some arbitrary bounded measurable nonzero function $\mathcal{D}(\mathbf{X}, U)$, which cannot take the form of $\alpha(U)^\top \mathbf{X}$ for any measurable function $\alpha(U)$. The Pitman local alternative hypothetical models
150 converge to the null one at the rate $n^{-1/2}$.

Let $\Delta(\mathbf{z}, u) = \mathbb{E}[Q^{-1}(U)E\{\tilde{\mathbf{X}}\mathcal{D}(\mathbf{X}, U) | U = u\}\tilde{\mathbf{X}}\mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] + \mathbb{E}\{\mathcal{D}(\mathbf{X}, U)\mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)\}$. This is corresponding large sample convergence property.

Theorem 3.3. *Under the regular conditions listed in the Appendix, when the local alternative (3.2) holds, we have the following results:*

- 155 (i) $E[\widehat{CR}_n(\mathbf{z}, u)]$ converges to $\Delta(\mathbf{z}, u)$ as $n \rightarrow \infty$;
(ii) The test statistic \mathcal{T}_n converges to $\int \{R(\mathbf{z}, u) + \Delta(\mathbf{z}, u)\}^2 dF(\mathbf{z}, u)$, as $n \rightarrow \infty$.

The drift function $\Delta(\mathbf{z}, u)$ can measure the difference of the estimated empirical process under the null hypothesis (2.3) and the alternative hypothesis (3.2). Thus, the test \mathcal{T}_n has nonignorable power for the alternative hypothesis (3.2). The property is unavailable for the local testing methods, 160 such as the weighted integrated squared distance (WISD) test and the test based on the U-statistic.

Finally, we consider the local alternative hypothetical models:

$$\mathcal{H}_{3n} : Y = \alpha(U)^\top \mathbf{X} + B_n \mathcal{D}(\mathbf{X}, U) + \varepsilon, \text{ a.s.} \quad (3.3)$$

with $n^{1/2}B_n \rightarrow \infty$ and an arbitrary bounded measurable nonzero function $\mathcal{D}(\mathbf{X}, U)$, which cannot take the form of $\alpha(U)^\top \mathbf{X}$ for any measurable function $\alpha(U)$.

Denote the critical value of the test by c_α , which satisfies $\Pr\{\mathcal{T}_n > c_\alpha\} \rightarrow \alpha$ as $n \rightarrow \infty$ under 165 \mathcal{H}_0 in (2.3). We have the results presented below.

Theorem 3.4. *Under the regular conditions listed in the Appendix, when the local hypothesis (3.3) with $n^{1/2}B_n \rightarrow \infty$ holds, we have the following results:*

- (i) $E[\widehat{CR}_n(\mathbf{z}, u)]$ converges to ∞ as $n \rightarrow \infty$;
(ii) The test statistic \mathcal{T}_n converges to ∞ as $n \rightarrow \infty$. Therefore, the power function $\Pr\{\mathcal{T}_n >$
170 $c_\alpha | \mathcal{H}_{3n}\}$ converges to 1 as $n \rightarrow \infty$.

The above theorem shows that the proposed test is consistent against the local alternative hypothetical models (3.3).

Following [14] and [15], to obtain the critical value of the proposed test, we resort to the wild bootstrap method to mimic the null distribution of the proposed test statistic. Let $\{V_i\}_{i=1}^n$ be the 175 random variable sequence with $E(V_i) = 0$, $\text{var}(V_i) = 1$ and $|V_i| < C$ for some finite constant C . The details of the wild bootstrap are as follows:

Step 1: Calculate the test statistic \mathcal{T}_n according to (2.7);

Step 2: Generate random variables $\{V_i\}_{i=1}^n$, and compute the bootstrap response variables:

$$Y_i^* = \hat{\alpha}_n(U_i)^\top \tilde{\mathbf{X}}_i + \{Y_i - \hat{\alpha}_n(U_i)^\top \tilde{\mathbf{X}}_i\} V_i, \quad i = 1, \dots, n. \quad (3.4)$$

Employ the bootstrap sample $\{(Y_i^*, \tilde{\mathbf{X}}_i, U_i), i = 1, 2, \dots, n\}$, and compute the test statistic \mathcal{T}_n ,
 180 which we denote by \mathcal{T}_n^* ;

Step 3: Repeat Step 2 B times, and obtain B values of \mathcal{T}_n^* , denoted by $\{\mathcal{T}_{n1}^*, \dots, \mathcal{T}_{nB}^*\}$. Take
 the $1 - \alpha$ sample quantile of $\{\mathcal{T}_{n1}^*, \dots, \mathcal{T}_{nB}^*\}$ as the critical value at the given level α .

By similar analyses of Theorem 1 of [15], we can obtain the result that the test statistic \mathcal{T}_n and
 the bootstrap test statistic \mathcal{T}_n^* have the same asymptotic distribution under the null hypothesis and
 185 the local hypothesis (3.2). Therefore, the critical value determined by the bootstrap method is a
 good substitute for the unknown true critical value, which ensures that the wild bootstrap method
 can maintain a nominal test size. Furthermore, the above wild bootstrap scheme is unrelated to
 the distributional information of the model error and is robust to the conditional heteroscedasticity
 of the model error. We can also find that the proposed test is data-driven, which means that a
 190 conclusion whether a varying-coefficient model can fit the data or not can be drawn directly from
 the data.

4. Numerical studies

4.1. Simulations

In this section, we conduct simulation studies to assess the performance of the proposed test.
 195 Recall that $\Sigma_e = E(\mathbf{e}\mathbf{e}^\top)$. We consider both cases that Σ_e is known and unknown. When Σ_e is
 unknown, it can be estimated by employing the replicate measurements [16, 17]. Let m_i be some
 integer larger than one for $i = 1, \dots, n$. Let $\tilde{\mathbf{X}}_{i1}, \dots, \tilde{\mathbf{X}}_{im_i}$ be m_i repeated observations of \mathbf{X}_i , which
 satisfy $\tilde{\mathbf{X}}_{ij} = \mathbf{X}_i + \mathbf{e}_{ij}$ for $i = 1, 2, \dots, n; j = 1, 2, \dots, m_i$. Here, we take $m_i = 3$ for $i = 1, \dots, n$. We
 calculate the mean of $\tilde{\mathbf{X}}_{i1}, \dots, \tilde{\mathbf{X}}_{im_i}$, denoted by $\bar{\tilde{\mathbf{X}}}_i$. Then, Σ_e can be estimated by the following:

$$\hat{\Sigma}_e = \frac{\sum_{i=1}^n \sum_{j=1}^{m_i} (\tilde{\mathbf{X}}_{ij} - \bar{\tilde{\mathbf{X}}}_i)(\tilde{\mathbf{X}}_{ij} - \bar{\tilde{\mathbf{X}}}_i)^\top}{\sum_{i=1}^n (m_i - 1)}. \quad (6.1)$$

200 We compute the empirical sizes and powers of the proposed test by considering the following three
 examples.

Example 4.1.1: We consider the model:

$$Y = U^2 X_1 + 4 \ln(U/2) X_2 + d \{ \sin((X_1 + X_2)^2) - \ln(|X_1 X_2|) \} + \varepsilon.$$

In this example, X_1 and X_2 follow $\mathcal{N}(1, 1)$, $\varepsilon \sim \mathcal{N}(0, 1)$, $U \sim \mathcal{U}(0, 2)$. The observed variables
 with measurement errors, \tilde{X}_1 and \tilde{X}_2 , satisfy $\tilde{X}_1 = X_1 + e_1$ and $\tilde{X}_2 = X_2 + e_2$ with $(e_1, e_2)^\top \sim$

205 $Laplace(0, \Sigma_e)$. We consider two choices of Σ_e : $\Sigma_1 = \text{diag}(0.5, 0)$ and $\Sigma_2 = \text{diag}(0.3, 0)$. Clearly, the variable X_2 is accurately observed. Let d_0 be 0, 0.5, 1.5, 2.5, 4.0. The constant d is chosen to be $d_0 n^{-1/3}$.

Example 4.1.2: The following model is considered:

$$Y = -UX_1 + (2 \exp(U) + 1)X_2 + \cos(1.5\sqrt{U})X_3 + d \exp(\sin(X_1 X_2) + X_3 - 2U) + \varepsilon.$$

Let X_1 and X_2 follow $\mathcal{N}(1, 1)$, $X_3 \sim \mathcal{U}(0, 2\pi)$, $\varepsilon \sim \mathcal{N}(0, 1)$, $U \sim \mathcal{U}(0, 2)$. The observed variables with measurement errors, \tilde{X}_j , satisfy $\tilde{X}_j = X_j + e_j$ for $j = 1, 2, 3$ with $(e_1, e_2, e_3)^\top \sim \mathcal{N}(0, \Sigma_e)$. We consider two choices of Σ_e : $\Sigma_1 = \text{diag}(0.5, 0, 0)$ and $\Sigma_2 = \text{diag}(0.3, 0, 0)$. Let d_0 be 0, 0.1, 0.2, 0.5, 1.0
210 and the constant d is chosen to be $d_0 n^{-1/2}$.

Example 4.1.3: We consider the following model:

$$Y = \log(3U)X_1 - 1.5 \sin(\pi U)X_2 + \sqrt{U}X_3 + \exp(\cos(U))X_4 \\ + d\{\cos(X_1 + X_3) + UX_2 X_4\} + \varepsilon.$$

Let the covariates X_1 and X_3 follow $\mathcal{N}(1, 1)$ and the covariates X_2 and X_4 follow $\mathcal{U}(0, 2\pi)$. Further we generate the model error variable ε and the covariate U from $\mathcal{N}(0, 1)$ and $\mathcal{U}(0, 1)$, respectively. The observed variables with measurement errors, \tilde{X}_j , satisfy $\tilde{X}_j = X_j + e_j$ for $j = 1, 2, 3, 4$ with $(e_1, e_2, e_3, e_4)^\top \sim \mathcal{N}(0, \Sigma_e)$. We consider two choices of Σ_e : $\Sigma_1 = \text{diag}(0.5, 0.5, 0, 0)$ and $\Sigma_2 =$
215 $\text{diag}(0.3, 0.3, 0, 0)$. The constant d is chosen to be 0, 0.2, 0.5, 1.0, 2.0.

In the above examples, when $d = 0$, the null hypothesis holds. The different values of d larger than zero mean that different alternative models are considered. We fix the sample size n to be 100 and 200. Take test levels to be 0.05 and 0.1. The times of bootstrap resample B is set to be 300. To estimate the regression coefficient, the Gaussian kernel function is selected.

220 **Bandwidth selection:** For the model checking problem, the choice of the associated bandwidth is a challenging problem. The proposed test involves a selection of bandwidth when estimating the coefficient function. To illustrate the impact of bandwidth on the proposed test, we choose five different bandwidths $h_n = C\hat{\sigma}(U)n^{-1/5}$ to compute the empirical sizes and powers with C being one of 0.6, 0.8, 1, 1.2, 1.4. Here, $\hat{\sigma}(U)$ is the standard deviation of U based on the sample
225 $\{U_1, \dots, U_n\}$. For the Gaussian kernel function, the bandwidth $h_n = 1.06\hat{\sigma}(U)n^{-1/5}$ is chosen by the rule of thumb [18]. The above five bandwidths are selected by considering both oversmoothing and undersmoothing. The results under the sample sizes 100 and 200 and the test level 0.05 are shown in Figure 1.

Here inserts Figure 1

230 From Figure 1, it can be observed that the test with the ROT (rule of thumb) bandwidth ($\approx \hat{\sigma}(U)n^{-1/5}$) can control Type I error. When $C = 0.6, 0.8, 1.2$, the empirical sizes are similar. But when $C = 1.4$, the empirical sizes tend to be larger than the test level. We also observed that the bandwidth has an impact on the empirical powers when the alternative models are close to the null hypothetical models corresponding to small values of d in all three examples. With the increase
235 in the values of d , the effect of the bandwidth on the empirical powers weakens.

As shown in [19] and [20], finding an optimal bandwidth for hypothesis testing is still an open problem. It is well-known that a maintenance of the significance level is important for the testing methods. Therefore, by considering the feasibility and rationality, we choose the bandwidth according to the following scheme. First, we take the bandwidth $h_n = C\hat{\sigma}(U)n^{-1/3}$ into account,
240 which satisfies the condition (C5) in the Appendix. Second, we compute the empirical sizes by considering the values of C varying from 1 to 3 at the 0.01 interval for all three examples based on 500 repetitions. From the simulated results, we find that $C = 2$ is a good choice to keep the empirical sizes close to the test levels. In the following simulation studies, we employ the bandwidth $h_n = 2\hat{\sigma}(U)n^{-1/3}$.

245 For the proposed method, the empirical sizes and powers are calculated based on 500 repetitions, which are listed in Tables 1-3. For comparison purposes, we apply the method of [13] to the errors-in-variables settings considered in this paper directly. It should be noted that this method is actually the naive method. The direct method \mathcal{T}_n^{direct} defined in (2.5) is also considered. The empirical sizes and powers of these two tests are calculated and listed in Tables 1-3.

Here insert Tables 1-3

250 From Tables 1-3, we can observe that the empirical sizes of the proposed method \mathcal{T}_n are close to the test levels for the case of Σ_e being known or unknown. These results demonstrate that the proposed procedure works satisfactorily for controlling Type I error. Additionally, the empirical powers of the proposed test increase with sample sizes and the values of d , which is reasonable. However, it can be found that the empirical sizes of the method of [13] and \mathcal{T}_n^{direct} are much larger
255 than the test levels. Clearly, these two methods cannot control Type I error and loss effect for the considered testing problem with errors in variables. Note that if a test cannot control Type I error, its high empirical powers are meaningless. Therefore, the superiority of the proposed test is

evident. As suggested by a reviewer, we have investigated the effect of m_i on the performance of the proposed test by making some additional simulation studies with $m_i = 6$. We find that the empirical sizes and powers of the proposed test are similar for both cases of $m_i = 3$ and $m_i = 6$. However, when practitioners use our test, if there are m_i repeated observations in the dataset, we suggest that all the m_i repeated observations should be used.

4.2. Real Data Analyses

In this section, we employ the proposed method to analyze two datasets: the Framingham heart dataset and Duchenne Muscular Dystrophy (DMD) dataset. For the sake of comparison, we also employ the method of [13] and the test method \mathcal{T}_n^{direct} to analyze these two datasets.

Example 4.2.1: Analysis of the data of the Framingham Heart Study The study includes 1615 males aged from 31 to 65 years. We set the response variable Y to be the average blood pressure in a fixed 2-year period. The covariates U , X_1 and X_2 denote age, the true cholesterol level, and the smoking status, respectively. The cholesterol level X_1 is measured with error. The observed cholesterol level \tilde{X}_1 has two replicate measurements for each subject. All variables except X_2 are standardized. The description of this dataset can be found in [17].

We aim to check the adequacy of the following varying-coefficient model for this dataset:

$$Y = X_1\alpha_1(U) + X_2\alpha_2(U) + \varepsilon.$$

Here, X_1 is measured with error e_1 , which satisfies $E(e_1) = 0$ and $\text{var}(e_1) > 0$. The variable X_2 is free of measurement error. Hence, the covariance matrix of the measurement errors is $\Sigma_e = \text{diag}(\text{var}(e_1), 0)$.

The variance of e_1 is estimated by applying the replicate measurements according to Equ.(6.1). We choose the bandwidth and kernel function similar to the method in the simulation studies. We employ the proposed method, the method of [13], and the method \mathcal{T}_n^{direct} in (2.5) to compute the P -values, which are shown to be 0.077, 0.090, and 0.168. Thus, the null hypothetical varying-coefficient model is not adequate for this dataset. The runtime for the above three methods is 125.56 seconds, 128.73 seconds and 125.91 seconds.

To illustrate the rationale of the above results, we plot the residual plots of the proposed method and the method of [13] in Figure 2. In Figure 2, we also plot the fitted and 95% confidence curves of the estimated model errors.

Here inserts Figure 2

From Figure 2, we observe that the estimated model errors of the proposed method and the method of [13] are distributed asymmetrically with regard to the abscissa axis. They are more dispersive above the horizontal axis and are more centralized below the abscissa axis. We can also
290 note that the fitted curves of the estimated residuals deviate from the abscissa axis. Therefore the conclusion that the varying-coefficient model cannot fit the data is reasonable.

Example 4.2.2: Analysis of the data of Duchenne Muscular Dystrophy In the following, we investigate the dataset of Duchenne Muscular Dystrophy (DMD) of 209 observations. [21] analyzed this dataset to determine whether the partially linear model is adequate. They
295 calibrated the measurement error with the help of auxiliary variables under the framework of additive measurement error model. It is known that the choice of the auxiliary variables is a little difficult and subjective. Here, we aim to check whether the varying-coefficient model is adequate for this dataset by setting the response variable Y and the covariates U , X_1 , and X_2 to be lactate dehydrogenase (LD), age, creatine kinase (CK), and hemopexin (H), respectively.

300 For this dataset, we assume that the variables CK and H are measured with error because they are measured from frozen serum instead of fresh serum, and their measurements are affected by environmental factors such as seasonality. The details can be found in [22]. Let \tilde{X}_1 and \tilde{X}_2 be the observed values of CK and H respectively. Then, $\tilde{X}_1 = X_1 + e_1$ and $\tilde{X}_2 = X_2 + e_2$. We assume two situations for Σ_e : $\Sigma_1 = \text{diag}(0.05, 0.05)$ and $\Sigma_2 = \text{diag}(0.07, 0.07)$. These assumptions mean
305 that the variables CK and H suffer from measurement errors with small variances. The other aim of these assumptions is to make the proposed method realizable.

All the data are standardized. The bootstrap procedure is repeated 1000 times. We compute the P -values of the proposed method, which are 0.852 and 0.980 for Σ_1 and Σ_2 , respectively. The P -values of the test $\mathcal{T}_n^{\text{direct}}$ in (2.5) are calculated, which are 0.521 and 0.938 for Σ_1 and Σ_2 ,
310 respectively. We also compute the P -value of the method in [13], which is 0.018. Therefore, it is concluded that the varying-coefficient model is not a good choice for the dataset when the measurement errors are ignored. However, when the influence of the measurement errors is eliminated, the varying-coefficient model can capture the relationship between the response and the covariates.

The runtime of the proposed test and the test $\mathcal{T}_n^{\text{direct}}$ with $\Sigma_e = \text{diag}(0.05, 0.05)$ is calculated
315 to be 1.957 seconds and 2.133 seconds, respectively. The runtime of the method of [13] is 2.047 seconds.

We plot residual plots, fitted residual curves and the 95% confidence curves of the estimated residuals of the proposed method and the method of [13] in Figure 3. From Figure 3, we can observe that for the method of [13], the estimated curve of the residuals deviates the abscissa axis. However, for the proposed method, the estimated curves of the residuals for Σ_1 and Σ_2 coincide with the abscissa axis. Clearly, Figure 3 verifies that the test results are reliable.

Here inserts Figure 3

5. Conclusion

In this work, based on an estimated empirical process, we propose a new test statistic, and establish several test procedures, to check the adequacy of the varying-coefficient models when some covariate is measured with error. The asymptotic properties of the test statistic were investigated for various local and omnibus tests. The simulation results indicate that the proposed test procedures are superior to the other two methods and can be applied regardless of whether the covariance of the measurement errors is known or unknown. We admit that this type of test has high power for some alternatives but may have power zero for other directions.

The regression model checking with classical measurement error is very challenging. The existing methods suffer from the heavy computational burden and the incapability to detect some alternative hypothetical models. The proposed method is computationally expedient and theoretically reliable. Though the test method based on the deconvolution is difficult to extend conveniently to semiparametric models, the proposed method can be extended to both semiparametric models and semiparametric models with complex structure. Further studies along this line of work can include the adequacy check of semiparametric regression models with classical measurement errors or with both classical measurement errors and other complex structure such as missingness, censorship or high-dimensional covariates.

Acknowledgment

The authors would like to thank Professor Park, the Associate Editor and two reviewers for their careful review and insightful comments that have led to significant improvement of this article. Liu's research was partially supported by the General Research Grants 15327216, Research Grants Council, Hong Kong. Xie's research was supported by NSFC No. 11771032 and the Science and Technology Project of Beijing Municipal Education Commission No. KM201710005032. Sun's

research was supported by the National Natural Science Foundation of China (Grant Nos. 11571340, U1430103), the Open Project of Key Laboratory of Big Data Mining and Knowledge Management, CAS.

References

- 350 [1] J. Fan, W. Zhang, Statistical methods with varying coefficient models, *Statistics and its Interface* (2008) 179–195.
- [2] B. U. Park, E. Mammen, Y. K. Lee, E. R. Lee, Varying coefficient regression models: a review and new developments, *International Statistical Review* 83 (2015) 36–64.
- [3] P. Hall, Y. Ma, Testing the suitability of polynomial models in errors-in-variables problems, 355 *The Annals of Statistics* 35 (2007) 2620–2638.
- [4] W. Song, Model checking in errors-in-variables regression, *Journal of Multivariate Analysis* 99 (2008) 2406–2443.
- [5] Y. Ma, J. D. Hart, R. Janicki, R. J. Carroll, Local and omnibus goodness-of-fit tests in classical measurement error models, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 73 (2011) 81–98. 360
- [6] G. Xu, S. Wang, A goodness-of-fit test of logistic regression models for case-control data with measurement error, *Biometrika* 98 (2011) 877–886.
- [7] W. González-Manteiga, R. M. Crujeiras, An updated review of goodness-of-fit tests for regression models, *Test* 22 (2013) 361–411.
- 365 [8] Z. Huang, Z. Pang, T. Hu, Testing structural change in partially linear single-index models with error-prone linear covariates, *Computational Statistics & Data Analysis* 59 (2013) 121–133.
- [9] W. Härdle, E. Mammen, Testing parametric versus nonparametric regression, *Annals of Statistics* 21 (1993) 1926–1947.
- 370 [10] Q. Li, S. Wang, A simple consistent bootstrap test for a parametric regression function, *Journal of Econometrics* 87 (1998) 145–165.

- [11] C. Niu, X. Guo, W. Xu, L. Zhu, Checking nonparametric component for partial linear regression model with missing response, *Journal of Statistical Planning and Inference* 168 (2016) 1–19.
- [12] J. You, Y. Zhou, G. Chen, Corrected local polynomial estimation in varying-coefficient models with measurement errors, *Canadian Journal of Statistics* 34 (2006) 391–410.
- 375 [13] W. L. Xu, L. X. Zhu, Goodness-of-fit testing for varying-coefficient models, *Metrika* 68 (2008) 129–146.
- [14] C.-F. J. Wu, Jackknife, bootstrap and other resampling methods in regression analysis, the *Annals of Statistics* 14 (1986) 1261–1295.
- [15] W. Stute, W. G. Manteiga, M. P. Quindimil, Bootstrap approximations in model checks for regression, *Journal of the American Statistical Association* 93 (1998) 141–149.
- 380 [16] R. J. Carroll, D. Ruppert, L. A. Stefanski, C. M. Crainiceanu, *Measurement error in nonlinear models: a modern perspective*, CRC press, 2006.
- [17] H. Liang, W. Härdle, R. J. Carroll, Estimation in a semiparametric partially linear errors-in-variables model, *The Annals of Statistics* 27 (1999) 1519–1535.
- 385 [18] W. K. Härdle, M. Müller, S. Sperlich, A. Werwatz, *Nonparametric and semiparametric models*, Springer Science & Business Media, 2012.
- [19] L. Zhu, K. Ng, Checking the adequacy of a partially linear model, *Statistica Sinica* 13 (2003) 763–781.
- [20] X. Zhu, G. Xu, L. Zhu, An adaptive-to-model test for partially parametric single-index models, *Statistics & Computing* (2015) 1–12.
- 390 [21] Z. Sun, X. Ye, L. Sun, Consistent test of error-in-variables partially linear model with auxiliary variables, *Journal of Multivariate Analysis* 141 (2015) 118–131.
- [22] E. Ziegel, *Data: A collection of problems from many fields for the student and research worker*, *Technometrics* 29 (1987) 502–503.
- 395 [23] D. Nolan, D. Pollard, Functional limit theorems for u-processes, *The Annals of Probability* 16 (1988) 1291–1298.

- [24] M. A. Arcones, B. Yu, Central limit theorems for empirical and u-processes of stationary mixing sequences, *Journal of Theoretical Probability* 7 (1994) 47–71.

Appendix

400 A.1. Assumptions

We begin this section by listing the conditions needed in the proofs of the theorems.

(C.1) The function $\alpha(u)$ has the continuous second derivative related to u .

(C.2) The matrix $Q(u) = E(\mathbf{X}\mathbf{X}^\top | U = u)$ is positive definite; $E|\varepsilon|^{2+\delta} < \infty$ and $E|e|^{2+\delta} < \infty$ with $\delta > 0$.

405 (C.3) The density of U , say $f_u(u)$, exists and satisfies $0 < \inf_u f_u(u) \leq \sup_u f_u(u) < \infty$.

(C.4) $\lambda(\cdot)$ is a bounded kernel function of order $j(\geq 2)$ with bounded support.

(C.5) $nh_n \rightarrow \infty$, $nh_n^4 \rightarrow 0$ and $h_n \rightarrow 0$ as $n \rightarrow \infty$.

Remark 2. Condition (C.1) is needed when Taylor expansion of $\alpha(u)$ is conducted. Condition (C.2) is necessary for the asymptotic normality of the model estimating procedure. Condition (C.3) aims at avoiding tedious proofs of the theorems. Conditions (C.4)–(C.5) are common conditions in the nonparametric estimates.

A.2. Estimation of $\alpha(u)$

Denote $\lambda_h(\cdot) = \lambda(\cdot/h_n)/h_n$, $\omega_u = \text{diag}(\lambda_h(U_1 - u), \dots, \lambda_h(U_n - u))$ and

$$V_u = \begin{pmatrix} \tilde{\mathbf{X}}_1^\top & \frac{U_1 - u}{h_n} \tilde{\mathbf{X}}_1^\top \\ \vdots & \vdots \\ \tilde{\mathbf{X}}_n^\top & \frac{U_n - u}{h_n} \tilde{\mathbf{X}}_n^\top \end{pmatrix}.$$

Similarly to [12], we define an estimator of the regression coefficient function, denoted by $\hat{\alpha}_n(u)$, as follows

$$\hat{\alpha}_n(u) = (\mathbf{I}_p \quad \mathbf{0}) \{V_u^\top \omega_u V_u - \Omega\}^{-1} V_u^\top \omega_u Y, \quad (\text{A.1})$$

where \mathbf{I}_p is $p \times p$ identity matrix and

$$\Omega = \sum_{i=1}^n \Sigma_e \otimes \begin{pmatrix} 1 & (U_i - u)/h_n \\ (U_i - u)/h_n & \{(U_i - u)/h_n\}^2 \end{pmatrix} \lambda_h(U_i - u).$$

Lemma 1. Under Conditions (C.1)-(C.5), under the alternative model (3.2), we have

$$\begin{aligned}
& (nh_n)^{1/2} \{ \hat{\alpha}_n(u) - \alpha(u) - \{ \mathbf{I}_p + Q^{-1}(u) \Sigma_e \} \alpha^{(2)}(u) \mu_2 h_n^2 \} \\
= & Q^{-1}(u) \left\{ \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda \left(\frac{U_i - u}{h_n} \right) \varepsilon_i - \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \mathbf{X}_i e_i^\top \lambda \left(\frac{U_i - u}{h_n} \right) \alpha(U_i)^\top \right\} \\
& + \sqrt{h_n} \mathbf{E} \{ \tilde{\mathbf{X}} \mathcal{D}(\mathbf{X}, U) | U = u \} + o_p(1)
\end{aligned} \tag{A.2}$$

where $\alpha^{(2)}(u)$ is the second derivative of $\alpha(u)$ related to u and $\mu_2 = \int u^2 \lambda(u) du$.

Proof: Step 1. Consider $V_u^\top \omega_u V_u - \Omega$. By some simple computations, we can obtain that

$$\frac{1}{n} V_u^\top \omega_u V_u = \frac{1}{nh_n} \sum_{i=1}^n \begin{pmatrix} \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top & \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top (U_i - u)/h_n \\ \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top (U_i - u)/h_n & \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top ((U_i - u)/h_n)^2 \end{pmatrix} \lambda \left(\frac{U_i - u}{h_n} \right).$$

By Conditions (C.4)-(C.5), we can prove that

$$\begin{aligned}
\frac{1}{nh_n} \sum_{i=1}^n \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top \lambda \left(\frac{U_i - u}{h_n} \right) &= \mathbf{E}[\tilde{\mathbf{X}} \tilde{\mathbf{X}}^\top | U = u] f_u(u) + o_p(1), \\
\frac{1}{nh_n} \sum_{i=1}^n \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top (U_i - u)/h_n \lambda \left(\frac{U_i - u}{h_n} \right) &= o_p(1),
\end{aligned}$$

and

$$\frac{1}{nh_n} \sum_{i=1}^n \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top ((U_i - u)/h_n)^2 \lambda \left(\frac{U_i - u}{h_n} \right) = \mu_2 \mathbf{E}[\tilde{\mathbf{X}} \tilde{\mathbf{X}}^\top | U = u] f_u(u) + o_p(1).$$

By the facts that $\tilde{\mathbf{X}} = \mathbf{X} + \mathbf{e}$ and \mathbf{e} is independent of U , it can be validated that $\mathbf{E}[\tilde{\mathbf{X}} \tilde{\mathbf{X}}^\top | U = u] =$

$\mathbf{E}(\mathbf{X} \mathbf{X}^\top | U = u) + \Sigma_e$. So it follows the result:

$$\frac{1}{n} V_u^\top \omega_u V_u = \{ \mathbf{E}(\mathbf{X} \mathbf{X}^\top | U = u) + \Sigma_e \} \otimes \begin{pmatrix} f_u(u) & 0 \\ 0 & \mu_2 f_u(u) \end{pmatrix} + o_p(1).$$

Similarly, we can prove that

$$\frac{1}{n} \Omega = \Sigma_e \otimes \begin{pmatrix} f_u(u) & 0 \\ 0 & \mu_2 f_u(u) \end{pmatrix} + o_p(1).$$

Therefore we have

$$\frac{1}{n} \{ V_u^\top \omega_u V_u - \Omega \} = \mathbf{E}(\mathbf{X} \mathbf{X}^\top | U = u) \otimes \begin{pmatrix} f_u(u) & 0 \\ 0 & \mu_2 f_u(u) \end{pmatrix} + o_p(1).$$

Step 2. Consider $(nh_n)^{1/2}\{\hat{\alpha}_n(u) - \alpha(u)\}$. Denote $\alpha(u) = (\alpha_1(u), \dots, \alpha_p(u))^\top$ and $\theta(u) = (\alpha_1(u), \dots, \alpha_p(u), h_n\beta_1(u), \dots, h_n\beta_p(u))^\top$, where $\beta_i(u)$ is the derivative of $\alpha_i(u)$ for $i = 1, \dots, p$.

Let $\hat{\theta}_n(u) = \{V_u^\top \omega_u V_u - \Omega\}^{-1} V_u^\top \omega_u Y$. Note that

$$\frac{1}{n} V_u^\top \omega_u Y = \frac{1}{nh_n} \sum_{i=1}^n \begin{pmatrix} \tilde{\mathbf{X}}_i Y_i \\ \tilde{\mathbf{X}}_i Y_i (U_i - u)/h_n \end{pmatrix} \lambda\left(\frac{U_i - u}{h_n}\right)$$

425 By some simple calculations, we can obtain that

$$\begin{aligned} & \sqrt{nh_n}\{\hat{\theta}_n(u) - \theta(u)\} \\ &= \left\{ \frac{1}{n} (V_u^\top \omega_u V_u - \Omega) \right\}^{-1} \frac{\sqrt{h_n}}{\sqrt{n}} \{V_u^\top \omega_u Y - (V_u^\top \omega_u V_u - \Omega)\theta(u)\} \\ &= \left\{ \frac{1}{n} (V_u^\top \omega_u V_u - \Omega) \right\}^{-1} \frac{\sqrt{h_n}}{\sqrt{n}} [\{V_u^\top \omega_u Y - V_u^\top \omega_u V_u \theta(u)\} + \Omega\theta(u)] \\ &=: \left\{ \frac{1}{n} (V_u^\top \omega_u V_u - \Omega) \right\}^{-1} (H_n + M_n). \end{aligned}$$

with

$$H_n =: \begin{cases} \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) [Y_i - \sum_{j=1}^p \{\alpha_j(u) + \beta_j(u)(U_i - u)\} \tilde{X}_{ij}] \\ \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i (U_i - u)/h_n \lambda\left(\frac{U_i - u}{h_n}\right) [Y_i - \sum_{j=1}^p \{\alpha_j(u) + \beta_j(u)(U_i - u)\} \tilde{X}_{ij}] \end{cases}$$

and

$$\begin{aligned} (\mathbf{I}_p \quad \mathbf{0}) \left\{ \frac{1}{\sqrt{nh_n}} (V_u^\top \omega_u V_u - \Omega) \right\}^{-1} M_n &= \sqrt{nh_n} \mathbf{E}^{-1}(\mathbf{X}\mathbf{X}^\top | U = u) \Sigma_e \alpha(u) \\ &+ o_p(1). \end{aligned}$$

Thus it yields

$$\begin{aligned} \sqrt{nh_n}\{\hat{\alpha}_n(u) - \alpha(u)\} &= \mathbf{E}^{-1}(\mathbf{X}\mathbf{X}^\top | U = u) G_n + \mathbf{E}^{-1}[\mathbf{X}\mathbf{X}^\top | U = u] \Sigma_e \alpha(u) \\ &+ o_p(1) \end{aligned} \tag{A.3}$$

with $G_n = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) [Y_i - \sum_{j=1}^p \{\alpha_j(u) + \beta_j(u)(U_i - u)\} \tilde{X}_{ij}]$.

430 **Step 3.** Consider G_n :

$$G_n = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) \{Y_i - \alpha(U_i)^\top \mathbf{X}_i\}$$

$$\begin{aligned}
& + \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) (\alpha(U_i)^\top \mathbf{X}_i - \alpha(U_i)^\top \tilde{\mathbf{X}}_i) \\
& + \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) [\alpha(U_i)^\top \tilde{\mathbf{X}}_i - \sum_{j=1}^p \{\alpha_j(u) + \beta_j(u)(U_i - u)\} \tilde{X}_{ij}] \\
= & \sum_{j=1}^3 G_{nj}
\end{aligned} \tag{A.4}$$

Step 3.1. Consider G_{n1} : Under the alternative model (3.2), we have $Y_i - \alpha(U_i)^\top \mathbf{X}_i = n^{-1/2}D(\mathbf{X}_i, U_i) + \varepsilon_i$ for $i = 1, \dots, n$. Then it yields

$$\begin{aligned}
G_{n1} & = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) [n^{-1/2}D(\mathbf{X}_i, U_i) + \varepsilon_i] \\
& = \sqrt{h_n} \mathbb{E}\{\tilde{\mathbf{X}} \mathcal{D}(\mathbf{X}, U) | U = u\} + \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) \varepsilon_i \\
& \quad + o_p(1).
\end{aligned} \tag{A.5}$$

Step 3.2. Consider G_{n2} :

$$\begin{aligned}
G_{n2} & = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n (\mathbf{X}_i + \mathbf{e}_i) \lambda\left(\frac{U_i - u}{h_n}\right) \alpha(U_i)^\top (\mathbf{X}_i - \tilde{\mathbf{X}}_i) \\
& = -\frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \mathbf{X}_i \mathbf{e}_i^\top \lambda\left(\frac{U_i - u}{h_n}\right) \alpha(U_i) - \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \mathbf{e}_i \mathbf{e}_i^\top \lambda\left(\frac{U_i - u}{h_n}\right) \alpha(U_i) \\
& = -\frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \mathbf{X}_i \mathbf{e}_i^\top \lambda\left(\frac{U_i - u}{h_n}\right) \alpha(U_i)^\top - \sqrt{nh_n} \Sigma_e \alpha(u) + o_p(1).
\end{aligned} \tag{A.6}$$

Step 3.3. Consider G_{n3} :

$$\begin{aligned}
G_{n3} & = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) \sum_{j=1}^p \alpha_j^{(2)}(u) (U_i - u)^2 \tilde{X}_{ij} + o_p(1) \\
& = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i^\top \alpha^{(2)}(u) (U_i - u)^2 \lambda\left(\frac{U_i - u}{h_n}\right) + o_p(1) \\
& = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n (\mathbf{X}_i + \mathbf{e}_i) (\mathbf{X}_i + \mathbf{e}_i)^\top \alpha^{(2)}(u) (U_i - u)^2 \lambda\left(\frac{U_i - u}{h_n}\right) + o_p(1) \\
& = \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n (\mathbf{X}_i \mathbf{X}_i^\top + 2\mathbf{e}_i \mathbf{X}_i^\top + \mathbf{e}_i \mathbf{e}_i^\top) \alpha^{(2)}(u) (U_i - u)^2 \lambda\left(\frac{U_i - u}{h_n}\right) + o_p(1) \\
& = \sqrt{nh_n} \{\mathbb{E}(\mathbf{X}\mathbf{X}^\top | U = u) + \Sigma_e\} \alpha^{(2)}(u) \mu_2 h_n^2 + o_p(1).
\end{aligned} \tag{A.7}$$

435 From (A.4)-(A.7), we have

$$\begin{aligned}
G_n &= \sqrt{h_n} \mathbb{E}\{\tilde{\mathbf{X}} \mathcal{D}(\mathbf{X}, U) | U = u\} + \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) \varepsilon_i \\
&\quad - \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \mathbf{X}_i \mathbf{e}_i^\top \lambda\left(\frac{U_i - u}{h_n}\right) \alpha(U_i)^\top - \sqrt{nh_n} \Sigma_e \alpha(u) \\
&\quad + \sqrt{nh_n} \{\mathbb{E}(\mathbf{X} \mathbf{X}^\top | U = u) + \Sigma_e\} \alpha^{(2)}(u) \mu_2 h_n^2 + o_p(1). \tag{A.8}
\end{aligned}$$

Further by (A.3), we can validate (A.2). \square

Proof of Theorem 2.1 The results of Theorem 2.1 can be obtained from Theorem 3.3 by setting $\mathcal{D}(\mathbf{X}, U) = 0$. We omit the details. \square

Proof of Theorem 3.2 We rewrite $Y_i = g(\mathbf{X}_i, U_i) + \varepsilon_i$ as $Y_i = \alpha(U_i)^\top \mathbf{X}_i + g(\mathbf{X}_i, U_i) - \alpha(U_i)^\top \mathbf{X}_i + \varepsilon_i$ for $i = 1, \dots, n$. Let $\eta_i = g(\mathbf{X}_i, U_i) - \alpha(U_i)^\top \mathbf{X}_i + \varepsilon_i$. By a similar method to prove Lemma 2.1, we can prove that

$$\begin{aligned}
&(nh_n)^{1/2} [\hat{\alpha}_n(u) - \alpha(u) - \{\mathbf{I}_p + Q^{-1}(u) \Sigma_e\} \alpha^{(2)}(u) \mu_2 h_n^2] \\
&= Q^{-1}(u) \left\{ \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \tilde{\mathbf{X}}_i \lambda\left(\frac{U_i - u}{h_n}\right) \eta_i - \frac{1}{\sqrt{nh_n}} \sum_{i=1}^n \mathbf{X}_i \mathbf{e}_i^\top \lambda\left(\frac{U_i - u}{h_n}\right) \alpha(U_i)^\top \right\} \\
&\quad + o_p(1). \tag{A.9}
\end{aligned}$$

By setting $\mathcal{D}(\mathbf{X}_i, U_i) = 0$ and replacing ε_i by η_i for $i = 1, \dots, n$, we can prove that under \mathcal{H}_{1n} in (3.1),

$$\begin{aligned}
\widehat{CR}_n(\mathbf{z}, u) &= n^{-1/2} \sum_{i=1}^n \eta_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\
&\quad + n^{-1/2} \sum_{i=1}^n \alpha(U_i)^\top (\mathbf{X}_i - \tilde{\mathbf{X}}_i) \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\
&\quad - n^{-1/2} \sum_{i=1}^n \tilde{\mathbf{X}}_i \eta_i \mathbb{E}[Q^{-1}(U) \tilde{\mathbf{X}} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u) | U = U_i] + o_p(1). \tag{A.10}
\end{aligned}$$

The proof of (A.10) is similar to the proof of (A.14) which is listed in the following. So we omit the details. Note that under (3.1), $n^{-1/2} \sum_{i=1}^n \eta_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) = \sqrt{n} \mathbb{E}[\eta \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] + o_p(1)$ and $\mathbb{E}[\eta \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] \neq 0$. So $n^{-1/2} \sum_{i=1}^n \eta_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \rightarrow \infty$ as $n \rightarrow \infty$. We can further prove that $n^{-1/2} \sum_{i=1}^n \alpha(U_i)^\top (\mathbf{X}_i - \tilde{\mathbf{X}}_i) \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) = O_p(1)$ and

$n^{-1/2} \sum_{i=1}^n \tilde{\mathbf{X}}_i \eta_i E[Q^{-1}(U) \tilde{\mathbf{X}} \mathbf{1}(\tilde{\mathbf{X}} < \mathbf{x}, U < u) | U = U_i] = n^{1/2} O_p(1)$. Thus we can obtain that $\mathcal{T}_n \rightarrow \infty$ as $n \rightarrow \infty$. And it is natural to obtain that $\Pr\{\mathcal{T}_n > c_\alpha | \mathcal{H}_{1n}\}$ converges to 1 as $n \rightarrow \infty$.

450

□

Proof of Theorem 3.3 We can decompose $\widehat{CR}_n(\mathbf{z}, u)$ into three parts:

$$\begin{aligned} \widehat{CR}_n(\mathbf{z}, u) &= n^{-1/2} \sum_{i=1}^n \{Y_i - \alpha(U_i)^\top \mathbf{X}_i\} \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &\quad + n^{-1/2} \sum_{i=1}^n \alpha(U_i)^\top (\mathbf{X}_i - \tilde{\mathbf{X}}_i) \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &\quad + n^{-1/2} \sum_{i=1}^n \{\alpha(U_i) - \hat{\alpha}(U_i)\}^\top \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &=: \sum_{j=1}^3 \widehat{CR}_{nj}(\mathbf{z}, u). \end{aligned} \tag{A.11}$$

Step 1 Consider $\widehat{CR}_{n1}(\mathbf{z}, u)$. Under \mathcal{H}_{2n} in (3.2), for $\widehat{CR}_{n1}(\mathbf{z}, u)$, we have

$$\begin{aligned} \widehat{CR}_{n1}(\mathbf{z}, u) &= n^{-1/2} \sum_{i=1}^n \{n^{-1/2} \mathcal{D}(\mathbf{X}_i, U_i) + \varepsilon_i\} \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &= n^{-1/2} \sum_{i=1}^n \varepsilon_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) + E[\mathcal{D}(\mathbf{X}, U) \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] \\ &\quad + o_p(1). \end{aligned} \tag{A.12}$$

Step 2 Consider $\widehat{CR}_{n3}(\mathbf{z}, u)$. By employing the results of Lemma 1, it yields

$$\begin{aligned} &\widehat{CR}_{n3}(\mathbf{z}, u) \\ &= n^{-1/2} \sum_{i=1}^n \{\mathbf{I}_p + Q^{-1}(U_i) \Sigma_e\} \alpha^{(2)}(u) \mu_2 h_n^2 \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &\quad + n^{-1/2} \sum_{i=1}^n Q^{-1}(U_i) \frac{1}{nh_n} \sum_{j=1}^n \tilde{\mathbf{X}}_j \lambda\left(\frac{U_j - U_i}{h_n}\right) \varepsilon_j \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &\quad - n^{-1/2} \sum_{i=1}^n Q^{-1}(U_i) \frac{1}{nh_n} \sum_{j=1}^n \tilde{\mathbf{X}}_j \mathbf{e}_j^\top \lambda\left(\frac{U_j - U_i}{h_n}\right) \alpha(U_j)^\top \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &\quad + n^{-1} \sum_{i=1}^n Q^{-1}(U_i) E\{\tilde{\mathbf{X}} \mathcal{D}(\mathbf{X}, U) | U = U_i\} \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) + o_p(1) \\ &=: \sum_{j=1}^4 \widehat{CR}_{n3,j}(\mathbf{z}, u). \end{aligned}$$

Step 2.1 Consider $\widehat{CR}_{n3,1}(\mathbf{z}, u)$. By the law of large numbers, we have $\widehat{CR}_{n3,1}(\mathbf{z}, u) = O_p(\sqrt{nh_n^2})$.

455 By Condition (C5), we can obtain that $\widehat{CR}_{n3,1}(\mathbf{z}, u) = o_p(1)$.

Step 2.2 Consider $\widehat{CR}_{n3,2}(\mathbf{z}, u)$. We first exchange the orders of the summations and then employ the fact that $\frac{1}{nh_n} \sum_{i=1}^n Q^{-1}(U_i) \lambda(\frac{U_j - U_i}{h_n}) \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) = E[Q^{-1}(U) \tilde{\mathbf{X}} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u) | U = U_j] + o_p(1)$. We can obtain that

$$\begin{aligned} \widehat{CR}_{n3,2}(\mathbf{z}, u) &= n^{-1/2} \sum_{j=1}^n \tilde{\mathbf{X}}_j \varepsilon_j \frac{1}{nh_n} \sum_{i=1}^n Q^{-1}(U_i) \lambda(\frac{U_j - U_i}{h_n}) \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &= n^{-1/2} \sum_{j=1}^n \tilde{\mathbf{X}}_j \varepsilon_j E[Q^{-1}(U) \tilde{\mathbf{X}} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u) | U = U_j] + o_p(1). \end{aligned}$$

Step 2.3 Consider $\widehat{CR}_{n3,3}(\mathbf{z}, u)$. Similarly, we can prove that

$$\begin{aligned} &\widehat{CR}_{n3,3}(\mathbf{z}, u) \\ &= -n^{-1/2} \sum_{i=1}^n Q^{-1}(U_i) \frac{1}{nh_n} \sum_{j=1}^n \mathbf{X}_j e_j^\top \lambda(\frac{U_j - U_i}{h_n}) \alpha(U_j)^\top \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &= -n^{-1/2} \sum_{j=1}^n \left\{ \frac{1}{nh_n} \sum_{i=1}^n Q^{-1}(U_i) \tilde{\mathbf{X}}_i^\top \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \lambda(\frac{U_j - U_i}{h_n}) \right\} \alpha(U_j) \mathbf{X}_j e_j^\top \\ &= -n^{-1/2} \sum_{j=1}^n E[Q^{-1}(U) \tilde{\mathbf{X}}^\top \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u) | U = U_j] \alpha(U_j) \mathbf{X}_j e_j^\top + o_p(1). \end{aligned}$$

460 **Step 2.4** Consider $\widehat{CR}_{n3,3}(\mathbf{z}, u)$. By the law of large numbers, it can be validated that

$$\begin{aligned} \widehat{CR}_{n3,4}(\mathbf{z}, u) &= n^{-1} \sum_{i=1}^n Q^{-1}(U_i) E\{\tilde{\mathbf{X}} \mathcal{D}(\mathbf{X}, U) | U = U_i\} \tilde{\mathbf{X}}_i \mathbf{1}(\mathbf{X}_{2i} < \mathbf{z}, U_i < u) \\ &=: E[Q^{-1}(U) E\{\tilde{\mathbf{X}} \mathcal{D}(\mathbf{X}, U) | U = u\} \tilde{\mathbf{X}} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] + o_p(1). \end{aligned}$$

Thus we have

$$\begin{aligned} &\widehat{CR}_{n3}(\mathbf{z}, u) \\ &= n^{-1/2} \sum_{j=1}^n \tilde{\mathbf{X}}_j \varepsilon_j E[Q^{-1}(U) \tilde{\mathbf{X}} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u) | U = U_j] \\ &\quad - n^{-1/2} \sum_{j=1}^n E[Q^{-1}(U_j) \tilde{\mathbf{X}}_j^\top \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u) | U = U_j] \alpha(U_j) \mathbf{X}_j e_j^\top \\ &\quad + E[Q^{-1}(U) E\{\tilde{\mathbf{X}} \mathcal{D}(\mathbf{X}, U) | U = u\} \tilde{\mathbf{X}} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] + o_p(1). \end{aligned} \tag{A.13}$$

Further we can obtain that

$$\begin{aligned} \widehat{CR}_n(\mathbf{z}, u) &= n^{-1/2} \sum_{i=1}^n \mathcal{IF}_{\mathbf{z}, u}(\mathbf{X}_i, \tilde{\mathbf{X}}_i, U_i, Y_i, \varepsilon_i, \mathbf{e}_i) \\ &\quad + \Delta(\mathbf{z}, u) + o_p(1) \end{aligned} \tag{A.14}$$

with $\mathcal{IF}_{\mathbf{z}, u}(\mathbf{X}, \tilde{\mathbf{X}}, U, Y, \varepsilon, e)$ and Δ defined in Sections 2 and 3. Because the indicator function is monotone, it is easy to prove that $G_{\mathbf{z}, u} = \{\mathcal{IF}_{\mathbf{z}, u}(\mathbf{X}, \tilde{\mathbf{X}}, U, Y, \varepsilon, e) : \mathbf{z} \in \mathcal{R}^p, u \in \mathcal{R}\}$ is a V-C class of functions. See [23]. By Theorem 3.1 of [24], we can show that $\widehat{CR}_n(\mathbf{z}, u)$ converges to a Gaussian process. Further by the continuous mapping theorem, we have proved the result for \mathcal{T}_n . \square

Proof of Theorem 3.4 We rewrite \mathcal{H}_{3n} as

$$Y_i = \alpha(U_i)^\top \mathbf{X}_i + n^{-1/2} \{n^{1/2} B_n D(\mathbf{X}_i, U_i)\} + \varepsilon_i, \quad i = 1, 2, \dots, n$$

and take $n^{1/2} B_n D(\mathbf{X}_i, U_i)$ as the new deviation function. By the similar method to prove (A.14), we can prove that

$$\widehat{CR}_n(\mathbf{z}, u) = n^{-1/2} \sum_{i=1}^n \mathcal{IF}_{\mathbf{z}, u}(\mathbf{X}_i, \tilde{\mathbf{X}}_i, U_i, Y_i, \varepsilon_i, \mathbf{e}_i) + \Delta^* + o_p(1).$$

with $\Delta^* = n^{1/2} B_n E[Q^{-1}(U) E\{\tilde{\mathbf{X}} D(\mathbf{X}, U) | U = u\} \tilde{\mathbf{X}} \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)] + n^{1/2} B_n E[D(\mathbf{X}, U) \mathbf{1}(\mathbf{X}_2 < \mathbf{z}, U < u)]$. Therefore we have that $E[\widehat{CR}_n(\mathbf{z}, u)]$ converges to ∞ , as $n \rightarrow \infty$. Furthermore, it can be proved that the test statistic \mathcal{T}_n converges to ∞ , as $n \rightarrow \infty$. \square

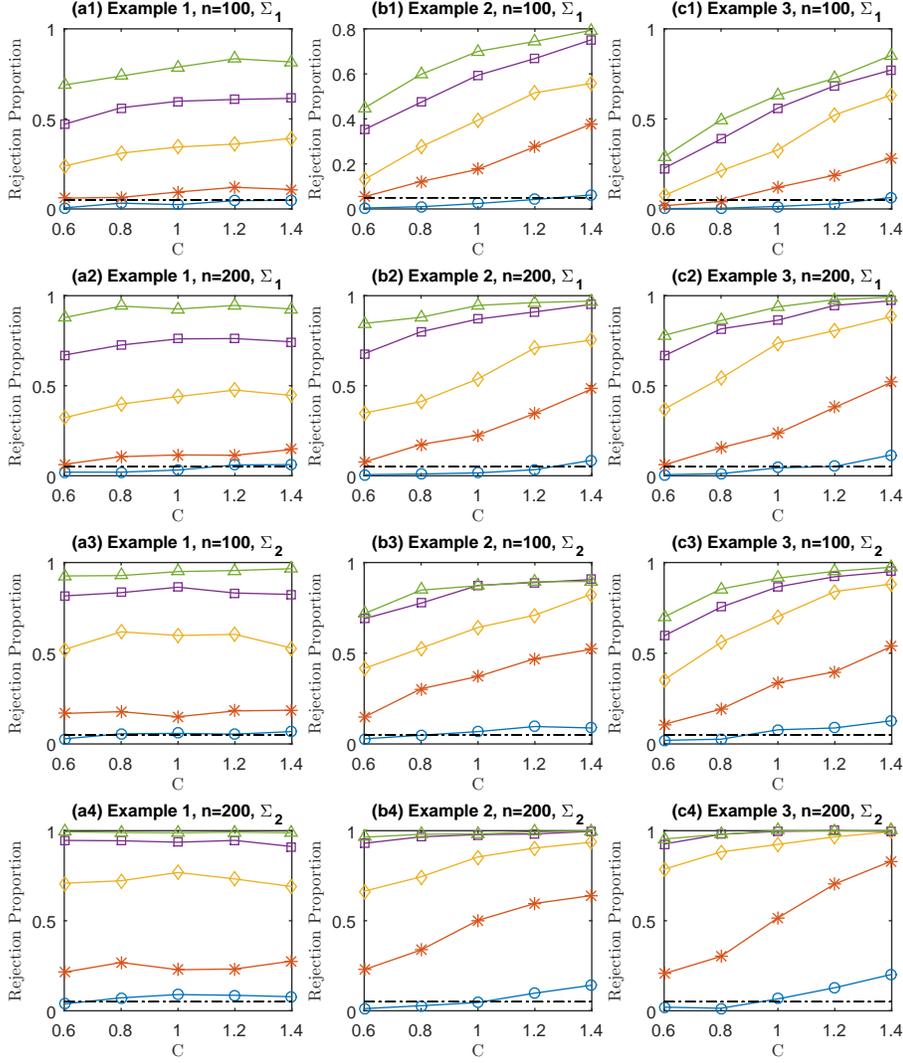


Figure 1: The solid-circle lines ($\text{---}\circ\text{---}$): the empirical sizes curves; the solid-asterisk lines ($\text{---}\ast\text{---}$), the solid-diamond lines ($\text{---}\diamond\text{---}$), the solid-square lines ($\text{---}\square\text{---}$), the solid-triangle lines ($\text{---}\triangle\text{---}$) are the empirical powers curves corresponding to the four values of d from small to large as shown in the descriptions of the Examples 4.1.1-4.1.3; the dash-dotted lines are the the horizontal line with the ordinate value 0.05. The horizontal ordinate C denotes the coefficient of the bandwidth $h_n = C\hat{\sigma}(U)n^{-1/5}$.

Table 1: Frequencies of rejecting the null hypothesis in Example 4.1.1 under different sample sizes and test levels. \mathcal{T}_n : the proposed test; \mathcal{T}_n^{direct} : the direct test defined in (2.5); \mathcal{T}_n^{XZ} : the test of Xu and Zhu (2008).

Σ_e	known						unknown					
	n	100	200	100	200		100	200		100	200	
d_0	\mathcal{T}_n	\mathcal{T}_n^{direct}	\mathcal{T}_n^{XZ}									
$\alpha=0.05$												
0.0 Σ_1	0.046	0.086	0.218	0.050	0.216	0.432	0.040	0.130	0.244	0.044	0.226	0.382
0.5	0.100	0.302	0.544	0.118	0.576	0.776	0.140	0.330	0.548	0.120	0.522	0.758
1.5	0.372	0.718	0.924	0.440	0.936	0.986	0.378	0.760	0.920	0.452	0.918	0.978
2.5	0.582	0.878	0.982	0.750	0.988	0.998	0.604	0.906	0.988	0.726	0.982	0.998
4.0	0.778	0.952	0.998	0.932	0.994	1.000	0.782	0.942	0.998	0.944	0.998	1.000
0.0 Σ_2	0.064	0.134	0.174	0.062	0.198	0.262	0.062	0.122	0.158	0.058	0.158	0.216
0.5	0.190	0.344	0.462	0.250	0.556	0.606	0.204	0.360	0.454	0.242	0.524	0.598
1.5	0.624	0.846	0.874	0.756	0.972	0.984	0.586	0.864	0.914	0.756	0.960	0.978
2.5	0.866	0.982	0.982	0.932	0.998	0.998	0.862	0.976	0.986	0.934	0.996	0.998
4.0	0.958	0.994	0.998	0.996	1.000	1.000	0.958	0.980	0.992	0.988	1.000	1.000
$\alpha=0.1$												
0.0 Σ_1	0.078	0.150	0.348	0.084	0.312	0.578	0.068	0.204	0.384	0.074	0.310	0.514
0.5	0.142	0.412	0.706	0.162	0.694	0.866	0.194	0.438	0.694	0.156	0.650	0.850
1.5	0.454	0.796	0.968	0.512	0.956	0.996	0.446	0.810	0.948	0.536	0.952	0.988
2.5	0.666	0.904	0.996	0.802	0.992	1.000	0.676	0.924	0.996	0.782	0.988	0.998
4.0	0.824	0.952	0.998	0.942	0.994	1.000	0.806	0.948	0.998	0.962	0.998	1.000
0.0 Σ_2	0.112	0.226	0.284	0.106	0.332	0.392	0.118	0.220	0.286	0.110	0.272	0.348
0.5	0.260	0.486	0.636	0.344	0.686	0.750	0.310	0.514	0.626	0.326	0.648	0.750
1.5	0.684	0.916	0.944	0.820	0.984	0.994	0.672	0.918	0.960	0.834	0.980	0.990
2.5	0.892	0.990	0.988	0.960	1.000	1.000	0.902	0.986	0.992	0.966	1.000	1.000
4.0	0.970	0.994	1.000	1.000	1.000	1.000	0.966	0.984	0.996	0.998	1.000	1.000

Table 2: Frequencies of rejecting the null hypothesis in Example 4.1.2 under different sample sizes and test levels. \mathcal{T}_n : the proposed test; \mathcal{T}_n^{direct} : the direct test in (2.5); \mathcal{T}_n^{XZ} : the test of Xu and Zhu (2008).

Σ_e	known						unknown					
	n	100			200			100			200	
d_0	\mathcal{T}_n	\mathcal{T}_n^{direct}	\mathcal{T}_n^{XZ}									
$\alpha=0.05$												
0.0 Σ_1	0.032	0.076	0.202	0.034	0.122	0.292	0.026	0.064	0.190	0.032	0.150	0.312
0.1	0.262	0.346	0.678	0.202	0.606	0.794	0.206	0.354	0.598	0.220	0.548	0.808
0.2	0.464	0.586	0.870	0.570	0.856	0.976	0.430	0.582	0.852	0.556	0.844	0.972
0.5	0.668	0.760	0.972	0.884	0.962	1.000	0.622	0.726	0.970	0.860	0.960	1.000
1.0	0.756	0.810	0.988	0.932	0.976	1.000	0.694	0.788	0.968	0.928	0.974	1.000
0.0 Σ_2	0.060	0.092	0.184	0.052	0.144	0.204	0.056	0.120	0.146	0.042	0.144	0.186
0.1	0.414	0.452	0.592	0.462	0.654	0.738	0.416	0.494	0.570	0.498	0.670	1.732
0.2	0.686	0.722	0.824	0.868	0.918	0.972	0.710	0.722	0.828	0.818	0.926	0.952
0.5	0.872	0.898	0.962	0.968	0.990	0.998	0.872	0.896	0.976	0.984	1.000	1.000
1.0	0.900	0.910	0.972	0.996	0.996	1.000	0.888	0.898	0.974	0.994	1.000	1.000
$\alpha=0.1$												
0.0 Σ_1	0.054	0.142	0.320	0.052	0.202	0.430	0.056	0.148	0.322	0.054	0.226	0.454
0.1	0.332	0.480	0.806	0.292	0.698	0.890	0.288	0.492	0.722	0.322	0.678	0.892
0.2	0.544	0.700	0.946	0.652	0.912	0.992	0.500	0.688	0.942	0.634	0.918	0.992
0.5	0.728	0.820	0.994	0.914	0.972	1.000	0.690	0.792	0.990	0.894	0.970	1.000
1.0	0.800	0.858	0.998	0.944	0.978	1.000	0.762	0.834	0.994	0.948	0.976	1.000
0.0 Σ_2	0.106	0.146	0.224	0.096	0.234	0.296	0.110	0.192	0.234	0.098	0.240	0.302
0.1	0.542	0.622	0.762	0.590	0.786	0.846	0.516	0.652	0.732	0.600	0.806	0.824
0.2	0.786	0.822	0.922	0.920	0.972	0.986	0.786	0.834	0.920	0.874	0.968	0.984
0.5	0.914	0.944	0.994	0.980	0.994	1.000	0.924	0.956	0.994	0.992	1.000	1.000
1.0	0.930	0.948	0.998	1.000	0.996	1.000	0.928	0.942	0.990	1.000	1.000	1.000

Table 3: Frequencies of rejecting the null hypothesis in Example 4.1.3 under different sample sizes and test levels. \mathcal{T}_n : the proposed test; \mathcal{T}_n^{direct} : the direct test in (2.5); \mathcal{T}_n^{XZ} : the test of Xu and Zhu (2008).

Σ_e	known						unknown					
	n	100			200			100			200	
d_0	\mathcal{T}_n	\mathcal{T}_n^{direct}	\mathcal{T}_n^{XZ}									
$\alpha=0.05$												
0.0 Σ_1	0.030	0.024	0.190	0.036	0.082	0.298	0.032	0.038	0.206	0.034	0.082	0.316
0.2	0.158	0.146	0.448	0.218	0.352	0.754	0.164	0.138	0.444	0.236	0.374	0.726
0.5	0.406	0.374	0.800	0.642	0.732	0.990	0.424	0.332	0.814	0.628	0.734	0.986
1.0	0.622	0.520	0.950	0.916	0.880	1.000	0.618	0.524	0.964	0.888	0.906	0.998
2.0	0.668	0.602	0.988	0.950	0.924	1.000	0.676	0.558	0.992	0.928	0.918	1.000
0.0 Σ_2	0.056	0.054	0.126	0.050	0.102	0.176	0.058	0.078	0.120	0.048	0.094	0.168
0.2	0.380	0.284	0.382	0.524	0.480	0.668	0.318	0.244	0.376	0.496	0.500	0.650
0.5	0.762	0.622	0.782	0.944	0.922	0.986	0.772	0.550	0.800	0.942	0.922	0.982
1.0	0.894	0.802	0.954	0.992	0.984	1.000	0.896	0.782	0.966	0.984	0.978	1.000
2.0	0.946	0.880	0.982	0.996	0.998	1.000	0.922	0.888	0.974	0.998	1.000	1.000
$\alpha=0.1$												
0.0 Σ_1	0.060	0.052	0.316	0.062	0.126	0.444	0.056	0.070	0.302	0.058	0.152	0.440
0.2	0.214	0.218	0.596	0.300	0.442	0.852	0.206	0.194	0.592	0.300	0.466	0.824
0.5	0.450	0.424	0.866	0.710	0.780	0.998	0.474	0.400	0.906	0.682	0.784	0.992
1.0	0.664	0.586	0.974	0.930	0.892	1.000	0.662	0.586	0.986	0.898	0.918	1.000
2.0	0.692	0.664	0.994	0.952	0.926	1.000	0.702	0.632	1.000	0.934	0.922	1.000
0.0 Σ_2	0.114	0.120	0.232	0.102	0.184	0.290	0.116	0.146	0.216	0.100	0.154	0.304
0.2	0.478	0.376	0.558	0.604	0.598	0.808	0.430	0.352	0.546	0.564	0.632	0.786
0.5	0.806	0.712	0.862	0.960	0.944	0.992	0.808	0.648	0.874	0.962	0.952	0.992
1.0	0.906	0.844	0.980	0.994	0.984	1.000	0.902	0.836	0.988	0.990	0.980	1.000
2.0	0.950	0.914	0.990	1.000	0.998	1.000	0.936	0.914	0.990	1.000	1.000	1.000

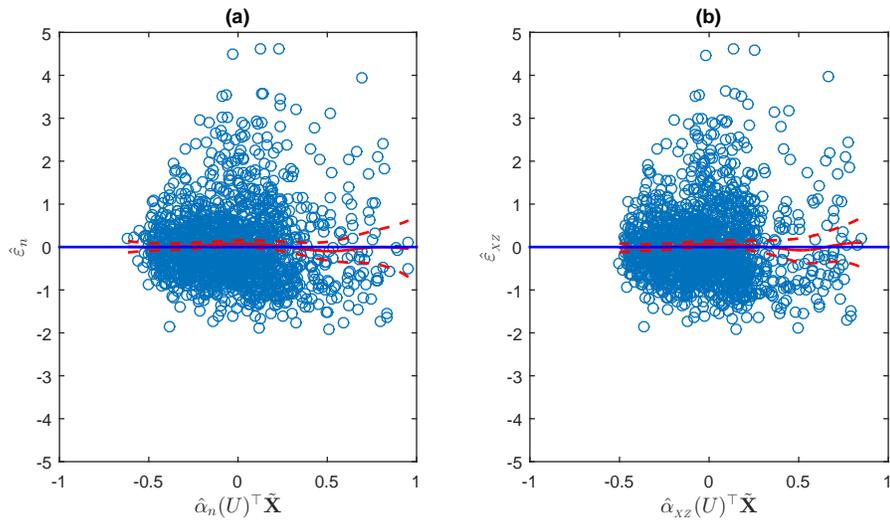


Figure 2: (a) Scatter plot of the calibrated model error estimator $\hat{\epsilon}_n$ versus $\hat{\alpha}_n^\top(U)\tilde{\mathbf{X}}$ for Example 4.2.1 in Section 4; (b) Scatter plot of the error estimator $\hat{\epsilon}_{XZ}$ versus $\hat{\alpha}_{XZ}^\top(U)\tilde{\mathbf{X}}$ for Example 4.2.1 in Section 4 for the method of [13].

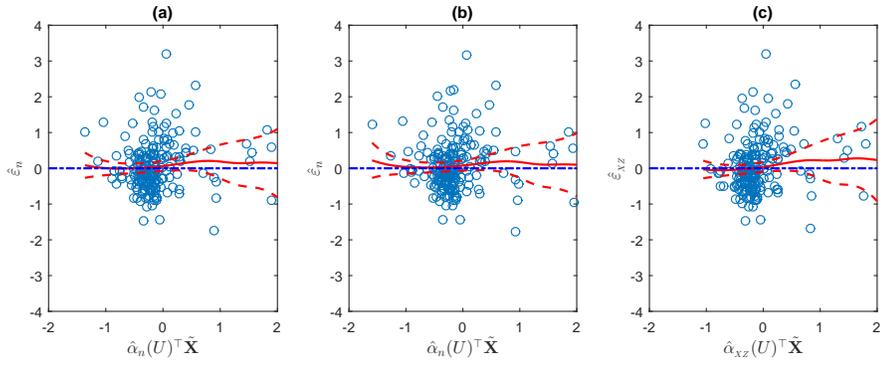


Figure 3: (a) Scatter plot of the calibrated model error estimator $\hat{\varepsilon}_n$ versus $\hat{\alpha}_n^\top(U)\tilde{\mathbf{X}}$ with $\Sigma_1 = \text{diag}(0.05, 0.05)$ for Example 4.2.2 in Section 4; (b) Scatter plot of the calibrated model error estimator $\hat{\varepsilon}_n$ versus $\hat{\alpha}_n^\top(U)\tilde{\mathbf{X}}$ with $\Sigma_2 = \text{diag}(0.07, 0.07)$ for Example 4.2.2 in Section 4; (c) Scatter plot of the model error estimator $\hat{\varepsilon}_{XZ}$ versus $\hat{\alpha}_{XZ}(U)^\top \tilde{\mathbf{X}}$ for Example 4.2.2 in Section 4 for the method of [13].