Derived neighborhoods and frontier orders

Xavier Daragon[a,b] and Michel Couprie[a,b] and
Gilles Bertrand[a,b]

(a) Laboratoire A^2SI, ESIEE, B.P. 99, 93162 Noisy-Le-Grand Cedex France
(b) IGM, Unité Mixte de Recherche CNRS-UMLV-ESIEE UMR 8049

Abstract

We study some structural and topological properties of the frontiers of objects in a certain class of discrete spaces, in the framework of simplicial complexes and partial orders. In a previous work, we introduced the notion of frontier order, which allows to define the frontier of any object in an n-dimensional space. The main goal of this paper is to exhibit the links which exist between frontier orders and the notion of derived neighborhood as introduced in the framework of piecewise linear topology. In particular, we prove that the derived subdivision of the frontier order of an object X in a "regular" n-dimensional space is equal to the frontier of the derived neighborhood of X, and that this frontier is a union of $(n-1)$-dimensional surfaces, for any dimension n.

Key words: partially ordered sets, simplicial complexes, discrete surfaces, frontier orders, derived neighborhoods

Introduction

In many applications stemming from digital image processing, geometrical modeling and computer graphics, the notion of frontier of discrete objects plays a central role.

We are interested in certain topological and structural properties of frontiers. In the continuous space $\mathbb{R}^n$, we remark that the boundaries of certain "well behaved" subsets of $\mathbb{R}^n$, such as convex n-polytopes, are topological $(n-1)$-manifolds. In the framework of piecewise linear topology, we may define an n-*dimensional space* as a simplicial complex which is a combinatorial n-manifold,

Email addresses: `x.daragon@esiee.fr` (Xavier Daragon), `m.couprie@esiee.fr` (Michel Couprie), `g.bertrand@esiee.fr` (Gilles Bertrand).

and we call *object* any subcomplex of this space. Then, it is possible to prove that the boundary of a derived neighborhood of any object is a combinatorial $(n-1)$-manifold [1].

Several purely discrete frameworks have been used in order to study topological properties of objects in discrete spaces (see e.g. [2–7]) Here, we follow an approach based on the notions of (abstract) simplicial complex and partial order [8–11]. Instead of combinatorial manifolds, we consider the notion of n-dimensional surfaces (or n-surfaces for short) which has been introduced by Evako et al. [3,12,13]. The notion of combinatorial manifold is complicated, in particular, the problem of recognizing a combinatorial manifold is difficult. On the opposite, the recognition of an n-surface is straightforward.

In previous works [14,15], we introduced the notion of frontier order, which allows to define the frontier of any object in an n-dimensional space. The main goal of this paper is to exhibit the links which exist between frontier orders and the notion of derived neighborhood. In particular, we prove that the derived subdivision of the frontier order of any object X is equal to the frontier of the derived neighborhood of X. Our second main result is a theorem which may be stated informally as follows: the frontier of the derived neighborhood of any object in an n-surface is a union of disjoint $(n-1)$-surfaces, for any n.

1 Partially ordered sets and simplicial complexes

Partially ordered sets

Let us first introduce the notations that we will use in this article. If X is a set and S a subset of X, when no confusion may occur we denote by $\overline{S}$ the complement of S in X. We write $S \subset X$ if S is a subset of X and $S \neq X$, we write $S \subseteq X$ if $S \subset X$ or $S = X$. If λ is a binary relation on X, *i.e.*, a subset of the cartesian product $X \times X$, the *inverse of* λ is the binary relation $\{(x,y) \in X \times X, (y,x) \in \lambda\}$. For any binary relation λ, $\lambda^\square$ is defined by $\lambda^\square = \lambda \setminus \{(x,x), x \in X\}$. For each x of X, $\lambda(x)$ denotes the set $\{y \in X, (x,y) \in \lambda\}$ and for any subset S of X, $\lambda(S)$ denotes the set $\{y \in \lambda(s), s \in S\}$.
An *order* [10,16–18], also called *partially ordered set* or *poset*, is a pair $|X| = (X, \alpha_X)$ where X is a set and α_X is a reflexive, antisymmetric and transitive binary relation on X. For example, the simplicial complex depicted in Fig. 1-1 may be interpreted as an order: the elements of this order are the triangles, the edges and the vertices, and the relation α_X is the inclusion relation. Let x be an element of X, the set $\alpha_X(x)$ is called the α_X*-adherence of* x. We denote by β_X the inverse of α_X and by θ_X the union of α_X and β_X. The set $\theta_X(x)$ is called the θ_X*-neighborhood of* x, or simply the *neighborhood of* x when no confusion may arise. We say that two elements x, y of X are *neighbors*, or

comparable, if $y \in \theta_X(x)$. If $y \in \alpha_X(x)$ then we say that y *is under* x and that x *is above* y.
Let x_0 and x_n be two elements of X, a *path from* x_0 *to* x_n *in* $|X|$ is a sequence $x_0, \ldots, x_n$ of elements of X such that for all $i \in [1 \ldots n]$, $x_i \in \theta_X(x_{i-1})$. A *connected component of* $|X|$ is a subset C of X such that for all $x, y \in C$, there exists a path from x to y in C, and which is maximal for this property.
Let x be an element of the order $|X|$, the *rank of* x *in* $|X|$ is the number $\rho(x, |X|)$ such that $\rho(x, |X|) = 0$ if $\alpha_X^\square(x) = \emptyset$ and $\rho(x, |X|) = \mathrm{Max}\{\rho(y, |X|) + 1, y \in \alpha_X^\square(x)\}$ otherwise. The *rank of* $|X|$ is the number $\rho(|X|)$ such that $\rho(|X|) = \mathrm{Max}\{\rho(x, |X|), x \in X\}$. Any element of an order is called a *point* or an *n-element*, n being the rank of this point.
An order $|X|$ is *countable* if X is countable, it is *locally finite* if, for each $x \in X$, $\theta_X(x)$ is a finite set. A *CF-order* is a countable locally finite order. In the following, we consider only CF-orders.
Let $|X| = (X, \alpha_X)$ and $|Y| = (Y, \alpha_Y)$ be two orders, $|X|$ and $|Y|$ are *order isomorphic* if there exists a bijection $f : X \to Y$ such that, for all $x_1, x_2 \in X$, $x_1 \in \alpha_X(x_2) \Leftrightarrow f(x_1) \in \alpha_Y(f(x_2))$.
If $|X| = (X, \alpha_X)$ is an order and S is a subset of X, the *sub-order of* $|X|$ *relative to* S is the order (S, α_S), with $\alpha_S = \alpha_X \cap (S \times S)$. When no confusion may arise, we also denote by $|S|$ the order (S, α_S).

Simplicial complexes

Let Λ be a finite set, any non-empty subset of Λ is called a *simplex*. A simplex s constituted of $(n+1)$ elements of Λ is called an *n-simplex*. Any non-empty subset of a simplex s is called a *face of* s. A *proper face of* s is a face of s which is not equal to s. Let X be a family of simplexes of Λ, we say that X is a *simplicial complex* if it is closed by inclusion, which means that, if s belongs to X, then any face of s also belongs to X. Let X be a non-empty simplicial complex, we say that X is a *(simplicial) n-complex* if all the simplexes of X are m-simplexes with $m \leq n$, and if at least one simplex of X is an n-simplex. The subset of Λ which is the union of all the simplexes of X is called the *support of* X. The simplicial complexes we just defined are often known as *abstract simplicial complexes*, as opposed to other notions of complexes based upon an underlying Euclidean space.

To any simplicial complex X, we can associate a canonical order $|X| = (X, \alpha_X)$ where α_X is the inclusion relation: $t \in \alpha_X(s)$ means that $t \subseteq s$. In this paper, we will often refer to the canonical order associated to a simplicial complex, especially when it allows simpler formulations or proofs. Let X be a simplicial complex and let $s \in X$. We observe that $\alpha_X(s)$ does not depend on X since any simplicial complex is closed by inclusion. Thus, we will often write α instead of α_X when discussing about simplicial complexes. We say that the simplicial complex X *is connected* if the order $|X|$ is connected. We can easily see that for any n-simplex s of X, for any $n \geq 0$, we have $\rho(s, |X|) = n$.

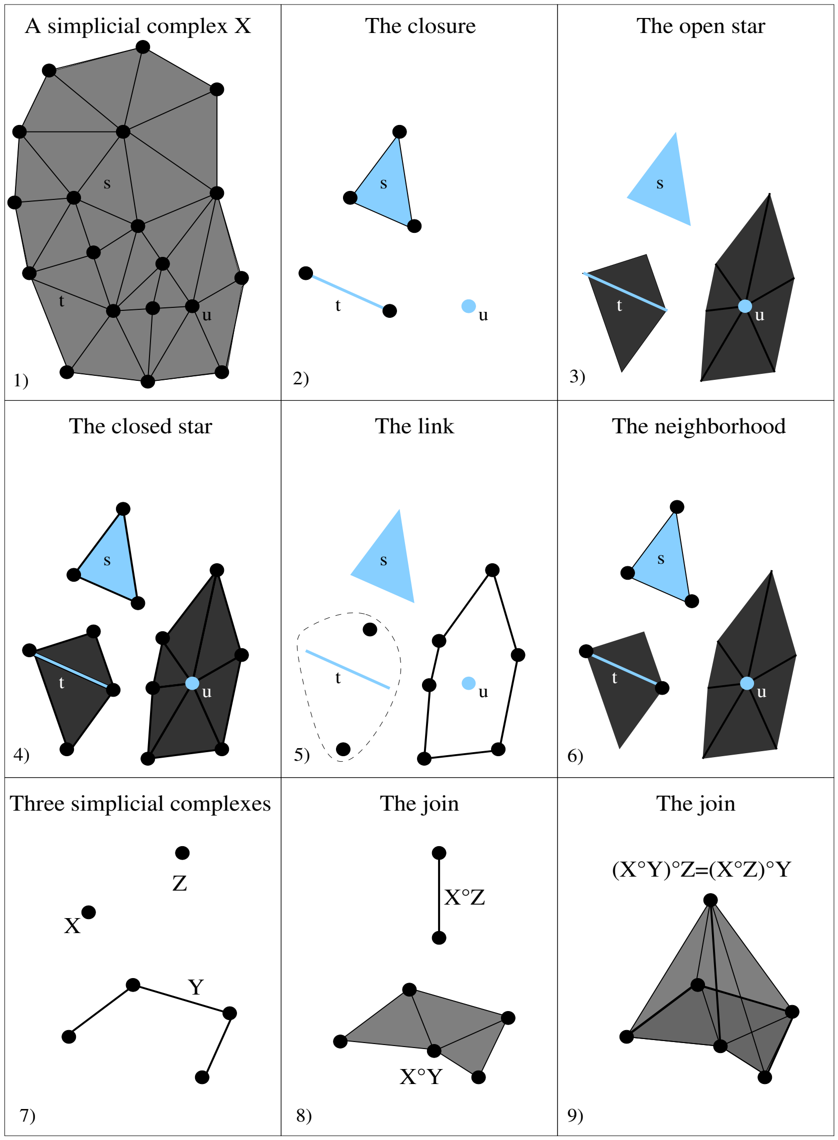


Fig. 1. Fundamental notions for simplicial complexes.
1) A simplicial complex X, in which s is a 2-simplex, t a 1-simplex and u a 0-simplex.
2) Depicts $\widehat{s}$, $\widehat{t}$ and $\widehat{u}$, which are equal to $\alpha_X(s)$, $\alpha_X(t)$ and $\alpha_X(u)$ respectively.
3) Depicts $star(s, X)$, $star(t, X)$ and $star(u, X)$, which are equal to $\beta_X(s)$, $\beta_X(t)$ and $\beta_X(u)$ respectively.
4) Depicts $\widehat{star}(s, X)$, $\widehat{star}(t, X)$ and $\widehat{star}(u, X)$, which are equal to $\alpha_X(\beta_X(s))$, $\alpha_X(\beta_X(t))$ and $\alpha_X(\beta_X(u))$ respectively.
5) Depicts s and $link(s, X)$ (which is empty), t and $link(t, X)$ (two isolated $0 - simplexes$) and u and $link(u, X)$.
6) Depicts $\theta_X(s)$, $\theta_X(t)$ and $\theta_X(u)$.
7) Depicts a 1-complex Y and two 0-complexes X and Z.
8) Depicts the 2-complex $X \circ Y$ and the 1-complex $X \circ Z$.
9) Depicts the 3-complex $(X \circ Y) \circ Z$, which is equal $(X \circ Z) \circ Y$.

The notions of boundary, open star, closed star, join and link are fundamental in the framework of simplicial complexes. We give below their definitions and their interpretations in terms of order. We show some illustrations in Fig. 1.

- Let s be a simplex, the *closure of* s, denoted by $\widehat{s}$, is the simplicial complex consisting of s and all its faces. In other words, $\widehat{s} = \alpha(s)$.
  By extension, if S is a set of simplexes, the *closure of* S denoted by $\widehat{S}$ is the union of the closures of its simplexes. In other words, $\widehat{S} = \alpha(S)$.
- Let s be a simplex, the *boundary of* s is constituted by all the proper faces of s, it is equal to $\alpha^{\Box}(s)$.
- Let s be a simplex of a simplicial complex X; the *(open) star of* s *in* X is defined as $star(s, X) = \{t \in X, s \subseteq t\}$. Thus $star(s, X)$ is equal to $\beta_X(s)$. The *closed star of* s *in* X is defined as the closure of the star of s in X. In terms of order, we have $\widehat{star}(s, X) = \alpha_X(\beta_X(s))$. Notice that the closed star is always a simplicial complex while the open star is not.
- Two simplexes are *joinable* if their intersection is empty. If s and t are joinable simplexes, the *simplicial join of* s *and* t is defined as $s \circ t = s \cup t$. Two simplicial complexes X and Y are said to be *joinable* if every simplex of X is joinable with every simplex of Y; thus X and Y are joinable if and only if the intersection of their supports is empty. If X and Y are joinable, the *(simplicial) join of* X *and* Y is defined as $X \circ Y = X \cup Y \cup \{s \circ t, s \in X, t \in Y\}$. It can easily be seen that the join of two simplicial complexes is always a simplicial complex, and that the join operation is associative and commutative.
- Let s be a simplex of a simplicial complex X; the *link of* s *in* X is defined as the set of all simplexes t in X such that the join of t and s belongs to X, that is, $link(s, X) = \{t \in X, s \circ t \in X\}$. It can be easily seen that the link of a simplex in a simplicial complex is always a (sometimes empty) simplicial complex. In terms of order relation, the link of s in X is order isomorphic to $\beta_X^{\Box}(s)$, as proved in [19].

2 Discrete surfaces

Definition of n-surfaces in the framework of orders

The main results of this article are based on a notion of n-dimensional discrete surface proposed by Evako, Kopperman and Mukhin [3,12,13]. Such n-dimensional surfaces have been proved to verify discrete analogs of the Jordan-Brouwer theorem in $\mathbb{Z}^2$ [20] and $\mathbb{Z}^3$ [21] equipped with the Khalimsky topology [16].

Let $|X| = (X, \alpha_X)$ be a non-empty CF-order.
- The order $|X|$ is a 0-*surface* if X is composed of exactly two points x and y

such that $y \notin \alpha_X(x)$ and $x \notin \alpha_X(y)$.
• The order $|X|$ is an *n-surface*, $n > 0$, if $|X|$ is connected and if, for each x in X, the order $|\theta_X^{\Box}(x)|$ is an $(n-1)$-surface.
For technical reasons, we will say that $|X|$ is a (-1)-*surface* if $X = \emptyset$.

Definition of n-surfaces in the framework of simplicial complexes

We say that a simplicial complex C is an *n-surface*, for any $n \in \mathbb{N}$, if the order $(C, \subseteq)$ is an n-surface. The following property shows that, in the framework of simplicial complexes, n-surfaces may be characterized by a simpler condition based on the link operator.

Property 1 *A non-empty simplicial complex C is an n-surface, $n > 0$, if and only if C is connected and, for each 0-simplex s in C, $link(s, C)$ is an $(n-1)$-surface.*

The proof of this property is based on the two following properties, which we also use later in this article:

Property 2 *Let $|X| = (X, \alpha_X)$ be an order. Then, $|X|$ is an n-surface if and only if, for any x in X, $|\alpha_X^{\Box}(x)|$ is a $(k-1)$-surface and $|\beta_X^{\Box}(x)|$ is an $(n-k-1)$-surface, with $k = \rho(x, |X|)$.*

Property 3 *Let S be an n-simplex, then $\alpha^{\Box}(S)$ is an $(n-1)$-surface.*

Prop. 1, Prop. 2 and Prop. 3 are proved in [19].

Theorems related to n-surfaces and simplicial complexes

The following theorem is an important tool for demonstrating properties related to n-surfaces in the framework of simplicial complexes. Results similar to Th. 4 have been obtained by Evako et al. [12] in a framework based on graphs, and by ourselves in the framework of orders [19].

Theorem 4 *Let the simplicial complexes C_1 and C_2 be, respectively, an n-surface and an m-surface ($n, m \geq 0$). Then the simplicial complex $C = C_1 \circ C_2$ is an $(n + m + 1)$-surface.*

Proof: Let us first consider the case where C_1 and C_2 are both 0-surfaces, then any point of C has a link composed of two isolated points, thus C is a 1-surface (the connectedness is obvious).
Assume now that the property is true for every n and m such that $n + m \leq d$, $d \geq 0$, and let us prove it for $(n + 1)$ and m (which, by symmetry, will also prove it for n and $(m + 1)$, and, by induction, for any $n, m \geq 0$):
• Let x be a 0-simplex of C, according to the definition of the join operator, x is either a 0-simplex of C_1 or a 0-simplex of C_2.

• If x is a simplex of C_1, then $link(x, C) = link(x, C_1) \circ C_2$ (see lemma 14 in the annex). Since $link(x, C_1)$ is an n-surface (by Prop. 1, C_1 being an $(n+1)$-surface) and C_2 is an m-surface (by hypothesis), $link(x, C)$ is an $(n+m+1)$-surface (by induction hypothesis).
• If x is a simplex of C_2, then $link(x, C) = link(x, C_2) \circ C_1$ (still according to lemma 14). Thus, either C_2 is a 0-surface, in which case $link(x, C) = C_1$ is an $(n+1) = (n+m+1)$-surface, or $link(x, C_2)$ is an $(m-1)$ surface, in which case $link(x, C)$ is an $(n+m+1)$-surface (by induction hypothesis).
• Moreover, the connectedness of C is guaranteed by the definition of the simplicial join, thus, by Prop. 1, C is an $(n+m+2)$-surface: the property is true for $(n+1)$ and m. $\square$

3 Subcomplex, border and frontier

Let X be a simplicial complex, and let Y be a subset of X. If Y is a simplicial complex then it is called a *subcomplex of* X.

Let X be a simplicial complex with support Λ, and let Y be a subcomplex of X, with support $\Lambda' \subseteq \Lambda$. We say that Y is a *full subcomplex of* X if every simplex of X which is a subset of Λ' also belongs to Y. The notions of subcomplex and full subcomplex are illustrated in Fig. 2.

One can easily verify the following property, which states that there is a unique full subcomplex associated to each subset of the support of a simplicial complex.

Property 5 *Let X be a simplicial complex of support Λ. Let Λ' be a subset of Λ. The subcomplex Y of X defined by $Y = \{y \in X, y \subseteq \Lambda'\}$ is the unique full subcomplex of X with support Λ'.*

Let X be a simplicial complex with support Λ, and let Y be a subcomplex of X, with support $\Lambda' \subseteq \Lambda$. The *simplicial complement of* Y *in* X, denoted by $\mathrm{compl}(Y, X)$ or simply by $\tilde{Y}$ when no confusion may occur, is the simplicial complex composed of all the simplexes of X which are subsets of $\Lambda \setminus \Lambda'$, that is, $\tilde{Y} = \mathrm{compl}(Y, X) = \{s \in X, s \subseteq \Lambda \setminus \Lambda'\}$. We can easily see that the previous expression indeed defines a simplicial complex, the support of which is $\Lambda \setminus \Lambda'$. The simplicial complement of Y can also be expressed as $\tilde{Y} = \{s \in X$, Y does not contain any face of $s\}$. The notion of simplicial complement is illustrated in Fig. 2-d,e.

We can deduce from Prop. 5 that the simplicial complement of any subcomplex of X is a full subcomplex of X. Furthermore, since $\tilde{\tilde{Y}} = \{s \in X, s \subseteq \Lambda'\}$, the following property also follows easily from Prop. 5 (see also Fig. 2-d,e).

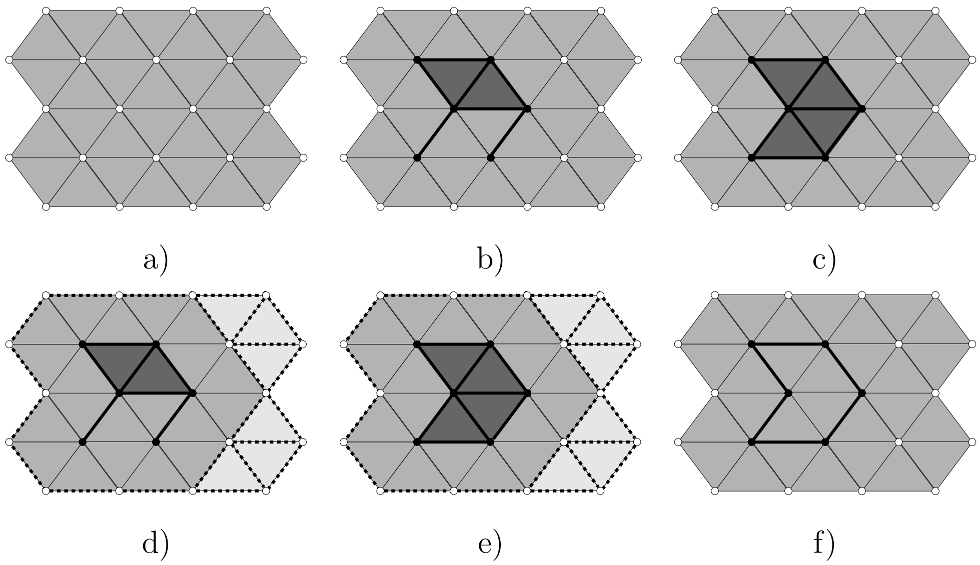
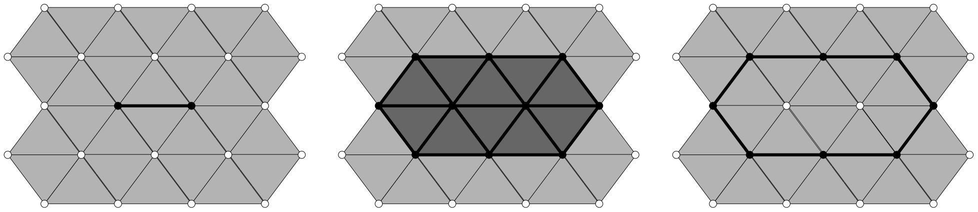


Fig. 2. Subcomplex, full subcomplex, simplicial complement and border.
a) A simplicial complex X.
b) A subcomplex Y_1 of X (black dots, bold edges and dark triangles), which is not a full subcomplex of X.
c) A full subcomplex Y_2 of X (black dots, bold edges and dark triangles).
d) The simplicial complement $\tilde{Y}_1$ of Y_1 (white dots, dotted edges and light triangles).
e) The simplicial complement $\tilde{Y}_2$ of Y_2 (white dots, dotted edges and light triangles), which is equal to $\tilde{Y}_1$ since Y_1 and Y_2 have the same support. Notice that $\tilde{\tilde{Y}}_2 = Y_2$ (see Prop. 6).
f) The border $\delta(Y_2)$ (black dots, bold edges).

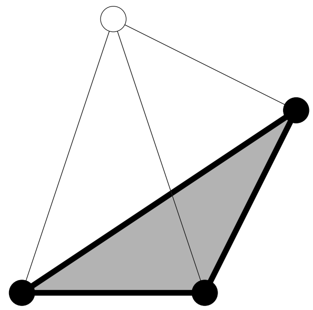
Property 6 *Let X be a simplicial complex, and let Y be a subcomplex of X. We have $\tilde{\tilde{Y}} = Y$ if and only if Y is a full subcomplex of X.*

Let X be a simplicial complex, and let Y be a subcomplex of X. The *border of Y in X* is the set of elements of Y which are neighbors of some element of $X \setminus Y$, in other words, the set $\delta(Y, X) = \{y \in Y, \theta_X(y) \cap (X \setminus Y) \neq \emptyset\}$. It may be easily seen that
$\delta(Y, X) = \{y \in Y, \beta_X(y) \cap (X \setminus Y) \neq \emptyset\} = Y \setminus \{y \in Y, \beta_X(y) \subseteq Y\}$.
When no confusion may occur, we omit the reference to X and we write $\delta(Y) = \delta(Y, X)$. It can easily be seen that the border of any subcomplex of X is a simplicial complex. In Fig. 2-f, we see the border of the subcomplex of Fig. 2-c.

We can see that any subcomplex Y of a subcomplex X gives birth to five remarkable sets of simplexes: Y, $\delta(Y)$, $\tilde{Y}$, $\delta(\tilde{Y})$ which are subcomplexes of X, and the reminder $X \setminus (Y \cup \tilde{Y})$ (in Fig. 2-d,e, this reminder is depicted by medium gray triangles and thin edges). We denote by $\Delta(Y, X)$, or simply by $\Delta(Y)$ when no confusion may occur, the set $X \setminus (Y \cup \tilde{Y})$. Obviously, $\Delta(Y)$ is not a simplicial complex, thus it is not a subcomplex of X. The order $|\Delta(Y)| = (\Delta(Y), \subseteq)$ is named the *frontier order relative to Y in X*. By abuse



a) b) c)



d)

Fig. 3. Simplicial neighborhood and its border.
a) A simplicial complex X (all the triangles, edges and vertices) and a full subcomplex Y of X (one bold edge and two black vertices).
b) In dark grey and bold black, $N(Y)$.
c) In bold black, $\delta(N(Y))$. We can see that $\delta(N(Y)) = \alpha_X(\beta_X(Y)) \setminus \beta_X(Y)$.
d) A complex X composed of the proper faces of a 3-simplex (tetraedron), and a subcomplex Y of S (in dark grey and bold black). We can see that $\delta(N(Y))$ is empty, while $\alpha_X(\beta_X(Y)) \setminus \beta_X(Y)$ is composed of one 0-simplex (in white).

of terminology, we also call *frontier order* the set $\Delta(Y)$. It should be noted that the notion of frontier order may be extended to any CF-order, and that this definition is equivalent, up to an order isomorphism, to the definition proposed in [15].

We can easily deduce from Prop. 6 that, if Y is a full subcomplex of X, then the frontier order $\Delta(Y)$ is "symmetrical" between Y and $\tilde{Y}$, that is, $\Delta(Y) = \Delta(\tilde{Y})$.

Let Y be a subcomplex of the simplicial complex X, the *simplicial neighborhood of* Y *in* X is defined as the union of the closed stars of the simplexes of Y in X, that is, $N(Y, X) = \bigcup_{s \in Y} \widehat{star}(s, X)$. When no confusion may occur, we write $N(Y) = N(Y, X)$. In terms of order relation, $N(Y, X) = \alpha_X(\beta_X(Y))$. The notion of simplicial neighborhood is illustrated in Fig. 3-a,b.

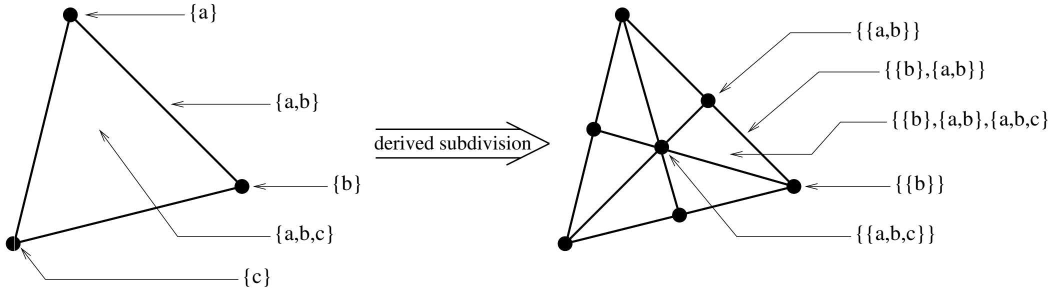


Fig. 4. Graphical illustration of the notion of derived subdivision. Left: the initial complex X composed of the closure of the simplex $\{a, b, c\}$. Right: the subdivision X^1 constituted by the chains of X.

4 Subdivision, derived neighborhoods and derived frontiers

In the previous section, we defined the border $\delta(Y)$ of a subcomplex Y of a complex X. We saw that $\delta(Y)$ is always a simplicial complex, but this border is not symmetrical between Y and $\tilde{Y}$, more precisely, $\delta(Y) \neq \delta(\tilde{Y})$. On the other hand, we introduced the frontier order of Y, which is symmetrical but which is not a simplicial complex. The subdivision operation will allow us to define the derived frontier, which is both a simplicial complex and symmetrical between Y and $\tilde{Y}$.

The notion of derived subdivision, that we present now, is especially interesting for us since it can be applied not only to simplicial complexes, but more generally to any partially ordered set.
Let $|X|$ be an order, a *chain of* $|X|$ is a fully ordered non-empty subset of X, *i.e.*, a non-empty subset Y of X such that any two elements of Y are comparable. An *n-chain* is a chain composed of $n + 1$ elements.
The *derived subdivision of* $|X|$ if the set, denoted by X^1, constituted by all the chains of $|X|$. The notion of derived subdivision is illustrated in Fig. 4. Notice that for any order (X, α_X), the derived subdivision X^1 is always a simplicial complex, the support of which is X. We also call X^1 the *chain complex of* X. Let X be a simplicial complex, the *derived subdivision of* X is the derived subdivision X^1 of the order $(X, \subseteq)$.

It can be easily verified that for any two orders $|Y|, |Z|$ we have $[Y \cap Z]^1 = Y^1 \cap Z^1$, but in general $[Y \cup Z]^1 \neq Y^1 \cup Z^1$ and $[Y \setminus Z]^1 \neq Y^1 \setminus Z^1$. Furthermore, if Y and Z are simplicial complexes, then we have $[Y \cap Z]^1 = Y^1 \cap Z^1$ and $[Y \cup Z]^1 = Y^1 \cup Z^1$, but in general $Y \setminus Z$ is not a simplicial complex.

Let X be a simplicial complex, and let Y be a subcomplex of X. The *derived neighborhood of* Y *in* X is defined as the simplicial neighborhood of Y^1 in X^1, that is: $N(Y^1, X^1) = \bigcup_{y^1 \in Y^1} \widehat{star}(y^1, X^1) = \alpha_{X^1}(\beta_{X^1}(Y^1))$ (see Fig. 5). When no confusion may occur, we simply write $N(Y^1) = N(Y^1, X^1)$.

Observe that $\beta_{X^1}(Y^1)$ is composed of the chains of X which contain at least one simplex of Y, that is,

$$\beta_{X^1}(Y^1) = \{c \in X^1, \exists y \in c, y \in Y\} \tag{1}$$

The following lemma gives us an expression of $N(Y^1)$ which will be useful in the sequel.

Lemma 7 *Let X be a simplicial complex, let Y be a subcomplex of X and let Λ' be the support of Y. Then, we have $N(Y^1) = \{c \in X^1, \forall x \in c, x \cap \Lambda' \neq \emptyset\}$.*

Proof: Observe that $N(Y^1) = \alpha_{X^1}(\beta_{X^1}(Y^1)) =$
$\{c \in X^1, \exists c' \in \beta_{X^1}(Y^1), c \subseteq c'\} =$
$\{c \in X^1, \exists c' \in X^1, \exists y \in Y, y \in c', c \subseteq c'\}$ (from (1)).
If c' is a chain of X^1 which contains y and includes c, then we see easily that $c \cup \{y\}$ is also a chain of X^1 which contains y and includes c, thus $N(Y^1) = \{c \in X^1, \exists y \in Y, c \cup \{y\} \in X^1\} = \{c \in X^1, \forall x \in c, x \cap \Lambda' \neq \emptyset\}$ (any x of c either is included in y or includes y, in both cases $x \cap \Lambda' \neq \emptyset$). $\square$

Let us now focus on the border of the neighborhood of a full subcomplex. We can see in Fig. 3-a,b,c a simple case, where $\delta(N(Y))$ can be expressed as $\alpha_X(\beta_X(Y)) \setminus \beta_X(Y)$. It can easily be proved that for any full subcomplex Y of a simplicial complex, we have $\delta(N(Y)) \subseteq \alpha_X(\beta_X(Y)) \setminus \beta_X(Y)$. The converse is false in general, see a counter-example in Fig. 3-d. The following lemma shows that the equality holds for the border of the derived neighborhood.

Lemma 8 *Let X be a simplicial complex and let Y be a full subcomplex of X. We have $\delta(N(Y^1)) = \alpha_{X^1}(\beta_{X^1}(Y^1)) \setminus \beta_{X^1}(Y^1)$.*

Proof: From the very definitions of the border and the simplicial neighborhood, we see that $\delta(N(Y^1)) = \alpha_{X^1}(\beta_{X^1}(Y^1)) \setminus A$, where
$A = \{c \in \alpha_{X^1}(\beta_{X^1}(Y^1)), \beta_{X^1}(c) \subseteq \alpha_{X^1}(\beta_{X^1}(Y^1))\}$.
We have to prove that $A = \beta_{X^1}(Y^1)$. Let $c \in \beta_{X^1}(Y^1)$, thus $\beta_{X^1}(c) \subseteq \beta_{X^1}(Y^1)$ and obviously $\beta_{X^1}(c) \subseteq \alpha_{X^1}(\beta_{X^1}(Y^1))$, thus $\beta_{X^1}(Y^1) \subseteq A$.
Conversely, let $c \in A$, and suppose that c does not belong to $\beta_{X^1}(Y^1)$. Let x be the lowest element of c. Let Λ' be the support of Y. From lemma 7 we know that $x \cap \Lambda' \neq \emptyset$. Moreover, since $c \notin \beta_{X^1}(Y^1)$, we can see (from (1)) that $x \in X \setminus Y$. Thus, x is not a 0-simplex of X and some 0-simplex $y_0 \in Y$ must exist such that $y_0 \subset x$. However, if every 0-simplex x_0 of X such that $x_0 \subset x$ were to belong to Y, since Y is a full subcomplex we would have $x \in Y$. Thus, some 0-simplex $x_0 \in X \setminus Y$ exists such that $x_0 \subset x$. Then, $\{x_0\} \cup c$ belongs to $\beta_{X^1}(c)$ (it obviously contains c, and since x is the lowest element of c, it is indeed a chain) but not to $\alpha_{X^1}(\beta_{X^1}(Y^1))$ (according to lemma 7,

since $x_0 \cap \Lambda' = \emptyset$), a contradiction. $\Box$

Notice that the latter property does not hold if Y is not a full subcomplex. A counter-example is given in Fig. 6.

From the previous lemma, we derive a property which highlights the symmetry of the border of $N(Y^1)$ between Y and $\tilde{Y}$ (see Fig. 5-d,e,f).

Property 9 *Let X be a simplicial complex and let Y be a full subcomplex of X. We have $\delta(N(Y^1)) = N(Y^1) \cap N(\tilde{Y}^1)$.*

Proof: Let Λ be the support of X, let Λ' be the support of Y, and let $\Lambda'' = \Lambda \setminus \Lambda'$. From lemma 8, we have $\delta(N(Y^1)) = \alpha_{X^1}(\beta_{X^1}(Y^1)) \setminus \beta_{X^1}(Y^1)$, thus $\delta(N(Y^1)) = N(Y^1) \cap [X^1 \setminus \beta_{X^1}(Y^1)]$. We see that:
$X^1 \setminus \beta_{X^1}(Y^1) = \{c \in X^1, \forall x \in c, x \not\subseteq \Lambda'\}$ (from (1)), thus
$X^1 \setminus \beta_{X^1}(Y^1) = \{c \in X^1, \forall x \in c, x \cap \Lambda'' \neq \emptyset\} = N(\tilde{Y}^1)$ (by lemma 7);
and thus $\delta(N(Y^1)) = N(Y^1) \cap N(\tilde{Y}^1)$. $\Box$

Let X be a simplicial complex, and let Y be a full subcomplex of X. Recall that the frontier order of Y in X has been defined as $\Delta(Y) = X \setminus (Y \cup \tilde{Y})$. The *derived frontier of Y in X* is defined as the derived subdivision of the frontier order of Y in X, that is: $[\Delta(Y)]^1$.

The following result shows a strong link between the notion of derived neighborhood and the notions of frontier order and derived frontier.

Theorem 10 *Let X be a simplicial complex and let Y be a full subcomplex of X. The border of the derived neighbohood of Y is equal to the derived frontier of Y, that is: $\delta(N(Y^1)) = [\Delta(Y)]^1$.*

Proof:
Let Λ be the support of X, let Λ' be the support of Y, and let $\Lambda'' = \Lambda \setminus \Lambda'$.
Using Prop. 9 and lemma 7 we see that $\delta(N(Y^1)) = N(Y^1) \cap N(\tilde{Y}^1) =$
$\{c \in X^1, \forall x \in c, x \cap \Lambda' \neq \emptyset\} \cap \{c \in X^1, \forall x \in c, x \cap \Lambda'' \neq \emptyset\} =$
$\{c \in X^1, \forall x \in c, x \cap \Lambda' \neq \emptyset$ and $x \cap \Lambda'' \neq \emptyset\} =$
$\{c \in X^1, \forall x \in c, x \notin Y$ and $x \notin \tilde{Y}\} =$
$[X \setminus (Y \cup \tilde{Y})]^1 = [\Delta(Y)]^1$. $\Box$

5 Derived neighborhoods and n-surfaces

In this section we present the second main result of this paper, which states that the border of the derived neighborhood of any full subcomplex of an n-surface is composed of disjoint $(n-1)$-surfaces.

The following theorem, which reveals a strong link between the structure of an order and the structure of its chain complex, will be used to obtain this result.

Property 11 *Let $|X|$ be an order. If $|X|$ is an n-surface then the simplicial complex X^1 is an n-surface.*

Proof: Let $|X|$ be a 0-surface, then X is of the form $\{a, b\}$, thus $X^1 = \{\{a\}, \{b\}\}$ is a 0-surface. Let us now suppose that the property is true for all k such that $0 \leq k < n$, and let us prove it for n. Since X^1 is a connected simplicial complex (the connectedness of X^1 is a direct consequence of the connectedness of X), it is sufficient (by Prop. 1) to prove that the *link* of any 0-simplex $s = \{x\}$ of X^1 is an $(n-1)$-surface. By the definition of the link, we have $link(s, X^1) = \{c \in X^1, c \circ s \in X^1\}$. Since $c \circ s$ is a chain, any element y of c is comparable to x. Note also that, if y is under x, then any z above x is also above y. So any chain of $link(s, X^1)$ can be expressed either as a chain of elements stricly under x, a chain of elements stricly above x, or as the join (union) of a chain of elements strictly under x and a chain of elements strictly above x (and any such chain obviously belongs to $link(s, X^1)$); thus: $link(s, X^1) = [\alpha_X^{\square}(x)]^1 \circ [\beta_X^{\square}(x)]^1$. By property 2, we know that $\alpha_X^{\square}(x)$ is a $(k-1)$-surface and that $\beta_X^{\square}(x)$ is an $(n-k-1)$-surface, with $k = \rho(x, |X|)$. Then, by induction hypothesis, $[\alpha_X^{\square}(x)]^1$ is a $(k-1)$-surface and $[\beta_X^{\square}(x)]^1$ is an $(n-k-1)$-surface, and by Th. 4, $link(s, X^1)$ is an $(n-1)$-surface. $\square$

Before proving our main result, let us first consider the case where the complex X is the boundary of an n-simplex.

Property 12 *Let S be an n-simplex with $n > 1$, let X be the boundary of S, and let Y be a full subcomplex of X. Then, $\delta(N(Y^1, X^1))$ is an $(n-2)$-surface.*

Proof:
Let Λ be the support of X, let Λ' be the support of Y, and let $\Lambda'' = \Lambda \setminus \Lambda'$. Let us first consider the case where S is a 2-simplex $\{a, b, c\}$. We can assume that $\Lambda' = \{a\}$ (the case $\Lambda' = \{b, c\}$ is similar) and then $\delta(N(Y^1, X^1)) = \{\{\{a, b\}\}, \{\{a, c\}\}\}$, which is a 0-surface.
Let us now suppose that the property is true for any i-simplex, with $2 \leq i < n$, and let us prove it for an n-simplex.
• We first need to prove that the link of any 0-simplex in $\delta(N(Y^1, X^1))$ is an $(n-3)$-surface. Let $s = \{x\}$ be such a 0-simplex. Remind that, according to Th. 10:

$$\delta(N(Y^1, X^1)) = \{c \in X^1, \forall z \in c, z \cap \Lambda' \neq \emptyset \text{ and } z \cap \Lambda'' \neq \emptyset\} \quad (2)$$

Thus x is a k-simplex of X such that $\Lambda' \cap x \neq \emptyset$ and $\Lambda'' \cap x \neq \emptyset$ (and obviously, $0 < k < n$).

By definition:
$link(s, \delta(N(Y^1, X^1))) = \{c \in \delta(N(Y^1, X^1)), c \circ s \in \delta(N(Y^1, X^1))\}$.
In other terms, $link(s, \delta(N(Y^1, X^1)))$ is composed by all the elements c of X^1 such that for all $z \in c$, we have $z \in \theta^\square_X(x)$, $\Lambda' \cap z \neq \emptyset$ and $\Lambda'' \cap z \neq \emptyset$. It should be noted that any element w of X above x verifies both $\Lambda' \cap w \neq \emptyset$ and $\Lambda'' \cap w \neq \emptyset$. So, since $[\beta^\square_X(x)]^1 \subseteq \delta(N(Y^1, X^1))$, any element of $link(s, \delta(N(Y^1, X^1)))$ can be expressed either as an element of $[\beta^\square_X(x)]^1$, an element of $\delta(N(Y^1, X^1)) \cap [\alpha^\square(x)]^1$, or as the simplicial join of an element of $[\beta^\square_X(x)]^1$ and an element of $\delta(N(Y^1, X^1)) \cap [\alpha^\square(x)]^1$. Thus,

$$link(s, \delta(N(Y^1, X^1))) = [\beta^\square_X(x)]^1 \circ (\delta(N(Y^1, X^1)) \cap [\alpha^\square(x)]^1) \tag{3}$$

Then (from (2)):

$$\begin{aligned}\delta(N(Y^1, X^1)) \cap [\alpha^\square(x)]^1 &= \{c \in X^1, \forall z \in c, z \cap \Lambda' \neq \emptyset, z \cap \Lambda'' \neq \emptyset\} \cap [\alpha^\square(x)]^1 \\ &= \{c \in [\alpha^\square(x)]^1, \forall z \in c, z \cap \Lambda' \neq \emptyset, z \cap \Lambda'' \neq \emptyset\} \\ &= \delta(N([Y \cap \alpha^\square(x)]^1, [\alpha^\square(x)]^1)) \qquad (4)\end{aligned}$$

Since X is an $(n-1)$-surface (Prop. 3), we deduce from Prop. 2 and Prop. 11 that $\beta^\square_X(x)$ and $[\beta^\square_X(x)]^1$ are $(n-k-2)$-surfaces. It can be easily verified that $Y \cap \alpha^\square(x)$ is a full subcomplex of $\alpha^\square(x)$, furthermore $\alpha^\square(x)$ is the boundary of a k-simplex with $k < n$. Thus, by induction hypothesis at rank $k < n$, $\delta(N([Y \cap \alpha^\square(x)]^1, [\alpha^\square(x)]^1))$ is a $(k-2)$-surface. Consequently, by (3), (4) and Th. 4, we deduce that the link of any 0-simplex of $\delta(N(Y^1, X^1))$ is an $(n-3)$-surface.
• We must now prove that $\delta(N(Y^1, X^1))$ is connected. Let s_i and s_j be two elements of $\delta(N(Y^1, X^1))$, let x_i be a simplex of s_i and let x_j be a simplex of s_j. Then, there exist four elements of Λ (not necessarily distinct) a, b, c and d such that $a \in x_i \cap \Lambda'$, $b \in x_i \cap \Lambda''$, $c \in x_j \cap \Lambda'$ and $d \in x_j \cap \Lambda''$. Then, it can be verified that $\{s_i$, $\{x_i\}$, $\{\{a,b\}, x_i\}$, $\{\{a,b\}\}$, $\{\{a,b\},\{a,b,c\}\}$, $\{\{a,b,c\}\}$, $\{\{b,c\},\{a,b,c\}\}$, $\{\{b,c\}\}$, $\{\{b,c\},\{b,c,d\}\}$, $\{\{b,c,d\}\}$, $\{\{c,d\},\{b,c,d\}\}$, $\{\{c,d\}\}$, $\{\{c,d\}, x_j\}$, $\{x_j\}$, $s_j\}$ is a path from s_i to s_j in $\delta(N(Y^1, X^1))$.
Since $\delta(N(Y^1, X^1))$ is connected and the link of each of its 0-simplexes is an $(n-3)$-surface, $\delta(N(Y^1, X^1))$ is an $(n-2)$-surface (by Prop. 1). $\square$

We can now prove the main result of this section.

Theorem 13 *Let X be a simplicial complex which is an n-surface, with $n > 0$, and let Y be a full subcomplex of X. Then, each connected component of $\delta(N(Y^1, X^1))$ is an $(n-1)$-surface.*

Proof:
Let Λ be the support of X, let Λ' be the support of Y, and let $\Lambda'' = \Lambda \setminus \Lambda'$. Let s be a 0-simplex of $\delta(N(Y^1, X^1))$, $s = \{x\}$ where x is a k-simplex of X, with

$0 < k \leq n$. The link of s in $\delta(N(Y^1, X^1))$ is constituted by all the elements c of X^1 such that for all $z \in c$, we have $z \in \theta_X^\square(x)$, $\Lambda' \cap z \neq \emptyset$ and $\Lambda'' \cap z \neq \emptyset$ (see the proof of Prop. 12). Each of those chains c can be expressed either as an element of $[\beta_X^\square(x)]^1$, an element of $\delta(N([Y \cap \alpha^\square(x)]^1, [\alpha^\square(x)]^1))$, or as the join of an element of $[\beta_X^\square(x)]^1$ and an element of $\delta(N([Y \cap \alpha^\square(x)]^1, [\alpha^\square(x)]^1))$ (see again the proof of Prop. 12).

- Since X is an n-surface, $\beta_X^\square(x)$ is an $(n-k-1)$-surface, and so is $[\beta_X^\square(x)]^1$.
- By Prop. 12, $\delta(N([Y \cap \alpha^\square(x)]^1, [\alpha^\square(x)]^1))$ is a $(k-2)$-surface.
- Thus, $link(s, \delta(N(Y^1, X^1))) = [\beta_X^\square(x)]^1 \circ \delta(N([Y \cap \alpha^\square(x)]^1, [\alpha^\square(x)]^1))$ is an $(n-2)$-surface by Th. 4.

Consequently, each connected component of $\delta(N(Y^1, X^1))$ is an $(n-1)$-surface.
$\square$

6 Conclusion

The results presented in this paper clarify the links between the notion of frontier order that we introduced in anterior articles and the notion of derived neighborhood as introduced in the framework of piecewise linear topology. Furthermore, they also constitute new results about derived neighborhoods, since the notion of n-surface had not been studied in this framework until now. In a forthcoming article, we deepen the discussion about different frameworks for discrete surfaces, in particular combinatorial manifolds, n-surfaces and pseudo-manifolds, and prove a theorem which establishes inclusion relations between these three classes of discrete n-dimensional surfaces (for any n).

Annex

Lemma 14 *Let C_1 and C_2 be simplicial complexes. Let x be an element of $C_1 \circ C_2$. If $x \in C_1$ (resp. $x \in C_2$), then $link(x, C_1 \circ C_2)$ is equal to $link(x, C_1) \circ C_2$ (resp. $C_1 \circ link(x, C_2)$). If $x = x_1 \circ x_2$, with $x_1 \in C_1$ and $x_2 \in C_2$, then $link(x, C_1 \circ C_2) = link(x_1, C_1) \circ link(x_2, C_2)$.*

Proof: From the definitions of the link and the join, we have:

$$\begin{aligned} link(x, C_1 \circ C_2) &= \{t \in C_1 \circ C_2, x \circ t \in C_1 \circ C_2\} \\ &= \{t \in C_1, x \circ t \in C_1 \circ C_2\} \cup \{t \in C_2, x \circ t \in C_1 \circ C_2\} \\ &\quad \cup \{t = t_1 \circ t_2, t_1 \in C_1, t_2 \in C_2, x \circ t_1 \circ t_2 \in C_1 \circ C_2\} \end{aligned}$$

Then, if $x \in C_1$, we obtain:

$$
\begin{aligned}
link(x, C_1 \circ C_2) &= \{t \in C_1, x \circ t \in C_1\} \cup \{t \in C_2\} \\
&\quad \cup \{t = t_1 \circ t_2, t_1 \in C_1, t_2 \in C_2, x \circ t_1 \in C_1\} \\
&= link(x, C_1) \cup C_2 \cup \{t = t_1 \circ t_2, t_1 \in link(x, C_1), t_2 \in C_2\} \\
&= link(x, C_1) \circ C_2
\end{aligned}
$$

Similarly, with $x \in C_2$ we would obtain $link(x, C_1 \circ C_2) = C_1 \circ link(x, C_2)$. Now, if $x = x_1 \circ x_2$, with $x_1 \in C_1$ and $x_2 \in C_2$, we have:

$$
\begin{aligned}
link(x, C_1 \circ C_2) &= \{t \in C_1, x_1 \circ t \in C_1\} \cup \{t \in C_2, x_2 \circ t \in C_2\} \\
&\quad \cup \{t = t_1 \circ t_2, t_1 \in C_1, t_2 \in C_2, (x_1 \circ x_2) \circ (t_1 \circ t_2) \in C_1 \circ C_2\} \\
&= \{t \in C_1, x_1 \circ t \in C_1\} \cup \{t \in C_2, x_2 \circ t \in C_2\} \\
&\quad \cup \{t = t_1 \circ t_2, t_1 \in C_1, (x_1 \circ t_1) \in C_1, t_2 \in C_2, (x_2 \circ t_2) \in C_2\} \\
&= link(x_1, C_1) \cup link(x_2, C_2) \\
&\quad \cup \{t = t_1 \circ t_2, t_1 \in link(x_1, C_1), t_2 \in link(x_2, C_2)\} \\
&= link(x_1, C_1) \circ link(x_2, C_2) \square
\end{aligned}
$$

References

[1] J. F. P. Hudson, Piecewise Linear Topology, W. A. Benjamin inc., 1969.

[2] T. Y. Kong, A. Rosenfeld, Digital topology: introduction and survey, Computer Vision, Graphics and Image Processing 48 (1989) 357–393.

[3] A. V. Ivashchenko, Representation of smooth surfaces by graphs, transformations of graphs which do not change the Euler characteristic of graphs, Discrete Mathematics 122 (1993) 219–133.

[4] E. Brisson, Representing geometric structures in d dimensions: topology and order, Discrete and Computational Geometry 9 (4) (1993) 387–426.

[5] P. Lienhardt, N-dimensional generalized combinatorial maps and cellular quasi-manifolds, Int. J. Comput. Geometry Appl. 4 (3) (1994) 275–324.

[6] V. A. Kovalevsky, A new concept for digital geometry, in: Shape in Pictures, Vol. 126 of NATO ASI Series F., 1994, pp. 37–51.

[7] R. Ayala, E. Domínguez, A. R. Francés, A. Quintero, Digital lighting functions, in: procs. Discrete Geometry for Computer Imagery, LNCS, Springer Verlag, Vol. 1347, 1997, pp. 139–150.

[8] J. W. Alexander, The combinatorial theory of complexes, Ann. Math. 31 (1930) 294–322.

[9] W. B. R. Lickorish, Simplicial moves on complexes and manifolds, Geometry And Topology Monograph, Proceedings of the KirbyFest 2 (1998) 229–320.

[10] P. Alexandroff, Diskrete Räume, Mat. Sbornik 2 (1937) 501–518.

[11] P. Alexandroff, Combinatorial topology, Dover Publications, 1998.

[12] A. V. Evako, R. Kopperman, Y. V. Mukhin, Dimensional properties of graphs and digital spaces, Jour. of Math. Imaging and Vision 6 (1996) 109–119.

[13] A. V. Evako, Dimension on discrete spaces, International Journal of Theoretical Physics 33 (1996) 1553–1568.

[14] X. Daragon, M. Couprie, G. Bertrand, Marching chains algorithm for Alexandroff-Khalimsky spaces, in: Vision Geometry XI, 2002, pp. 51–62.

[15] X. Daragon, M. Couprie, G. Bertrand, Discrete frontiers, in: Springer (Ed.), Discrete Geometry for Computer Imagery, Vol. 2886 of LNCS, 2003, pp. 236–245.

[16] E. D. Khalimsky, On topologies of generalized segments, Soviet Mat. Doklady 10 (1969) 1508–1511.

[17] G. Bertrand, New notions for discrete topology, in: Springer (Ed.), Discrete Geometry for Computer Imagery, Vol. 1568 of LNCS, 1999, pp. 218–228.

[18] G. Bertrand, M. Couprie, A model for digital topology, in: Springer (Ed.), Discrete Geometry for Computer Imagery, Vol. 1568 of LNCS, 1999, pp. 229–241.

[19] X. Daragon, M. Couprie, G. Bertrand, Discrete surfaces and frontier orders, Tech. Rep. IGM2004-05, Université de Marne-la-Vallée (2004).

[20] E. D. Khalimsky, R. Kopperman, P. R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topology Appl. 36 (1990) 1–17.

[21] R. Kopperman, P. R. Meyer, R. G. Wilson, A Jordan surface theorem for three-dimensional digital spaces, Discrete Computational Geometry 6 (1991) 155–161.

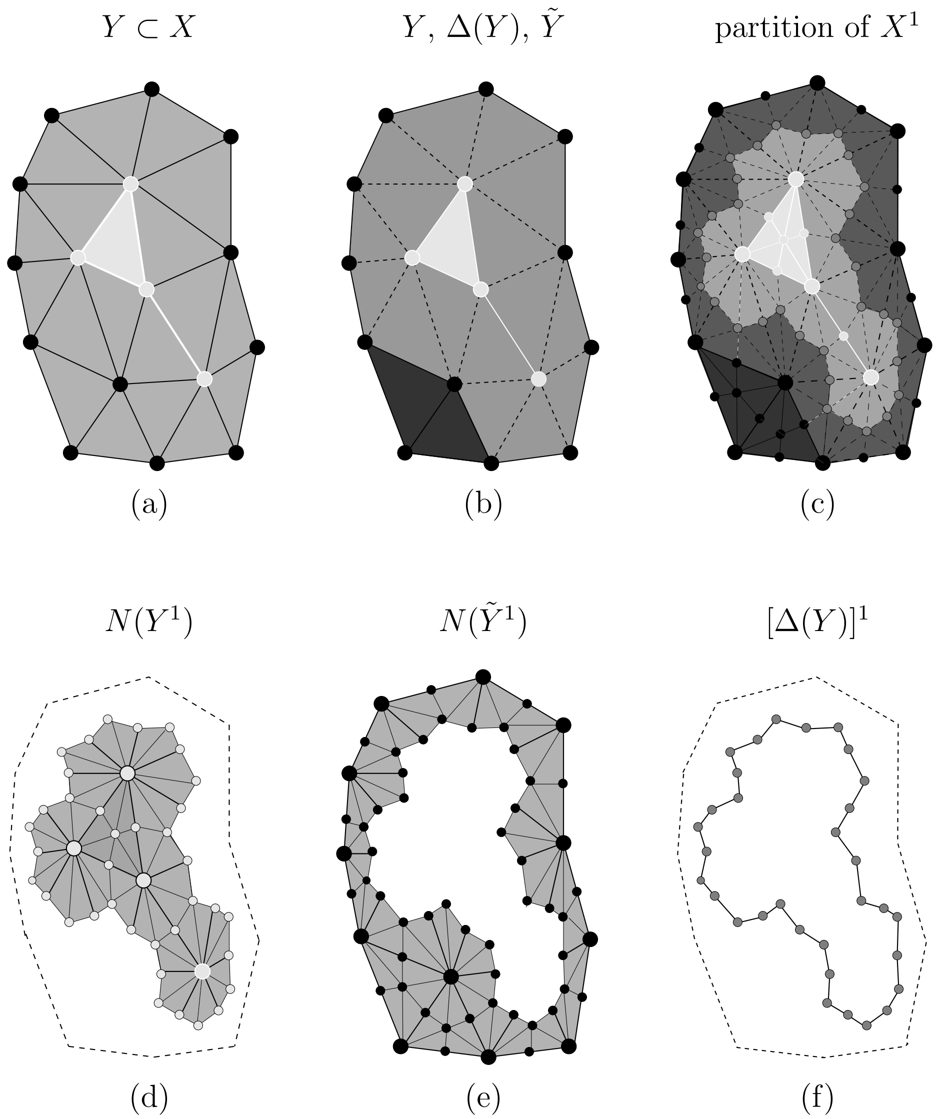


Fig. 5. Example based upon a full subcomplex.
a) A simplicial complex X and a full subcomplex Y of X.
b) Partition of X between Y (light gray, white edges), its simplicial complement $\tilde{Y}$ (dark gray, black lines) and the set $\Delta(Y)$ (average gray, dashed lines), which is not a simplicial complex).
c) The derived subdivision X^1 of X. In light gray and white: Y^1, in dark gray and black (with solid edges): $\tilde{Y}^1$.
d) The derived neighborhood $N(Y^1)$.
e) The derived neighborhood $N(\tilde{Y}^1)$.
f) The derived frontier of Y, since Y is a full subcomplex of X we have:
$[\Delta(Y)]^1 = \delta(N(Y^1)) = N(Y^1) \cap N(\tilde{Y}^1) = \alpha_{X^1}(\beta_{X^1}(Y^1)) \setminus \beta_{X^1}(Y^1)$.

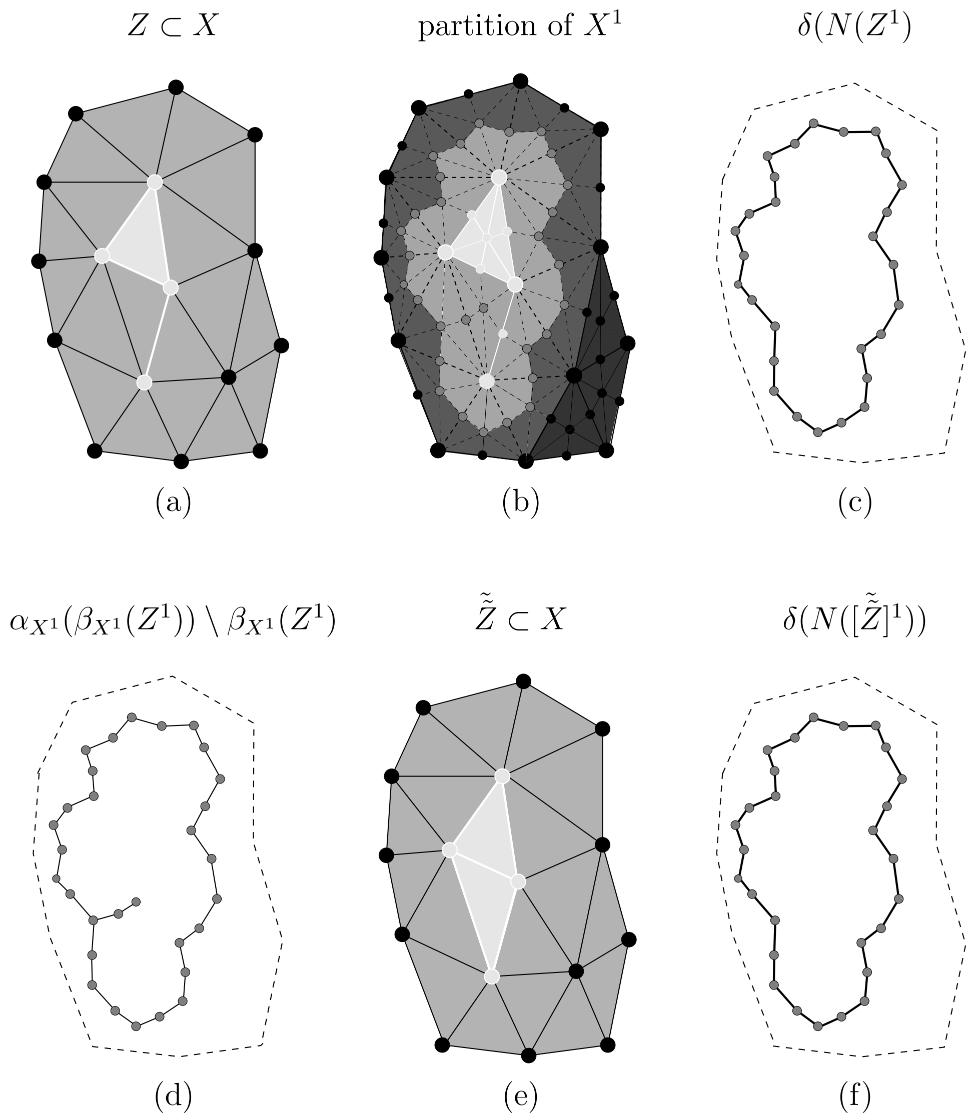


Fig. 6. The case of a non-full subcomplex.
a) The simplicial complex X, and a subcomplex Z of X which is not full.
b) The derived subdivision X^1 of X. In light gray and white: Z^1, in dark gray and black (with solid edges): $\tilde{Z}^1$.
c) The border $\delta(N(Z^1))$.
d) $\alpha_{X^1}(\beta_{X^1}(Z^1)) \setminus \beta_{X^1}(Z^1)$, which differs from $\delta(N(Z^1))$.
e) The simplicial complex and the full subcomplex $\tilde{\tilde{Z}}$ of X, which is the unique full subcomplex of X having the same support as Z.
f) The border $\delta(N([\tilde{\tilde{Z}}]^1))$, which is equal to $\alpha_{X^1}(\beta_{X^1}([\tilde{\tilde{Z}}]^1)) \setminus \beta_{X^1}([\tilde{\tilde{Z}}]^1)$ since $\tilde{\tilde{Z}}$ is a full subcomplex of X. We notice also that $\delta(N([\tilde{\tilde{Z}}]^1)) = \delta(N(Z^1))$