

# A Convex Programming Solution Based Debiased Estimator for Quantile with Missing Response and High-dimensional Covariables

Miaomiao Su and Qihua Wang

*Academy of Mathematics and Systems Science, Chinese Academy of Sciences*

*University of Chinese Academy of Sciences*

E-mail: [smm@amss.ac.cn](mailto:smm@amss.ac.cn); [qhwang@amss.ac.ac.cn](mailto:qhwang@amss.ac.ac.cn)

*Abstract:*

This paper is concerned with the estimating problem of response quantile with high dimensional covariates when response is missing at random. Some existing methods define root- $n$  consistent estimators for the response quantile. But these methods require correct specifications of both the conditional distribution of response given covariates and the selection probability function. In this paper, a debiased method is proposed by solving a convex programming. The estimator obtained by the proposed method is asymptotically normal given a correctly specified parametric model for the condition distribution function, without the requirement to specify the selection probability function. Moreover, the proposed estimator can be asymptotically more efficient than the existing estimators. The proposed method is evaluated by a simulation study and is illustrated by a real data example.

*Key words and phrases:* high dimension, missing at random, marginal response quantile, optimal weights, selection probability function

## 1. Introduction

The estimation and inference problem with missing responses is an important topic in statistics and has been studied extensively. It may define a biased estimator and lead to a loss of efficiency by simply ignoring the subjects with missing responses. This inspires the development of some approaches, including the imputation, inverse probability weighting and doubly robust methods. See, for example, Rosenbaum and Rubin (1983), Hahn (1998), Hirano et al. (2003), Cao et al. (2009), Rotnitzky et al. (2012), Firpo (2007), Wang and Qin (2010), Hu et al. (2011), Zhang et al. (2011), and Markus and Blaise (2013). Many early literature established asymptotic theory on estimation and inference problem with missing responses in the classical setting where the dimension of covariable vector is a constant. However, the asymptotic results established in the classical setting may not hold in the high dimensional setting when dimension  $p$  of the covariable vector diverges with the sample size  $n$  and even possibly is larger than  $n$ . With the help of some important techniques in high dimension, such as the Lasso (Tibshirani (1996)), adaptive-lasso (Zou (2006)), elastic-net lasso (Zou and Hastie (2005)), there has been considerable recent developments on estimation and inference problems when the response is missing at random with high dimensional covariates.

There are some important advances for statistical inference on the response mean. Farrell (2015) extended the augmented inverse probability weighting approach in classical setting to high dimensions by incorporating the regularized penalization to both the outcome regression model and the selection probability function simultaneously, and proposed an asymptotically normal estimator for the response mean. Although the estimator proposed by Farrell (2015) can be used to make inference on the response mean, it crucially relies on correct specifications of both the outcome regression model and the selection probability function. To alleviate the conditions on model specification of unknown functions, Athey et al. (2018) proposed an approximate residual balancing debiasing method and obtained a  $\sqrt{n}$ -consistent estimator for the response mean with a correctly specified linear model on the outcome regression model without the requirement to specify the selection probability function. However, it is somewhat unfortunate that the asymptotic normality has not been proved for the estimator. It should be mentioned that the debiasing techniques are also used by Javanmard and Montanari (2014), Van de Geer et al. (2014) and Zhang and Zhang (2014) for estimating regression coefficients.

Another important issue is the estimation of the marginal response quantile. In this paper, we consider the estimation of the marginal re-

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sponse quantile with response missing at random and high dimensional covariate vector. There are many researches focusing on quantile regression (He et al., 2016; Belloni et al., 2019; Pietrosanu et al., 2021; Han et al., 2019) with low dimensional or high dimensional covariate vector. However, one can not resort to the conditional quantile regression to obtain an estimator for the marginal response quantile directly. To our knowledge, the only estimator for the marginal response quantile that is shown to be  $\sqrt{n}$  asymptotically normal is proposed in Belloni et al. (2017). The  $\sqrt{n}$ -consistency of the estimator in Belloni et al. (2017) needs that both the conditional distribution of the response given covariates and the selection probability function are correctly specified. This motivates us to propose a new method for estimation of the marginal response quantile. This method defines an asymptotically normal estimator for the response quantile in the setting where the condition distribution function is assumed to be a correctly specified parametric model, without the requirement to specify the selection probability function. Moreover, the proposed estimator can be asymptotically more efficient since the asymptotic variance of the proposed estimator is less than or equal to that in Belloni et al. (2017). This method consists of the following three steps. First, we assume a single index model for the conditional distribution of the response given covariates and es-

establish a conditional distribution based estimating equation. Second, we make an adjustment on the equation by adding the difference between the weighted empirical distribution and the weighted conditional distribution to the estimating function. Third, we solve the adjusted estimation equation to obtain the proposed estimator. The weight in the second step is obtained by solving a convex programming which makes the variance of the proposed estimator attain minimum and constrains its bias such that it is  $\sqrt{n}$ -consistent.

The rest of this paper is organized as follows. In Section 2, we develop a debiased estimating method by solving a convex programming. All assumptions and asymptotic properties are stated in Section 3. Section 4 provides an equivalent easy-to-implement method for calculating the weights. Section 5 presents some simulation studies to examine the finite sample performance of the proposed method. The real data application is reported in Section 6. Outlines of the proofs of the main theorems are presented in the Appendix and the technical details are relegated to the supplementary material.

## 2. Methodology

We first introduce some frequently used notations. For positive sequences  $a_n$  and  $b_n$ , let  $b_n \asymp a_n$  denote  $\lim_{n \rightarrow \infty} a_n^{-1} b_n = c$  for a positive constant  $c$ . Let  $b_n \gtrsim a_n$  denote  $\lim_{n \rightarrow \infty} a_n^{-1} b_n \geq c$  and  $b_n \lesssim a_n$  denote  $\lim_{n \rightarrow \infty} a_n^{-1} b_n \leq c$ . For a  $p$ -dim vector  $v$ , we let  $\|v\|_0$  denote the number of non-zero elements in  $v$ ,  $\|v\|_\infty = \max\{|v_1|, \dots, |v_p|\}$ ,  $\|v\|_q = (\sum_{i=1}^p |v_i|^q)^{1/q}$  for  $1 \leq q < \infty$  and  $|v| = (|v_1|, \dots, |v_p|)^\top$ . For two  $p$ -dim vectors  $u$  and  $v$ , let  $u^\top v = \sum_{j=1}^p u_j v_j$  and  $u \odot v = (u_1 v_1, \dots, u_p v_p)^\top$ . For a matrix  $A \in \mathbb{R}^{m \times n}$ ,  $\|A\|_\infty$  denotes the largest absolute value of the elements in  $A$ . For a  $p$ -dim vector  $v$  and a set  $S \subseteq \{1, \dots, p\}$ , let  $|S|$  denote the number of elements in  $S$ ,  $v_S$  the vector in which  $v_{Sj} = v_j$  if  $j \in S$ ,  $v_{Sj} = 0$  if  $j \notin S$ , and  $v_{-S}$  the vector removing the elements of  $v$  corresponding to the index in  $S$ .

Let  $Y$  be the response variable and  $X$  the  $p$  dimension covariable vector. The population  $\tau$ -quantile of  $Y$  is defined as

$$q_0(\tau) \equiv \inf\{y : F_Y(y) \geq \tau\},$$

where  $F_Y(\cdot)$  is the distribution function of  $Y$  and  $0 < \tau < 1$  is a constant. For ease of notation, we write  $q_0(\tau)$  to be  $q_0$  hereafter. We consider the case where  $X$  is always observed and  $Y$  is missing. Let  $\delta$  denote the binary missing indicator for  $Y$ ; that is,  $\delta = 1$  if  $Y$  is observed; otherwise,

$\delta = 0$ . Throughout this paper, we assume that the response is missing at random, that is,  $P(\delta = 1 | Y, X) = P(\delta = 1 | X)$ , a commonly used missing mechanism in literature of statistical inference with missing data (Rosenbaum and Rubin (1983)). Statistical inference on  $q_0$  cannot proceed without further restrictions when  $p$  diverges with  $n$ . Hence, we impose the following structure on the conditional density of response given covariates  $f_{Y|X}(y | X = x) = f(y, x^\top \beta)$  with  $f$  being a known function and  $\beta$  a  $p$ -dim model parameter vector. The true unknown parameter vector is denoted by  $\beta^0$ . Although  $p$  may be larger than  $n$ , it is often true that only  $\|\beta^0\|_0$  among  $p$  covariates have nonzero coefficients and  $\|\beta^0\|_0$  is fixed or diverges much slower than  $n$ . The corresponding conditional distribution function is denoted by  $h(y, x^\top \beta) = \int_{-\infty}^y f(u, x^\top \beta) du$ . Without loss of generality, we assume that the intercept is zero and all covariates are centered (See, for example, Section 2.2 in Buhlmann and van de Geer (2011), page 8).

Suppose we have  $n$  independent and identically distributed observations

$$(X_i, \delta_i, Y_i), i = 1, \dots, n,$$

where some of  $Y_i$ 's are missing and  $X_i$ 's are completely observed. It is noted that  $E[h(y, X^\top \beta^0)] = F_Y(y)$ . A natural way to estimate  $q_0$  is to solve

$$\frac{1}{n} \sum_{i=1}^n h(q, X_i^\top \hat{\beta}) = \tau, \quad (2.1)$$

where  $\hat{\beta}$  is given by the following Lasso method with complete case (CC) analysis

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} - \sum_{i=1}^n \delta_i \log(f(Y_i, X_i^\top \beta)) + \lambda \|\beta\|_1 \quad (2.2)$$

and  $\lambda \asymp \sqrt{\log(p)/n}$ . The solution of (2.1) is denoted by  $\tilde{q}$ .  $\tilde{q}$  is typically not a  $\sqrt{n}$ -consistent estimator of  $q_0$  since  $\hat{\beta}$  is generally not  $\sqrt{n}$ -consistent due to its high dimensionality. Therefore, to define a  $\sqrt{n}$ -consistent estimator for  $q_0$ , we consider a modified estimation equation of (2.1), which is given as follows

$$\frac{1}{n} \sum_{i=1}^n h(q, X_i^\top \hat{\beta}) + \sum_{\{i:\delta_i=1\}} w_i \left\{ I[Y_i \leq q] - h(q, X_i \hat{\beta}) \right\} = \tau, \quad (2.3)$$

where  $w_i$  for  $1 \leq i \leq n$  are data-dependent weights. The exact solution of (2.3) may not exist due to the non-smoothness of the estimating function. However, just as discussed in Han et al. (2019) and Zhang et al. (2011), any value of  $q$  that minimizes the absolute value of the difference between two sides of (2.3) can be taken as our estimate and the practical impact of this arbitrariness are negligible in large samples. With a given  $w = (w_1, w_2, \dots, w_n)$ , an estimator for  $q_0$  can be obtained by solving (2.3), which is denoted by  $\hat{q}_w$ . If  $w_i$  for  $1 \leq i \leq n$  are taken to be zero, we have  $\hat{q}_w = \tilde{q}$ , which is not  $\sqrt{n}$ -consistent. If the selection probability function  $\pi(x) = E[\delta | X = x]$  is specified by a parametric model  $g(x^\top \gamma)$  and then  $w_i$

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is taken as the inverse of  $g(X_i^\top \hat{\gamma})$  with  $\hat{\gamma}$  obtained by the lasso method, the resulting estimator is the augmented inverse probability weighted estimator in Belloni et al. (2017). When only one of  $f_{Y|X}(Y | X = x)$  and  $\pi(x)$  is correctly specified, the augmented inverse probability weighted estimator may not be  $\sqrt{n}$ -consistent to  $q_0$  due to the same reason as  $\tilde{q}$ , where  $\hat{\beta}$  and  $\hat{\gamma}$  are generally not  $\sqrt{n}$ -consistent due to high dimension of  $X$ . This is different from the classical case where the dimension of  $X$  is a constant. However, its asymptotic normality is proved when both  $f_{Y|X}(Y | X = x)$  and  $\pi(x)$  are correctly specified. Clearly, the weights play a crucial role for the asymptotic property of the adjusted estimating equation estimator defined by (2.3). Hence, we propose a debiased method by constructing optimal weights, which are obtained by solving a convex programming, such that not only the resulting estimator is asymptotically normal but also its asymptotic variance attains minimum. This method hence may define an asymptotically more efficient estimator compared to the existing ones and avoids the requirement to specify  $\pi(x)$ .

This method consists of the following steps:

Step 1 Calculate pilot estimators  $\hat{\beta}$  and  $\tilde{q}$  by solving (2.2) and (2.1), respectively.

Step 2 With pilot estimators  $\hat{\beta}$  and  $\tilde{q}$ , construct the debiasing weight  $\hat{w}$  as

follows

$$\hat{w} = \underset{w}{\operatorname{argmin}} \sum_{\{i:\delta_i=1\}} w_i^2 h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta})) \quad (2.4)$$

$$\text{s.t.} \quad \left\| \frac{1}{n} \sum_{i=1}^n \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i - \sum_{\{i:\delta_i=1\}} w_i \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i \right\|_\infty \leq \Delta, \quad (2.5)$$

$$\sum_{\{i:\delta_i=1\}} w_i = 1, \quad (2.6)$$

where  $\Delta$  is a suitable tuning parameter tending to zero.

Step 3 Replace  $w$  in (2.3) by  $\hat{w}$  and solve the following modified equation

$$\frac{1}{n} \sum_{i=1}^n h(q, X_i^\top \hat{\beta}) + \sum_{\{i:\delta_i=1\}} \hat{w}_i \left\{ I[Y_i \leq q] - h(q, X_i \hat{\beta}) \right\} = \tau. \quad (2.7)$$

The proposed estimator  $\hat{q}$  is the solution of (2.7). The optimization of the convex programming in Step 2 is to obtain weights such that  $\hat{q}$  is root- $n$  consistent, and its asymptotic variance attains minimum which is discussed in the following section.

### 3. Asymptotic Properties

In order to investigate asymptotic properties of the proposed estimator, we first list the following conditions.

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(A.1) (i) The parameter spaces  $\mathcal{Q}$  for  $q$  is compact, and  $q_0$  is in the interior of  $\mathcal{Q}$ .

(ii)  $Y$  is a real value continuous random variable with strictly increasing cumulative distribution function.

(A.2)  $\inf_x \pi(x) > 0$ .

(A.3) (i)  $0 < \inf_x |h(q_0, x^\top \beta^0)| \leq \sup_x |h(q_0, x^\top \beta^0)| < 1$ .

(ii)  $\sup_{(q,u)} |\dot{h}_u(q, u)| < \infty$ , where  $\dot{h}_u(q, u) = \partial h(q, u) / \partial u$ .

(iii) There exists positive constants  $L_1$  and  $L_2$  such that  $|\dot{h}_u(q_1, u_1) - \dot{h}_u(q_2, u_2)| \leq L_1 (|q_1 - q_2| + |u_1 - u_2|)$  and  $|f(q_1, u_1) - f(q_2, u_2)| \leq L_2 (|q_1 - q_2| + |u_1 - u_2|)$ .

(A.4) (i)  $\max_{1 \leq j \leq p} |X_j| \leq K$  almost surely, where  $K$  is a positive constant.

(ii)  $\Sigma \equiv E[XX^\top] \in \mathbb{R}^{p \times p}$  satisfies that  $\Lambda_{\max} \leq c_{\max} < \infty$  and  $\Lambda_{\min} \geq c_{\min} > 0$ , where  $\Lambda_{\min}$  and  $\Lambda_{\max}$  are the smallest and the largest eigenvalues of  $\Sigma$  respectively, and  $c_{\max}$  and  $c_{\min}$  are positive constants.

(A.5) Let  $D = (X, Y, \delta)$  and  $\rho_\beta(x, \tilde{y}) = -\delta \log(f(y, x^\top \beta))$  with  $\tilde{y} = (y, \delta)$  be a loss function which is assumed to be a convex function in  $\beta$ . (i) There exists positive constants  $c$  and  $c_m$  such that  $E[\rho_\beta(D) - \rho_{\beta^0}(D)] \geq$

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$c_m \|\beta - \beta^0\|_2^2$  holds for all  $\beta$  with  $\|\beta - \beta^0\|_1 \leq c$ . (ii) For all  $\beta$  and  $\tilde{\beta}$ ,  $|\rho_\beta(x, \tilde{y}) - \rho_{\tilde{\beta}}(x, \tilde{y})| \leq L_\rho |x^\top \beta - x^\top \tilde{\beta}|$ , where  $L_\rho$  is a positive constant not depending on  $\tilde{y}$ .

**Remark 1.** Condition (A.1) is often assumed for quantile estimation, and (A.1)(ii) ensures the identifiability of  $q_0$ . See, for example, Firpo (2007), Han et al. (2019). Condition (A.2) is fundamental in the missing problem, which means that each individual with the covariates values has positive probability to be observed. Condition (A.3) puts some requirements on the data-generating model. (A.3)(i) assumes that  $P(Y \leq q_0 \mid X = x)$  is bounded away from zero and one. It is reasonable since  $q_0$  is the  $\tau$ -quantile of  $Y$  with  $0 < \tau < 1$ . (A.3)(ii) is a boundness assumption on the partial derivative of  $h(q, u)$  and (A.3)(iii) assumes  $h(q, u)$  and  $f(q, u)$  satisfy Lipschitz conditions. Condition (A.4)(i) is a commonly used condition in literature of high dimensions. See, for example, Assumption A in Van de Geer (2008), (C3) in Van de Geer et al. (2014), etc. Condition (A.5) (i) and (ii) are commonly used conditions for the consistency of lasso estimators. See, for example, Assumption L, B in Van de Geer (2008), margin condition of Theorem 6.4 and conditions of Theorem 14.5 in Buhlmann and van de Geer (2011), (A.3) and (A.4) in Van de Geer et al. (2014). Condition (A.5) (i) holds, for example, when  $\rho_\beta(x, \tilde{y})$  is twice differentiable on  $\beta$ , and the ex-

pectation of the second derivative is larger than some positive constant.

Define  $A = (\dot{h}_u(q_0, X^\top \beta^0) X^\top, 1)^\top \in \mathbb{R}^{(p+1) \times 1}$  and

$$\eta^* = \operatorname{argmin}_{\eta \in \mathbb{R}^{(p+1)}} \frac{1}{4} E \left[ \frac{\eta^\top \delta A A^\top \eta}{h(q_0, X^\top \beta^0)(1 - h(q_0, X^\top \beta^0))} \right] - E[A^\top] \eta. \quad (3.8)$$

The following theorem states that the proposed method defines an asymptotically normal estimator of  $q_0$ .

**Theorem 1.** *Assume  $s^4 \log(p) = o(\sqrt{n})$  where  $s = \|\beta^0\|_0 \vee \|\eta^*\|_0$  and the smallest eigenvalue of  $E[AA^\top]$  is bounded away from zero. Under conditions (A.1)–(A.5), if  $\lambda \asymp \sqrt{\log(p)/n}$  and  $\Delta \asymp n^{-5/16}(\log(p))^{1/8}$ , then we have*

$$\sqrt{n} \sigma^{-1} (\hat{q} - q_0) \xrightarrow{d} N(0, 1), \quad (3.9)$$

where  $\sigma^2 = T^{-2}V$ ,  $T = E[f(q_0, X_i^\top \beta^0)]$  and

$$V = \frac{1}{4} E \left[ \frac{\eta^{*\top} \delta A A^\top \eta^*}{h(q_0, X^\top \beta^0)(1 - h(q_0, X^\top \beta^0))} \right] + \operatorname{Var}(h(q_0, X^\top \beta^0)).$$

The sparsity condition  $s^4 \log(p) = o(\sqrt{n})$  seems somewhat stronger than that used by Belloni et al. (2017). However, Belloni et al. (2017) need to specify both  $f_{Y|X}(Y | X = x)$  and  $\pi(x)$  correctly to prove the asymptotic normality of the augmented inverse probability weighted estimator, and the proposed method defines an asymptotically normal estimator without specifying  $\pi(x)$ . In addition, the proposed method can define a more efficient

estimator than that of Belloni et al. (2017). Next, we show this by analyzing  $V$  and comparing it with the asymptotic variance in Belloni et al. (2017). For simplicity, denote  $\tau(X) = h(q_0, X^\top \beta^0)(1 - h(q_0, X^\top \beta^0))$ .

Note that  $\eta^*$  can be explicitly written as follows

$$\eta^* = 2 \left( E \left[ \frac{\delta A A^\top}{\tau(X)} \right] \right)^{-1} E[A].$$

Then we have

$$V = E[A^\top] \left( E \left[ \frac{\delta A A^\top}{\tau(X)} \right] \right)^{-1} E[A] + \text{Var} (h(q_0, X^\top \beta^0)).$$

Let  $e(X) = \tau(X)^{-1}A$  and

$$r^*(X) = e(X)^\top (E [\delta \tau(X) e(X) e(X)^\top])^{-1} E[\delta \tau(X) e(X) \pi(X)^{-1}]. \quad (3.10)$$

Then we have

$$V = E [\delta \tau(X) r^*(X)^2] + \text{Var} (h(q_0, X^\top \beta^0)).$$

The asymptotic variance derived by Belloni et al. (2017) is  $\sigma_b^2 = T^{-2}V_b$

where

$$V_b = E [\delta \tau(X) r(X)^2] + \text{Var} (h(q_0, X^\top \beta^0))$$

and  $r(X) = \pi(X)^{-1}$ . Let  $\mathcal{F} = \{f : E[\delta \tau(X) f(X)^2] \leq \infty\}$  and define the inner product on  $\mathcal{F}$  by  $\langle f_1, f_2 \rangle_\# = E[\delta \tau(X) f_1(X) f_2(X)]$ . Then  $\mathcal{F}$  is

a Hilbert space with respect to  $\langle \cdot, \cdot \rangle_{\#}$  and the norm induced by the inner product satisfies  $\|f\|_{\#}^2 = E[\delta\tau(X)f(X)^2]$ . Then we have

$$V = \|r^*\|_{\#}^2 + \text{Var} \left( h(q_0, X^{\top} \beta^0) \right).$$

and

$$V_b = \|r\|_{\#}^2 + \text{Var} \left( h(q_0, X^{\top} \beta^0) \right).$$

The form of  $r^*(X)$  in (3.10) indicates that  $r^*(X)$  is the projection of  $r(X)$  on the space spanned by  $e(X)$ . Hence we have  $\|r^*\|_{\#}^2 \leq \|r\|_{\#}^2$  and  $\sigma^2 \leq \sigma_b^2$ . The inequality holds if  $r(X)$  is not in the space spanned by  $e(X)$ . Note that  $V_b$  is actually the semiparametric efficiency bound established in (Firpo, 2007). Compared to Firpo (2007), we make an extra parametric assumption on  $f_{Y|X}(y | X = x)$ , our results indicate that this parametric assumption may induce a smaller efficiency bound.

The asymptotic variance of the proposed estimator can be consistently estimated as follows. Define  $\hat{\sigma}^2 = (\hat{T})^{-2} \hat{V}$ , where  $\hat{T} = \frac{1}{n} \sum_{i=1}^n f(\tilde{q}, X_i^{\top} \hat{\beta})$  and  $\hat{V} = \hat{V}_1 + \hat{V}_2$  with

$$\hat{V}_1 = n \sum_{\{i:\delta_i=1\}} \hat{w}_i^2 h(\tilde{q}, X_i^{\top} \hat{\beta}) \left\{ 1 - h(\tilde{q}, X_i^{\top} \hat{\beta}) \right\}$$

and

$$\hat{V}_2 = \frac{1}{n} \sum_{i=1}^n \left[ h(\tilde{q}, X_i^{\top} \hat{\beta})^2 - \left\{ \frac{1}{n} \sum_{i=1}^n h(\tilde{q}, X_i^{\top} \hat{\beta}) \right\}^2 \right].$$

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**Theorem 2.** *Under conditions of Theorem 1, we have  $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$ .*

The consistent variance estimation  $\hat{\sigma}^2$  depends on  $\hat{w}$  by  $\hat{V}_1$  only and  $\hat{V}_1$  is just the objective function in (2.4)–(2.6) with  $w_i$  replaced by  $\hat{w}_i$  for  $i = 1, 2, \dots, n$ . This makes that the variance estimation and hence the asymptotic variance attains the minimum. The constraint in (2.5) controls the bias of the resulting estimator.

#### 4. Derivation of the optimal weights

After establishing the theoretical properties of the proposed estimator, we next focus on the computation of the weight  $\hat{w}$ . From Step 1 and Step 2 of the proposed method, we need to make concrete choice for the tuning parameter  $\lambda$  and  $\Delta$ . We apply the 10-fold cross validation (CV) method to choose  $\lambda$ . For the selection of tuning parameter  $\Delta$ , since the asymptotic normality of  $\hat{q}$  implies  $E[(\hat{q} - q_0)^2] = \sigma^2/n + r(\Delta)$  when  $\hat{q}$  is uniform square integrate, where  $r(\Delta) = o(n^{-1})$  is the second-order term of mean square error of  $\hat{q}$ . Hence  $\Delta$  affects the second-order term of the mean square error of the estimator and hence its selection might not be so critical. According to Theorem 1, we take  $\Delta = cn^{-5/16}(\log(p))^{1/8}$ , where  $c$  is a positive constant. We set  $c$  to be 0.10 which has led to good finite sample performance in our simulations, and if the optimizing problem is infeasible, update  $c$  by

0.11, 0.12, 0.13,  $\dots$  in turn until the constraints (2.5) and (2.6) have feasible points. Furthermore, to ensure the numerical implementation, we provide an equivalent easy-to-implement alternative to compute  $\hat{w}$ . According to Athey et al. (2018), we can establish a 1 : 1 mapping between the optimizing problem (2.4)–(2.6) and the following optimizing problem

$$\begin{aligned} \hat{w} = \operatorname{argmin}_w & \left[ (1 - \hat{\zeta}) \sum_{\{i:\delta_i=1\}} w_i^2 h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta})) \right. \\ & \left. + \hat{\zeta} \left\| \frac{1}{n} \sum_{i=1}^n \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i - \sum_{\{i:\delta_i=1\}} w_i \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i \right\|_\infty^2 \right] \quad (4.11) \\ \text{s.t.} & \quad \sum_{\{i:\delta_i=1\}} w_i = 1, \end{aligned}$$

where

$$\begin{aligned} \hat{\zeta} = \operatorname{argmax}_{\zeta \in [0,1]} \min_w & \left[ \sum_{\{i:\delta_i=1\}} w_i^2 h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta})) \right. \\ & \left. + \frac{\zeta}{(1 - \zeta)} \left\| \frac{1}{n} \sum_{i=1}^n \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i - \sum_{\{i:\delta_i=1\}} w_i \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i \right\|_\infty^2 - \frac{\zeta}{(1 - \zeta)} \Delta^2 \right] \\ \text{s.t.} & \quad \sum_{\{i:\delta_i=1\}} w_i = 1 \quad (4.12) \end{aligned}$$

is the solution to the dual problem of (2.4)–(2.6). Note that (4.11) is equal to

$$\begin{aligned} \hat{w} = \operatorname{argmin}_w \min_{\Gamma} & \left[ (1 - \hat{\zeta}) \sum_{\{i:\delta_i=1\}} w_i^2 h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta})) + \hat{\zeta} \Gamma^2 \right] \\ \text{s.t.} & \left\| \frac{1}{n} \sum_{i=1}^n \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i - \sum_{\{i:\delta_i=1\}} w_i \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i \right\|_{\infty} \leq \Gamma, \\ & \sum_{\{i:\delta_i=1\}} w_i = 1, \end{aligned} \tag{4.13}$$

and the optimizing problem (4.13) is a quadratic programming. Then we can obtain  $\hat{w}$  by using the `solve.QP` from the `quadprog` package.

The calculation of  $\hat{\zeta}$  can be achieved by the following three steps.

Step 1 Consider a set of values  $G = \{0, 0.01, \dots, 0.98, 0.99\}$  for  $\zeta$  and denote the  $l$ -th value by  $\zeta_l$ .

Step 2 For each  $\zeta_l$  in  $G$ , calculate a weight by (4.13) with  $\hat{\zeta}$  replaced by  $\zeta_l$ , and denote the weight by  $\hat{w}(\zeta_l)$ .

Step 3 With  $\hat{w}(\zeta_l)$  for all  $\zeta_l \in G$ , according to (4.12), approximate  $\hat{\zeta}$  by

$$\begin{aligned} \hat{\zeta} = \operatorname{argmax}_{\zeta_l \in G} & \left[ \sum_{\{i:\delta_i=1\}} \hat{w}(\zeta_l)_i^2 h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta})) \right. \\ & + \frac{\zeta_l}{(1 - \zeta_l)} \left\| \frac{1}{n} \sum_{i=1}^n \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i - \sum_{\{i:\delta_i=1\}} \hat{w}(\zeta_l)_i \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i \right\|_{\infty}^2 \\ & \left. - \frac{\zeta_l}{(1 - \zeta_l)} \Delta^2 \right]. \end{aligned}$$

Then according to (4.11), the solution to the optimizing problem (2.4)–(2.6) is given by  $\hat{w} = \hat{w}(\hat{\zeta})$ . In contrast to Athey et al. (2018), which simply takes  $\hat{\zeta} = 0.5$  for every  $\Delta$ , we provide a data-adaptive and reasonable computing way to obtain  $\hat{\zeta}$ .

## 5. Simulation Study

To evaluate the numerical performance of the proposed method, we conducted a simulation study with the design similar to Tan (2020) and calculated the augmented inverse probability weighting (AIPW) estimator  $\hat{q}_{aipw}$  due to Belloni et al. (2017) as a comparison. Unless otherwise specified, a logistic model of  $\delta$  versus  $X$  and a standard normal linear conditional distribution of  $Y$  given  $X$  were assumed for  $\pi(X)$  and  $f_{Y|X}(y | X)$ , respectively.

Let  $X = (X_1, \dots, X_p)$ , where  $X_j$  for  $j = 1$  and  $2$  was generated from a uniform distribution  $U(-5, 5)$  and  $X_j$  for  $j = 3, \dots, p$  from a truncated normal with mean  $0$ , variance  $1/2$  and truncation constant  $5$ . In addition, let  $X^\dagger = (X_1^\dagger, \dots, X_p^\dagger)$ , where  $X_j^\dagger = X_j - X_j^2 + 2X_j^3$  for  $j = 1, 2, 3$  and  $4$  and  $X_j^\dagger = X_j$  for  $j = 5, \dots, p$ . Consider the median case ( $\tau = 0.5$ ) under the following two data-generating processes (DGP):

(DGP1) Generate  $Y$  given  $X$  from a normal distribution

$$N(0.25X_1 + 0.125X_2 + 0.25X_3 + 0.125X_4, 1)$$

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and generate  $\delta$  given  $X$  from a Bernoulli distribution with

$$P(\delta = 1 | X) = \frac{\exp(1 - 0.25X_1^\dagger - 0.125X_2^\dagger - 0.25X_3^\dagger - 0.125X_4^\dagger)}{1 + \exp(1 - 0.25X_1^\dagger - 0.125X_2^\dagger - 0.25X_3^\dagger - 0.125X_4^\dagger)}.$$

(DGP2) Generate  $Y$  given  $X$  as in DGP1 but generate  $\delta$  given  $X$  from a Bernoulli distribution with

$$P(\delta = 1 | X) = \frac{\exp(1 - 0.25X_1 - 0.125X_2 - 0.25X_3 - 0.125X_4)}{1 + \exp(1 - 0.25X_1 - 0.125X_2 - 0.25X_3 - 0.125X_4)}.$$

Depending on above data generation processes where  $\pi(X)$  involves two completely different sets of regressors,  $\pi(X)$  is misspecified under DGP1 and correctly specified under DGP2.

For each of the two DGPs, the simulation was conducted based on 1000 replications with sample size of  $n = 200, 400$  and  $800$  and covariates number of  $p = \frac{1}{4}n, \frac{1}{2}n, n$  and  $2n$ , respectively. From the 1000 simulated values of  $\hat{q}_{aipw}$  and  $\hat{q}$ , we computed the Monte Carlo bias (Bias), standard deviation (SD), root mean square error (RMSE). For nominal confidence level  $1 - \alpha = 0.95$ , we evaluated the coverage probabilities (CP) of the confidence intervals. The simulation results are reported in Table 1 and Table 2 for DGP1 and DGP2, respectively. In addition, we compared the estimated standard deviation of  $\hat{q}$  (ESD) based on the asymptotic variance given by theorem 2 with the Monte Carlo standard deviation based on 1000 repetitions, which are reported in Table 3.

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[Insert Table 1, Table 2 and Table 3 about here.]

From Table 1, Table 2 and Table 3, we have the following observations.

- (i) In the case where  $\pi(X)$  is misspecified,  $\hat{q}$  outperforms  $\hat{q}_{aipw}$  in terms of Bias, RMSE and CP for all combinations of  $n$  and  $p$ , especially when  $p$  diverges with  $n$  at a relatively large rate. Although  $\hat{q}_{aipw}$  has generally slightly smaller SD than  $\hat{q}$ , its Bias is approximately 5 times as large as that of  $\hat{q}$ . In addition, the coverage probability based on of the AIPW estimator is considerably lower than the nominal level 95%. On the contrary, the proposed estimator performs well with coverage probabilities generally closing to 0.95, which is expected. This could be explained by the fact that the root- $n$  asymptotic normality of  $\hat{q}_{aipw}$  requires that  $\pi(X)$  is correctly specified and the requirement is not satisfied in this case. The fact also implies that the AIPW method cannot be used to make statistical inference for  $q_0$  in the case where  $\pi(X)$  is misspecified. On the contrary, the calculation of the proposed estimator  $\hat{q}$  does not involve  $\pi(X)$  and hence the asymptotic normality of  $\hat{q}$  is robust to the misspecification of  $\pi(X)$ .
- (ii) In the case where  $\pi(X)$  is correctly specified, both  $\hat{q}$  and  $\hat{q}_{aipw}$  perform well while the standard deviations of  $\hat{q}$  are generally smaller than those of  $\hat{q}_{aipw}$ , which is in agreement with the asymptotics in theorem 2.

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(iii) The estimated standard deviations are close to the empirical standard deviations for the proposed estimator  $\hat{q}$ .

Overall, our theoretical results are supported by the simulation studies. Although  $\hat{q}_{aipw}$  has comparable performance to  $\hat{q}$  in the case where  $\pi(X)$  is correctly specified, it is hard to specify a correct model for  $\pi(X)$  in practice. Hence the proposed method is more trustworthy and hence recommended.

In addition, our simulation results indicate that the proposed estimator performs fairly well even if this sparsity condition  $s = o(n^{1/8}(\log(p))^{-1/4})$  is violated. This implies that the sparsity condition may be weakened. In Section S3 of the supplementary material, we weaken the sparsity condition while maintaining the  $\sqrt{n}$  consistency via data splitting

## 6. Real Data Analysis

We provide an application to analyzing a medical dataset collected on 2139 HIV-infected subjects enrolled in AIDS clinical Trial Group Protocol 175 (ACTG 175). The original data were collected by Hammer et al. (1996). ACTG 175 is a randomized clinical trial where patients are randomized to four antiretroviral regimens in: zidovudine (ZDV) only, ZDV+didanosine (ddI), ZDV+zalcitabine (ddC), and ddI only. Following the analysis in Davidian et al. (2005), we consider two groups: the group with ZDV alone

(control) and the group with the other three therapies (treatment). The dataset contains  $n = 2139$  patients,  $n_0 = 532$  of whom participated in the control group and  $n_1 = 1607$  of whom participated in the treated group (*treat*: 0=control). This study evaluates the treatment effect by the change in CD4 count from baseline to  $96 \pm 5$  weeks ( $CD4_{96}$ ) which is a measure of immunologic status. Previous work analyzed the dataset by the average treatment effect (see, Davidian et al. (2005), Han (2014) and Han et al. (2019)). Our main interest is the median treatment effect  $m = m_1 - m_0$ , where  $m_1$  and  $m_0$  are the median of  $CD4_{96} \mid treat = c$  with  $c = 1$  and 0 respectively.

However, there are 797 subjects whose  $CD4_{96}$  are missing (*r*: 0=missing) due to dropout from the study. At the baseline and during the follow-up, 23 covariates ( $X$ ) correlated with  $CD4_{96}$  are obtained. There may be interactions between covariates  $X$ . To employ the proposed method, we treat the observed covariates and their two-way interactions as a new covariate vector. Specifically, we denote it by a vector  $U$  whose  $j$ -th component  $U_j = X_j$  for  $1 \leq j \leq 23$ , and  $U_j = X_l X_{j-23l+l(l-1)/2}$  for  $[23l - (l-1)(l-2)/2] < j \leq [23(l+1) - l(l-1)/2]$  and  $l = 1, 2, \dots, 23$ . The dimension of  $U$  is 299. As analyzed in Section 3, the selection probability function is unknown and difficult to specify correctly, in which case AIPW

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method performs poorly and cannot be used to make inference, and hence we apply our method to the real data analysis only here. We consider a standard normal distribution for  $CD4_{96} \mid U, treat = c$  with  $c = 0, 1$ . The confidence intervals for  $m_1$ ,  $m_0$  and  $m$  obtained via the proposed method are reported in Table 4.

[Insert Table 4 about here.]

From Table 4, it can be seen that people who received three newer treatments had a potential median effect of  $\hat{m} = 48$  compared with those who received ZDV alone. Moreover, this effect is significant since the 95% confidence interval does not contain 0.

**Supplementary Materials** Supplementary materials are available online, which contain the lemmas that used in the proofs of Theorem 1 and Theorem 2, some additional simulation results for Section 5 and further explorations about the sparsity condition.

## Appendix. Proof of Main Results

Appendix contains proofs of Theorem 1 and Theorem 2. Note that constant  $c$  may vary from lines and all of them are positive.

*Proof of Theorem 1.* For the simplicity of illustration, let

$$\hat{F}_n(q) \triangleq \frac{1}{n} \sum_{i=1}^n h(q, X_i^\top \hat{\beta}) + \sum_{\{i:\delta_i=1\}} \hat{w}_i \hat{\xi}_i(q),$$

where  $\hat{\xi}_i(q) = I[Y_i \leq q] - h(q, X_i^\top \hat{\beta})$ . An outline of the proof of Theorem 1

is as follows. According to mean value theorem, it follows that

$$\hat{F}_n(\hat{q}) - \hat{F}_n(q_0) = \frac{1}{n} \sum_{i=1}^n f(\bar{q}, X_i^\top \hat{\beta}) (\hat{q} - q_0) + \sum_{\{i:\delta_i=1\}} \hat{w}_i \left\{ \hat{\xi}_i(\hat{q}) - \hat{\xi}_i(q_0) \right\}, \quad (6.14)$$

where  $\bar{q}$  is between  $q_0$  and  $\hat{q}$ . On the one hand, we show that

$$\frac{1}{n} \sum_{i=1}^n f(\bar{q}, X_i^\top \hat{\beta}) (\hat{q} - q_0) = \left\{ E[f(q_0, X^\top \beta^0)] + o_p(1) \right\} (\hat{q} - q_0) \quad (6.15)$$

and

$$\sum_{\{i:\delta_i=1\}} \hat{w}_i \left\{ \hat{\xi}_i(\hat{q}) - \hat{\xi}_i(q_0) \right\} = o_p(|\hat{q} - q_0|) + o_p(n^{-1/2}). \quad (6.16)$$

On the other hand, we show that

$$\begin{aligned} \hat{F}_n(\hat{q}) - \hat{F}_n(q_0) &= -\frac{1}{n} \sum_{\{i:\delta_i=1\}} \frac{A_i^\top \eta^* \xi_i(q_0)}{2h(q_0, X_i^\top \beta^0)(1 - h(q_0, X_i^\top \beta^0))} \\ &\quad - \left\{ \frac{1}{n} \sum_{i=1}^n h(q_0, X_i^\top \beta^0) - E[h(q_0, X^\top \beta^0)] \right\} \\ &\quad + o_p(n^{-1/2}), \end{aligned} \quad (6.17)$$

where  $A_i = (\dot{h}_u(q_0, X_i^\top \beta^0) X_i^\top, 1)^\top \in \mathbb{R}^{(p+1) \times 1}$ ,  $\xi_i(q) = I[Y_i \leq q] - h(q, X_i^\top \beta^0)$  and  $\eta^*$  is defined in (3.8). (6.15)–(6.17) together with (6.14) implies the fol-

lowing asymptotic representation

$$\begin{aligned}
& - \{E[f(q_0, X^\top \beta^0)] + o_p(1)\} (\hat{q} - q_0) \\
&= \frac{1}{n} \sum_{\{i:\delta_i=1\}} \frac{A_i^\top \eta^* \xi_i(q_0)}{2h(q_0, X_i^\top \beta^0)(1 - h(q_0, X_i^\top \beta^0))} \\
&+ \left\{ \frac{1}{n} \sum_{i=1}^n h(q_0, X_i^\top \beta^0) - E[h(q_0, X^\top \beta^0)] \right\} + o_p(n^{-1/2}).
\end{aligned} \tag{6.18}$$

Then the main result (3.9) in Theorem 1 is proved by the central limit theorem and Slutsky's theorem.

(a) First, we prove (6.15). Lemma 1 in the supplementary material proves that

$$\|\hat{\beta} - \beta^0\|_1 = O_p\left(\|\beta^0\|_0 \sqrt{\frac{\log(p)}{n}}\right). \tag{6.19}$$

Then we have  $\|\hat{\beta} - \beta^0\|_1 = o_p(1)$  by the assumption on  $\|\beta^0\|_0$ . If the consistency of  $\hat{q}$  is proved, (6.15) follows immediately by the law of large numbers. To prove the consistency of  $\hat{q}$ , on the basis of Theorem 5.9 in Van der Vaart (2000) and Condition (A.1), it suffices to check that

$$\sup_q \left| \hat{F}_n(q) - E[h(q, X^\top \beta^0)] \right| = o_p(1). \tag{6.20}$$

Let  $F_n(q) \triangleq n^{-1} \sum_{i=1}^n h(q, X_i^\top \beta^0)$ . Then we can write

$$\begin{aligned}
& \sup_q \left| \hat{F}_n(q) - E[h(q, X_1^\top \beta^0)] \right| \\
& \leq \sup_q \left| \hat{F}_n(q) - F_n(q) \right| + \sup_q \left| F_n(q) - E[h(q, X^\top \beta^0)] \right| \\
& \triangleq U_{n1} + U_{n2}.
\end{aligned} \tag{6.21}$$

First,  $U_{n2} = o_p(1)$  can be proved according to Lemma 2.4 in Newey and McFadden (1986). Next, we prove  $U_{n1} = o_p(1)$ . By mean value theorem, we have

$$\begin{aligned}
U_{n1} &\leq \sup_q \left| \frac{1}{n} \sum_{i=1}^n \dot{h}_u(q, X_i^\top \tilde{\beta}) X_i^\top (\hat{\beta} - \beta^0) \right| \\
&\quad + \sup_q \left| \sum_{\{i:\delta_i=1\}} \hat{w}_i \dot{h}_u(q, X_i^\top \bar{\beta}) X_i^\top (\hat{\beta} - \beta^0) \right| \\
&\quad + \sup_q \left| \sum_{\{i:\delta_i=1\}} \hat{w}_i \xi_i(q) \right| \\
&\triangleq U_{n11} + U_{n12} + U_{n13},
\end{aligned} \tag{6.22}$$

where  $\tilde{\beta}$  and  $\bar{\beta}$  are between  $\beta^0$  and  $\hat{\beta}$ . Then the problem reduces to show that  $U_{n1i} = o_p(1)$  for  $i = 1, 2$  and  $3$ .

By Conditions (A.3)(ii), (A.4)(i) and (6.19), we have

$$U_{n11} \leq c \left\| \hat{\beta} - \beta^0 \right\|_1 = o_p(1)$$

and

$$U_{n12} \leq c \sum_{\{i:\delta_i=1\}} |\hat{w}_i| \left\| \hat{\beta} - \beta^0 \right\|_1 = o_p \left( \sum_{\{i:\delta_i=1\}} |\hat{w}_i| \right).$$

Then  $U_{n12} = o_p(1)$  is proved if we can show that  $\sum_{\{i:\delta_i=1\}} |\hat{w}_i| = O_p(1)$ .

Lemma 3 in the supplementary material proves that the constraints (2.5)

and (2.6) can be satisfied with probability tending to 1 by taking  $w_i$  to be

$\tilde{w}_i = \pi(X_i)^{-1} / \sum_{\{i:\delta_i=1\}} \pi(X_i)^{-1}$ . Recalling the definition of  $\hat{w}$ , note that

$\hat{w}$  not only satisfies the constraints (2.5) and (2.6), but also minimizes the

objective function in (2.4). Then we have

$$\sum_{\{i:\delta_i=1\}} \hat{w}_i^2 h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta})) \leq \sum_{\{i:\delta_i=1\}} \tilde{w}_i^2 h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta})). \quad (6.23)$$

By Condition (A.2), the right side of (6.23) is  $O_p(n^{-1})$ . This together with Condition (A.3)(i) proves  $\sum_{\{i:\delta_i=1\}} \hat{w}_i^2 = O_p(n^{-1})$ . Since  $\left(\sum_{\{i:\delta_i=1\}} |\hat{w}_i|\right)^2 \leq n \sum_{\{i:\delta_i=1\}} \hat{w}_i^2$  by Cauchy-Schwartz inequality, it follows that  $\sum_{\{i:\delta_i=1\}} |\hat{w}_i| = O_p(1)$ .

By (6.22), it remains to show that  $U_{n13} = o_p(1)$ . By the Lagrange multiplier method, Lemma 4 in the supplementary material provides an alternative representation of  $\hat{w}_i$  as follows

$$\hat{w}_i = \frac{1}{2n} \frac{\hat{A}_i^\top \hat{\eta}}{h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta}))},$$

where  $\hat{A}_i = (\dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i^\top, 1)^\top \in \mathbb{R}^{(p+1) \times 1}$  and

$$\hat{\eta} = \underset{\eta \in \mathbb{R}^{(p+1)}}{\operatorname{argmin}} \frac{1}{4n} \sum_{i=1}^n \frac{\eta^\top \delta_i \hat{A}_i \hat{A}_i^\top \eta}{h(\tilde{q}, X_i^\top \hat{\beta})(1 - h(\tilde{q}, X_i^\top \hat{\beta}))} - \frac{1}{n} \sum_{i=1}^n \hat{A}_i^\top \eta + \|\eta_{-(p+1)}\|_1 \Delta.$$

Define

$$w_i^* = \frac{1}{2n} \frac{A_i^\top \eta^*}{h(q_0, X_i^\top \beta^0)(1 - h(q_0, X_i^\top \beta^0))} \quad (6.24)$$

for  $i \in \{i : \delta_i = 1\}$ . Then we can write

$$U_{n13} \leq \sup_q \left| \sum_{\{i:\delta_i=1\}} (\hat{w}_i - w_i^*) \xi_i(q) \right| + \sup_q \left| \sum_{\{i:\delta_i=1\}} w_i^* \xi_i(q) \right| \quad (6.25)$$

$$\triangleq J_{n1} + J_{n2}.$$

According to the restrictions on the sparsity  $s$  and the rate of  $\Delta$ , the conditions of Lemma 6 in the supplementary material is satisfied. Then Lemma 6 shows that

$$\|\hat{w} - w^*\|_1 = o_p(1), \quad (6.26)$$

which implies  $J_{n1} = o_p(1)$ . By Theorem 37 in Pollard (1984) and Lemma 6 in the supplementary material, we have  $J_{n2} = o_p(1)$ . These together with (6.25) prove  $U_{n13} = o_p(1)$ . Then the consistency of  $\hat{q}$  is proved.

(b) Next, we prove (6.16). It is noted that

$$\begin{aligned} \left| \sum_{\{i:\delta_i=1\}} \hat{w}_i \left( \hat{\xi}_i(\hat{q}) - \hat{\xi}_i(q_0) \right) \right| &\leq \left| \sum_{\{i:\delta_i=1\}} w_i^* \left\{ \hat{\xi}_i(\hat{q}) - \hat{\xi}_i(q_0) \right\} \right| \\ &\quad + \left| \sum_{\{i:\delta_i=1\}} (\hat{w}_i - w_i^*) \left\{ \hat{\xi}_i(\hat{q}) - \hat{\xi}_i(q_0) \right\} \right| \\ &\triangleq R_{n1} + R_{n2}. \end{aligned} \quad (6.27)$$

By some algebras and mean value theorem, it follows that

$$\begin{aligned}
|R_{n1}| &\leq \left| \sum_{\{i:\delta_i=1\}} w_i^* \{\xi_i(\hat{q}) - \xi_i(q_0)\} \right| \\
&\quad + \left| \sum_{\{i:\delta_i=1\}} w_i^* \left\{ h(\hat{q}, X_i^\top \hat{\beta}) - h(q_0, X_i^\top \hat{\beta}) - [h(\hat{q}, X_i^\top \beta^0) - h(q_0, X_i^\top \beta^0)] \right\} \right| \\
&\leq \sup_{|q-q_0|=O(\iota_n)} \left| \sum_{\{i:\delta_i=1\}} w_i^* \{\xi_i(q) - \xi_i(q_0)\} \right| \\
&\quad + \left| \sum_{\{i:\delta_i=1\}} w_i^* \left[ f(\bar{q}, X_i^\top \hat{\beta}) - f(\check{q}, X_i^\top \beta^0) \right] (\hat{q} - q_0) \right| \\
&\triangleq R_{n11} + R_{n12},
\end{aligned} \tag{6.28}$$

where  $\bar{q}$  and  $\check{q}$  are between  $q_0$  and  $\hat{q}$ . Denote  $\iota_n$  the convergence rate of  $\hat{q}$ . According to the Theorem 37 in Pollard (1984) and Lemma 6 in the supplementary material, we have

$$R_{n11} = o_p(n^{-1/2} \vee \iota_n). \tag{6.29}$$

By Conditions (A.3)(iii) and (A.4)(i), we have

$$\begin{aligned}
R_{n12} &\leq c \sum_{\{i:\delta_i=1\}} |w_i^*| \left\{ |\bar{q} - \check{q}| + |X_i^\top (\hat{\beta} - \beta^0)| \right\} |\hat{q} - q_0| \\
&\leq c \sum_{\{i:\delta_i=1\}} |w_i^*| \left\{ |\bar{q} - \check{q}| + \left\| \hat{\beta} - \beta^0 \right\|_1 \right\} |\hat{q} - q_0|.
\end{aligned}$$

Similar to the previous result, we can show that  $\sum_{\{i:\delta_i=1\}} |w_i^*| = O_p(1)$ .

This together with (6.19) and the consistency of  $\hat{q}$  proves

$$R_{n12} = o_p(\iota_n). \tag{6.30}$$

Equations (6.28), (6.29) and (6.30) prove

$$R_{n1} = o_p(n^{-1/2}) + o_p(t_n). \quad (6.31)$$

By some algebras, it is easy to show that

$$\begin{aligned} R_{n2} &\leq \left| \sum_{\{i:\delta_i=1\}} (\hat{w}_i - w_i^*) \{\xi_i(\hat{q}) - \xi_i(q_0)\} \right| \\ &\quad + \left| \sum_{\{i:\delta_i=1\}} \{\hat{w}_i - w_i^*\} \left\{ h(\hat{q}, X_i^\top \hat{\beta}) - h(q_0, X_i^\top \hat{\beta}) \right\} \right| \\ &\quad + \left| \sum_{\{i:\delta_i=1\}} \{\hat{w}_i - w_i^*\} \left\{ h(\hat{q}, X_i^\top \beta^0) - h(q_0, X_i^\top \beta^0) \right\} \right| \\ &\triangleq R_{n21} + R_{n22} + R_{n23}. \end{aligned} \quad (6.32)$$

By Cauchy-Schwartz inequality, we have

$$\begin{aligned} R_{n21} &\leq \left\{ n \sum_{\{i:\delta_i=1\}} (\hat{w}_i - w_i^*)^2 \right\}^{1/2} \left\{ \frac{1}{n} \sum_{i=1}^n [\xi_i(\hat{q}) - \xi_i(q_0)]^2 \right\}^{1/2} \\ &\leq \left\{ n \sum_{\{i:\delta_i=1\}} (\hat{w}_i - w_i^*)^2 \right\}^{1/2} \left\{ \sup_{|q-q_0|=O(t_n)} \frac{1}{n} \sum_{i=1}^n [\xi_i(q) - \xi_i(q_0)]^2 \right\}^{1/2}. \end{aligned} \quad (6.33)$$

Notice that

$$\begin{aligned}
& \sup_{|q-q_0|=O(\iota_n)} \frac{1}{n} \sum_{i=1}^n \{\xi_i(q) - \xi_i(q_0)\}^2 \\
\leq & \sup_{|q-q_0|=O(\iota_n)} \left\{ \frac{1}{n} \sum_{i=1}^n \{\xi_i(q) - \xi_i(q_0)\}^2 - E [\{\xi_i(q) - \xi_i(q_0)\}^2] \right\} \\
& + \sup_{|q-q_0|=O(\iota_n)} E [\{\xi_i(q) - \xi_i(q_0)\}^2] \\
= & \sup_{|q-q_0|=O(\iota_n)} \left\{ \frac{1}{n} \sum_{i=1}^n \{\xi_i(q) - \xi_i(q_0)\}^2 - E [\{\xi_i(q) - \xi_i(q_0)\}^2] \right\} + O(\iota_n) \\
= & O_p(n^{-1/2} \vee \iota_n)
\end{aligned} \tag{6.34}$$

by Theorem 37 in Pollard (1984). In addition, Lemma 6 in the supplementary material proves that

$$n \|\hat{w} - w^*\|_2^2 = o_p(n^{-1/2}). \tag{6.35}$$

Equations (6.33), (6.34) and (6.35) imply

$$R_{n21} = o_p\left((n^{-1/4})(n^{-1/4} \vee \iota_n^{1/2})\right) = o_p\left((n^{-1/4} \vee \iota_n^{1/2})^2\right) = o_p(n^{-1/2} \vee \iota_n). \tag{6.36}$$

By mean values theorem and Condition (A.3)(ii), we have

$$R_{n22} \leq c \|\hat{w} - w^*\|_1 |\hat{q} - q_0|.$$

This together with (6.26) proves

$$R_{n22} = o_p(|\hat{q} - q_0|). \tag{6.37}$$

Similarly, it can be proved that

$$R_{n23} = o_p(|\hat{q} - q_0|). \quad (6.38)$$

Then equations (6.32), (6.36), (6.37) and (6.38) prove

$$R_{n2} = o_p(n^{-1/2}) + o_p(\iota_n). \quad (6.39)$$

Relations (6.27), (6.31) and (6.39) together prove (6.16).

(c) Finally, we prove (6.17). Note that  $\hat{F}_n(\hat{q}) = \tau = E[h(q_0, X^\top \beta^0)]$ .

Then we have

$$\begin{aligned} \hat{F}_n(\hat{q}) - \hat{F}_n(q_0) &= E[h(q_0, X^\top \beta^0)] - \hat{F}_n(q_0) \\ &= \{E[h(q_0, X^\top \beta^0)] - F_n(q_0)\} - \{\hat{F}_n(q_0) - F_n(q_0)\}. \end{aligned} \quad (6.40)$$

By mean value theorem and some algebras, it follows that

$$\begin{aligned} &\hat{F}_n(q_0) - F_n(q_0) \\ &= \left[ \frac{1}{n} \sum_{i=1}^n \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i^\top - \sum_{\{i:\delta_i=1\}} \hat{w}_i \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i^\top \right] (\hat{\beta} - \beta^0) \\ &\quad + \frac{1}{n} \sum_{i=1}^n \left[ \dot{h}_u(q_0, X_i^\top \tilde{\beta}) - \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) \right] X_i^\top (\hat{\beta} - \beta^0) \\ &\quad - \sum_{\{i:\delta_i=1\}} \hat{w}_i \left[ \dot{h}_u(q_0, X_i^\top \tilde{\beta}) - \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) \right] X_i^\top (\hat{\beta} - \beta^0) \\ &\quad + \sum_{\{i:\delta_i=1\}} (\hat{w}_i - w_i^*) \xi_i(q_0) \\ &\quad + \sum_{\{i:\delta_i=1\}} w_i^* \xi_i(q_0) \triangleq \sum_{i=1}^5 L_{ni}, \end{aligned} \quad (6.41)$$

where  $\tilde{\beta}$  is between  $\beta^0$  and  $\hat{\beta}$ . Note that

$$|L_{n1}| \leq \left\| \frac{1}{n} \sum_{i=1}^n \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i^\top - \sum_{\{i:\delta_i=1\}} \hat{w}_i \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i^\top \right\|_\infty \left\| \hat{\beta} - \beta^0 \right\|_1.$$

By the definition of  $\hat{w}$ , we have

$$\left\| \frac{1}{n} \sum_{i=1}^n \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i^\top - \sum_{\{i:\delta_i=1\}} \hat{w}_i \dot{h}_u(\tilde{q}, X_i^\top \hat{\beta}) X_i^\top \right\|_\infty \leq \Delta.$$

This together with (6.19) proves  $L_{n1} = O_p\left(\Delta \|\beta^0\|_0 \sqrt{\log(p)/n}\right)$ . By the requirements on  $\Delta$  and assumptions on  $\|\beta^0\|_0$ , we have

$$L_{n1} = o_p(n^{-1/2}). \quad (6.42)$$

By Condition (A.3)(ii), it follows that

$$\begin{aligned} |L_{n2}| &\leq c \frac{1}{n} \sum_{i=1}^n \left\{ |\tilde{q} - q_0| + |X_i^\top (\hat{\beta} - \beta^0)| \right\} |X_i^\top (\hat{\beta} - \beta^0)| \\ &\leq c |\tilde{q} - q_0| \left\| \hat{\beta} - \beta^0 \right\|_1 + c \left\| \hat{\beta} - \beta^0 \right\|_1^2 \end{aligned}$$

and

$$\begin{aligned} |L_{n3}| &\leq c \sum_{\{i:\delta_i=1\}} |\hat{w}_i| \left\{ |\tilde{q} - q_0| + |X_i^\top (\hat{\beta} - \beta^0)| \right\} |X_i^\top (\hat{\beta} - \beta^0)| \\ &\leq c \sum_{\{i:\delta_i=1\}} |\hat{w}_i| \left( |\tilde{q} - q_0| \left\| \hat{\beta} - \beta^0 \right\|_1 + \left\| \hat{\beta} - \beta^0 \right\|_1^2 \right). \end{aligned}$$

Lemma 2 in the supplementary material proves that

$$\tilde{q} - q_0 = O_p\left(\sqrt{\|\beta^0\|_0 \frac{\log(p)}{n}}\right).$$

These together with (6.19) and  $\sum_{\{i:\delta_i=1\}} |\hat{w}_i| = O_p(1)$  prove

$$|L_{ni}| = O_p \left( \left\| \beta^0 \right\|_0^2 \frac{\log(p)}{n} \right) = o_p(n^{-1/2}), \quad i = 2, 3. \quad (6.43)$$

by the assumption on  $\|\beta^0\|_0$ . Lemma 7 in the supplementary material proves that  $L_{n4} = o_p(n^{-1/2})$ . This together with (6.41), (6.42) and (6.43) proves

$$\hat{F}_n(q_0) - F_n(q_0) = \sum_{\{i:\delta_i=1\}} w_i^* \xi_i(q_0). \quad (6.44)$$

Recalling the definition of  $w^*$  in (6.24), relations (6.40) and (6.44) together prove (6.17).

The proof of Theorem 1 is then completed.  $\square$

*Proof of Theorem 2.* We first prove  $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ . Denote

$$V_1 = \frac{1}{4} E \left[ \frac{\delta \eta^{*\top} A A^\top \eta^*}{h(q_0, X^\top \beta^0)(1 - h(q_0, X^\top \beta^0))} \right]$$

and  $V_2 = \text{Var}(h(q_0, X^\top \beta^0))$ . On the basis of Slutsky's theorem, it suffices to show that  $\hat{V}_1 \xrightarrow{p} V_1$ ,  $\hat{V}_2 \xrightarrow{p} V_2$  and  $\hat{T} \xrightarrow{p} T_0$ . Since  $\tilde{q} - q_0 = o_p(1)$  and  $\|\hat{\beta} - \beta^0\|_1 = o_p(1)$ , then  $|\hat{T} - T_0| = o_p(1)$  and  $|\hat{V}_2 - V_2| = o_p(1)$  are proved by the law of large numbers.  $|\hat{V}_1 - V_1| = o_p(1)$  is proved by Lemma 8 in the supplementary material.  $\square$

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Table 1: Bias, SD, RMSE and CP of relevant estimators based on 1000 repetitions.

|                                |                   | Bias             |           | SD               |           | RMSE             |           | CP               |           |
|--------------------------------|-------------------|------------------|-----------|------------------|-----------|------------------|-----------|------------------|-----------|
|                                |                   | $\hat{Q}_{aipw}$ | $\hat{q}$ | $\hat{Q}_{aipw}$ | $\hat{q}$ | $\hat{Q}_{aipw}$ | $\hat{q}$ | $\hat{Q}_{aipw}$ | $\hat{q}$ |
| DGP1: $\pi(X)$ is misspecified |                   |                  |           |                  |           |                  |           |                  |           |
| $n = 200$                      | $p = \frac{n}{4}$ | -0.241           | -0.042    | 0.177            | 0.196     | 0.299            | 0.201     | 0.383            | 0.952     |
|                                | $p = \frac{n}{2}$ | -0.300           | -0.065    | 0.170            | 0.206     | 0.345            | 0.216     | 0.238            | 0.941     |
|                                | $p = n$           | -0.338           | -0.070    | 0.167            | 0.212     | 0.377            | 0.223     | 0.165            | 0.933     |
|                                | $p = 2n$          | -0.374           | -0.104    | 0.161            | 0.197     | 0.407            | 0.222     | 0.119            | 0.906     |
| $n = 400$                      | $p = \frac{n}{4}$ | -0.177           | -0.027    | 0.130            | 0.155     | 0.219            | 0.158     | 0.401            | 0.937     |
|                                | $p = \frac{n}{2}$ | -0.218           | -0.039    | 0.124            | 0.157     | 0.251            | 0.162     | 0.250            | 0.941     |
|                                | $p = n$           | -0.261           | -0.048    | 0.128            | 0.160     | 0.290            | 0.167     | 0.170            | 0.941     |
|                                | $p = 2n$          | -0.133           | -0.023    | 0.091            | 0.114     | 0.161            | 0.116     | 0.417            | 0.941     |
| $n = 800$                      | $p = \frac{n}{4}$ | -0.299           | -0.075    | 0.122            | 0.145     | 0.323            | 0.163     | 0.086            | 0.924     |
|                                | $p = \frac{n}{2}$ | -0.161           | -0.034    | 0.090            | 0.119     | 0.185            | 0.123     | 0.253            | 0.923     |
|                                | $p = n$           | -0.192           | -0.048    | 0.089            | 0.122     | 0.211            | 0.131     | 0.161            | 0.927     |
|                                | $p = 2n$          | -0.215           | -0.059    | 0.084            | 0.110     | 0.231            | 0.124     | 0.088            | 0.908     |

Table 2: Bias, SD, RMSE and CP of relevant estimators based on 1000 repetitions.

|                                       |                   | Bias             |           | SD               |           | RMSE             |           | CP               |           |
|---------------------------------------|-------------------|------------------|-----------|------------------|-----------|------------------|-----------|------------------|-----------|
|                                       |                   | $\hat{Q}_{aipw}$ | $\hat{q}$ | $\hat{Q}_{aipw}$ | $\hat{q}$ | $\hat{Q}_{aipw}$ | $\hat{q}$ | $\hat{Q}_{aipw}$ | $\hat{q}$ |
| DGP2: $\pi(X)$ is correctly specified |                   |                  |           |                  |           |                  |           |                  |           |
| $n = 200$                             | $p = \frac{n}{4}$ | -0.033           | -0.027    | 0.132            | 0.130     | 0.136            | 0.133     | 0.927            | 0.936     |
|                                       | $p = \frac{n}{2}$ | -0.028           | -0.017    | 0.139            | 0.138     | 0.142            | 0.139     | 0.898            | 0.909     |
|                                       | $p = n$           | -0.042           | -0.025    | 0.134            | 0.134     | 0.141            | 0.137     | 0.895            | 0.920     |
|                                       | $p = 2n$          | -0.043           | -0.023    | 0.131            | 0.130     | 0.138            | 0.132     | 0.914            | 0.931     |
| $n = 400$                             | $p = \frac{n}{4}$ | -0.014           | -0.012    | 0.097            | 0.097     | 0.098            | 0.098     | 0.921            | 0.911     |
|                                       | $p = \frac{n}{2}$ | -0.021           | -0.017    | 0.093            | 0.092     | 0.095            | 0.093     | 0.920            | 0.932     |
|                                       | $p = n$           | -0.022           | -0.015    | 0.098            | 0.096     | 0.100            | 0.098     | 0.905            | 0.921     |
|                                       | $p = 2n$          | -0.031           | -0.021    | 0.094            | 0.095     | 0.099            | 0.097     | 0.912            | 0.915     |
| $n = 800$                             | $p = \frac{n}{4}$ | -0.012           | -0.012    | 0.069            | 0.069     | 0.070            | 0.070     | 0.926            | 0.926     |
|                                       | $p = \frac{n}{2}$ | -0.013           | -0.013    | 0.068            | 0.067     | 0.069            | 0.069     | 0.922            | 0.921     |
|                                       | $p = n$           | -0.016           | -0.015    | 0.067            | 0.067     | 0.069            | 0.068     | 0.930            | 0.932     |
|                                       | $p = 2n$          | -0.023           | -0.019    | 0.070            | 0.069     | 0.073            | 0.072     | 0.894            | 0.897     |

Table 3: Comparison of SD and ESD of the proposed estimator  $\hat{q}$ 

|           |                   | DGP1  |       | DGP2  |       |
|-----------|-------------------|-------|-------|-------|-------|
|           |                   | SD    | ESD   | SD    | ESD   |
| $n = 200$ | $p = \frac{n}{4}$ | 0.196 | 0.204 | 0.130 | 0.123 |
|           | $p = \frac{n}{2}$ | 0.206 | 0.214 | 0.138 | 0.122 |
|           | $p = n$           | 0.212 | 0.220 | 0.134 | 0.121 |
|           | $p = 2n$          | 0.197 | 0.196 | 0.130 | 0.121 |
| $n = 400$ | $p = \frac{n}{2}$ | 0.155 | 0.152 | 0.097 | 0.087 |
|           | $p = \frac{n}{2}$ | 0.157 | 0.155 | 0.092 | 0.087 |
|           | $p = n$           | 0.160 | 0.162 | 0.096 | 0.087 |
|           | $p = 2n$          | 0.145 | 0.145 | 0.095 | 0.086 |
| $n = 800$ | $p = \frac{n}{2}$ | 0.114 | 0.115 | 0.069 | 0.062 |
|           | $p = \frac{n}{2}$ | 0.119 | 0.117 | 0.067 | 0.062 |
|           | $p = n$           | 0.122 | 0.123 | 0.067 | 0.062 |
|           | $p = 2n$          | 0.110 | 0.109 | 0.069 | 0.062 |

Table 4: Estimates and confidence intervals (CI) of  $m_1$ ,  $m_0$  and  $m$ 

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|          | $m_1$          | $m_0$          | $m$         |
|----------|----------------|----------------|-------------|
| Estimate | 308            | 260            | 48          |
| CI       | (292.1, 323.9) | (241.7, 278.3) | (21.6,74.4) |

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