An improved upper bound on the adjacent vertex distinguishing chromatic index of a graph

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Abstract

An adjacent vertex distinguishing coloring of a graph G is a proper edge coloring of G such that any pair of adjacent vertices are incident with distinct sets of colors. The minimum number of colors needed for an adjacent vertex distinguishing coloring of G is denoted by $\chi'_a(G)$. In this paper, we prove that $\chi'_a(G) \leqslant \frac{5}{2}(\Delta+2)$ for any graph G having maximum degree Δ and no isolated edges. This improves a result in [S. Akbari, H. Bidkhori, N. Nosrati, r-Strong edge colorings of graphs, Discrete Math. 306 (2006), 3005-3010], which states that $\chi'_a(G) \leqslant 3\Delta$ for any graph G without isolated edges.

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1 Introduction

All graphs considered in this paper are finite and without self-loops or multiple edges. In order to avoid trivialities, we also assume that every graph has no isolated vertices.

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Let $V(G)$ and $E(G)$ denote the vertex and the edge sets of G, respectively. Let $N_G(v)$ denote the set of neighbors of v in G and $d_G(v) = |N_G(v)|$ the degree of v in G. A vertex v is called a *k-vertex* if $d_G(v) = k$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of a vertex in G, respectively. An *edge k-coloring* of a graph G is a function $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that any two incident edges receive different colors. The *chromatic index*, denoted by $\chi'(G)$, of a graph G is the smallest integer k such that G has an edge k-coloring. Given an edge k-coloring ϕ of G, we use $C_\phi(v)$ to denote the set of colors assigned to edges incident to a vertex v. We call $C_\phi = \cup_{v \in V(G)} C_\phi(v)$ the color set of ϕ. The coloring ϕ is called an *adjacent vertex distinguishing edge coloring* if $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v. A graph G is *normal* if it contains no isolated edges. Clearly, G has an adjacent vertex distinguishing edge coloring if and only if G is normal. The *adjacent vertex distinguishing chromatic index* $\chi'_a(G)$ of a graph G is the smallest integer k such that G has an adjacent vertex distinguishing edge k-coloring.

Zhang, Liu and Wang [20] first introduced and investigated the adjacent vertex distinguishing edge coloring (*adjacent strong edge coloring* in their terminology) of graphs. They proposed the following conjecture.

Conjecture 1 *If a connected normal graph G is different from a 5-cycle and satisfies $|V(G)| \geqslant 3$, then $\chi'_a(G) \leqslant \Delta(G) + 2$.*

Balister et al. [4] confirmed Conjecture 1 for all normal graphs G that are bipartite or satisfy $\Delta(G) = 3$. In particular, we need the following statement in the sequel.

Theorem 1.1 *For any normal graph G with $\Delta(G) \leqslant 3$, $\chi'_a(G) \leqslant 5$.*

They further proved that $\chi'_a(G) \leqslant \Delta(G) + O(\log k)$, where k is the (vertex) chromatic number of the normal graph G. It follows from Brooks' Theorem that $\chi'_a(G) \leqslant 2\Delta(G)$ for G with sufficiently large $\Delta(G)$. Hatami [12] showed that every normal graph G with $\Delta(G) > 10^{20}$ has $\chi'_a(G) \leqslant \Delta(G) + 300$ by the probabilistic method. Edwards et al. [11] proved that $\chi'_a(G) \leqslant \Delta(G) + 1$ if G is a planar bipartite normal graph with $\Delta(G) \geqslant 12$. Wang and Wang [18] verified Conjecture 1 for a class of graphs with small maximum average degree. Their results were further extended by Hocquard and Montassier [13, 14]. Recently, it has been characterized in [19] which of the two cases $\chi'_a(G) = \Delta(G)$ and $\chi'_a(G) = \Delta(G) + 1$ holds for a K_4-minor-free normal graph G with $\Delta(G) \geqslant 5$.

An adjacent vertex distinguishing edge coloring of a graph G is a special case of a *vertex distinguishing edge coloring*, which requires that every pair of vertices be incident with distinct color sets. This more general notion was introduced by Burris and Schelp [9], and independently by Horňák and Soták [15], and Černý et al. [10] (under the name *observability*). The reader is referred to [2, 3, 5–8, 17] for relevant results.

The aim of this paper is to improve the following upper bound obtained in [1].

Theorem 1.2 *For any normal graph G, $\chi'_a(G) \leqslant 3\Delta(G)$.*

The proof of our main theorem in Section 2 is based on an edge-partition result. The details will be supplied in the last section. In Section 3, the new upper bound is further reduced for regular graphs.

2 An improved upper bound

For a graph G and any $S \subseteq E(G)$, the *edge-induced* subgraph $G[S]$ is the subgraph of G whose edge set is S and whose vertex set consists of all end vertices of edges in S. We only deal with subgraphs that are edge-induced subgraphs unless otherwise stated. For a subgraph H of G, we use $\overline{H}$ to denote the edge-induced subgraph $G[E(G)\backslash E(H)]$ and call it the *complement* of H in G. An *edge-partition* of a graph G into subgraphs $G_1, G_2, \ldots, G_m$ is a decomposition of G that satisfies $V(G) = \cup_{i=1}^m V(G_i)$, $E(G) = \cup_{i=1}^m E(G_i)$ and $E(G_i) \cap E(G_j) = \emptyset$ for any pair $i \neq j$. Clearly, a subgraph H of G together with its complement $\overline{H}$ constitute an edge-partition of G. This edge-partition is said to be induced by the subgraph H. The proof of the following is deferred to Section 4.

Theorem 2.1 *Let G be a normal graph with $\Delta(G) \geqslant 6$. Then there is an edge-partition of G induced by a subgraph H such that the following conditions hold.*

1. *Both H and $\overline{H}$ are normal.*

2. *$\Delta(H) \leqslant 3$.*

3. *$\Delta(\overline{H}) \leqslant \Delta(G) - 2$.*

Theorem 2.2 *Let G be a normal graph with $\Delta(G) \geqslant 4$. Then there is an edge-partition of G into subgraphs $G_0, G_1, \ldots, G_k$, $k \leqslant \lfloor \Delta(G)/2 \rfloor - 2$, such that the following hold.*

1. *Every* G_i *is a normal subgraph.*

2. $\Delta(G_i) \leqslant 3$ *for* $1 \leqslant i \leqslant k$.

3. $\Delta(G_0) \leqslant 5$.

Proof. The proof proceeds by induction on $\Delta(G)$. If $\Delta(G) \leqslant 5$, the result holds trivially. Let G be a normal graph with $\Delta(G) \geqslant 6$. By Theorem 2.1, there is an edge-partition of G induced by a subgraph H such that both H and $\overline{H}$ are normal, $\Delta(H) \leqslant 3$ and $\Delta(\overline{H}) \leqslant \Delta(G) - 2$. Clearly, $\Delta(\overline{H}) \geqslant 3$. If $\Delta(\overline{H}) = 3$, then $\Delta(G) = 6$. Let $G_0 = H$ and $G_1 = \overline{H}$. If $\Delta(\overline{H}) \geqslant 4$, by the induction hypothesis, there is an edge-partition of $\overline{H}$ into subgraphs $G_0, G_1, \ldots, G_k$, $k \leqslant \lfloor \Delta(\overline{H})/2 \rfloor - 2$, such that properties 1, 2 and 3 hold. Now let $G_{k+1} = H$. Then $G_0, G_1, \ldots, G_k, G_{k+1}$ form an edge-partition of G. Note that $k + 1 \leqslant \lfloor \Delta(\overline{H})/2 \rfloor - 2 + 1 \leqslant \lfloor (\Delta(G) - 2)/2 \rfloor - 1 = \lfloor \Delta(G)/2 \rfloor - 2$ and we are done. ■

Lemma 2.3 *If a normal graph* G *has an edge-partition into two normal subgraphs* G_1 *and* G_2*, then* $\chi'_a(G) \leqslant \chi'_a(G_1) + \chi'_a(G_2)$.

Proof. For $i = 1, 2$, let ϕ_i be an adjacent vertex distinguishing edge coloring of G_i satisfying $|C_{\phi_i}| = \chi'_a(G_i)$ and $C_{\phi_1} \cap C_{\phi_2} = \emptyset$. The union of ϕ_1 and ϕ_2 forms a proper edge coloring ϕ of G with color set $C_{\phi_1} \cup C_{\phi_2}$. Let $uv \in E(G)$ with $d_G(u) = d_G(v)$. Since $E(G_1) \cap E(G_2) = \emptyset$, we may assume that $uv \in E(G_1) \setminus E(G_2)$ with $d_{G_1}(u) \geqslant d_{G_1}(v)$. Since G_1 is normal, uv is not an isolated edge of G_1, i.e., $d_{G_1}(u) \geqslant 2$. By definition of ϕ_1, there exists a $c \in C_{\phi_1}(u) \setminus C_{\phi_1}(v)$. Since $C_{\phi_1} \cap C_{\phi_2} = \emptyset$, it follows that $c \in C_\phi(u) \setminus C_\phi(v)$, and hence $C_\phi(u) \neq C_\phi(v)$. Consequently, $\chi'_a(G) \leqslant |C_{\phi_1} \cup C_{\phi_2}| = |C_{\phi_1}| + |C_{\phi_2}| = \chi'_a(G_1) + \chi'_a(G_2)$. ■

Theorem 2.4 *If* G *is a normal graph, then* $\chi'_a(G) \leqslant \frac{5}{2}(\Delta(G) + 2)$.

Proof. The result can be derived immediately from Theorem 1.1 when $\Delta(G) \leqslant 3$. Now assume that $\Delta(G) \geqslant 4$. By Theorem 2.2, there is an edge-partition of G into subgraphs $G_0, G_1, \ldots, G_k$, $k \leqslant \lfloor \Delta(G)/2 \rfloor - 2$, such that properties 1, 2 and 3 hold. Using Lemma 2.3 and Theorem 1.1 repeatedly, we have

$$\begin{aligned} \chi'_a(G) &\leqslant \chi'_a(G_0) + \chi'_a(G_1) + \cdots + \chi'_a(G_k) \\ &\leqslant \chi'_a(G_0) + 5k \\ &\leqslant \chi'_a(G_0) + 5(\lfloor \Delta(G)/2 \rfloor - 2). \end{aligned}$$

By Theorem 2.2, $\Delta(G_0) \leqslant 5$. It follows from Theorem 1.2 that $\chi'_a(G) \leqslant 15 + 5(\lfloor \Delta(G)/2 \rfloor - 2) \leqslant \frac{5}{2}(\Delta(G)+2)$. $\blacksquare$

3 Regular graphs

Theorem 2.4 can be further improved for regular graphs. We first establish an auxiliary edge-partition lemma. We need the following well-known result of Vizing [16] on chromatic index.

Theorem 3.1 *For every graph G, $\chi'(G) \leqslant \Delta(G) + 1$.*

Lemma 3.2 *Let G be a regular graph of degree $r \geqslant 5$. Then there is an edge-partition of G into normal subgraphs $G_1, G_2, \ldots, G_k$ such that one of the following conditions holds.*

1. *If $r \equiv 2 \pmod 3$, then $k = (r+1)/3$ and $\Delta(G_i) \leqslant 3$ for $1 \leqslant i \leqslant k$.*
2. *If $r \equiv 1 \pmod 3$, then $k = (r-1)/3$, $\Delta(G_i) \leqslant 4$ for $1 \leqslant i \leqslant 2$ and $\Delta(G_i) \leqslant 3$ for $3 \leqslant i \leqslant k$.*
3. *If $r \equiv 0 \pmod 3$, then $k = r/3$ and $\Delta(G_1) \leqslant 4$ and $\Delta(G_i) \leqslant 3$ for $2 \leqslant i \leqslant k$.*

Proof. By Theorem 3.1, $E(G)$ can be partitioned into $r+1$ disjoint color classes $E_1, E_2, \ldots, E_{r+1}$ such that each E_i is a matching of G. Let H be a subgraph of G edge-induced by m, $3 \leqslant m \leqslant r$, of these color classes. Obviously, $\Delta(H) \leqslant m$. For any given vertex v of G, exactly one color is not used on any edge incident with v since G is r-regular. Therefore $d_H(v) \geqslant 2$, and hence H is a normal graph.

If $r \equiv 2 \pmod 3$, let $k = (r+1)/3$. Then we define $G_1 = G[E_1 \cup E_2 \cup E_3]$, $G_2 = G[E_4 \cup E_5 \cup E_6]$, $\ldots$, $G_k = G[E_{r-1} \cup E_r \cup E_{r+1}]$. Then $G_1, G_2, \ldots, G_k$ form an edge-partition of G satisfying condition 1.

If $r \equiv 1 \pmod 3$, let $k = (r-1)/3$. Then we define $G_1 = G[E_1 \cup E_2 \cup E_3 \cup E_4]$, $G_2 = G[E_5 \cup E_6 \cup E_7 \cup E_8]$, $G_3 = [E_9 \cup E_{10} \cup E_{11}]$, $\ldots$, $G_k = G[E_{r-1} \cup E_r \cup E_{r+1}]$. Then $G_1, G_2, \ldots, G_k$ form an edge-partition of G satisfying condition 2.

If $r \equiv 0 \pmod 3$, let $k = r/3$. Then we define $G_1 = G[E_1 \cup E_2 \cup E_3 \cup E_4]$, $G_2 = G[E_5 \cup E_6 \cup E_7]$, $G_3 = [E_8 \cup E_9 \cup E_{10}]$, $\ldots$, $G_k = G[E_{r-1} \cup E_r \cup E_{r+1}]$. Then $G_1, G_2, \ldots, G_k$ form an edge-partition of G satisfying condition 3. $\blacksquare$

Theorem 3.3 *Let G be a regular graph of degree $r \geqslant 2$. Then $\chi'_a(G) \leqslant (5r+37)/3$.*

Proof. If $2 \leqslant r \leqslant 4$, the result follows from Theorems 1.1 and 1.2. Assume that $r \geqslant 5$. By Lemma 3.2, there is an edge-partition of G into normal subgraphs $G_1, G_2, \ldots, G_k$ such that one of the stated conditions 1, 2 or 3 holds.

If condition 1 holds, by Lemma 2.3, Theorems 1.1 and 1.2, we have $\chi'_a(G) \leqslant \sum_{i=1}^{k} \chi'_a(G_i) \leqslant 5k = 5(r+1)/3 < (5r+37)/3$.

If condition 2 holds, then $\chi'_a(G) \leqslant \chi'_a(G_1) + \chi'_a(G_2) + \sum_{i=3}^{k} \chi'_a(G_i) \leqslant 12 + 12 + 5(k-2) = 5(r-1)/3 + 14 = (5r+37)/3$.

If condition 3 holds, then $\chi'_a(G) \leqslant \chi'_a(G_1) + \sum_{i=2}^{k} \chi'_a(G_i) \leqslant 12 + 5(k-1) = 5r/3 + 7 < (5r+37)/3$. ■

Note that the upper bound in Theorem 3.3 is better than the upper bound in Theorem 2.4 when $r \geqslant 14$.

4 Proof of Theorem 2.1

We devote this section to a complete proof of Theorem 2.1.

Assume that G is a normal graph with $\Delta(G) \geqslant 6$. We abbreviate $\Delta(G)$ and $d_G(v)$ to Δ and $d(v)$, respectively. Let $\mathcal{H}(G)$ be the collection of subgraphs M of G that satisfy the following conditions.

1. $\Delta(M) \leqslant 3$.
2. If $d(v) = \Delta$, then $d_M(v) \geqslant 2$.
3. If $d(v) = \Delta - 1$, then $d_M(v) \geqslant 1$.

We first show that $\mathcal{H}(G) \neq \emptyset$. By Theorem 3.1, $E(G)$ can be partitioned into $\Delta + 1$ disjoint color classes $E_1, E_2, \ldots, E_{\Delta+1}$ such that each E_i is a matching of G. Let $M = G[E_1 \cup E_2 \cup E_3]$. Then $\Delta(M) \leqslant 3$. For a Δ-vertex x of G, at most one among E_1, E_2, E_3 contains no edge incident with x. For a $(\Delta-1)$-vertex y of G, at most two among E_1, E_2, E_3 contain no edge incident with y. Thus $M \in \mathcal{H}(G)$.

For any $M \in \mathcal{H}(G)$, it is easy to see that $\Delta(\overline{M}) \leqslant \Delta - 2$. Now let $I(M)$ and $I(\overline{M})$ denote the sets of isolated edges of M and $\overline{M}$, respectively, and write $i(M) = |I(M)|$ and $i(\overline{M}) = |I(\overline{M})|$. Among all subgraphs M that attain the minimum for $i(M) + i(\overline{M})$, we pick and fix an H that has minimum number of edges.

We are going to show that the edge-partition of G induced by this H satisfies conditions 1, 2 and 3 of Theorem 2.1. If $i(H) + i(\overline{H}) = 0$, then we are done. Now we assume that $i(H) + i(\overline{H}) > 0$.

We first classify some of the vertices of G into two types.

A vertex $v \in V(G)$ is classified as *type-I* if $1 \leqslant d_H(v) \leqslant 2$, $d(v) \geqslant \Delta - 1$, and for every $u \in N_{\overline{H}}(v)$, one of the following three conditions holds.

(1) $d_H(u) = 3$.

(2) $d_H(u) = d_{\overline{H}}(u) = 2$.

(3) $d_H(u) \leqslant 1$, $d_{\overline{H}}(u) = 2$, and, for the unique $w \in N_{\overline{H}}(u) \setminus \{v\}$, both $d_{\overline{H}}(w) = 1$ and $d_H(w) = 3$.

Claim 1. *Suppose that $vv' \in I(H)$ with $d(v) \geqslant d(v')$. Then $d(v) = \Delta - 1$ and v is a type-I vertex.*

Proof. Since $H \in \mathcal{H}(G)$ and vv' is an isolated edge of H, $d_H(v) = 1$ and $d(v) \leqslant \Delta - 1$. If $d(v) \leqslant \Delta - 2$, then $H' = H \setminus \{vv'\} \in \mathcal{H}(G)$. Note that $i(H') = i(H) - 1$ and $i(\overline{H'}) \leqslant i(\overline{H})$ since $vv' \notin I(\overline{H'})$. The subgraph H' contradicts the choice of H. Consequently, $d(v) = \Delta - 1$.

Assume to the contrary that v is not a type-I vertex. Then there exists a particular $u \in N_{\overline{H}}(v)$ that satisfies none of (1), (2) or (3). Thus, the following three statements hold for this u.

(a) $d_H(u) \neq 3$, and hence $d_H(u) \leqslant 2$.

(b) If $d_H(u) = 2$, then $d_{\overline{H}}(u) \neq 2$.

(c) If $d_H(u) \leqslant 1$ and $d_{\overline{H}}(u) = 2$, then, for the unique $w \in N_{\overline{H}}(u) \setminus \{v\}$, $d_{\overline{H}}(w) = 1$ implies $d_H(w) \neq 3$, and hence $d_H(w) \leqslant 2$.

Define $H' = H \cup \{uv\}$ for case (b) or when $d_{\overline{H}}(w) \neq 1$ for case (c). Define $H' = H \cup \{uv, uw\}$ when $d_{\overline{H}}(w) = 1$ for case (c). It is easy to check that $H' \in \mathcal{H}(G)$. Since $d_{\overline{H'}}(v) = d(v) - d_{H'}(v) = (\Delta - 1) - 2 > 2$, no new isolated edge is created in $\overline{H'}$. Yet $i(H') = i(H) - 1$. This contradicts the choice of H. ■

A vertex $u \in V(G)$ is classified as *type-II* if $d_H(u) = 3$, or $d_H(u) = d_{\overline{H}}(u) = 2$, and for every $v \in N_H(u)$, one of the following two conditions holds.

(4) $1 \leqslant d_H(v) \leqslant 2$ and $d(v) \geqslant \Delta - 1$.

(5) $d_H(v) = 2$, $d(v) < \Delta - 1$, and, for the unique $w \in N_H(v) \setminus \{u\}$, both $d_H(w) = 1$ and $d(w) = \Delta - 1$.

Claim 2. *Suppose that $uu' \in I(\overline{H})$ with $d(u) \geqslant d(u')$. Then $d_H(u) = 3$ and u is a type-II vertex.*

Proof. Since uu' is an isolated edge of $\overline{H}$ and G has no isolated edges, it follows that $d_H(u) \geqslant 1$. If $d_H(u) \leqslant 2$, then $H' = H \cup \{uu'\} \in \mathcal{H}(G)$. Note that $i(H') \leqslant i(H)$

and $i(\overline{H'}) = i(\overline{H}) - 1$. The subgraph H' contradicts the choice of H. Consequently, $d_H(u) = 3$.

Assume to the contrary that u is not a type-II vertex. Then there exists a particular $v \in N_H(u)$ that satisfies neither (4) nor (5). Thus, the following two statements hold for this v.

(d) If $1 \leqslant d_H(v) \leqslant 2$, then $d(v) < \Delta - 1$.

(e) If $d_H(v) = 2$, $d(v) < \Delta - 1$, then, for the unique $w \in N_H(v) \setminus \{u\}$, $d_H(w) = 1$ implies $d(w) \neq \Delta - 1$, and hence $d(w) < \Delta - 1$.

If $d_H(v) = 1$ or $d_H(v) = 2$ and $d_H(w) \geqslant 2$, let $H' = H \setminus \{uv\}$. If $d_H(v) = 2$ and $d_H(w) = 1$, let $H' = H \setminus \{uv, vw\}$. Thus, the subgraph $H' \in \mathcal{H}(G)$ and satisfies $i(H') \leqslant i(H)$ and $i(\overline{H'}) = i(\overline{H}) - 1$, contradicting the choice of H. ■

We observe that no vertex can be classified both as type-I and type-II since $1 \leqslant d_H(z) \leqslant 2$ and $d(z) \geqslant \Delta - 1 \geqslant 5$ for a type-I vertex z, while $d_H(w) = 3$ or $d_H(w) = d_{\overline{H}}(w) = 2$ for a type-II vertex w.

An H-*chain* emanating from a vertex u is a path from u to a $v \in N_H(u)$ when v satisfies (4), or through v to the unique $w \in N_H(v) \setminus \{u\}$ when v satisfies (5). We write $u \to x$ for an H-chain emanating from u and terminating at x. An $\overline{H}$-*chain* emanating from a vertex v is a path from v to a $u \in N_{\overline{H}}(v)$ when u satisfies (1) or (2), or through u to the unique $w \in N_{\overline{H}}(u) \setminus \{v\}$ when u satisfies (3). We write $v \rightsquigarrow y$ for an $\overline{H}$-chain emanating from v and terminating at y. A path P of G is called an *alternating chain* if P is a concatenation of H-chains and $\overline{H}$-chains such that they appear alternately and the terminating vertex of one chain is the emanating vertex of the next chain.

Claim 3. *If $vv' \in I(H)$ satisfies $d(v) \geqslant d(v')$, then the two ends of each H-chain or $\overline{H}$-chain of an alternating chain P beginning with v are of different types.*

Proof. Let $v_0 = v$. By Claim 1, v_0 is a type-I vertex. By the definition of an alternating chain, we may assume that P is $v_0 \rightsquigarrow u_1 \to v_1 \rightsquigarrow \cdots \to v_{s-1} \rightsquigarrow u_s$ or P is $v_0 \rightsquigarrow u_1 \to v_1 \rightsquigarrow \cdots \rightsquigarrow u_s \to v_s$, where $s \geqslant 1$. It suffices to prove by induction that $v_1, v_2, \ldots, v_s$ are type-I vertices and $u_1, u_2, \ldots, u_s$ are type-II vertices. Equivalently, for each $1 \leqslant k \leqslant s$, the following statements (A) and (B) are true.

(A) If $v_1, v_2, \ldots, v_{k-1}$ are type-I vertices and $u_1, u_2, \ldots, u_{k-1}$ are type-II vertices, then u_k is a type-II vertex.

(B) If $v_1, v_2, \ldots, v_{k-1}$ are type-I vertices and $u_1, u_2, \ldots, u_k$ are type-II vertices, then v_k is a type-I vertex.

In order to show (A), assume to the contrary that u_k is not a type-II vertex. Since $v_{k-1} \rightsquigarrow u_k$ and v_{k-1} is a type-I vertex, $d_H(u_k) = 3$, or $d_H(u_k) = d_{\overline{H}}(u_k) = 2$. Then there exists a vertex $x \in N_H(u_k)$ such that the following two statements hold for this x.

(d$'$) If $1 \leqslant d_H(x) \leqslant 2$, then $d(x) < \Delta - 1$.

(e$'$) If $d_H(x) = 2$, $d(x) < \Delta - 1$, then, for the unique $y \in N_H(x) \setminus \{u_k\}$, $d_H(y) = 1$ implies $d(y) < \Delta - 1$.

Since $v_0, v_1, \ldots, v_{k-1}$ are type-I vertices by the induction hypothesis, $1 \leqslant d_H(v_i) \leqslant 2$ and $d(v_i) \geqslant \Delta - 1$ for all $0 \leqslant i \leqslant k-1$. Since $d_H(x) = 3$, or $d(x) < \Delta - 1$, it follows that $x \notin \{v_0, v_1, \ldots, v_{k-1}\}$. We next show that $x \notin \{u_1, u_2, \ldots, u_{k-1}\}$.

Assume to the contrary that there is an index i $(i < k)$ such that $x = u_i$. Since u_i is a type-II vertex and $u_k \in N_H(u_i)$, it follows that $d_H(u_k) \leqslant 2$. We have already known that $d_H(u_k) = 3$, or $d_H(u_k) = d_{\overline{H}}(u_k) = 2$. Hence, $d_H(u_k) = 2$ and $d(u_k) = 4$. Let $z \in N_H(u_k) \setminus \{u_i\}$. Define

$$H' = (H \cup \bigcup_{j=0}^{i-1} E(v_j \rightsquigarrow u_{j+1})) \setminus (S \cup \bigcup_{j=1}^{i-1} E(u_j \to v_j)),$$

where $S = \{u_i u_k, u_k z\}$ if $d_H(z) = 1$; or $S = \{u_i u_k\}$ otherwise. It is straightforward to check that $H' \in \mathcal{H}(G)$ such that $i(H') = i(H) - 1$ and $i(\overline{H'}) = i(\overline{H})$, which contradicts the choice of H.

Suppose that $d_H(x) = 1$ or $d_H(x) = 2$ and $d_H(y) > 1$ in (e$'$). If $d_H(u_k) = 3$, then let $H' = H \setminus \{x u_k\}$. It is obvious that $H' \in \mathcal{H}(G)$. Since $x u_k$ is adjacent to an edge in $v_{k-1} \rightsquigarrow u_k$, $x u_k$ can not be an isolated edge of $\overline{H'}$. Thus, $i(H') = i(H)$ and $i(\overline{H'}) = i(\overline{H})$. However, $|E(H')| = |E(H)| - 1$, which contradicts the choice of H. If $d_H(u_k) = d_{\overline{H}}(u_k) = 2$, define

$$H' = (H \cup \bigcup_{i=0}^{k-1} E(v_i \rightsquigarrow u_{i+1})) \setminus (\bigcup_{i=1}^{k-1} E(u_i \to v_i) \cup \{x u_k\}).$$

Note that $d_{H'}(u_i) = d_H(u_i)$ and $d_{H'}(v_i) = d_H(v_i)$ for $1 \leqslant i \leqslant k$, $d_{H'}(v_0) = d_H(v_0) + 1 = 2$, $d_{\overline{H'}}(v_0) = (\Delta - 1) - 2 \geqslant 3$, and hence $v' v_0 \notin I(H')$. It follows that $i(H') = i(H) - 1$ and $i(\overline{H'}) = i(\overline{H})$, which contradicts the choice of H.

Next consider the case $d_H(y) = 1$ in (e$'$). Then $y \notin \{v_0, v_1, \ldots, v_{k-1}\}$ since $d(y) < \Delta - 1$; $y \notin \{u_1, u_2, \ldots, u_{k-1}\}$ for each type-II vertex u_i $(1 \leqslant i \leqslant k-1)$ has $d_H(u_i) \geqslant 2$.

Define

$$H' = (H \cup \bigcup_{i=0}^{k-1} E(v_i \rightsquigarrow u_{i+1})) \setminus (\bigcup_{i=1}^{k-1} E(u_i \to v_i) \cup \{xy, xu_k\}).$$

Then $H' \in \mathcal{H}(G)$. Reasoning as before, we see that $i(H') = i(H) - 1$ and $i(\overline{H'}) = i(\overline{H})$, which contradicts the choice of H.

To prove (B), assume to the contrary that v_k is not a type-I vertex. Since $u_k \to v_k$ and u_k is a type-II vertex, $1 \leqslant d_H(v_k) \leqslant 2$ and $d(v_k) \geqslant \Delta - 1$. Then there exists a vertex $x \in N_{\overline{H}}(v_k)$ such that the following three statements hold for this x.

(a′) $d_H(x) \neq 3$, and hence $d_H(x) \leqslant 2$.

(b′) If $d_H(x) = 2$, then $d_{\overline{H}}(x) \neq 2$.

(c′) If $d_H(x) \leqslant 1$ and $d_{\overline{H}}(x) = 2$, then, for the unique $y \in N_{\overline{H}}(x) \setminus \{v_k\}$, $d_{\overline{H}}(y) = 1$ implies $d_H(y) \leqslant 2$.

Since $u_1, u_2, \ldots, u_k$ are type-II vertices by the induction hypothesis, we see that for $1 \leqslant i \leqslant k$, either $d_H(u_i) = 3$ or $d_H(u_i) = d_{\overline{H}}(u_i) = 2$. Therefore, $x \notin \{u_1, u_2, \ldots, u_k\}$.

We next show that $x \notin \{v_0, v_1, \ldots, v_{k-1}\}$. Assume to the contrary that there is an index i ($0 \leqslant i \leqslant k-1$) such that $x = v_i$. Since v_i is a type-I vertex and $v_k \in N_{\overline{H}}(v_i)$, it follows that $d_H(v_k) = 3$ or $d_H(v_k) = d_{\overline{H}}(v_k) = 2$. However, $d_H(v_k) \leqslant 2$ and $d(v_k) \geqslant \Delta - 1 \geqslant 5$ since $u_k \to v_k$. We have reached a contradiction.

Now assume $d_{\overline{H}}(y) = 1$ in (c′). Then $y \notin \{u_1, u_2, \ldots, u_k\}$. We also have $y \notin \{v, v_1, \ldots, u_{k-1}\}$, for otherwise it would imply $d_{\overline{H}}(y) \geqslant 2$. Define

$$H' = (H \cup S \cup \bigcup_{i=0}^{k-1} E(v_i \rightsquigarrow u_{i+1})) \setminus \bigcup_{i=1}^{k-1} E(u_i \to v_i),$$

where $S = \{xy, xv_k\}$ when $d_{\overline{H}}(y) = 1$ for case (c′); $S = \{xv_k\}$ for case (b′) or when $d_{\overline{H}}(y) \neq 1$ for case (c′). It is easy to check that $H' \in \mathcal{H}(G)$ such that $i(H') = i(H) - 1$ and $i(\overline{H'}) = i(\overline{H})$. This contradicts the choice of H. ■

Claim 4. *If $uu' \in I(\overline{H})$ satisfies $d(u) \geqslant d(u')$, then the two ends of each H-chain or $\overline{H}$-chain of an alternating chain P beginning with u are of different types.*

Proof. Let $u_1 = u$ which is a type-II vertex by Claim 2. By the definition of an alternating chain, we may assume that P is $u_1 \to v_1 \rightsquigarrow u_2 \to \cdots \rightsquigarrow u_s \to v_s$ or P is $u_1 \to v_1 \rightsquigarrow u_2 \to \cdots \to v_{s-1} \rightsquigarrow u_s$, where $s \geqslant 1$. Similar to the proof of Claim 3, we may argue that, for each $1 \leqslant k \leqslant s$, the following statements (C) and (D) are true.

(C) If $u_1, u_2 \ldots, u_k$ are type-II vertices and $v_1, v_2, \ldots, v_{k-1}$ are type-I vertices, then v_k is a type-I vertex.

(D) If $u_1, u_2 \ldots, u_{k-1}$ are type-II vertices and $v_1, v_2, \ldots, v_{k-1}$ are type-I vertices, then u_k is a type-II vertex.

The proof of (B) in Claim 3 can be adapted to show the validity of (C). Here we define

$$H' = (H \cup S \cup \bigcup_{i=1}^{k-1} E(v_i \rightsquigarrow u_{i+1})) \setminus \bigcup_{i=1}^{k-1} E(u_i \to v_i),$$

where $S = \{xy, xv_k\}$ if $d_{\overline{H}}(y) = 1$; $S = \{xv_k\}$ if $d_{\overline{H}}(y) > 1$.

The proof of (A) in Claim 3 can be adapted to show the validity of (D). Here we define

$$H' = (H \cup \bigcup_{i=1}^{k-1} E(v_i \rightsquigarrow u_{i+1})) \setminus (S \cup \bigcup_{i=1}^{k-1} E(u_i \to v_i)),$$

where $S = \{xy, xu_k\}$ if $d_{\overline{H}}(y) = 1$; $S = \{xu_k\}$ if $d_{\overline{H}}(y) > 1$.

In both cases, $d_{H'}(u_1) = 3 - 1 = 2$ and $d_{\overline{H'}}(u_1) = 2$. It is easy to check that $H' \in \mathcal{H}(G)$ such that $i(H') = i(H)$ and $i(\overline{H'}) = i(\overline{H}) - 1$. This contradicts the choice of H. ■

Now we are ready to derive contradictions from the assumption $i(H) + i(\overline{H}) > 0$.

Case 1 $i(H) > 0$.

Suppose that $v_0 v' \in I(H)$ with $d(v_0) \geqslant d(v')$. Let $\mathcal{C}(v_0)$ be the set of alternating chains of G beginning with the vertex v_0. By Claims 1 and 3, $\mathcal{C}(v_0)$ is a nonempty set. Let $V_{\mathrm{I}}(P)$ and $V_{\mathrm{II}}(P)$, respectively, be the sets of type-I vertices and type-II vertices on an alternating path $P \in \mathcal{C}(v_0)$. Define $V_{\mathrm{I}} = \cup\{V_{\mathrm{I}}(P) \mid P \in \mathcal{C}(v_0)\}$ and $V_{\mathrm{II}} = \cup\{V_{\mathrm{II}}(P) \mid P \in \mathcal{C}(v_0)\}$.

For any vertex $w \in V_{\mathrm{II}}$, if $x \in N_H(w)$, then either $x \in V_{\mathrm{I}}$, or $d_H(x) = 2$ and the unique vertex $y \in N_H(x) \setminus \{w\}$ satisfies that $d_H(y) = 1$ and $y \in V_{\mathrm{I}}$. Thus

$$\sum_{z \in V_{\mathrm{I}}} d_H(z) \geqslant \sum_{w \in V_{\mathrm{II}}} d_H(w).$$

Since each vertex of V_{I} has degree at most two in H, and each vertex of V_{II} has degree at least two in H, we have

$$2|V_{\mathrm{I}}| \geqslant \sum_{z \in V_{\mathrm{I}}} d_H(z) \geqslant \sum_{w \in V_{\mathrm{II}}} d_H(w) \geqslant 2|V_{\mathrm{II}}|.$$

Thus, $|V_{\mathrm{I}}| \geqslant |V_{\mathrm{II}}|$.

For any $z \in V_{\rm I}$, we have $d_H(z) \leqslant 2$ and $d(z) \geqslant \Delta - 1$, and hence $d_{\overline{H}}(z) \geqslant \Delta - 3$. From $d_H(v_0) = 1$ and $d(v_0) = \Delta - 1$, we know $d_{\overline{H}}(v_0) = \Delta - 2$. Hence,

$$\sum_{z \in V_{\rm I}} d_{\overline{H}}(z) = d_{\overline{H}}(v_0) + \sum_{z \in V_{\rm I} \setminus \{v_0\}} d_{\overline{H}}(z) \geqslant |V_{\rm I}|(\Delta - 3) + 1.$$

For any $w \in V_{\rm II}$, we see that $d_H(w) = 3$ or $d_H(w) = d_{\overline{H}}(w) = 2$. Thus $\Delta \geqslant 6$ implies

$$\sum_{w \in V_{\rm II}} d_{\overline{H}}(w) \leqslant |V_{\rm II}|(\Delta - 3).$$

Then $|V_{\rm I}| \geqslant |V_{\rm II}|$ implies

$$\sum_{w \in V_{\rm II}} d_{\overline{H}}(w) < \sum_{z \in V_{\rm I}} d_{\overline{H}}(z).$$

However, for $z \in V_{\rm I}$ and for each $x \in N_{\overline{H}}(z)$, either $x \in V_{\rm II}$, or $d_{\overline{H}}(x) = 2$ and the unique vertex $y \in N_{\overline{H}}(x) \setminus \{w\}$ has $d_{\overline{H}}(y) = 1$ and $y \in V_{\rm II}$. We get a contradictory consequence

$$\sum_{w \in V_{\rm II}} d_{\overline{H}}(w) \geqslant \sum_{z \in V_{\rm I}} d_{\overline{H}}(z).$$

Case 2 $i(\overline{H}) > 0$.

Suppose that $u_1u' \in I(\overline{H})$ with $d(u_1) \geqslant d(u')$. Let $\mathcal{D}(u_1)$ be the set of alternating chains of G beginning with the vertex u_1. By Claims 2 and 4, $\mathcal{D}(u_1)$ is a nonempty set. Let $V_{\rm I}(P)$ and $V_{\rm II}(P)$, respectively, be the sets of type-I vertices and type-II vertices on an alternating path $P \in \mathcal{D}(u_1)$. Define $V_{\rm I} = \cup\{V_{\rm I}(P) \mid P \in \mathcal{D}(u_1)\}$ and $V_{\rm II} = \cup\{V_{\rm II}(P) \mid P \in \mathcal{D}(u_1)\}$.

Similar to the proof of Case 1, we have that $|V_{\rm I}| \geqslant |V_{\rm II}|$ and

$$|V_{\rm I}|(\Delta - 3) \leqslant \sum_{z \in V_{\rm I}} d_{\overline{H}}(z) \leqslant \sum_{w \in V_{\rm II}} d_{\overline{H}}(w).$$

However, since $d_{\overline{H}}(u_1) = 1$ and $\Delta \geqslant 6$, we get

$$\sum_{w \in V_{\rm II}} d_{\overline{H}}(w) = d_{\overline{H}}(u_1) + \sum_{w \in V_{\rm II} \setminus \{u_1\}} d_{\overline{H}}(w) < |V_{\rm II}|(\Delta - 3).$$

A contradiction is produced. This completes the proof of Theorem 2.1. ■

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