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# Clique cycle-transversals in distance-hereditary graphs

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Abstract. A cycle-transversal of a graph G is a subset  $T \subseteq V(G)$  such that  $T \cap V(C) \neq \emptyset$  for every cycle C of G. A clique cycle-transversal, or cct for short, is a cycle-transversal which is a clique. Recognizing graphs which admit a cct can be done in polynomial time; however, no structural characterization of such graphs is known. We characterize distance-hereditary graphs admitting a cct in terms of forbidden induced subgraphs. This extends similar results for chordal graphs and cographs.

# 1 Introduction

A cycle-transversal of a graph G is a subset  $T \subseteq V(G)$  such that  $T \cap V(C) \neq \emptyset$  for every cycle C of G. When T is a clique, we say that T is a clique cycle-transversal or simply cct. A graph admits a cct if and only if it can be partitioned into a complete subgraph and a forest; by this reason such a graph is also called a  $(\mathcal{C}, \mathcal{F})$ -graph in [3].

Finding a minimum cycle-transversal in a graph is NP-hard due to a general result in [12], which says that the problem of finding the minimum number of vertices of a graph Gwhose deletion results in a subgraph satisfying a hereditary property  $\pi$  on induced subgraphs is NP-hard. This result implies the NP-hardness of other problems involving cycletransversals, for instance the problem of finding a minimum odd cycle-transversal (which is equivalent to finding a maximum induced bipartite subgraph), or the problem of finding a minimum triangle-transversal (which is equivalent to finding a maximum induced trianglefree subgraph). Odd cycle-transversals are interesting due to their connections to perfect graph theory; in [11], an O(mn) algorithm is developed to find odd cycle-transversals, for a fixed integer k, in graphs with bounded degree; among other results, they describe a polynomial-time algorithm for finding minimum  $C_4$ -transversals in graphs with maximum degree three. Graphs admitting a cct can be recognized in polynomial time, as follows. Note first that  $(\mathcal{C}, \mathcal{F})$ -graphs form a subclass of (2, 1)-graphs (graphs whose vertex set can be partitioned into two stable sets and one clique). The strategy for recognizing a  $(\mathcal{C}, \mathcal{F})$ -graph G initially checks whether G is a (2, 1)-graph, which can be done in polynomial time (see [2]). If so, then test, for each candidate clique Q of a (2, 1)-partition of G, if G - Q is acyclic (which can be done in linear time). If the test fails for all cliques Q, then G is not a  $(\mathcal{C}, \mathcal{F})$ -graph, otherwise G is a  $(\mathcal{C}, \mathcal{F})$ -graph. To conclude the argument, we claim that the number of candidate cliques Q is polynomial. Since G is a (2, 1)-graph, let (B, Q) be a (2, 1)-partition of V(G) where B induces a bipartite subgraph and Q is a clique. Let (B', Q') be another (2, 1)-partition of V(G). Then  $|Q' \setminus Q| \leq 2$  and  $|Q \setminus Q'| \leq 2$ , otherwise G[B] or G[B'] would contain a triangle, which is impossible. Therefore, we can generate in polynomial time all the other candidate cliques Q' from Q. This is the same argument used to count sparse-dense partitions (for more details see [7]). Although recognizing graphs admitting a cct can be done in polynomial time, no structural characterization of such graphs is known, even for perfect graphs.

A similar sparse-dense partition argument can be employed to show that an interesting superclass of  $(\mathcal{C}, \mathcal{F})$ -graphs, namely graphs admitting a clique triangle-transversal, can also be recognized in polynomial time. Such graphs are also known in the literature as (1, 2)-split graphs. A characterization of this class is given in [13], where it has been proved that there are 350 minimal forbidden induced subgraphs for (1, 2)-split graphs. When Gis a perfect graph, being a (1, 2)-split graph is equivalent to being a (2, 1)-graph: observe that a perfect graph G contains a clique triangle-transversal if and only if G contains a clique that intersects all of its odd cycles. In [4] and [9], respectively, characterizations by forbidden induced subgraphs of cographs and chordal graphs which are (1, 2)-split graphs are presented.

Deciding whether a distance-hereditary graph admits a cct can be done in linear-time using the clique-width approach, since the existence of a cct can be represented by a Monadic Second Order Logic (MSOL) formula using only predicates over vertex sets [6,10]. However, no structural characterization for distance-hereditary graphs admitting a cct was known. In order to fill this gap, in this note we describe a characterization of distance-hereditary graphs with cct in terms of forbidden induced subgraphs.

# 2 Background

In this work, all graphs are finite, simple and undirected. Given a graph G = (V(G), E(G)), we denote by  $\overline{G}$  the complement of G. For  $V' \subseteq V(G)$ , G[V'] denotes the subgraph of G induced by V'. Let  $X = (V_X, E_X)$  and  $Y = (V_Y, E_Y)$  be two graphs such that  $V_X \cap V_Y = \emptyset$ . The operations "+" and " $\cup$ " are defined as follows: the *disjoint union*  $X \cup Y$ , sometimes referred simply as graph union, is the graph with vertex set  $V_X \cup V_Y$ and edge set  $E_X \cup E_Y$ ; the *join* X + Y is the graph with vertex set  $V_X \cup V_Y$  and edge set  $E_X \cup E_Y \cup \{xy \mid x \in V_X, y \in V_Y\}.$ 

Let  $N(x) = \{y \mid y \neq x \text{ and } xy \in E\}$  denote the open neighborhood of x and let  $N[x] = \{x\} \cup N(x)$  denote the closed neighborhood of x. A *cut-vertex* is a vertex x such that  $G[V \setminus \{x\}$  has more connected components than G. A *block* (or 2-connected component) of G is a maximal induced subgraph of G having no cut-vertex. A block is *nontrivial* if it contains a cycle; otherwise it is *trivial*.

For a set  $\mathcal{F}$  of graphs, G is  $\mathcal{F}$ -free if no induced subgraph of G is in  $\mathcal{F}$ .

Vertices x and y are true twins (false twins, respectively) in G if N[x] = N[y] (N(x) = N(y), respectively).

Adding a true twin (false twin, pendant vertex, respectively) y to vertex x in graph G means that for G and  $y \notin V(G)$ , a new graph G' is constructed with  $V(G') = V(G) \cup \{y\}$  and  $E(G') = E(G) \cup \{xy\} \cup \{uy \mid u \in N(x)\}$  ( $E(G') = E(G) \cup \{uy \mid u \in N(x)\}$ ,  $E(G') = E(G) \cup \{xy\}$ , respectively).

The complete (resp. edgeless) graph with n vertices is denoted by  $K_n$  (respectively  $I_n$ ). The graphs  $K_1$  and  $K_3$  are called *trivial graph* and *triangle*, respectively. The *chordless* cycle (chordless path, respectively) with n vertices is denoted by  $C_n$  ( $P_n$ , respectively). The graph  $C_n$  ( $\overline{C_n}$ , respectively) for  $n \geq 5$  is a hole (anti-hole, respectively).

The *house* is the graph with vertices a, b, c, d, e and edges ab, bc, cd, ad, ae, be. The *gem* is the graph with vertices a, b, c, d, e and edges ab, bc, cd, ae, be, ce, de. The *domino* is the graph with vertices a, b, c, d, e, h and edges ab, bc, cd, ad, be, eh, ch.

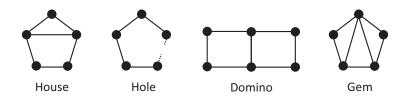


Figure 1: House, hole, domino, and gem.

If H is an induced subgraph of G then we say that G contains H, otherwise G is H-free. A clique (resp. stable or independent set) is a subset of vertices inducing a complete (resp. edgeless) subgraph. A universal vertex is a vertex adjacent to all the other vertices of the graph. A split graph is a graph whose vertex set can be partitioned into a stable set and a clique. It is well known that G is a split graph if an only if G is  $(2K_2, C_4, C_5)$ -free.

A star is a graph whose vertex set can be partitioned into a stable set and a universal vertex. A bipartite graph is a graph whose vertex set can be partitioned into two stable sets. A cograph is a graph containing no  $P_4$ . A chordal graph is a graph containing no  $C_k$ , for  $k \ge 4$ . A distance-hereditary graph is a graph in which the distances in any connected induced subgraph are the same as they are in the original graph. A threshold graph is a graph that can be constructed from a one-vertex graph by repeated applications of the

following two operations: (a) addition of a single isolated vertex to the graph; (b) addition of a single universal vertex to the graph. It is well known that G is a threshold graph if an only if G is  $(2K_2, C_4, P_4)$ -free.

If  $T \cap V(C) \neq \emptyset$  for cycle C, we say that T covers C.

# 3 The forbidden subgraph characterization

The following well-known characterization of distance-hereditary graphs, also called *HHDG*free graphs, will be fundamental for our result:

**Theorem 1** [1] The following are equivalent for any graph G:

- (i) G is a distance-hereditary graph.
- (ii) G can be generated from a single vertex by repeatedly adding a pendant vertex, a false twin, or a true twin, respectively.
- (iii) G is (house, hole, domino, gem)-free. (See Figure 1.)

Let G = (V, E) be a graph, and for a vertex  $x \in V$ , let  $N^k(x) = \{y \in V \mid dist_G(x, y) = k\}$ for  $k \ge 0$  denote the distance levels in G with respect to x. For k = 2, let  $R = V \setminus (N[x] \cup N^2(x))$ . The following are useful properties of distance-hereditary graphs:

**Proposition 1** Let G be distance hereditary and  $u, v \in N^2(x)$ .

- (i) If  $uv \in E(G)$  or  $uv \notin E(G)$  but connected by a path in  $N^2(x) \cup R$  then  $N(u) \cap N(x) = N(v) \cap N(x)$ .
- (ii) If N(x) is a stable set then for  $u, v \in N^2(x)$  with  $uv \notin E$ ,  $N(u) \cap N(x)$  and  $N(v) \cap N(x)$  do not overlap.

**Proof.** (i): Since G is (house, hole)-free, u, v cannot have incomparable neighborhoods in N(x). Moreover, since G is (house, hole, domino, gem)-free, the neighborhoods of u and v in N(x) cannot properly contain one another, which shows Proposition 1.

(*ii*): Let us suppose, by contradiction, that u and v overlap. In this case, let a, b, c be vertices in N(x) such that  $au \in E(G)$ ,  $av \notin E(G)$ ,  $bu, bv \in E(G)$ ,  $cu \notin E(G)$  and  $cv \in E(G)$ . Then G[x, a, b, c, u, v] induces a domino, which is a contradiction.

**Theorem 2** Let G be a distance-hereditary graph. Then G admits a clique cycle transversal if and only if G is  $(G_1, \ldots, G_{12})$ -free.

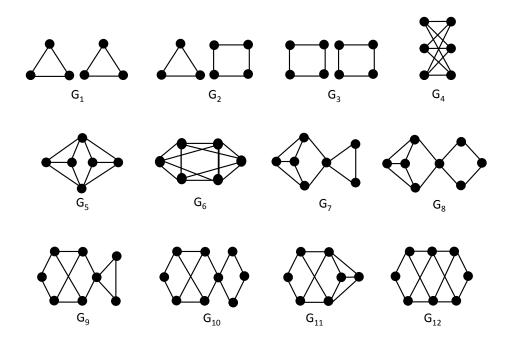


Figure 2: Forbidden subgraphs for distance-hereditary graphs with cct.

**Proof.** It is easy to see that  $G_1, \ldots, G_{12}$  from Figure 2 have no cct. For the converse direction, let G' be a distance-hereditary  $(G_1, \ldots, G_{12})$ -free graph. By Theorem 1, G' results, starting with a single vertex, by repeatedly applying one of the three operations in Theorem 1 (*ii*). Since adding a pendant vertex y to a vertex x in G does not create cycles with y, we can restrict ourselves to the following two cases: G' results from G by either adding a true twin or a false twin y to vertex x in G, and in both cases, we have to show that G' has a cct.

We can inductively assume that G has a cct Q. The vertex set V(G) can be partitioned into  $\{x\} \cup N(x) \cup N^2(x) \cup R$ . Let  $Q_1 = Q \cap N(x)$ ,  $Q_2 = Q \cap N^2(x)$  and  $N_1(x) = N(x) \setminus Q_1$ .

### 3.1 Case 1: y is a true twin to x.

Let G' result from G by adding a true twin y to x in G. In this case, the possible cycles with y in G' are triangles xya for  $a \in N(x)$ , triangles yab for  $a, b \in N(x), ab \in E(G)$ , and  $C_4$ 's yabc for  $a, b \in N(x), ab \notin E(G), c \in N^2(x)$ . If  $x \in Q$  or, more generally,  $Q \subseteq N[x]$ , then  $Q \cup \{y\}$  is a cct of G'. Thus we have to consider the case  $x \notin Q$ . Since for a triangle yab also xab is a triangle which is covered by Q, the triangle yab is covered by Q, and similarly for the  $C_4$  yabc where xabc is a  $C_4$  in G covered by Q. Thus, we only have to deal with triangles xya.

**Claim 1** If  $x \notin Q$  then G[N(x)] is a split graph with partition  $(N_1(x), Q_1)$ .

Proof of Claim 1. For each edge  $ab \in G[N(x)]$ , xab is a triangle. Hence, a or b is in Q and  $N_1(x)$  is a stable set. Since  $Q_1$  is a clique, the claim follows.

Since G' is  $(G_1, G_2, G_3)$ -free, R induces a cycle-free subgraph in G.

Case 1.1:  $G[N^2(x) \cup R]$  is cycle-free.

Claim 2 If  $x \notin Q$  then G' has a cct.

Proof of Claim 2. Since Q is a clique and  $G[N^2(x) \cup R]$  is cycle-free,  $Q_2$  contains at most two vertices. If  $Q_2 = \emptyset$  then  $Q \cup \{y\}$  is a cct of G'. If a vertex  $u \in Q_2$  has no neighbors in  $N_1(x)$  then every cycle containing u also contains a vertex of  $Q_1$ , i.e.,  $Q \setminus \{u\}$  is still a cct of G. Thus, assume without loss of generality that every vertex in  $Q_2$  has a neighbor in  $N_1(x)$ .

If  $Q_2 = \{u, v\}$ , the neighborhood of  $Q_2$  in  $N_1(x)$  cannot contain two vertices a and b, otherwise by Claim 1 and Proposition 1 vertices x, y, a, b, u, v induce  $G_5$ . Hence u and v have precisely one neighbor  $a \in N_1(x)$  which must be adjacent to all vertices of  $Q_1$ , otherwise if a misses a vertex  $b \in Q_1$  then vertices x, y, a, b, u, v induce  $G_5$ . Therefore  $(Q \setminus \{u, v\}) \cup \{a, y\}$  is a cct of G'.

If  $Q_2 = \{u\}$ , we consider two subcases:

(i) Vertex u has a neighbor  $a \in N_1(x)$  which misses some vertex  $b \in Q_1$ . Then every cycle C in G containing a, u but no vertex of  $Q_1$  must also contain x. This is shown as follows: If C does not contain x then, by Proposition 1, C is either a triangle *auv* with  $v \in N^2(x)$  or a  $C_4$  *aucv* with  $c \in N_1(x), v \in N^2(x)$ . In the former case by using Proposition 1, vertices x, y, a, b, u, v induce  $G_5$ . The latter case cannot occur since the existence of cycle *aucv* implies the existence of cycle *axcv* in G, not covered by Q. This implies that  $(Q \setminus \{u\}) \cup \{x, y\}$  is a cct of G'.

(*ii*) Every neighbor  $a \in N_1(x)$  of u sees all vertices in  $Q_1$ . Then  $Q \cup \{a\}$  is a cct of G for some  $a \in N_1(x)$  and, since by Claim 1  $N_1(x)$  is a stable set, every other neighbor  $a' \in N_1(x)$ of u misses some vertex in  $Q_1 \cup \{a\}$ . By applying a similar argument as in (*i*), every cycle C in G containing a', u but no vertex of  $Q_1 \cup \{a\}$  must also contain x. We conclude that  $(Q \setminus \{u\}) \cup \{x, y, a\}$  is a cct of G'. This completes the proof of Claim 2.

Case 1.2:  $G[N^2(x) \cup R]$  is not cycle-free.

We now assume that  $G[N^2(x) \cup R]$  contains a cycle C. This cycle can be one of the following types (see Figure 3):

- $(A_1)$  C has exactly one vertex u in  $N^2(x)$ , and C is a  $C_4$ .
- $(A_2)$  C has exactly one vertex u in  $N^2(x)$ , and C is a  $C_3$ .
- $(B_1)$  C has exactly two vertices u, v in  $N^2(x), uv \in E(G)$ , and C is a  $C_4$ .
- $(B_2)$  C has exactly two vertices u, v in  $N^2(x), uv \in E(G)$ , and C is a  $C_3$ .
- (B<sub>3</sub>) C has exactly two vertices u, v in  $N^2(x), uv \notin E(G)$ , and C is a  $C_4$ .
- $(C_1)$  C is a  $C_4$  with exactly three vertices u, v, w in  $N^2(x)$  (which form a  $P_3$  in  $N^2(x)$ ).
- $(D_1) \ C \text{ is a } C_3 \text{ in } N^2(x).$
- $(D_2)$  C is a  $C_4$  in  $N^2(x)$ .

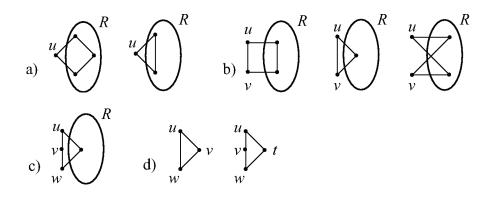


Figure 3: Cycles in  $G[N^2(x) \cup R]$ .

Claim 3 N(x) is a clique.

Proof of Claim 3. Suppose to the contrary that there are  $a, b \in N(x)$  with  $ab \notin E(G)$ . Since G' is  $(G_1, G_2)$ -free, a and b must see each cycle in  $G[N^2(x) \cup R]$ . If C is a cycle of type  $(A_1)$  or  $(A_2)$  in  $G[N^2(x) \cup R]$ , i.e., with exactly one vertex u in  $N^2(x)$  then a and b see u and we obtain  $G_7$  or  $G_8$  - contradiction. If C is of type  $(B_3)$  with  $u, v \in N^2(x)$ ,  $uv \notin E(G)$ , then both a and b have to see C, and if not both a and b see both u and v then there is either a hole or domino or  $G_8$ . Thus a and b see both u and v, i.e., there is  $G_{11}$  - contradiction.

We analyze the remaining cases by considering the following situation: If a sees vertex u and b sees vertex  $v \neq u$  in  $N^2(x)$  such that there exists a path linking u and v in  $N^2(x)$  then by Proposition 1, a and b see a common edge u'v'; but then G' contains  $G_5$  with x, y, a, b, u', v' - contradiction. This shows Claim 3.

Let  $N_C^2(x)$  denote the set of all vertices in  $N^2(x)$  which are contained in cycles of subgraph  $G[N^2(x) \cup R]$ . Since G is  $(G_1, G_2, G_3)$ -free, there is only one connected component in  $G[N_C^2(x) \cup R]$ . In addition, if  $a \in N(x)$  then there is a triangle xya, and a must see every cycle in  $G[N_C^2(x) \cup R]$ .

**Claim 4** Every vertex in N(x) sees every vertex in  $Q_2$ .

Proof of Claim 4. Since G' is  $(G_1, G_2, G_3)$ -free, any vertex  $a \in N(x)$  sees at least one vertex u in each cycle of  $G[N^2(x) \cup R]$ . Since by Claim 3, N(x) is a clique, and by Proposition 1, all vertices in  $Q_2$  have the same neighborhood in N(x), and vertex u sees a, all vertices in  $Q_2$  see all vertices in N(x) which shows Claim 4.

We conclude that if there is a cycle in  $G[N^2(x) \cup R]$  and  $x \notin Q$  then  $N(x) \cup Q_2$  is a cct of G', which finishes the proof in Case 1.

### **3.2** Case 2: y is a false twin to x.

Let G' result from G by adding a false twin y to x in G. We again inductively suppose that G has a cct Q. The possible cycles with y in G' are triangles yab for  $a, b \in N(x), ab \in E(G), C_4$ 's yabc for  $a, b \in N(x), ab \notin E(G), c \in N^2(x)$ , and  $C_4$ 's xyab for  $a, b \in N(x)$ .

If |N(x)| = 1 then Q is also a cct of G'. Now assume that  $|N(x)| \ge 2$ .

Recall that V(G) is partitioned into  $\{x\} \cup N(x) \cup N^2(x) \cup R$ , and since G' is  $(G_1, G_2, G_3)$ -free, R induces a cycle-free subgraph in G'.

The fact below strengthens Claim 1.

Claim 5 G'[N(x)] is a threshold graph.

Proof of Claim 5. Since G' is distance hereditary, N(x) is  $P_4$ -free, and since G' is  $G_5$ - and  $G_6$ -free, N(x) is  $2K_2$ - and  $C_4$ -free, i.e., G'[N(x)] is a threshold graph which shows Claim 5.  $\diamond$ 

Case 2.1:  $G[N^2(x) \cup R]$  is cycle-free.

We are going to show that also in this case, G' has a cct.

Recall that G has a cct Q, and let  $Q_1 = Q \cap N(x)$ ,  $Q_2 = Q \cap N^2(x)$ ,  $N_1(x) = N(x) \setminus Q_1$ . As in Claim 2,  $Q_2$  can contain at most two vertices, and we can assume that every vertex in  $Q_2$  has a neighbor in  $N_1(x)$ . Moreover, if  $Q_2 \neq \emptyset$  then  $x \notin Q$ .

Case 2.1.1:  $|Q_2| = 2$ .

Let  $Q_2 = \{u, v\}$ ; recall that by Proposition 1, u and v have the same neighborhood in  $N_1(x)$ . We distinguish between three subcases:

(i) If u, v have three neighbors a, b, c in (the stable set)  $N_1(x)$  then vertices x, y, u, a, b, c(vertices x, y, v, a, b, c, respectively) induce  $G_4$  in G', which is impossible.

(*ii*) If u, v have exactly two neighbors a, b in  $N_1(x)$  then there is no other vertex  $c \in N_1(x)$ , otherwise x, y, a, b, c, u, v induce  $G_{11}$  in G'. In addition, since G' is  $G_4$ -free and by Claim 5, G'[N(x)] is a threshold graph, either a or b sees all vertices of  $Q_1$ , otherwise if a misses a' and b misses b' in  $Q_1$ , respectively, then either G' contains  $G_4$  (if we can choose a' = b') or there is a  $P_4 ab'a'b$  in N(x). Suppose that a sees all vertices of  $Q_1$ . Then every cycle in G' containing y also contains some vertex in  $Q \cup \{a\}$ , showing that  $Q \cup \{a\}$  is a cct of G'.

(*iii*) If u, v have exactly one neighbor a in  $N_1(x)$ , we analyze the neighborhood of a. If a misses some vertex  $b \in Q_1$  then there is no other vertex  $c \in N_1(x)$  (otherwise x, y, a, b, c, u, v induce  $G_{11}$  if  $bc \notin E(G)$  or c, b, x, u, a induce house if  $bc \in E(G)$ ), and hence every cycle in G' containing y also contains some vertex in Q, i.e., Q is still a cct of G'. If a sees all vertices in  $Q_1$ , every cycle containing u or v also contains some vertex in  $Q_1 \cup \{a\}$ , and hence  $(Q \setminus \{u, v\}) \cup \{a, y\}$  is a cct of G'.

### Case 2.1.2: $|Q_2| = 1$ .

Let  $Q_2 = \{u\}$ . Since G' is  $G_4$ -free, u has at most two neighbors in  $N_1(x)$ . If u has two neighbors a, b in  $N_1(x)$  then, by Claim 5 and since G' is  $G_4$ -free, one of them, say a, must see all vertices in  $Q_1$ , and this means that every cycle containing u also contains some vertex in  $Q_1 \cup \{a\}$  (recall that  $G[N^2(x) \cup R]$  is cycle-free), i.e.,  $(Q \setminus \{u\}) \cup \{a, y\}$  is a cct of G'. If u has precisely one neighbor a in  $N_1(x)$  and a sees all vertices in  $Q_1$ , again  $(Q \setminus \{u\}) \cup \{a, y\}$  is a cct of G'; otherwise, a misses a vertex b in  $Q_1$ , and the analysis is as follows:

(i') If  $N_1(x)$  consists only of vertex *a* then every cycle in G' containing *y* also contains a vertex of  $Q_1$ , and hence *Q* is a cct of G'.

(*ii'*) If  $N_1(x)$  contains a vertex  $c \neq a$ , we must have  $bc \notin E(G)$  (otherwise x, a, b, c, u induce a house). We show that there is no cycle C in G containing a, u but no vertex of  $Q_1$ . If there is such a cycle C then by Proposition 1, it must be a triangle *auv* with  $v \in N^2(x)$ , and then vertices x, y, a, b, c, u, v induce graph  $G_{11}$ , or it is a  $C_4$  *auvw* with  $u, v \in N^2(x)$ and  $w \in R$  but then there is a  $G_{12}$  or domino in G'. We conclude that  $(Q \setminus \{u\}) \cup \{y\}$  is a cct of G'.

### Case 2.1.3: $Q_2 = \emptyset$ .

In this case,  $Q \subseteq N[x]$ . If there is a cct Q of G with  $x \notin Q$  then  $Q \cup \{y\}$  is a cct of G' and we are done. So we have to show that in Case 2.1.3, G has a cct Q with  $Q \subseteq N(x)$ .

In G, there are two types of cycles containing x: Triangles xab with  $a, b \in N(x)$  and  $C_4$ 's xabc with  $a, b \in N(x)$  and  $c \in N^2(x)$ . Recall that by Claim 5, N(x) induces a threshold graph and in particular is partitioned into a clique  $Q_1$  and a stable set  $N_1(x)$ . Then  $Q_1$  (and in general, every maximal clique in N(x)) covers every triangle xab since  $ab \in E$ .

The case of  $C_4$  with x in G is more involved. Assume that there is a  $C_4$  xabc with

 $a, b \in N(x)$  and  $c \in N^2(x)$  which is not covered by  $Q_1$ . If vertex *a* (vertex *b*, respectively) sees all vertices in  $Q_1$  then the clique  $Q_1 \cup \{a\}$  ( $Q_1 \cup \{b\}$ , respectively) covers *xabc* as well. Otherwise both *a* and *b* have non-neighbors in  $Q_1$ . Since by Claim 5, N(x) is  $P_4$ free, *a* and *b* have a common non-neighbor, say *d*, in  $Q_1$ . It follows that  $cd \notin E$  (otherwise x, y, c, a, b, d induce  $G_4$ ). If *a* and *b* miss another vertex  $d' \in Q_1$  then x, y, d, d', a, b, c induce  $G_{11}$ , a contradiction. Thus one of *a* and *b*, say *a*, has at most one non-neighbor, say *d*, in  $Q_1$ , and since N(x) is a threshold graph, without loss of generality, the neighborhood of *b* in  $Q_1$  is contained in the neighborhood of *a* in  $Q_1$ ; in particular, *b* misses *d*, and, as above, *c* misses *d*. Let *e* be a neighbor of *d* in  $N^2(x)$ .

**Claim 6** Let e be a neighbor of d in  $N^2(x)$ . Then e misses a, b and c. (with a, b, c and d as described above).

Proof of Claim 6.

We begin by observing that if  $ce \in E$ , by Proposition 1 (i), c and e have the same neighbors in N(x) which is impossible since e sees d and c misses d. Therefore  $ce \notin E(G)$ . In this case, by Proposition 1 (ii), if e sees one of a and b, it must see both of them but now, x, y, e, a, b, d induce  $G_4$  - a contradiciton. Then e must also miss both a and b.

Suppose that  $Q'_1 := (Q_1 \setminus \{d\}) \cup \{a\}$  is not a cct of G. Note that  $N(x) \setminus Q'_1$  is stable. Then there is a cycle in G whose only vertex from  $Q_1$  is d. Obviously, if C is a cycle containing dand an edge in N(x) then  $Q'_1$  covers C since  $N(x) \setminus Q'_1$  is stable. Thus, we have to consider cycles without an edge in N(x).

First consider a  $C_3 duv$  with  $u, v \in N^2(x)$ . Then by Claim 6, u and v miss a, b and c, and now, together with y, G' contains  $G_9$ , a contradiction. If d is in a  $C_4 duvw$  with  $u, v \in N^2(x)$ and  $w \in R$  then very similarly, together with y, G' contains  $G_{10}$ , a contradiction. Thus, dis not contained in any of such cycles.

If C is a  $C_4$  with  $d, z \in N(x)$  and  $u, v \in N^2(x)$  then, again by Claim 6, u and v miss a, b and c. Then z must see a and b, otherwise there is a house or  $G_{12}$  in G', together with y, but then xadzu induce a house, a contradiction.

This also happens when d is in a  $C_4$  with x and no vertex from  $Q'_1$ . This final contradiction shows that in Case 2.1.3, there is a cct Q of G without x, and thus, there is a cct  $Q \cup \{y\}$ in G'.

Case 2.2:  $G[N^2(x) \cup R]$  is not cycle-free.

As in Case 1, we now assume that  $G[N^2(x) \cup R]$  contains a cycle C (which implies that in this case,  $x \notin Q$  holds).

Claim 7 N(x) is  $I_3$ -free.

Proof of Claim 7. Assume that  $G[N^2(x) \cup R]$  is not cycle-free and N(x) contains a stable set of three vertices  $a_1, a_2, a_3$ . Let S be a maximal stable set in N(x) containing  $a_1, a_2, a_3$ .

Recall that by Proposition 1, for vertices  $u, v \in N^2(x)$  in the same connected component of  $G[N^2(x) \cup R]$ , their neighborhoods in S are equal. In addition, no vertex  $u \in N^2(x)$ sees at least three vertices in S, otherwise  $G_4$  is contained in G'. Thus, for every pair of vertices  $u, v \in N^2(x), N(u) \cap S = N(v) \cap S \subseteq \{a_1, a_2\}$  holds.

If  $u \in N^2(x)$  is in a cycle of type  $(A_1)$  or  $(A_2)$  then  $x, y, a_1, a_2, a_3, u$  and the vertices of the remaining cycle induce  $G_2, G_3, G_9$  or  $G_{10}$ . If the cycle with  $u, v \in N^2(x)$  is of type  $(B_1)$ , there is a house; if it is of type  $(B_2)$  or  $(D_1)$ , there is  $G_2$  or  $G_{11}$ ; if of type  $(B_3)$ , there is  $G_3$  or  $G_{12}$ ; and finally, if of type  $(C_1)$  or  $(D_2)$ , there is  $G_3$  or  $G_{11}$ . This shows Claim 7.

We conclude that if y is a false twin to x and  $G[N^2(x) \cup R]$  is not cycle-free then by Claim 7,  $N_1(x)$  contains at most two vertices. If  $|N_1(x)| \leq 1$  then Q is a cct of G'. If  $N_1(x) = \{a, b\}$  then by Claims 5 and 7, one of them, say a, sees all vertices in  $Q_1$ . By Proposition 1, either  $Q_2 \cup \{a\}$  is a clique, and then  $Q \cup \{a\}$  is a cct of G', or a sees no vertex of  $Q_2$ , and then let C be a cycle in  $G[N^2(x) \cup R]$  and let  $u \in V(C) \cap Q_2$ . Since xyab is a  $C_4$  and G is  $(G_2, G_3)$ -free, there is an edge linking xyab and C. By Proposition 1, we conclude that b sees all vertices in  $C \cap N^2(x)$  and, therefore, in  $Q_2$ . Now if there is some  $a' \in Q_1$  then x, a, a', b, u induce either a house or a gem – a contradiction. Therefore  $Q_1 = \emptyset$  and  $Q'_2 \cup \{b\}$  is a cct of G'. This finishes the proof in Case 2 and thus also the proof of Theorem 2.

As a direct consequence of Theorem 2, we obtain another proof of a result in [3]:

**Corollary 1** If G is a cograph then G admits a clique cycle-transversal if and only if G is  $(G_1, \ldots, G_6)$ -free.

**Proof.** Graphs  $G_1$  to  $G_6$  admit no cct. Conversely, G is also  $(G_7, \ldots, G_{12})$ -free (because all of them contain  $P_4$ ). Since every cograph is a distance-hereditary graph, by Theorem 2 the corollary follows.

**Corollary 2** Let G be a distance-hereditary graph. Then G is a (2,1)-graph if and only if G is  $(G_1, G_5, G_6, G_7)$ -free.

**Proof.** Graphs  $G_1, G_5, G_6$  and  $G_7$  are not (2, 1)-graphs. Conversely, assume that G is  $(G_1, G_5, G_6, G_7)$ -free and G is not a (2, 1)-graph. Let G' be a minimal induced subgraph of G which is not a (2, 1)-graph. Note that being a (2, 1)-graph is equivalent to admitting a clique that intersects every odd cycle. Thus G' does not admit a cct. By Theorem 2, G' is isomorphic to one of the graphs  $G_1, G_2, \ldots, G_{12}$ . Since  $G_2, G_3, G_4, G_8, G_9, G_{10}, G_{11}$ , and  $G_{12}$  are (2, 1)-graphs, it follows that G contains  $G_1, G_5, G_6$ , or  $G_7$  as an induced subgraph.

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