# A generalization of Hungarian method and Hall's theorem with applications in wireless sensor networks

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**Abstract**

In this paper, we consider various problems concerning quasi-matchings and semi-matchings in bipartite graphs, which generalize the classical problem of determining a perfect matching in bipartite graphs. We prove a vast generalization of Hall's marriage theorem, and present an algorithm that solves the problem of determining a lexicographically minimum $g$-quasi-matching (that is a set $F$ of edges in a bipartite graph such that in one set of the bipartition every vertex $v$ has at least $g(v)$ incident edges from $F$, where $g$ is a so-called need mapping, while on the other side of the bipartition the distribution of degrees with respect to $F$ is lexicographically minimum). We also present an application in designing an optimal CDMA-based wireless sensor networks.

**Keywords:** matching, quasi-matching, semi-matching, flow, Hungarian method, augmenting path
**AMS subject classification (2000)**: 05C70, 68R10, 05C90

## 1 Introduction

Problems related to matchings and factors belong to the classical and intensively studied problems in graph theory. We refer to the monograph of Lovász and Plummer [10] from over 20 years ago which is still one of the most comprehensive surveys on the topic. Since the seminal paper of P. Hall [7] containing a characterization of perfect matchings in bipartite graphs, many generalizations and variations of matchings and factors in (bipartite) graphs have been considered. Let us mention the concepts of 2-matchings, weighted matchings and $f$-factors [10]. At least

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as much interest has been given to algorithmic issues related to matchings, where a similarly influential role is played by the famous max-flow min-cut theorem of Ford and Fulkerson [5], cf. [10]. The research in the area is still vivid, which is in part due to its applicability. Notably applications often require special properties and yield different variants of existing concepts which were not previously covered by the theory. In this paper, we introduce and study the so-called $f, g$-quasi-matching as a natural generalization of matchings in bipartite graphs.

When modeling CDMA-based wireless sensor networks with graphs [3, 9], the following routing problem was encountered (naturally, it can appear in any communication network with similar features). The topology of the network is given by the nodes (in our case sensor units) that are able to communicate among each other with respect to physical limitations and their mutual distance. There is a special vertex, the sink, represented by a fixed station with relatively large computational capabilities. In our model, we assume that nodes are also fixed, and they can also communicate with the sink, depending on the mentioned limitations. This yields the initial rooted graph, in which we wish to pass information from nodes to the root. While nearby nodes communicate directly with the sink, other (remote) nodes can pass information to the sink by using other nodes as communication devices. For the purpose of energy saving and latency, the number of hops from a given node to the station must be as small as possible. The overall aim is to design a routing protocol, by which each node in the network transfers information to the sink as quickly as possible. Translating our problem to graphs, we wish to find a spanning tree in a given rooted graph using only edges that connect two different distance-levels with respect to the root. There are many such trees obtainable by an ordinary BFS-algorithm, yet they may have vertices with relatively large degree, which can cause both communication delay and large energy consumption of these nodes. Since the life-time of the network depends on its weakest nodes, such situations need to be avoided. See [1] for more on wireless sensor networks and their routing protocols. We remark that finding a spanning tree with the smallest maximum degree in a non-rooted graphs is a rather well studied problem (see [6] and the references therein), yet it does not have much connection with the problem on rooted graphs.

Our situation can be quickly translated to the following optimization problem. *Given a rooted graph, find a spanning tree with maximum degree as small as possible.* Another more general problem follows from the requirement that more than one path from a node to the sink is needed, either to provide robustness against possible node failures or to avoid communication delay due to collisions at more frequent nodes. Hence alternative paths need to be determined in advance. Then the problem is to *find a spanning subgraph with maximum degree as small as possible in which each vertex has $k$ neighbors in the neighboring distance level that is closer to the root.* More generally, if we have a traffic estimation at the nodes, then the number of neighbors in the lower level can be assigned to each vertex individually. By concentrating solely on two neighboring levels, the problem is to find a spanning subgraph in a bipartite graph such that, in one set of the partition, the degrees of vertices are prescribed: they can be 1 (derived from the original problem), have a fixed degree $k$ (for the so-called multipath routing), or they can be determined by an arbitrary function that corresponds to estimated traffic at the nodes. In the other set of the bipartition, we are either aiming at the minimization of the largest degree (optimization problem), or we are also facing some constraints on degrees of vertices (decision problem). We will address both of these problems.

A variation of the first (and the simplest) of the mentioned problems was considered in [8], with motivation arising from some task scheduling. The authors introduced the so-called semi-matchings which coincide with spanning forests in bipartite graphs and their objective was the reduction of a certain cost-function that is connected to the maximum degree of a forest.

We present a solution to the more general problem of determining an optimal quasi-matchings, where on one side of the bipartition degrees of vertices with respect to a quasi-matching obey specified lower bounds, while on the other side not only the maximum degree of vertices is minimized, but also their degree distribution is lexicographically minimum. As it turns out, the resulting algorithm is on-line, in the sense that an increase or decrease of a lower bound by one in a vertex, after the semi-matching has been built, requires only one additional step to obtain an optimal semi-matching of the graph with new bounds.

In the next section, we fix the notation and present the main problems, expressed in the language of graph theory. In Section 3, the Hungarian method is extended to the above mentioned problem of finding a lexicographically minimum quasi-matching in a bipartite graph that yields an efficient algorithm for the original problem. This algorithm is presented as an off-line algorithm, although it can be interpreted as an on-line algorithm when only additions of vertices or the increase of the prescribed lower bounds occur. It is extended in Section 4 to the case when the prescribed lower bound decreases (or the vertex is deleted). In Section 5, we consider a decision version of the most general problem that comes from the above discussion. We prove a characterization of bipartite graphs that admit a spanning subgraph in which for the degrees of vertices of one of the sets in the partition arbitrary lower bounds are imposed, while in the other set of the partition degrees of vertices with respect to the spanning subgraph need to obey arbitrarily specified upper bounds. This result is a vast generalization of the famous Hall's marriage theorem.

# 2 Quasi-matchings in bipartite graphs

This section introduces the terminology used throughout the paper. We also characterize minimum semi-matchings and establish their various properties concerning optimality.

**Definition 1** *Let $G = A + B$ be a bipartite graph. Given a positive integer $k$, a set $F \subseteq E(G)$ is a $k$-*quasi-matching *of $Y \subseteq B$, if every element of $Y$ has at least $k$ incident edges from $F$. A 1-quasi-matching of $Y$ in which every element of $Y$ has exactly* 1 *incident edge from $F$ is called a* semi-matching.

**Definition 2** *Let $G = A + B$ be a bipartite graph and $g\colon B \to \mathbb{N}$ a mapping. For a vertex $v \in B$ we call $g(v)$ the* need *of $v$, and for any $Y \subseteq B$, the* need *of $Y$ is $g(Y) = \sum_{v \in Y} g(v)$. A set $F \subseteq E(G)$ is a $g$-*quasi-matching *of $Y \subseteq B$ if every element $v$ of $Y$ has at least $g(v)$ incident edges from $F$. Next, for a mapping $f\colon A \to \mathbb{N}$, and a vertex $u \in A$ we call $f(u)$ the* capacity *of $u$, and for any $X \subseteq A$, the* capacity *of $X$ is $f(X) = \sum_{u \in X} f(u)$. A set $F \subseteq E(G)$ is an $f,g$-*quasi-matching *of $A + B$ if every element $v$ of $Y$ has at least $g(v)$ incident edges from $F$, and every element $u$ of $X$ has at most $f(u)$ incident edges from $F$.*

Note that a $g$-quasi-matching of $B$ with a constant need function, $g(v) = k$, for all $v \in B$, is a $k$-quasi-matching of $B$.

**Definition 3** *Let $G = A + B$ be a bipartite graph and $F \subseteq E(G)$. For a vertex $v \in V(G)$, the $F$-degree of $v$, $d_F(v)$ is the degree of $v$ in $G[F]$. The* degree *of $F$ is the maximum degree in $G[F]$ of a vertex from $A$.*

Note that a matching of $Y \subseteq B$ is a semi-matching of $Y$ with degree equal to 1. We are interested in the following two problems.

**Problem 1** *Given a bipartite graph $G = A + B$ and a need function $g$ on $B$, find a $g$-quasi-matching of $B$ with minimum degree.*

**Problem 2** *Given a bipartite graph $G = A + B$, is there an $f, g$-quasi-matching of $A + B$?*

We solve the first problem by generalizing Hungarian method in Section 3 and the second one by giving a characterization that generalizes Hall's theorem in Section 5.

**Definition 4** *Let $G = A + B$ be a bipartite graph, let $F \subseteq E(G)$, and $X \subseteq A$. Let $d_F(X)$ be the sequence $d_1, d_2, \ldots, d_{|X|}$ of $F$-degrees of vertices from $X$, where $d_1 \geq d_2 \geq \cdots \geq d_{|X|}$. For $Y \subseteq B$, we define $d_F(Y) = d_F(N(Y))$.*

The following definition applies to all types of quasi-matchings (integer, $g$-quasi-matchings and $f, g$-quasi-matchings).

**Definition 5** *Let $G = A + B$ be a bipartite graph, let $F, F'$ be two quasi-matchings of $Y \subseteq B$. Then $F$ is (lexicographically) greater than $F'$, if $d_F(Y)$ is lexicographically greater than $d_{F'}(Y)$. A quasi-matching $F$ of $Y \subseteq B$ that is not greater than any other quasi-matching of $Y$ is a* minimum *quasi-matching of $Y$.*

Clearly, a minimum quasi-matching of $B$ has a minimum degree. It is also easy to see that in a minimum $g$-quasi-matching all vertices in $B$ have $F$-degree equal to their need. Thus, to solve Problem 1, we propose

**Problem 3** *Given a bipartite graph $G = A + B$ and a need function $g\colon B \to \mathbb{N}$, find a (lexicographically) minimum $g$-quasi-matching of $B$.*

An on-line algorithm for solving Problem 3 is one of the major contributions of this paper. We start with the following easy lemma. (Recall that the *pigeonhole or Dirichlet principle* states that given a set of $t$ objects that are placed into boxes, and there are $s$ boxes available, then there will be a box containing at least $\lceil \frac{t}{s} \rceil$ objects.)

**Lemma 6** *Let $G = A + B$ be a bipartite graph, $g\colon B \to \mathbb{N}$ a need function, and $F$ a $g$-quasi-matching of $B$. Let $X \subseteq A$, with $|X| = k$, and let $Y = N(X)$ be the set of their neighbors. Let $t$ be the number of edges with one end-vertex from $Y$ and the other from $A - X$, and let $g(Y) = t + dk + r$, where $0 \leq r < k$ and $d \geq 0$. Then $d_F(X)$ is lexicographically greater or equal to the distribution with $r$ integers $d + 1$ and $k - r$ integers $d$.*

**Proof.** Note that $d_F(X)$ is (lexicographically) the smallest only if all edges with one end-vertex from $Y$ and the the other from $A - X$ are in $F$. We may thus assume without loss of generality that this is the case. Hence $\sum_{x \in X} d_F(x) = dk + r$.

If $r = 0$, then either $d_F(X)$ consists of precisely $k$ integers $d$ or $d_F(x)$ contains at least one integer strictly greater than $d$. Both distributions are lexicographically greater or equal to the distribution with $k$ integers $d$.

So suppose $r > 0$ implying $k > r \geq 1$. By applying Dirichlet's principle, either $X$ contains a vertex $a$ with $d_F(a) > d + 1 \geq 1$ (in which case $d_F(X)$ is lexicographically greater than the distribution with the largest degree $d + 1$) or there are $r$ vertices in $X$ with $F$-degree $d + 1$ and $k - r$ vertices in $X$ with $F$-degree $d$. The claim follows. $\square$

**Definition 7** *Let $G = A + B$ be a bipartite graph and $F \subseteq E(G)$ a set of edges. A* (forward) $F$-alternating path *from a vertex $a \in A$ to a vertex $a' \in A$ in $G$ is a path $P$ such that every internal vertex of $P$ is in $P$ incident with one edge in $F$ and another not in $F$, and that $a$ is in $P$ incident with $F$, but $a'$ is not. A path $P$ from a vertex $a \in A$ to a vertex $a' \in A$ in $G$ is a* backward $F$-alternating path *if the reversed path on the same edges from $a'$ to $a$ is a (forward) $F$-alternating path. An $F$-*augmenting path *$P$ in $G$ is a path from a vertex $b \in B$ to a vertex $a \in A$, such that $P - b$ is an $F$-alternating path from $a'$ to $a$, and the edge $a'b$ is not in $F$.*

Note that by performing $F$-exchange $F' = F \oplus E(P)$ of edges in an $F$-alternating path $P$ from $a \in A$ to $a' \in A$, the degree of $a$ decreases by one ($d_{F'}(a) = d_F(a) - 1$), the degree of $a'$ increases by one ($d_{F'}(a') = d_F(a) + 1$), and all other quasi-matching-degrees remain as in $F$.

**Definition 8** *Let $G = A + B$ be a bipartite graph, $F$ a quasi-matching of $Y \subseteq B$ and $P$ an $F$-alternating path from $a \in A$ to $a' \in A$. The* decline *of $P$ is $dc(P) = d_F(a) - d_F(a')$.*

**Definition 9** *Let $G = A + B$ be a bipartite graph, $F \subseteq E(G)$, and $a \in A$. The* $a$-section *of $G$ is a maximal subgraph $G_a = X_a + Y_a \subseteq G$, such that there is an $F$-alternating path $P_{a'}$ from $a$ to every $a' \in X_a$ and $Y_a = N_F(X_a)$ is the set of $F$-neighbors of $X_a$. Furthermore, $F_a$ is the set of edges in $F$ incident with $X_a$.*

Thus defined $a$-sections play a crucial role in our proof of the following characterization of minimum $g$-quasi-matchings.

**Theorem 10** *Let $G = A + B$ be a bipartite graph, $g \colon B \to \mathbb{N}$ a need function and $F$ a $g$-quasi-matching of $B$. Then $F$ is a minimum $g$-quasi-matching of $B$ if and only if any $F$-alternating path in $G$ has decline at most* 1.

**Proof.** Suppose there is an $F$-alternating path $P$ in $G$ whose decline is at least two. By performing an $F$-exchange of edges on $P$, we get a $g$-quasi-matching $F'$, such that $F$ is lexicographically greater than $F'$, a contradiction.

The converse is by induction on $g(B) = \sum_{y \in B} g(y)$. Assume that all $F$-alternating paths in $G$ have decline at most 1. Let $a \in A$ be a vertex with the largest $F$-degree in $G$, and let $H = X + Y$ be the $a$-section in $G$. Note that for any $a' \in X$,

$$d_F(a) - 1 \leq d_F(a') \leq d_F(a).$$

Also note that by definition of the $a$-section (maximality), any edge connecting a vertex from $Y$ to a vertex from $A - X$ is in $F$. Let $t$ be the number of edges connecting a vertex from $Y$ to a vertex from $A - X$. Then by letting $|X| = k$ and $d = d_F(a)$, we easily infer that $g(Y) = t + k(d-1) + r$, where $r$ is the number of vertices in $X$ with $F$-degree equal to $d$. By Lemma 6, the distribution $d_F(X)$ coincides with the lexicographically minimum degree distribution of a $g$-quasi-matching. Hence, if $X = A$ (and so $t = 0$), the proof is complete.

Thus, suppose that $X \neq A$. Let $Y' = N(A - X)$, and note that $Y \cup Y' = B$, while $Y \cap Y'$ may be nonempty. Let $F'$ be the restriction of $F$ to the edges with one endvertex in $A - X$, and set $F'' = F - F'$ (i.e. $F''$ contains edges from $F$ that have one endvertex in $X$). We set a need mapping $g'$ of $Y'$ with $g'(v) = g(v) - d_{F''}(v)$ for any $v \in Y'$. Now, any $F'$-alternating path in $(X - A) + Y'$ has decline at most one because $F'$ is just the restriction of $F$. As $g'(Y') < g(B)$ we infer by induction hypothesis that $F'$ is a (lexicographically) minimum $g'$-quasi-matching of $Y'$.

Let $Q$ be a minimum $g$-quasi-matching. Hence $d_Q(A)$ is not greater than $d_F(A)$. In addition we infer by Lemma 6 that the distribution $d_Q(X)$ is at least $d_F(X)$, that is, there is at least

$r$ vertices from $X$ whose $Q$-degree is $d$. Let $p$, $p \geq r$ be the number of vertices in $X$ whose $Q$-degree is $d$. Denote by $Q''$ the set of edges from $Q$ that have one endvertex in $X$, and let $Q' = Q - Q''$. Now we introduce a need mapping $g''$ on $Y'$ by setting $g''(v) = g(v) - d_{Q''}(v)$ for any $v \in Y'$. Note that $g''(Y') = g'(Y') - (p - r)$, and so

$$\sum_{u \in A-X} d_{Q'}(u) = \sum_{u \in A-X} d_{F'}(u) - (p - r). \tag{1}$$

Note also that $g'(u) \geq g''(u)$ for any $u \in Y'$. Since $Q'$ is clearly a minimum $g''$-quasi-matching of $(A - X) + Y'$ we infer (again by induction hypothesis) that it has no alternating paths with decline more than 1.

We gradually increase the $g''$-quasi-matching $Q'$ of $(A - X) + Y'$ to a $g'$-quasi-matching by using the following procedure that consists of $p - r$ steps. We denote by $Q_i$ the quasi-matching in the $i$-th step of the procedure (and set $Q_0 = Q'$). In each step we obtain $Q_i$ from $Q_{i-1}$ by taking a vertex $u \in Y'$ with $g''(u) < g'(u)$, for which $d_{Q_{i-1}}(u) < g'(u)$. Let $P$ be an augmenting path from $u$ to a vertex $a_i$ of smallest possible $Q_{i-1}$-degree in $A - X$. Then we set $Q_i = Q_{i-1} \oplus E(P)$. Note that all vertices from $A - X$ on $P$ have degree $d_{Q_i}(a_i)$ because $P - u$ is a forward $Q_{i-1}$-alternating path, having decline exactly 1 (unless $a_i$ is already a neighbor of $u$). From this we quickly infer that there are no $Q_i$-alternating path with decline more than 1, provided there were no such $Q_{i-1}$-alternating paths. In the last step we get a $g'$-quasi-matching $Q_{p-r}$ which thus has no alternating paths with decline more than 1. By induction hypothesis $Q_{p-r}$ is a minimum $g'$-quasi-matching of $(A - X) + Y'$ hence its degree distribution in $A - X$ coincides with $d_{F'}(A - X)$.

From (1) we find that $d_{Q'}(A - X)$ is the smallest possible (noting that it can be obtained from $d_{F'}(A - X)$ by taking off $p - r$ units from vertex degrees in $A - X$) if there are exactly $p - r$ vertices in $A - X$ with $Q'$-degree $d - 1$ and whose $F'$-degree is $d$ (in all other cases, the number of vertices with $F'$-degree equal to $d$ is less than the sum of $p - r$ and the number of vertices with $Q'$-degree equal to $d$, which would in turn imply that $d_F(A)$ is strictly smaller than $d_Q(A)$). Now, this implies that in other vertices of $A - X$ the distributions of $d_{Q'}$ and $d_{F'}$ are the same. Combined with distributions of degrees in $X$ we derive that $d_F(A) = d_Q(A)$, and so $F$ is a minimum $g$-quasi-matching as well. $\square$

The 1-quasi-matchings alias semi-matchings were studied also in [8]. In order to connect our results to theirs, we adopt the following definition.

**Definition 11** *Let $G = A+B$ be a bipartite graph, $F$ a semi-matching of $B$, and $f : \mathbb{R}^+ \to \mathbb{R}$ a strictly (weakly) convex function. Then the function $\mathrm{cost}_f$, defined as $\sum_{i=1}^{|A|} f(d_F(a_i))$ is called a strict (weak) cost function for $f$.*

In [8], the strictly convex function $\ell(n) = \frac{1}{2}n(n+1)$ is emphasized. It is interesting in task scheduling, as it measures total latency of uniform tasks on a single machine. It is also proved that a semi-matching $F$ has minimum $\mathrm{cost}_\ell(F)$ if and only if any $F$-alternating path in $G = A + B$ has decline at most 1. By Theorem 10, $F$-alternating paths in $G$ have such property if and only if $F$ is (lexicographically) minimum semi-matching of $B$. The special case of Theorem 10 where the need function is constant 1 combined with the results from [8] leads to the following equivalent characteristics of the (lexicographically) minimum semi-matching.

**Corollary 12** *Let $G = A + B$ be a bipartite graph, $F$ a semi-matching of $B$, and $f : \mathbb{R}^+ \to \mathbb{R}$ a strictly convex function. Then the following are equivalent:*

*(i) $F$ is (lexicographically) minimium semi-matching of $B$.*

*(ii)* *Any $F$-alternating path in $G$ has decline at most 1.*

*(iii)* *$F$ has minimum $\mathrm{cost}_\ell(F)$ for $\ell(n) = \frac{1}{2}n(n+1)$.*

*(iv)* *$F$ has minimum $\mathrm{cost}_f(F)$.*

*(v)* *$L_p$-norm, $1 \le p < \infty$, of the vector $X = (d_F(a_1), \ldots, d_F(a_{|A|}))$ is minimal.*

*(vi)* *The variance of the vector $X = (d_F(a_1), \ldots, d_F(a_{|A|}))$ is minimal.*

**Proof.** The equivalence $(i) \iff (ii)$ follows from the Theorem 10. Furthermore, $(ii)$ is equivalent to $(iii)$ ([8], Theorem 3.1) and $(iv)$ ([8], Theorem 3.5). Finally, $(iii)$ is equivalent to $(v)$ ([8], Theorem 3.9) and $(vi)$ ([8], Theorem 3.10). □

Every property of the above Theorem 12 implies that $F$ has minimum $\mathrm{cost}_f(F)$ for every weakly convex function $f$ ([8], Theorem 3.5) and that $L_\infty$-norm of the vector $X = (d_F(a_1), \ldots, d_F(a_{|A|}))$ is minimal ([8], Theorem 3.12). In both cases, the converse is not true.

**Corollary 13** *Let $G = A + B$ be a bipartite graph and let $F$ be a (lexicographically) minimium semi-matching of $B$. Then there exists a maximum matching $M \subseteq F$ in $G$.*

**Proof.** Follows directly from Theorem 12 and Theorem 3.7 from [8]. □

The converse of Corollary 13 does not hold (see [8]).

# 3 Generalized Hungarian method

In this section, we solve Problem 3 with an algorithm of complexity $O(g(B)|E(G)|)$. We use the fact that quasi-matchings are a generalization of matchings: if we restrict ourselves to quasi-matchings with degree one, our method is a generalization of the Hungarian method of augmenting paths for finding maximum matchings in bipartite graphs.

Let $B = \{b_1, \ldots, b_n\}$ and $B_\ell = \{b_1, \ldots, b_\ell\}$, $\ell = 1, \ldots, n$. Define a mapping $g_i\colon B \to \mathbb{N}$ with $g_0(b) = 0$, for all $b \in B$, $\ell_1 = 1$, $\ell_i = \max\{j \mid g_{i-1}(b_j) \neq 0\}$ for $i > 1$, and

$$g_i(b) = \begin{cases} g_{i-1}(b) + 1; & b = b_{\ell_i} \text{ and } g_{i-1}(b) < g(b), \\ 1; & b = b_{\ell_i+1} \text{ and } g_{i-1}(b_{\ell_i}) = g(b_{\ell_i}), \\ g_{i-1}(b); & \text{otherwise} \end{cases}$$

for every $1 \le i \le g(B)$. Note that for simplicity we assume $g(b) > 0$ for all $b \in B$. We propose to find a minimum $g$-quasi-matching $F$ of $B$ using an iterative algorithm that gradually extends an $g_i$-quasi-matching $F_i$ of $B_\ell$ using an $F_{i-1}$-augmenting path $P_{i-1}$ from $b_\ell$ to $a \in A$ with smallest $d_{F_{i-1}}(a)$. By induction, we argue that $F_i$ is a minimum $g_i$-quasi-matching of $B_\ell$, thus the final $F_i$ is a minimum $g$-quasi-matching of corresponding $B_\ell = B$.

**Lemma 14** *Let $G = A + B$ be a bipartite graph and $a \in A$. Using the notation of Algorithm 1, the following holds:*

$$d_{F_i}(a) = \begin{cases} d_{F_{i-1}}(a) + 1; & \text{if } a \text{ is the A}-\text{endvertex of } P_{i-1}, \\ d_{F_{i-1}}(a); & \text{otherwise}. \end{cases}$$

**Algorithm 1** Iterative construction of a minimum $g$-quasi-matching of $B$.

**Parameter** $G = A + B$**:** a bipartite graph with $B = \{b_1, \ldots, b_n\}$.
**Output** $F$**:** a minimum $g$-quasi-matching of $B$.
  Set $i = 0$, $\ell = 0$.
  Set $F_i = \emptyset$, $B_\ell = \emptyset$, $G_\ell = \emptyset$.
  **while** $\ell \leq n$ **do**
    $\ell = \ell + 1$.
    Set $B_\ell = B_{\ell-1} \cup \{b_\ell\}$.
    Set $G_\ell = G[B_{\ell-1} \cup A]$.
    $c = 0$.
    **while** $c < g(b_\ell)$ **do**
      $i = i + 1$, $c = c + 1$.
      Set $P_{i-1}$ to be an $F_{i-1}$-augmenting path in $G_\ell$ from $b_\ell$ to $a \in A$ with smallest possible degree $d_{F_{i-1}}(a)$.
      Set $F_i = F_{i-1} \oplus E(P_{i-1})$.
    **end while**
  **end while**
  **return** $F_i$.

**Proof.** The Lemma is obviously true for every vertex $a \in A \setminus V(P_{i-1})$. Since $F_i = F_{i-1} \oplus E(P_{i-1})$, $e \in F_{i-1} \cap E(P_{i-1})$ implies that $e \notin F_i$. Similarly, for every $e \in E(P_{i-1}) \setminus F_{i-1}$ we have $e \in F_i$. Therefore, the number of $F_i$-edges at an internal $P_{i-1}$ vertex $a$ is the same as the number of $F_{i-1}$-edges at $a$. However, if $a$ is the $A$-endvertex of $P_{i-1}$, then its only $P_{i-1}$ incident edge is not in $F_{i-1}$ but is in $F_i$, so $d_{F_i}(a) = d_{F_{i-1}}(a) + 1$. □

**Theorem 15** *Let $G = A + B$ be a bipartite graph. Using the notation of Algorithm 1, $F_i$ is a minimum $g_i$-quasi-matching of $B_\ell$ in $G_\ell$ for $i = 1, \ldots, n$.*

**Proof.** For $i = 1$, we have $\ell = 1$, $B_1 = \{b_1\}$ and $g_1(b_1) = 1$. Let $a$ be any vertex from $N(b_1)$. Then $F_1 = P_0 = b_1 a$ is a minimum $g_1$-quasi-matching of $B_1$ in $G_1$.

Suppose now that $F_{i-1}$ is a minimum $g_{i-1}$-quasi-matching of $B' = B_\ell$ (or $B' = B_{\ell-1}$) in $G' = G_\ell$ (or $G' = G_{\ell-1}$). We claim that $F_i = F_{i-1} \oplus E(P_{i-1})$ is a minimum $g_i$-quasi-matching of $B_\ell$ in $G_\ell$. If this is not the case, then Theorem 10 yields an $F_i$-alternating path $P$ in $G_\ell$ from $a' \in A$ to $a'' \in A$ with decline $d_{F_i}(a') - d_{F_i}(a'') \geq 2$. Note that every $F_i$-alternating subpath of $P$ from a vertex $a \in A$ leads to $a''$ and every backward $F_i$-alternating subpath leads to $a'$.

Consider first the case for $E(P) \cap E(P_{i-1}) = \emptyset$. Then an edge $e$ of $P$ is in $F_{i-1}$ if and only if it is in $F_i$. For the rest of the proof let $a$ denote the endvertex of $P_{i-1}$. We distinguish three cases:

**Case A1:** $a \notin \{a', a''\}$.
Lemma 14 implies that $d_{F_i}(a') = d_{F_{i-1}}(a')$ and $d_{F_i}(a'') = d_{F_{i-1}}(a'')$. Thus, $P$ is an $F_{i-1}$-alternating path from $a'$ to $a''$ in $G'$ with decline at least 2. A contradiction to Theorem 10, since $F_{i-1}$ is a minimum $g_{i-1}$-quasi-matching of $B'$ in $G'$.

**Case A2:** $a = a'$.
Lemma 14 implies $d_{F_i}(a') = d_{F_{i-1}}(a') + 1$ and $d_{F_i}(a'') = d_{F_{i-1}}(a'')$. Let $v$ be the common vertex of the paths $P$ and $P_{i-1}$ closest to $b_\ell$ in $P_{i-1}$. Then $Q = b_\ell P_{i-1} v P a''$ (resp. $Q = b_\ell P a''$ for $v = b_\ell$) is an $F_{i-1}$-augmenting path in $G_\ell$. Since $P_{i-1}$ in $G_\ell$ is chosen so that $d_{F_{i-1}}(a)$ is minimum, we have

$$d_{F_{i-1}}(a) = d_{F_{i-1}}(a') \leq d_{F_{i-1}}(a'')$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \leq 0.$$

This contradicts the assumption $d_{F_i}(a') - d_{F_i}(a'') \geq 2$, as

$$d_{F_{i-1}}(a') + 1 - d_{F_{i-1}}(a'') \geq 2$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \geq 1.$$

**Case A3:** $a = a''$.
In this case, Lemma 14 implies that $P$ is an $F_{i-1}$-alternating path from $a'$ to $a''$ in $G'$ with $d_{F_i}(a') = d_{F_{i-1}}(a')$ and $d_{F_i}(a'') = d_{F_{i-1}}(a'') + 1$. The inequality $d_{F_i}(a') - d_{F_i}(a'') \geq 2$ yields

$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') - 1 \geq 2$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \geq 3.$$

Hence, $P$ is an $F_{i-1}$-alternating path in $G'$ with decline at least 3. But this is again impossible by Theorem 10 and minimality of $F_{i-1}$.

It remains to examine the case $E(P) \cap E(P_{i-1}) \neq \emptyset$.
**Case B1:** $a \notin \{a', a''\}$.
Let $v$ be the common vertex of $P$ and $P_{i-1}$ closest to $b_\ell$ in $P_{i-1}$. Then $Q = b_\ell P_{i-1} v P a''$ (resp. $Q = b_\ell P a''$ for $v = b_\ell$) is an $F_{i-1}$-augmenting path in $G_\ell$. The choice of $P_{i-1}$ implies $d_{F_{i-1}}(a) \leq d_{F_{i-1}}(a'')$.
Let $v'$ be the common vertex of $P$ and $P_{i-1}$ closest to $a'$ in $P$. Then $R = a' P v' P_{i-1} a$ is an $F_{i-1}$-alternating path in $G_\ell$. Since Lemma 14 implies

$$2 \leq d_{F_i}(a') - d_{F_i}(a'') = d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \leq d_{F_{i-1}}(a') - d_{F_{i-1}}(a),$$

$R$ is a path with $F_{i-1}$-decline at least two, another contradiction to Theorem 10 and minimality of $F_{i-1}$.
**Case B2:** $a = a'$.
Let $Q$ be the $F_{i-1}$-augmenting path in $G_\ell$ from $b_\ell$ to $a''$ as in case B1. The existence of such a path ensures that $d_{F_{i-1}}(a) - d_{F_{i-1}}(a'') \leq 0$. But this is not possible, since Lemma 14 implies

$$2 \leq d_{F_i}(a') - d_{F_i}(a'') = d_{F_i}(a) - d_{F_i}(a'') = d_{F_{i-1}}(a) + 1 - d_{F_{i-1}}(a'')$$

and hence $d_{F_{i-1}}(a) - d_{F_{i-1}}(a'') \geq 1$.
**Case B3:** $a = a''$.
Let $R$ be the $F_{i-1}$-alternating path in $G_\ell$ from $a'$ to $a$ constructed as in case B1. We claim that $R$ has decline at least three. From $d_{F_i}(a') - d_{F_i}(a'') \geq 2$ and Lemma 14, we deduce that

$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') - 1 \geq 2$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a'') \geq 3$$
$$d_{F_{i-1}}(a') - d_{F_{i-1}}(a) \geq 3.$$

But this contradicts the minimality of $F_{i-1}$.
We conclude that $F_i$ is a minimum $g_i$-quasi-matching of $B_\ell$ in $G_\ell$. □

By setting $\ell = n$, Theorem 15 proves correctness of the Algorithm 1.

**Corollary 16** *Algorithm 1 finds a minimum $g$-quasi-matching of $B$ and has time-complexity $O(g(B)|E(G)|)$, where $g(B)$ is the need of $B$.*

**Proof.** As $B_n = B$, Theorem 15 establishes that $B$ is a minimum $g$-quasi-matching of $B$. The path $P_i$ can be found using an augmented Hungarian method: the algorithm performs a breadth-first search from the vertex $b_\ell$ in such way, that if the vertex whose neighbors are examined is in $A$, then the search proceeds along its $F_{i-1}$ incident edges only, but from vertices of $B$, the search proceeds along the non-$F_{i-1}$-incident edges only. The search tree $T$ produced in this manner has exchanging levels of non-$F_{i-1}$ and $F_{i-1}$ edges, and in $T$ there is a unique $F_{i-1}$-augmenting path from any vertex to $B$. This path starting at a vertex $a \in A$ of minimum $F_{i-1}$-degree is the path $P_{i-1}$ required for Algorithm 1. The whole tree $T$ (and thus the augmenting path $P_{i-1}$ can be constructed in $O(|E|)$ time. As there are $g(B) = \sum_{b \in B} g(b)$ iterations, the overall complexity of Algorithm 1 is $O(g(B)|E(G)|)$. □

Note that Theorem 10 can be applied to prune the tree constructed in the generalized Hungarian method in such a way, that the search tree contains vertices of one $F_{i-1}$-degree only. If $d$ is the minimum $F_{i-1}$-degree of a neighbor of $b_i$, then $P_{i-1}$ need not contain any vertex of degree $d+1$. Furthermore, as soon as a vertex of $F_{i-1}$-degree $d-1$ is encountered, we can assume that this is the terminating vertex of $P_{i-1}$. These observations do not improve the theoretical complexity of the algorithm (in the worst case, for instance when $G$ has a perfect matching, we still need to consider $O(|E(G)|)$ edges at each iteration), but they could considerably improve any practical implementation.

# 4 On-line application of Algorithm 1

Note that each step of Algorithm 1 can be viewed as a part of an on-line procedure, where the need of a vertex, denoted $b_\ell$, increases by one. In particular, this allows for immediate application of this algorithm to the on-line setting — to rearrange it for the on-line addition of a new vertex $v$ with need $g(v)$, one only needs to perform one step of the outer **while** loop (hence the inner **while** loop which takes $O(|E(G)|)$ time is performed $g(v)$ times).

However, the full on-line setting, as presented in [2], also allows for removal of the vertices of $B$, i.e. an on-line event is not just appearance of a new vertex, but also disappearance of an existing vertex. In our setting, this would correspond to a wireless sensor malfunction or running out of battery, and in the task-scheduling setting of [2], this corresponds to a task being removed from the schedule or the number of required machines for the task being decreased.

Algorithm 2 describes how to augment an existing minimum quasi-matching when the need of a single vertex $b \in B$ decreases by one to obtain an optimal quasi-matching with respect to the new need function. As above, if $b$ disappears, then this algorithm simply needs to be performed $g(v)$ times.

Let $G = A + B$ be a bipartite graph with $B = \{b_1, \ldots, b_n\}$, and let $b \in B$, say $b = b_k$ for some $k$. If $g \colon B \to \mathbb{N}$ is a need function of $B$, then we denote by $g_b$ the mapping from $B$ to $\mathbb{N}$ with $g_b(b_i) = g(b_i)$ for $i \neq k$, and $g_b(b) = g(b) - 1$.

**Theorem 17** *Let $G = A + B$ be a bipartite graph and $F$ a minimum $g$-quasi-matching of $B$ in $G$. Using the notation and assumptions of Algorithm 2, $F'$ is a minimum $g_b$-quasi-matching of $B$ in $G$.*

**Proof.** By Theorem 10, we need to prove that every $F'$-alternating path has decline at most 1 in $G$. Note that every $F$-alternating path has decline at most 1 in $G$, since $F$ is minimum by assumption. There are two cases in the algorithm that we deal with separately.

Suppose first there is no such backward $F$-alternating path $P$ in $G$ from $a' \in A_b$ to $a'' \in A$ with $d_F(a'') = d_F(a') + 1$. Then $F' = F - ab$, and note that $d_F(A) = d_{F'}(A)$ except in $a$

**Algorithm 2** Obtaining a minimum $g_b$-quasi-matching from a minimum $g$-quasi-matching in $G = A + B$.

**Parameter** $G = A + B$**:** a bipartite graph with $B = \{b_1, \ldots, b_n\}$ and need function $g \colon B \to \mathbb{N}$.
**Parameter** $F$**:** a minimum $g$-quasi-matching of $B$ in $G$.
**Parameter** $b$**:** a vertex of $B$.
**Output** $F'$**:** a minimum $g_b$-quasi-matching in $G$.

```
Set A_b be the set of F-neighbors of b.
Set a ∈ A_b be the vertex with largest F-degree in A_b.
if there is a backward F-alternating path P in G from a' ∈ A_b to a'' ∈ A with d_F(a'') = d_F(a') + 1 then
  set F' = F ⊕ P − a'b
else
  set F' = F − ab.
end if
return F'.
```

where $d_{F'}(a) = d_F(a) - 1$. Hence, if there is any $F'$-alternating path with decline greater than 1, it ends in $a$. Now, no such violating path could start with a vertex from $A_b$, since $a$ has the largest $F$-degree among these vertices. And also, no such violating path could start in any other vertex $a''$ of $A$, because that would mean there is a backward $F$-alternating path in $G$ from $a \in A_b$ to $a'' \in A$ with $d_F(a'') = d_F(a') + 1$, contrary to our assumption.

Secondly, suppose there exists a backward $F$-alternating path in $G$ from $a' \in A_b$ to $a'' \in A$ with $d_F(a'') = d_F(a') + 1$, and let $P$ be a shortest such path. Then $F' = F \oplus P - a'b$, and we have $d_F(A) = d_{F'}(A)$ except in $a''$ where $d_{F'}(a'') = d_F(a'') - 1$. By the choice of $P$ and the fact that there are no $F$-alternating paths with decline more than one, we infer that $d_F(v) = d_F(a')$ for all vertices $v \in A$ on $P \setminus \{a''\}$. Hence, for all vertices $v \in A$ on $P$ ($a''$ included), we have $d_{F'}(v) = d_{F'}(a')$. For the purpose of contradiction let us suppose there is a violating $F'$-alternating path $P'$ from $\hat{a}$ to $\tilde{a}$. Since $F'$ and $F$ differ only on $P$, we infer that $P'$ must intersect $P$ in some vertex of $A$. This readily implies that $d_{F'}(\hat{a}) \leq d_{F'}(a') + 1$ and $d_{F'}(\tilde{a}) \geq d_{F'}(a') - 1$. Since $P'$ is violating, we infer that in fact $d_{F'}(\hat{a}) = d_{F'}(a') + 1$ and $d_{F'}(\tilde{a}) = d_{F'}(a') - 1$ so that the decline of $P'$ with respect to $F'$ is exactly 2. Now, we can easily find that there is an $F$-alternating path from $a''$ to $\tilde{a}$ in $G$ whose decline equals 2, which is a contradiction with $F$ being a minimum $g$-quasi-matching. □

From Theorem 17 and previous discussion, we infer that the augmented Hungarian method presented in this paper can be applied to the on-line problem of constructing an optimal quasi-matching of $B$ with the set $A$ fixed, when the vertices of $B$ either appear or disappear one at a time. Each on-line step assures optimality of the current quasi-matching in $O(g(v)|E(G)|)$ steps. Moreover, a similar approach could be used for on-line setting, where the vertices of $A$ can appear or disappear. When a vertex of $A$ of $F$-degree $d$ is removed, its $F$-neighbors from $B$ loose the degree with respect to a quasi-matching, which can be iteratively recovered, resulting in a patching algorithm of complexity $O(d|E(G)|)$. On the other hand, when an $A$-vertex of $G$-degree $d$ is added, up to $d$ vertices can be assigned to it, again resulting in a $O(d|E(G)|)$ algorithm per on-line step. These (rather technical) issues are treated in greater detail in a sequel paper [4], which is oriented towards the mentioned application.

Note that our adaptation of Hungarian method is, when reduced to semi-matchings and only addition of $b$-vertices, the same as in [8]. However, our proof of correctness differs in that we explicitly maintain minimality of the constructed semi-matching (in fact, even an arbitrary

$g$-quasi-matching), after each addition (or removal) of a vertex. Furthermore, the set of possible alternating paths with decline at least two is in our approach narrowed to the vertex that is added to or removed from the graph, resulting in an efficient on-line version of the algorithm.

# 5 Generalized Hall's marriage theorem

In this section, we present a solution to Problem 2 by characterizing bipartite graphs $A + B$ with given $f\colon A \to \mathbb{N}$ and $g\colon B \to \mathbb{N}$ that admit an $f, g$-quasi-matching. The result is a vast generalization of Hall's theorem.

A *network* $N = (V, A)$ is a digraph with a nonnegative capacity $c(e)$ on each edge $e$, and with two distinguished vertices: *source* $s$ and *sink* $t$ (usually, $s$ has only outgoing, and $t$ has only ingoing arcs). A *flow* $g$ assigns a value $fl(e)$ to each edge $e$. A flow $fl$ is *feasible* if for each edge $e$, $0 \le fl(e) \le c(e)$ and the *conservation (Kirchhoff's) law* is fulfilled: for every vertex $v \in V(N) \setminus \{s, t\}$,

$$\sum_{vx \in A(N)} fl(vx) = \sum_{xv \in A(N)} fl(xv).$$

The *value* of a flow $fl$ is $\sum_{sx \in A(N)} fl(sx)$, which is equal to $\sum_{xt \in A(N)} fl(xt)$. The famous Ford-Fulkerson (or max-flow min-cut) theorem states that the maximum value of a feasible flow in $N$ coincides with the minimum capacity of a cut in $N$. (Where *cut* is the set of arcs from $S$ to $T$ in a $S,T$ partition of $N$ (i.e. $s \in S, t \in T$), and its *capacity* is the sum of the $c$-values of its edges). More on this well-known problem and theorem can be found for instance in [10, 11]. One of the several proofs of the famous Hall's marriage theorem uses the max-flow min-cut theorem, and in our generalization of Hall's theorem, we will follow similar lines.

**Definition 18** *Let $G = A + B$ be a bipartite graph, $f\colon A \to \mathbb{N}$ an availability function, and $Y \subseteq B$. For $x \in A$, let $d_Y(x) = |\{y \in Y : xy \in E(G)\}|$, that is the number of neighbors of $x$ from $Y$. For $X \subset A$, let $f(X, Y) = \sum_{x \in X} \min\{f(x), d_Y(x)\}$ denote the* relative availability *of $X$ with respect to $f$ and $Y$. In particular, for $x \in X$, we write $f(\{x\}, Y)$ as $f(x, Y)$ (which is the least of $f(x)$ and $d_Y(x)$).*

Intuitively, the relative availability of $X$ with respect to $f$ and $Y$ presents the maximum number of edges going from $X$ that can be used to cover $Y$.

**Theorem 19** *Let $G = A + B$ be a bipartite graph, with $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$, a mapping $f\colon A \to \mathbb{N}$, and $g\colon B \to \mathbb{N}$. Then $G$ has an $f, g$-quasi-matching of $A + B$ if and only if for every $Y \subseteq B$,*

$$f(N(Y), Y) \ge g(Y). \tag{2}$$

**Proof.** Suppose there is a subset $Y \subset B$ such that $\sum_{u \in N(Y)} f(u, Y) = f(N(Y), Y) < g(Y) = \sum_{v \in Y} g(v)$. Let $F$ be an arbitrary $g$-quasi-matching of $B$ in $G$. The vertices of $Y$ altogether must have at least $g(Y)$ $F$-neighbors. As the relative availability of their neighbors $N(Y)$ is less than $g(Y)$, we derive by the pigeon-hole principle that there will be a vertex $u \in N(Y)$ such that $d_F(u) > f(u)$. Hence $F$ is not an $f, g$-quasi-matching, which readily implies (since $F$ was arbitrarily chosen) that no $f, g$-quasi-matching exists.

For the converse, let $f(N(Y), Y) \ge g(Y)$ hold for all $Y \subseteq B$. We introduce two additional vertices: $a$ that is connected to all vertices $a_i \in A$, and $b$, connected to all $b_j \in B$. Construct a digraph $G'$, by choosing a direction of all edges from $G$ as follows: from $a$ to each $a_i \in A$, from vertices of $A$ to their neighbors in $B$, and from each $b_j$ to $b$. Next, construct a network out of the digraph $G'$, by setting flow capacities $c : E(G') \to \mathbb{N}$ as follows: $c(aa_i) = f(a_i)$, $c(a_ib_j) = 1$ (for $a_ib_j \in E(G)$), and $c(b_jb) = g(b_j)$. Note that there exists a flow of size $g(B)$ in $G'$ if and only if there exists an $f, g$-quasi-matching of $A + B$. By max-flow min-cut theorem, the maximum flow value coincides with the minimum cut capacity in the network $G'$.

Let $C$ be a minimum cut in the network, and let $Z$ be the set of vertices from $B$ for which $b_jb \in C$. Let $Y = B \setminus Z$. Since $C$ is a cut, for every vertex $b_j \in Y$ and every neighbor $a_i$ of $b_j$, we have either $a_ib_j \in C$ or $aa_i \in C$ (since $C$ is minimum, we may assume that both does not happen). Denote by $K$ the set of vertices $a_i$ from $N(Y)$ such that $aa_i \in C$ and let $L = N(Y) \setminus K$. For $b_j \in Y$, let $m_j$ denote the number of its neighbors in $L$ (which coincides with the number of its incident edges that are from $C$). Note that

$$\sum_{j, b_j \in Y} m_j = \sum_{a_i \in L} d_Y(a_i) \geq f(L, Y).$$

Now,

$$\begin{aligned}
|C| &= g(Z) + f(K) + \sum_{j, b_j \in Y} m_j \\
&\geq g(Z) + f(K, Y) + f(L, Y) \\
&\geq g(Z) + f(N(Y), Y) \\
&\geq g(Z) + g(Y) = g(B)
\end{aligned}$$

where in the last inequality (2) is used. The result now readily follows. □

The theorem has several corollaries. We state the most obvious. First, if $f$ is not involved, i.e. if $f(u) = d(u)$ for all $u \in A$, then $f(N(Y), Y) = \sum_{u \in N(Y)} d_Y(u) = \sum_{v \in Y} d(v)$, and (2) turns into a much simpler condition $\sum_{v \in Y} d(v) \geq g(Y)$ for every $Y \subseteq B$.

If we want that each vertex in $A$ covers only one vertex from $B$, that is $f(u) = 1$ for all $u \in A$, we get $f(N(Y), Y) = \sum_{u \in N(Y)} 1 = |N(Y)|$, and the condition (2) reads $|N(Y)| \geq g(Y)$ for every $Y \subseteq B$. If, in addition, $g(v) = 1$ for all $v \in B$, we get $|N(Y)| \geq |Y|$ for all $Y \subseteq B$ which is exactly Hall's condition. On the other hand, this implies that $A + B$ has a perfect matching of vertices from $B$. Thus Hall's theorem is a corollary of Theorem 19.

One of the common formulations of Hall's theorem is in terms of systems of distinct representatives. Let us formulate also Theorem 19 in this sense.

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a family of sets, with $S = \cup_{i=1}^m A_i = \{b_1, \ldots, b_n\}$, and let there be mappings $f \colon \mathcal{A} \to \mathbb{N}$, and $g \colon S \to \mathbb{N}$. We say that the family A has a *(lower) system of $f, g$-representatives* if to every set $A_i \in \mathcal{A}$ we associate at most $f(A_i)$ representatives from $S$, and every vertex $b_j \in S$ is a representative of at least $g(b_j)$ sets from $\mathcal{A}$. In this terminology, Theorem 19 reads as follows.

**Corollary 20** *A family of sets $\mathcal{A}$ has a lower system of $f, g$-representatives if and only if for every subset $Y \subseteq S$ we have*

$$\sum_{A_i \in \mathcal{A}} \min\{f(A_i), |A_i \cap Y|\} \geq \sum_{b_j \in Y} g(b_j).$$

By duality, since the interpretation of the roles of sets and vertices in Theorem 19 can be reversed, we have another corollary expressed in similar terms. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a family of sets, with $S = \cup_{j=1}^{n} B_j = \{a_1, \ldots, a_m\}$, and let there be mappings $f\colon S \to \mathbb{N}$, and $g\colon \mathcal{B} \to \mathbb{N}$. We say that the family $\mathcal{B}$ has *an upper system of $f, g$-representatives* if to every set $B_j \in \mathcal{B}$, we associate at least $g(B_j)$ representatives from $S$, and every vertex $a_i \in S$ is a representative of at most $f(a_i)$ sets from $\mathcal{B}$. In this terminology, we infer from Theorem 19:

**Corollary 21** *A family of sets $\mathcal{B}$ has an upper system of $f, g$-representatives if and only if for every subfamily $Y \subseteq \mathcal{B}$ we have*

$$\sum_{a_i \in S} \min\{f(a_i), |Y(a_i)|\} \geq \sum_{B_j \in Y} g(B_j),$$

*where $Y(a_i) = \{B_j \in Y : a_i \in B_j\}$ (i.e. $|Y(a_i)|$ is the number of sets from the family $Y$ that contain $a_i$).*

From the above corollaries, one can easily find formulations when one or both of the mappings $f, g$ is not involved or is constant (say, equal to 1). The resulting formulations are mostly easier and nicer as the above and could also be applicable.

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