# On Ramsey numbers of complete graphs with dropped stars 

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October 18, 2014


#### Abstract

Let $r(G, H)$ be the smallest integer $N$ such that for any 2-coloring (say, red and blue) of the edges of $K_{n}, n \geqslant N$, there is either a red copy of $G$ or a blue copy of $H$. Let $K_{n}-K_{1, s}$ be the complete graph on $n$ vertices from which the edges of $K_{1, s}$ are dropped. In this note we present exact values for $r\left(K_{m}-K_{1,1}, K_{n}-K_{1, s}\right)$ and new upper bounds for $r\left(K_{m}, K_{n}-K_{1, s}\right)$ in numerous cases. We also present some results for the Ramsey number of Wheels versus $K_{n}-K_{1, s}$.


Keywords: Ramsey numbers; graph Ramsey numbers.
MSC2010: 05C55; 05D10.

## 1. Introduction

Let $G$ and $H$ be two graphs. Let $r(G, H)$ be the smallest integer $N$ such that for any 2-coloring (say, red and blue) of the edges of $K_{n}, n \geqslant N$ there is either a red copy of $G$ or a blue copy of $H$. Let $K_{n}-K_{1, s}$ be the complete graph on $n$ vertices from which the edges of $K_{1, s}$ are dropped. We notice that $K_{n}-K_{1,1}=K_{n}-e$ (the complete graph on $n$ vertices from which an edge is dropped) and $K_{n}-K_{1,2}=K_{n}-P_{3}$ (the complete graph on $n$ vertices from which a path on three vertices is dropped).
In this note we investigate $r\left(K_{m}-e, K_{n}-K_{1, s}\right)$ and $r\left(K_{m}, K_{n}-K_{1, s}\right)$ for a variety of integers $m, n$ and $s$. In the next section, we prove our main result (Theorem (1). In Section 3, we will present exact values for $r\left(K_{m}-e, K_{n}-K_{1, s}\right)$ when $n=3$ or 4 and some values of $m$ and $s$. In Section 4 new upper bounds for $r\left(K_{m}, K_{n}-P_{3}\right)$ for several integers $m$ and $n$ are given. In Section 5, we give new upper bounds for $r\left(K_{m}, K_{n}-K_{1, s}\right)$ when $m, s \geqslant 3$ and several values of $n$. In Section 6, we present some equalities for $r\left(K_{4}, K_{n}-K_{1, s}\right)$ extending the validity of some results given in [3]. Finally, in Section 7, we will present results concerning the Ramsey number of the Wheel $W_{5}$ versus $K_{n}-K_{1, s}$. We present exact values for $r\left(W_{5}, K_{6}-K_{1, s}\right)$ when $s=3$ and 4 and the equalities $r\left(W_{5}, K_{n}-K_{1, s}\right)=r\left(W_{5}, K_{n-1}\right)$ when $n=7$ and 8 for some values of $s$.
Some known values/bounds for specific $r\left(K_{m}, K_{n}\right)$ needed for this paper are given in the Appendix.

## 2. Main Result

Let $G$ be a graph and denote by $G^{v}$ the graph obtained from $G$ to which a new vertex $v$, incident to all the vertices of $G$, is added. Our main result is the following

Theorem 1. Let $n$ and $s$ be positive integers. Let $G_{1}$ be any graph and let $N$ be an integer such that $N \geqslant r\left(G_{1}^{v}, K_{n}\right)$. If $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil \geqslant r\left(G_{1}, K_{n+1}-K_{1, s}\right)$ then $r\left(G_{1}^{v}, K_{n+1}-K_{1, s}\right) \leqslant$ $N$.

[^0]Proof. Let $K_{N}$ be a complete graph on $N$ vertices and consider any 2-coloring of the edges of $K_{N}$ (say, red and blue). We shall show that there is either a $G_{1}^{v}$ red or a $K_{n+1}-K_{1, s}$ blue. Since $N \geqslant r\left(G_{1}^{v}, K_{n}\right)$ then $K_{N}$ has a red $G_{1}^{v}$ or a blue $K_{n}$. In the former case we are done, so let us suppose that $K_{N}$ admit a blue $K_{n}$, that we will denote by $H$. We have two cases.

Case 1) There exists a vertex $u \in V\left(K_{N} \backslash H\right)$ such that $\left|N_{H}^{r}(u)\right| \leqslant s$ where $N_{H}^{r}(u)$ is the set of vertices in $H$ that are joined to $u$ by a red edge. In this case, we may construct the blue graph $G^{\prime}=K_{n+1}-K_{1,\left|N_{H}^{r}(u)\right|}$, this is done by taking $H$ (containing $n$ vertices) and vertex $u$ together with the blue edges between $u$ and the vertices of $H$. Now, since $\left|N_{H}^{r}(u)\right| \leqslant s$ then the graph $K_{n+1}-K_{1, s}$ is contained in $G^{\prime}$ (and thus we found a blue $\left.K_{n+1}-K_{1, s}\right)$.

Case 2) $\left|N_{H}^{r}(u)\right|>s$ for every vertex $u \in V\left(K_{N} \backslash H\right)$. Then we have that the number of red edges $\{x, y\}$ with $x \in V(H)$ and $y \in V\left(K_{N} \backslash H\right)$ is at least $(N-n)(s+1)$. So, by the pigeon hole principle, we have that there exists at least one vertex $v \in V(H)$ such that $d_{K_{N} \backslash H}^{r}(v) \geqslant\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil$, where $d_{K_{N} \backslash H}^{r}(v)=\left|N_{K_{N} \backslash H}^{r}(v)\right|$ and $N_{K_{N} \backslash H}^{r}(v)$ denotes the set of vertices in $K_{N} \backslash H$ incident to $v$ with a red edge. But since $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil \geqslant$ $r\left(G_{1}, K_{n+1}-K_{1, s}\right)$ then the graph induced by $N_{K_{N} \backslash H}^{r}(v)$ has either a blue $K_{n+1}-K_{1, s}$ (and we are done) or a red $G_{1}$ to which we add vertex $v$ to find a red $G^{v}$ as desired.

## 3. Some exact values for $r\left(K_{m}-e, K_{n}-K_{1, s}\right)$

Let $s \geqslant 1$ be an integer. We clearly have that

$$
r\left(K_{3}-e, K_{m}\right) \leqslant r\left(K_{3}-e, K_{m+1}-K_{1, s}\right) .
$$

Since

$$
r\left(K_{3}-e, K_{m+1}-K_{1, s}\right) \leqslant r\left(K_{3}-e, K_{m+1}-e\right)
$$

and (see [10])

$$
r\left(K_{3}-e, K_{m}\right)=r\left(K_{3}-e, K_{m+1}-e\right)=2 m-1
$$

then

$$
r\left(K_{3}-e, K_{m+1}-K_{1, s}\right)=2 m-1 \text { for each } s=1, \ldots, m-1 .
$$

### 3.1. Case $m=4$.

## Corollary 1.

(a) $r\left(K_{4}-e, K_{5}-K_{1,3}\right)=11$.
(b) $r\left(K_{4}-e, K_{6}-K_{1, s}\right)=16$ for any $3 \leqslant s \leqslant 4$.
(c) $r\left(K_{4}-e, K_{7}-K_{1, s}\right)=21$ for any $4 \leqslant s \leqslant 5$.

Proof. (a) It is clear that $r\left(K_{4}-e, K_{4}\right) \leqslant r\left(K_{4}-e, K_{5}-K_{1,3}\right)$. Since $r\left(K_{4}-e, K_{4}\right)=11$ (see [10]) then $11 \leqslant r\left(K_{4}-e, K_{5}-K_{1,3}\right)$. We will now show that $r\left(K_{4}-e, K_{5}-K_{1,3}\right) \leqslant 11$. By taking $N=11, s=3$ and $n=4$, we have that $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil=\left\lceil\frac{4 \times 7}{4}\right\rceil=7=$ $r\left(K_{3}-e, K_{5}-K_{1,3}\right)$ and so, by Theorem 回, we have $r\left(K_{4}-e, K_{5}-K_{1,3}\right) \leqslant 11$, and the result follows.
The proofs for (b) and (c) are analogues. We just need to check that conditions of Theorem [1 are satisfied by taking : $N=r\left(K_{4}-e, K_{5}\right)=16$ for (b) and $N=r\left(K_{4}-\right.$ $\left.e, K_{6}\right)=21$ for (回).

We notice that Corollary 1 (a) is claimed in [8] without a proof. Corollary [(b) can also be obtained by using that $r\left(K_{4}-e, K_{6}-P_{3}\right)=16$ [9] since $16=r\left(K_{4}-e, K_{6}-P_{3}\right) \geqslant$ $r\left(K_{4}-e, K_{6}-K_{1, s}\right) \geqslant r\left(K_{4}-e, K_{5}\right)=16$ for $s \in\{3,4\}$. Corollary $\mathbb{\square}(\mathbb{C})$ was first posed by Hoeth and Mengersen [9. The best known upper bounds for $r\left(K_{4}-e, K_{7}-K_{1,3}\right)$ and $r\left(K_{4}-e, K_{7}-P_{3}\right)$ are obtained by applying the following classical recursive formula :

$$
\begin{equation*}
r\left(K_{m}-e, K_{n}-K_{1, s}\right) \leqslant r\left(K_{m-1}-e, K_{n}-K_{1, s}\right)+r\left(K_{m}-e, K_{n-1}-K_{1, s}\right) \tag{1}
\end{equation*}
$$

Hence

$$
r\left(K_{4}-e, K_{7}-K_{1,3}\right) \leqslant r\left(K_{3}-e, K_{7}-K_{1,3}\right)+r\left(K_{4}-e, K_{6}-K_{1,3}\right)=11+16=27
$$

and

$$
r\left(K_{4}-e, K_{7}-P_{3}\right) \leqslant r\left(K_{3}-e, K_{7}-P_{3}\right)+r\left(K_{4}-e, K_{6}-P_{3}\right)=11+16=27
$$

We are able to improve the above upper bounds.
Corollary 2. $21 \leqslant r\left(K_{4}-e, K_{7}-K_{1,3}\right) \leqslant 22$.
Proof. It is clear that $r\left(K_{4}-e, K_{6}\right) \leqslant r\left(K_{4}-e, K_{7}-K_{1,3}\right)$. Since $r\left(K_{4}-e, K_{6}\right)=21$ (see [10]), then $21 \leqslant r\left(K_{4}-e, K_{7}-K_{1,3}\right)$. We will now show that $r\left(K_{4}-e, K_{7}-K_{1,3}\right) \leqslant 22$. By taking $N=22, s=3$ and $n=6$, we have that $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil=\left\lceil\frac{4 \times 16}{6}\right\rceil=11=$ $r\left(K_{3}-e, K_{7}-K_{1,3}\right)$ and so, by Theorem 1, we have that $r\left(K_{4}-e, K_{7}-K_{1,3}\right) \leqslant 22$, and the result follows.

The above upper bound improves the previously best known one, given by $r\left(K_{4}-e, K_{7}-\right.$ $\left.K_{1,3}\right) \leqslant 27$.
3.2. Case $m=5$. The following equality is claimed in [8] without a proof.

Corollary 3. $r\left(K_{5}-e, K_{5}-K_{1,3}\right)=19$.
Proof. It is clear that $r\left(K_{5}-e, K_{4}\right) \leqslant r\left(K_{5}-e, K_{5}-K_{1,3}\right)$. It is known that $r\left(K_{5}-e, K_{4}\right)=$ 19 (see [10]), then $19 \leqslant r\left(K_{5}-e, K_{5}-K_{1,3}\right)$. We will now show that $r\left(K_{5}-e, K_{5}-K_{1,3}\right) \leqslant$ 19. By Corollary 1, we have that $r\left(K_{4}-e, K_{5}-K_{1,3}\right)=11$. Then, by taking $N=19$, $s=3$ and $n=4$, we have that $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil=\left\lceil\frac{4 \times 15}{4}\right\rceil=15>r\left(K_{4}-e, K_{5}-K_{1,3}\right)=11$ and so, by Theorem $\mathbb{1}$, we have $r\left(K_{5}-e, K_{5}-K_{1,3}\right) \leqslant 19$, and the result follows.

Corollary 4. $r\left(K_{5}-e, K_{6}-K_{1, s}\right)=r\left(K_{5}-e, K_{5}\right)$ for $s=3,4$.
Proof. It is clear that $r\left(K_{5}-e, K_{5}\right) \leqslant r\left(K_{5}-e, K_{6}-K_{1, s}\right)$ for all $s \geqslant 1$. Let us now prove that $r\left(K_{5}-e, K_{5}\right) \geqslant r\left(K_{5}-e, K_{6}-K_{1, s}\right)$ for $s=3,4$. Since $r\left(K_{5}-e, K_{6}-K_{1,4}\right) \leqslant r\left(K_{5}-\right.$ $\left.e, K_{6}-K_{1,3}\right)$ then it is sufficient to prove that $r\left(K_{5}-e, K_{6}-K_{1,3}\right) \leqslant r\left(K_{5}-e, K_{5}\right)$. For, let $N=r\left(K_{5}-e, K_{5}\right) \geqslant 30$ (this lower bound was proved by Exoo [6]). Since $N \geqslant 30$ then if $s=3$ and $n=5$ we obtain that $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil \geqslant\left\lceil\frac{4 \times 25}{5}\right\rceil=20>17 \geqslant r\left(K_{4}-e, K_{6}-K_{1,3}\right)$ (see [10] or Corollary (B) for the last inequality). So, by Theorem [1 we obtain that $r\left(K_{5}-e, K_{6}-K_{1,3}\right) \leqslant N=r\left(K_{5}-e, K_{5}\right)$.

We notice that in the case $s=2$, if $r\left(K_{5}-e, K_{5}\right) \geqslant 32$ then we may obtain that $r\left(K_{5}-e, K_{6}-K_{1,2}\right)=r\left(K_{5}-e, K_{5}\right)$ (by using the same arguments as above). It is known that $r\left(K_{5}-e, K_{5}\right) \geqslant 30$.

## 4. New upper bounds for $r\left(K_{m}, K_{n}-P_{3}\right)$

In this section we will apply our main result to give new upper bounds for $r\left(K_{m}, K_{n}-P_{3}\right)$ in numerous cases. The value of $r\left(K_{n}, K_{m}-P_{3}\right)$ have already been studied in some cases. In [1, 4], it is proved that $r\left(K_{5}, K_{5}-P_{3}\right)=25$ and in [5] it is shown that $r\left(K_{4}, K_{5}-P_{3}\right)=$ $r\left(K_{4}, K_{4}\right)=18$.
Let us first notice that, by taking $G_{1}=K_{m}$ in Theorem 1, we obtain
Corollary 5. Let $N$ be an integer such that $N \geqslant r\left(K_{m+1}, K_{n}\right)$. If $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil \geqslant$ $r\left(K_{m}, K_{n+1}-K_{1, s}\right)$ then $r\left(K_{m+1}, K_{n+1}-K_{1, s}\right) \leqslant N$.

The case when $m=3$ has already been studied in [2] where it is proved that

$$
r\left(K_{3}, K_{n+1}-K_{1, s}\right)=r\left(K_{3}, K_{n}\right) \text { if } n \geqslant s+1>(n-1)(n-2) /(r(3, n)-n) .
$$

As a consequence, we have

$$
\begin{array}{ll}
r\left(K_{3}, K_{6}-P_{3}\right)=r\left(K_{3}, K_{5}\right) & (\text { with } n=5 \text { and } s=2), \\
r\left(K_{3}, K_{7}-K_{1,3}\right)=r\left(K_{3}, K_{6}\right) & \text { (with } n=6 \text { and } s=3), \\
r\left(K_{3}, K_{10}-K_{1, s}\right)=r\left(K_{3}, K_{9}\right) & \text { (with } n=9 \text { for any } 2 \leqslant s \leqslant 9),  \tag{2}\\
r\left(K_{3}, K_{11}-K_{1, s}\right)=r\left(K_{3}, K_{10}\right) & \text { (with } n=10 \text { for any } 3 \leqslant s \leqslant 10) .
\end{array}
$$

4.1. Results on $r\left(K_{m}, K_{5}-P_{3}\right)$. In [3, Theorem 4], it was shown that if $n \geqslant m \geqslant 3$ and $m+n \geqslant 8$, then
(3) $r\left(K_{m+1}-K_{1, m-p}, K_{n+1}-K_{1, n-q}\right)=r\left(K_{m}, K_{n}\right)$ where $p=\left\lceil\frac{m}{n-1}\right\rceil$ and $q=\left\lceil\frac{n}{m-1}\right\rceil$.

This result implies the following
Corollary 6. Let $n \geqslant m \geqslant 3$ and $m+n \geqslant 8$ and let $p=\left\lceil\frac{m}{n-1}\right\rceil$ and $q=\left\lceil\frac{n}{m-1}\right\rceil$. Then,

$$
r\left(K_{m}, K_{n+1}-K_{1, n-q}\right)=r\left(K_{m+1}-K_{1, m-p}, K_{n}\right)=r\left(K_{m}, K_{n}\right) .
$$

Proof. We clearly have

$$
r\left(K_{m}, K_{n}\right) \leqslant r\left(K_{m}, K_{n+1}-K_{1, n-q}\right) \leqslant r\left(K_{m+1}-K_{1, m-p}, K_{n+1}-K_{1, n-q}\right) \stackrel{(3)}{=} r\left(K_{m}, K_{n}\right)
$$

and thus $r\left(K_{m}, K_{n+1}-K_{1, n-q}\right)=r\left(K_{m}, K_{n}\right)$ (the proof for $r\left(K_{m+1}-K_{1, m-p}, K_{n}\right)=$ $r\left(K_{m}, K_{n}\right)$ is similar).

By taking $m=n=4$ (and thus $q=2$ ) in Corollary 6 we have that

$$
r\left(K_{4}, K_{5}-P_{3}\right)=r\left(K_{4}, K_{4}\right)=18 .
$$

It is also known [1] that

$$
r\left(K_{5}, K_{5}-P_{3}\right)=r\left(K_{5}, K_{4}\right)=25,
$$

and, by Corollary 6, we have

$$
\begin{array}{ll}
r\left(K_{6}, K_{4}-P_{3}\right)=r\left(K_{6}, K_{3}\right)=18 & \text { (with } m=5 \text { and } n=3), \\
\left.r\left(K_{7}, K_{4}-P_{3}\right)=r\left(K_{7}, K_{3}\right)=23 \quad \text { (with } m=6 \text { and } n=3\right), \\
\left.r\left(K_{8}, K_{4}-P_{3}\right)=r\left(K_{8}, K_{3}\right)=28 \quad \text { (with } m=7 \text { and } n=3\right),  \tag{4}\\
\left.r\left(K_{9}, K_{4}-P_{3}\right)=r\left(K_{9}, K_{3}\right)=36 \quad \text { (with } m=8 \text { and } n=3\right), \\
\left.r\left(K_{10}, K_{4}-P_{3}\right)=r\left(K_{10}, K_{3}\right) \leqslant 43 \quad \text { (with } m=9 \text { and } n=3\right) .
\end{array}
$$

The best known upper bounds of $r\left(K_{n}, K_{5}-P_{3}\right)$ for $n \geqslant 6$ are obtained by applying the following classical recursive formula :

$$
\begin{equation*}
r\left(K_{m}, K_{n}-K_{1, s}\right) \leqslant r\left(K_{m-1}, K_{n}-K_{1, s}\right)+r\left(K_{m}, K_{n-1}-K_{1, s}\right) . \tag{5}
\end{equation*}
$$

By using（4），we obtain

$$
\begin{aligned}
r\left(K_{6}, K_{5}-P_{3}\right) & \leqslant r\left(K_{5}, K_{5}-P_{3}\right)+r\left(K_{6}, K_{4}-P_{3}\right)=25+r\left(K_{6}, K_{3}\right)=25+18=43, \\
r\left(K_{7}, K_{5}-P_{3}\right) & \leqslant r\left(K_{6}, K_{5}-P_{3}\right)+r\left(K_{7}, K_{4}-P_{3}\right)=43+23=66, \\
r\left(K_{8}, K_{5}-P_{3}\right) & \leqslant r\left(K_{7}, K_{5}-P_{3}\right)+r\left(K_{8}, K_{4}-P_{3}\right) \\
& \leqslant r\left(K_{6}, K_{5}-P_{3}\right)+r\left(K_{7}, K_{4}-P_{3}\right)+28=43+23+28=94, \\
r\left(K_{9}, K_{5}-P_{3}\right) & \leqslant r\left(K_{8}, K_{5}-P_{3}\right)+r\left(K_{9}, K_{4}-P_{3}\right)=94+36=130, \\
r\left(K_{10}, K_{5}-P_{3}\right) & \leqslant r\left(K_{9}, K_{5}-P_{3}\right)+r\left(K_{10}, K_{4}-P_{3}\right) \\
& \leqslant r\left(K_{8}, K_{5}-P_{3}\right)+r\left(K_{9}, K_{4}-P_{3}\right)+43=94+36+43=173 .
\end{aligned}
$$

We are able to improve all the above upper bounds．

## Corollary 7.

（a）$r\left(K_{6}, K_{5}-P_{3}\right) \leqslant 41$ ．
（b）$r\left(K_{7}, K_{5}-P_{3}\right) \leqslant 61$ ．
（c）$r\left(K_{8}, K_{5}-P_{3}\right) \leqslant 85$ ．
（d）$r\left(K_{9}, K_{5}-P_{3}\right) \leqslant 117$ ．
（e）$r\left(K_{10}, K_{5}-P_{3}\right) \leqslant 159$ ．
Proof．（（a））It is known that $r\left(K_{6}, K_{4}\right) \leqslant 41$ ．Then，by taking $N=41, s=2$ and $n=4$ ， we have that $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil=\left\lceil\frac{3 \times 37}{4}\right\rceil=28>r\left(K_{5}, K_{5}-P_{3}\right)=25$ and so，by Corollary 國， the result follows．

The proofs for the rest of the cases are analogues．We just need to check that conditions are satisfied by taking：$N=61 \geqslant r\left(K_{7}, K_{4}\right)$ for（b）$N=85>84 \geqslant r\left(K_{8}, K_{4}\right)$ for（（C）， $N=117>115 \geqslant r\left(K_{9}, K_{4}\right)$ for（d）and $N=159>149 \geqslant r\left(K_{10}, K_{4}\right)$ for（回）．

By applying recursion（5）to $r\left(K_{11}, K_{5}-P_{3}\right)$ one may obtain that $r\left(K_{11}, K_{5}-P_{3}\right) \leqslant 224$ if the old known values are used in the recursion，and it can be improved to $r\left(K_{11}, K_{5}-P_{3}\right) \leqslant$ 210 by using the new values given in Corollary 7．The latter beats the upper bound $r\left(K_{11}, K_{5}-P_{3}\right) \leqslant 215$ obtained via Corollary 5

We can also use Corollary 5 to give the following equality．
Corollary 8．If $37 \leqslant r\left(K_{6}, K_{4}\right)$ then $r\left(K_{6}, K_{5}-P_{3}\right)=r\left(K_{6}, K_{4}\right)$ ．
Proof．It is clear that $r\left(K_{6}, K_{4}\right) \leqslant r\left(K_{6}, K_{5}-P_{3}\right)$ ．We show that $r\left(K_{6}, K_{5}-P_{3}\right) \leqslant$ $r\left(K_{6}, K_{4}\right)$ ．Let $N=r\left(K_{6}, K_{4}\right) \geqslant 37$ ．Since $N \geqslant 37$ and by taking $s=2$ and $n=4$ we have $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil \geqslant\left\lceil\frac{3 \times 33}{4}\right\rceil=25=r\left(K_{5}, K_{5}-P_{3}\right)$ ，and so，by Corollary 駸，$r\left(K_{6}, K_{5}-P_{3}\right) \leqslant$ $N=r\left(K_{6}, K_{4}\right)$ ．

It is known that $36 \leqslant r\left(K_{6}, K_{4}\right)$ ．In the case when $r\left(K_{6}, K_{4}\right)=36$ the above result might not hold．

4．2．Results on $r\left(K_{m}, K_{6}-P_{3}\right)$ ．Since $r\left(K_{3}, K_{5}\right)=14$ then，by（2）we have $r\left(K_{3}, K_{6}-\right.$ $\left.P_{3}\right)=14$［7］．So，by（5），we have

$$
r\left(K_{4}, K_{6}-P_{3}\right) \leqslant r\left(K_{3}, K_{6}-P_{3}\right)+r\left(K_{4}, K_{5}-P_{3}\right)=14+18=32 .
$$

Moreover，it is known that the upper bound is strict if the terms of the right side are even，which is our case，and so，$r\left(K_{4}, K_{6}-P_{3}\right) \leqslant 31$ ．

## Corollary 9.

（a） $25 \leqslant r\left(K_{4}, K_{6}-P_{3}\right) \leqslant 27$ ．
（b）$r\left(K_{5}, K_{6}-P_{3}\right) \leqslant 49$ ．
（c）$r\left(K_{6}, K_{6}-P_{3}\right) \leqslant 87$ ．

Proof. ((a)) We clearly have that $25=r\left(K_{4}, K_{5}\right) \leqslant r\left(K_{4}, K_{6}-P_{3}\right)$. It is known that $r\left(K_{4}, K_{5}\right)=25$. We take $N=27>r\left(K_{4}, K_{5}\right), s=2$ and $n=5$. So, $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil=$ $\left\lceil\frac{3 \times 22}{5}\right\rceil=14=r\left(K_{3}, K_{6}-P_{3}\right)$ and so, by Corollary 鲟, $r\left(K_{4}, K_{6}-P_{3}\right) \leqslant 27$.
The proofs for (b) and (©) are analogues. We just need to check that conditions of Corollary 5 are satisfied by taking: $N=49 \geqslant r\left(K_{5}, K_{5}\right)$ for (b) and $N=87 \geqslant r\left(K_{6}, K_{5}\right)$ for ( (C).

The recursive formula (5) gives now (by using the new above values) $r\left(K_{7}, K_{6}-P_{3}\right) \leqslant 148$ (before, by using the old values, it gave 158). This new upper bound beats the upper bound $r\left(K_{7}, K_{6}-P_{3}\right) \leqslant 149$ obtained by Corollary 5 .

### 4.3. Results on $r\left(K_{m}, K_{n}-P_{3}\right)$ for a variety of $m$ and $n$.

Corollary 10. For each $3 \leqslant m \leqslant 5$ and each $7 \leqslant n \leqslant 16$, we have that $r\left(K_{m}, K_{n}-P_{3}\right) \leqslant$ $u(m, n)$, where the value of $u(m, n)$ is given in the $(m, n)$ entry of the below table (the value between parentheses is the best previously known upper bound).

| $m \backslash n$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 |  |  |  |  | $44(47)$ | $52(59)$ | $61(72)$ | $70(86)$ | $80(101)$ | $91(117)$ |
| 4 | $41(49)$ | $61(72)$ |  | $115(136)$ | $154(183)$ | $199(242)$ | $253(319)$ | $313(405)$ | $383(506)$ | $466(623)$ |
| 5 | $87(105)$ | $143(177)$ | $222(277)$ |  |  |  |  |  |  |  |

Proof. We just need to check that conditions of Corollary 5 are satisfied by taking: $N=$ $41 \geqslant r\left(K_{4}, K_{6}\right)$ for $u(4,7), N=87 \geqslant r\left(K_{5}, K_{6}\right)$ for $u(5,7), N=61 \geqslant r\left(K_{4}, K_{7}\right)$ for $u(4,8), N=143 \geqslant r\left(K_{5}, K_{7}\right)$ for $u(5,8), N=222>216 \geqslant r\left(K_{5}, K_{8}\right)$ for $u(5,9)$, $N=115 \geqslant r\left(K_{4}, K_{9}\right)$ for $u(4,10), N=47>42 \geqslant r\left(K_{3}, K_{10}\right)$ for $u(3,11), N=154>$ $149 \geqslant r\left(K_{4}, K_{10}\right)$ for $u(4,11), N=52>51 \geqslant r\left(K_{3}, K_{11}\right)$ for $u(3,12), N=199>$ $191 \geqslant r\left(K_{4}, K_{11}\right)$ for $u(4,12), N=61>59 \geqslant r\left(K_{3}, K_{12}\right)$ for $u(3,13), N=253>$ $238 \geqslant r\left(K_{4}, K_{12}\right)$ for $u(4,13), N=70>69 \geqslant r\left(K_{3}, K_{13}\right)$ for $u(3,14), N=313>$ $291 \geqslant r\left(K_{4}, K_{13}\right)$ for $u(4,14), N=80>78 \geqslant r\left(K_{3}, K_{14}\right)$ for $u(3,15), N=383>$ $349 \geqslant r\left(K_{4}, K_{14}\right)$ for $u(4,15), N=91>88 \geqslant r\left(K_{3}, K_{15}\right)$ for $u(3,16), N=466>417 \geqslant$ $r\left(K_{4}, K_{15}\right)$ for $u(4,16)$.

## 5. Some bounds for $r\left(K_{m}, K_{n}-K_{1, s}\right)$ When $s \geqslant 3$

Here, we will focus our attention to upper bounds for $r\left(K_{m}, K_{n}-K_{1,3}\right)$ that yields to upper bounds for $r\left(K_{m}, K_{n}-K_{1, s}\right)$ when $s \geqslant 4$ since

$$
r\left(K_{m}, K_{n}-K_{1, s}\right) \leqslant r\left(K_{m}, K_{n}-K_{1,3}\right) \text { for all } s \geqslant 4
$$

5.1. Results on $r\left(K_{m}, K_{6}-K_{1,3}\right)$. In [3] it was proved that $r\left(K_{5}, K_{6}-K_{1,3}\right)=r\left(K_{5}, K_{5}\right) \leqslant$ 49. So by (5) we have

$$
r\left(K_{6}, K_{6}-K_{1,3}\right) \leqslant r\left(K_{5}, K_{6}-K_{1,3}\right)+r\left(K_{6}, K_{5}-K_{1,3}\right)=49+41=90
$$

Corollary 11. For each $6 \leqslant m \leqslant 15$, we have that $r\left(K_{m}, K_{6}-K_{1,3}\right) \leqslant u(m)$, where the value of $u(m)$ is given in the below table (the value between parentheses is the best previously known upper bound).

| $m$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{u}$ | $87(90)$ | $143(151)$ | $216(235)$ | $316(350)$ | $442(499)$ | $633(690)$ | $848(928)$ | $1139(1219)$ | $1461(1568)$ |

Proof. It follows by Corollary 5 and by taking $N$ as the best known upper bound of $r\left(K_{n}, K_{5}\right)$ for each $n=6, \ldots, 15$.

We notice that by using similar arguments as above, we could prove that $r\left(K_{6}, K_{6}-\right.$ $\left.K_{1,3}\right)=r\left(K_{6}, K_{5}\right)$ if $66 \leqslant r\left(K_{6}, K_{5}\right)$.
5.2. Results on $r\left(K_{m}, K_{7}-K_{1,3}\right)$. In [2] it was proved that $r\left(K_{3}, K_{7}-K_{1,3}\right)=18$. Since $r\left(K_{3}, K_{6}\right)=18$ then, by (2) we have $r\left(K_{3}, K_{7}-K_{1,3}\right)=18$. So, by (5), we have

$$
r\left(K_{4}, K_{7}-K_{1,3}\right) \leqslant r\left(K_{3}, K_{7}-K_{1,3}\right)+r\left(K_{4}, K_{6}-K_{1,3}\right)=18+25=43 .
$$

Corollary 12. For each $4 \leqslant m \leqslant 11$, we have that $r\left(K_{m}, K_{7}-K_{1,3}\right) \leqslant u(m)$, where the value of $u(m)$ is given in the below table (the value between parentheses is the best previously known upper bound).

| $m$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{u}$ | $41(43)$ | $87(90)$ | $165(180)$ | $298(331)$ | $495(566)$ | $780(916)$ | $1175(1415)$ | $1804(2105)$ |

Proof. It follows by Corollary 5, by taking $s=3$ and $N$ equals to the best known upper bound for $r\left(K_{n}, K_{6}\right)$ when $n=5,6,7,8,9,11$ and $N=1175>1171 \geqslant r\left(K_{10}, K_{6}\right)$ when $n=10$. For instance, for (1) we take $N=41 \geqslant r\left(K_{4}, K_{6}\right), s=3$ and $n=6$. Then, $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil=\left\lceil\frac{4 \times 35}{6}\right\rceil=24>r\left(K_{3}, K_{7}-K_{1,3}\right)$ and, by Corollary 5, $r\left(K_{4}, K_{7}-\right.$ $\left.K_{1,3}\right) \leqslant 41$.

## 6. More equalities

From (3) we have that $r\left(K_{4}, K_{n+1}-K_{1, s}\right)=r\left(K_{4}, K_{n}\right)$ if $s \geqslant n-\left\lceil\frac{n}{3}\right\rceil$. The latter yields to the following equalities.

$$
\begin{array}{lll}
r\left(K_{4}, K_{7}-K_{1, s}\right)=r\left(K_{4}, K_{6}\right) \text { if } s \geqslant 4, & r\left(K_{4}, K_{8}-K_{1, s}\right)=r\left(K_{4}, K_{7}\right) \text { if } s \geqslant 5, \\
r\left(K_{4}, K_{9}-K_{1, s}\right)=r\left(K_{4}, K_{8}\right) \text { if } s \geqslant 5, & r\left(K_{4}, K_{10}-K_{1, s}\right)=r\left(K_{4}, K_{9}\right) \text { if } s \geqslant 6, \\
r\left(K_{4}, K_{11}-K_{1, s}\right)=r\left(K_{4}, K_{10}\right) \text { if } s \geqslant 6, & r\left(K_{4}, K_{12}-K_{1, s}\right)=r\left(K_{4}, K_{11}\right) \text { if } s \geqslant 7, \\
r\left(K_{4}, K_{13}-K_{1, s}\right)=r\left(K_{4}, K_{12}\right) \text { if } s \geqslant 8, & r\left(K_{4}, K_{14}-K_{1, s}\right)=r\left(K_{4}, K_{13}\right) \text { if } s \geqslant 8, \\
r\left(K_{4}, K_{15}-K_{1, s}\right)=r\left(K_{4}, K_{14}\right) \text { if } s \geqslant 9, & r\left(K_{4}, K_{16}-K_{1, s}\right)=r\left(K_{4}, K_{15}\right) \text { if } s \geqslant 10 .
\end{array}
$$

We are able to extend all these equalities for further values of $s$.

## Corollary 13.

(a) $r\left(K_{4}, K_{7}-K_{1, s}\right)=r\left(K_{4}, K_{6}\right)$ for $s=3$. (b) $r\left(K_{4}, K_{8}-K_{1, s}\right)=r\left(K_{4}, K_{7}\right)$ for $s=3,4$.
(c) $r\left(K_{4}, K_{9}-K_{1, s}\right)=r\left(K_{4}, K_{8}\right)$ for $s=4$. (d) $r\left(K_{4}, K_{10}-K_{1, s}\right)=r\left(K_{4}, K_{9}\right)$ for $s=4,5$.
(e) $r\left(K_{4}, K_{11}-K_{1, s}\right)=r\left(K_{4}, K_{10}\right)$ for $s=5$. (f) $r\left(K_{4}, K_{12}-K_{1, s}\right)=r\left(K_{4}, K_{11}\right)$ for $s=6$.
(g) $r\left(K_{4}, K_{13}-K_{1, s}\right)=r\left(K_{4}, K_{12}\right)$ for $s=6,7$. (h) $r\left(K_{4}, K_{14}-K_{1, s}\right)=r\left(K_{4}, K_{13}\right)$ for $s=7$.
(i) $r\left(K_{4}, K_{15}-K_{1, s}\right)=r\left(K_{4}, K_{14}\right)$ for $s=8$. (j) $r\left(K_{4}, K_{16}-K_{1, s}\right)=r\left(K_{4}, K_{15}\right)$ for $s=9$.

Proof. (回) Since $r\left(K_{4}, K_{6}\right) \geqslant 36$ it follows that $r\left(K_{4}, K_{7}-K_{1,3}\right) \geqslant 36$ and by (2), we have $r\left(K_{3}, K_{7}-K_{1,3}\right)=r\left(K_{3}, K_{6}\right)=18$. Let us take $N=r\left(K_{4}, K_{6}\right) \geqslant 36, s=3$ and $n=6$. So, $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil \geqslant\left\lceil\frac{4 \times 30}{6}\right\rceil=20>r\left(K_{3}, K_{7}-K_{1,3}\right)=18$ and the result follows by Corollary 5 .
The proofs for the rest of the cases are analogues. We just need to check that conditions of Corollary 13 are satisfied by taking: $N=r\left(K_{4}, K_{7}\right) \geqslant 49$ and checking that $r\left(K_{3}, K_{8}-\right.$ $\left.K_{1,3}\right)=r\left(K_{3}, K_{7}\right)=23$ for (b),$N=r\left(K_{4}, K_{8}\right) \geqslant 58$ and checking that $r\left(K_{3}, K_{9}-K_{1,4}\right)=$ $r\left(K_{3}, K_{8}\right)=28$ for (뜨) and so on.

We notice that, by using the same arguments as above, we could improve cases (目) and (g) by showing that $r\left(K_{4}, K_{11}-K_{1,4}\right)=r\left(K_{4}, K_{10}\right)$ when $r\left(K_{4}, K_{10}\right) \neq 92$ and $r\left(K_{4}, K_{13}-\right.$ $\left.K_{1,5}\right)=r\left(K_{4}, K_{12}\right)$ when $r\left(K_{4}, K_{12}\right) \neq 128$.
In view of Corollary 13, we may pose the following question,
Question 1. Let $n \geqslant 7$ be an integer. For which integer $s$ the equality $r\left(K_{4}, K_{n}-K_{1, s}\right)=$ $r\left(K_{4}, K_{n-1}\right)$ holds?

Or more ambitious, in view of [3, Theorem 4], we may pose the following,

Question 2. Let $m \geqslant 4$ and $n \geqslant 7$ be integers. For which integer $s \leqslant n-1$ the equality $r\left(K_{m}, K_{n}-K_{1, s}\right)=r\left(K_{m}, K_{n-1}\right)$ holds?

## 7. Wheels versus $K_{n}-K_{1, s}$

In this section we obtain further relating results by applying Theorem 1 to other graphs. Indeed, we may consider $G_{1}$ as the cycle on $n-1$ vertices $C_{n-1}$, and thus $G_{1}^{v}$ will be the wheel $W_{n}$ by taking the new vertex $v$ incident to all the vertices of $C_{n-1}$.

## Corollary 14.

(a) $r\left(W_{5}, K_{6}-K_{1, s}\right)=27$ for $s=3,4,5$.
(b) $r\left(W_{5}, K_{7}-K_{1, s}\right)=r\left(W_{5}, K_{6}\right)$ for $s=4,5,6$.
(c) $r\left(W_{5}, K_{8}-K_{1, s}\right)=r\left(W_{5}, K_{7}\right)$ for $s=4,5,6,7$.

Proof. (回) It is clear that $r\left(W_{5}, K_{5}\right) \leqslant r\left(W_{5}, K_{6}-K_{1, s}\right)$ for any $1 \leqslant s \leqslant 5$. Since $r\left(W_{5}, K_{5}\right)=27$ (see [10]), then $27 \leqslant r\left(W_{5}, K_{6}-K_{1, s}\right)$. We will now show that $r\left(W_{5}, K_{6}-\right.$ $\left.K_{1, s}\right) \leqslant 27$ for $3 \leqslant s \leqslant 5$. By taking $N=27, s \geqslant 3$ and $n=5$, we have that $\left\lceil\frac{(s+1)(N-n)}{n}\right\rceil \geqslant\left\lceil\frac{4 \times 22}{5}\right\rceil=18=r\left(C_{4}, K_{6}\right) \geqslant r\left(C_{4}, K_{6}-K_{1, s}\right)$ and so, by Theorem 亿 , we have $r\left(W_{5}, K_{6}-K_{1, s}\right) \leqslant 27$, and the result follows.
The proofs for (b) and (드) are analogues. We just need to check that conditions of Theorem 1 are satisfied by taking: $N=r\left(W_{5}, K_{6}\right) \geqslant 33$ for (b) and $N=r\left(W_{5}, K_{7}\right) \geqslant 43$ for (ㄸ) (see [10] for the lower bounds of $r\left(W_{5}, K_{6}\right)$ and $r\left(W_{5}, K_{7}\right)$ ).

## Acknowledgments

The authors would like to thank the anonymous referees for their useful remarks and comments.

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## Appendix

The following table was obtained from [10].

|  | $K_{3}$ | $K_{4}$ | $K_{5}$ | $K_{6}$ | $K_{7}$ | $K_{8}$ | $K_{9}$ | $K_{10}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $K_{3}$ | 6 | 9 | 14 | 18 | 23 | 28 | 36 | $[40,42]$ |
| $K_{4}$ |  | 18 | 25 | $[36,41]$ | $[49,61]$ | $[58,84]$ | $73,115]$ | $[92,149]$ |
| $K_{5}$ |  |  | $[43,49]$ | $[58,87]$ | $[80,143]$ | $[101,216]$ | $[126,316]$ | $[144,442]$ |
| $K_{6}$ |  |  |  | $[102,165]$ | $[113,298]$ | $[132,495]$ | $[169,780]$ | $[179,1171]$ |

TABLE 1. Some known bounds and values of $r\left(K_{m}, K_{n}\right)$.


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