# A NOTE ON THE RELATIONSHIP BETWEEN THE GRAPHICAL TRAVELING SALESMAN POLYHEDRON, THE SYMMETRIC TRAVELING SALESMAN POLYTOPE, AND THE METRIC CONE (SHORT COMMUNICATION) 

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#### Abstract

In this short communication, we observe that the Graphical Traveling Salesman Polyhedron is the intersection of the positive orthant with the Minkowski sum of the Symmetric Traveling Salesman Polytope and the polar of the metric cone. This follows almost trivially from known facts. There are two reasons why we find this observation worth communicating none-the-less: It is very surprising; it helps to understand the relationship between these two important families of polyhedra.


## 1. Introduction

The Symmetric Traveling Salesman Polytope is the convex hull of all characteristic vectors of edge sets of cycles (i.e., circuits) on the vertex set $V_{n}:=\{1, \ldots, n\}$ (in other words, Hamiltonian cycles in the complete graph with vertex set $V_{n}$ ). For the formal definition, denote by $E_{n}$ the set of all two-element subsets of $V_{n}$. This is the set of all possible edges of a graph with vertex set $V_{n}$. The Symmetric Traveling Salesman Polytope is then the following set:

$$
S_{n}:=\operatorname{conv}\left\{\chi^{C} \mid C \text { is the edge set of a Hamiltonian cycle with vertex set } V_{n}\right\} \subset \mathbb{R}^{E_{n}} .
$$

Here, for an edge set $F, \chi^{F}$ is the characteristic vector in $\mathbb{R}^{E_{n}}$ with $\chi_{e}^{F}=1$ if $e \in F$, and zero otherwise. The importance of the Symmetric Traveling Salesman Polytope comes mainly, but not exclusively, from its use in the solution of the so-called Symmetric Traveling Salesman Problem, which consists in finding a Hamiltonian cycle of minimum cost.

The Graphical Traveling Salesman Polyhedron is the convex hull of all characteristic vectors of edge multi-sets of connected Eulerian multi-graphs on the vertex set $V_{n}$. A multi-graph with vertex set $V_{n}$ has as its edge set a sub-multi-set of $E_{n}$, which is to say that our multi-graphs can have parallel edges but no loops. By defining, for any multiset $F$ of edges of $K_{n}$, its characteristic vector $\chi^{F} \in \mathbb{R}^{E_{n}}$ in such a way that $\chi_{e}^{F}$ counts the number of occurrences of $e$ in $F$, the Graphical Traveling Salesman Polyhedron is formally defined as
$P_{n}:=\operatorname{conv}\left\{\chi^{F} \mid F\right.$ is the edge multi-set of a connected Eulerian multi-graph

$$
\text { with vertex set } \left.V_{n}\right\} \subset \mathbb{R}^{E_{n}} \text {. }
$$

Ever since the seminal work of Naddef \& Rinaldi [4, 5] on the two polyhedra, $P_{n}$ is considered to be an important tool for investigating the facets of $S_{n}$. Moreover, in works

[^0]of Carr [2] and Applegate, Bixby, Chvàtal \& Cook [1], $P_{n}$ has been used algorithmically in contributing to solution schemes for the Symmetric Traveling Salesman Problem.

Numerous authors have expressed how close the connection between Graphical and Symmetric Traveling Salesman Polyhedra is. The most basic justification for this opinion is the fact that $S_{n}$ is a face of $P_{n}$ - consisting of all points $x$ whose "degree" is two at every vertex: $\sum_{v \neq u} x_{u v}=2$ for all $u \in V_{n}$. However, the connections are far deeper (see [3] or [6] and the references therein). In this short communication, we contribute the following surprising geometric observation to the issue of the relationship between these two polyhedra:

Theorem. $P_{n}$ is the intersection of the positive orthant with the Minkowski sum of $S_{n}$ and the polar $C_{n}^{\Delta}$ of the metric cone $C_{n}$ :

$$
\begin{equation*}
P_{n}=\left(S_{n}+C_{n}^{\Delta}\right) \cap \mathbb{R}_{+}^{E_{n}} \tag{1}
\end{equation*}
$$

The metric cone consists of all $a \in \mathbb{R}^{E_{n}}$ which satisfy the triangle inequality:

$$
\begin{equation*}
a_{u v} \leq a_{u w}+a_{w v} \tag{2}
\end{equation*}
$$

for all pairwise distinct vertices $u, v, w \in V_{n}$. Consequently, its polar is generated as a cone by the vectors (we abbreviate $\chi^{\{e\}}$ to $\chi^{e}$ )

$$
\begin{equation*}
\chi^{u w}+\chi^{w v}-\chi^{u v} \tag{3}
\end{equation*}
$$

The proof of this theorem is an application of three or four known facts or techniques in the area of Symmetric and Graphical Traveling Salesman polyhedra.

## 2. Proof

We start with showing that $P_{n} \subset\left(S_{n}+C_{n}^{\Delta}\right) \cap \mathbb{R}_{+}^{E_{n}}$. While $P_{n} \subset \mathbb{R}_{+}^{E_{n}}$ holds trivially, $P_{n} \subset S_{n}+C_{n}^{\Delta}$ follows from an argument of [5], which we reproduce here for the sake of completeness.

Let $x \in \mathbb{Z}_{+}^{E_{n}}$ be a the characteristic vector of the edge multi-set of a connected Eulerian multi-graph $G$ with vertex set $V_{n}$. We prove by induction on the number $m$ of edges of $G$, that $x$ can be written as a sum of a cycle and a number of vectors (3). If $m=n$, then there is nothing to prove. Let $m \geq n+1$. There exists a vertex $w$ of degree at least four in $G$. We distinguish two cases. The easy case occurs when $G \backslash w$ is still connected. Here, we let $u$ and $v$ be two arbitrary (possibly identical) neighbors of $w$. By either replacing the edges $u w$ and $w v$ of $G$ with the new edge $u v$, if $u \neq v$, or deleting $u w$ and $w v$, if $u=v$, one obtains a connected Eulerian multi-graph $G^{\prime}$ with fewer edges than $G$. The change in the vector $x$ amounts to subtracting the expression (3) : $x^{\prime}=x-\left(\chi^{u w}+\chi^{w v}-\chi^{u v}\right)$, if $u \neq v$, and $x^{\prime}=x-\left(\chi^{u w}+\chi^{w v}\right.$, if $u=v$. In the slightly more difficult case when the graph $G \backslash w$ has at least two connected components, we can let $u$ and $v$ be two neighbors of $w$ in distinct components of $G \backslash w$. This makes sure that the graph $G^{\prime}$ is still connected. We conclude by induction that $x^{\prime}$, and hence $x$, can be written as a sum of a cycle and a number of vectors (3).

We now prove $P_{n} \supset\left(S_{n}+C_{n}^{\Delta}\right) \cap \mathbb{R}_{+}^{E_{n}}$. For this, we show that any inequality which is facet-defining for $P_{n}$ is valid for $\left(S_{n}+C_{n}^{\Delta}\right) \cap \mathbb{R}_{+}^{E_{n}}$.

We again invoke an argument from [5]: Naddef \& Rinaldi have shown ${ }^{11}$ that the inequalities defining facets of $P_{n}$ fall into one of two categories: the non-negativity inequalities

[^1]$x_{e} \geq 0$, with $e \in E_{n}$ (or positive scalar multiples thereof), or inequalities whose coefficient vectors satisfy the triangle inequality (2). We reproduce the proof of this statement.

First recall that an inequality $a \cdot x \geq \alpha$ is said to be dominated by another inequality $b \cdot x \geq \beta$, if the face defined by the first inequality is contained in the face defined by the second inequality.

Suppose that $a \cdot x \geq \alpha$ is not dominated by a non-negativity inequality (it need not be define a facet, though), and let $u, v, w$ be three distinct vertices in $V_{n}$. Then there exists an $x \in \mathbb{Z}_{+}^{E_{n}}$ defining the edge multi-set of a connected Eulerian multi-graph $G$ which has an edge between $u$ and $v$, such that $a \cdot x=\alpha$. If we replace the edge $u v$ of $G$ by the two edges $u w$ and $w v$, then we obtain a connected Eulerian multi-graph, whose edge multi-set is given, in terms of its characteristic vector, by $x^{\prime}:=x+\chi^{u w}+\chi^{w v}-\chi^{u v}$. Now $a \cdot x^{\prime} \geq \alpha$, implies $a_{u w}+a_{w v}-a_{u v} \geq 0$, i.e., the triangle inequality.

We now conclude the proof of the inclusion $P_{n} \supset\left(S_{n}+C_{n}^{\Delta}\right) \cap \mathbb{R}_{+}^{E_{n}}$. Let $a \cdot x \geq \alpha$ be an inequality which is facet-defining for $P_{n}$. First note that the non-negativity inequalities are clearly satisfied by the right hand side of (1). Hence, using what we have just discussed, let us assume that $a$ satisfies the triangle inequality. This means that $a$ is a member of the metric cone $C_{n}$. Consequently, the inequality $a \cdot x \geq 0$ is valid for $C_{n}^{\Delta}$. Further, since $S_{n} \subset P_{n}$, the inequality $a \cdot x \geq \alpha$ is clearly valid for $S_{n}$. Hence the inequality is valid for $S_{n}+C_{n}^{\Delta}$.

This concludes the proof of the theorem.
Note that, en passant, we have proved the following. If we define $P_{n}^{\prime}$ to be the set of all $y \in \mathbb{R}^{E_{n}}$ which satisfy $a \cdot y \geq \alpha$ for every inequality $a \cdot x \geq \alpha$ defining a facet of $P_{n}$ but not being a scalar multiple of a non-negativity inequality, then we have $S_{n}+C_{n}^{\Delta} \subset P_{n}^{\prime}$.

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[^1]:    ${ }^{1}$ In fact, Proposition 2.2 of [5] states that the facet-defining inequalities for $P_{n}$ fall into three classes one of which is the class of non-negativity inequalities and the other two satisfy the triangle inequality.

