# Clique cycle-transversals in distance-hereditary graphs 

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#### Abstract

A cycle-transversal of a graph $G$ is a subset $T \subseteq V(G)$ such that $T \cap V(C) \neq \emptyset$ for every cycle $C$ of $G$. A clique cycle-transversal, or $c c t$ for short, is a cycle-transversal which is a clique. Recognizing graphs which admit a cct can be done in polynomial time; however, no structural characterization of such graphs is known. We characterize distancehereditary graphs admitting a cct in terms of forbidden induced subgraphs. This extends similar results for chordal graphs and cographs.


## 1 Introduction

A cycle-transversal of a graph $G$ is a subset $T \subseteq V(G)$ such that $T \cap V(C) \neq \emptyset$ for every cycle $C$ of $G$. When $T$ is a clique, we say that $T$ is a clique cycle-transversal or simply cct. A graph admits a cct if and only if it can be partitioned into a complete subgraph and a forest; by this reason such a graph is also called a $(\mathcal{C}, \mathcal{F})$-graph in [3].
Finding a minimum cycle-transversal in a graph is NP-hard due to a general result in [12], which says that the problem of finding the minimum number of vertices of a graph $G$ whose deletion results in a subgraph satisfying a hereditary property $\pi$ on induced subgraphs is NP-hard. This result implies the NP-hardness of other problems involving cycletransversals, for instance the problem of finding a minimum odd cycle-transversal (which is equivalent to finding a maximum induced bipartite subgraph), or the problem of finding a minimum triangle-transversal (which is equivalent to finding a maximum induced trianglefree subgraph). Odd cycle-transversals are interesting due to their connections to perfect graph theory; in [11], an $O(m n)$ algorithm is developed to find odd cycle-transversals with bounded size. In [8], the authors study the problem of finding $C_{k}$-transversals, for a fixed integer $k$, in graphs with bounded degree; among other results, they describe a polynomial-time algorithm for finding minimum $C_{4}$-transversals in graphs with maximum degree three.

Graphs admitting a cct can be recognized in polynomial time, as follows. Note first that $(\mathcal{C}, \mathcal{F})$-graphs form a subclass of $(2,1)$-graphs (graphs whose vertex set can be partitioned into two stable sets and one clique). The strategy for recognizing a $(\mathcal{C}, \mathcal{F})$-graph $G$ initially checks whether $G$ is a ( 2,1 )-graph, which can be done in polynomial time (see [2]). If so, then test, for each candidate clique $Q$ of a $(2,1)$-partition of $G$, if $G-Q$ is acyclic (which can be done in linear time). If the test fails for all cliques $Q$, then $G$ is not a $(\mathcal{C}, \mathcal{F})$-graph, otherwise $G$ is a $(\mathcal{C}, \mathcal{F})$-graph. To conclude the argument, we claim that the number of candidate cliques $Q$ is polynomial. Since $G$ is a $(2,1)$-graph, let $(B, Q)$ be a $(2,1)$-partition of $V(G)$ where $B$ induces a bipartite subgraph and $Q$ is a clique. Let ( $B^{\prime}, Q^{\prime}$ ) be another (2,1)-partition of $V(G)$. Then $\left|Q^{\prime} \backslash Q\right| \leq 2$ and $\left|Q \backslash Q^{\prime}\right| \leq 2$, otherwise $G[B]$ or $G\left[B^{\prime}\right]$ would contain a triangle, which is impossible. Therefore, we can generate in polynomial time all the other candidate cliques $Q^{\prime}$ from $Q$. This is the same argument used to count sparse-dense partitions (for more details see [7]). Although recognizing graphs admitting a cct can be done in polynomial time, no structural characterization of such graphs is known, even for perfect graphs.

A similar sparse-dense partition argument can be employed to show that an interesting superclass of $(\mathcal{C}, \mathcal{F})$-graphs, namely graphs admitting a clique triangle-transversal, can also be recognized in polynomial time. Such graphs are also known in the literature as $(1,2)$-split graphs. A characterization of this class is given in [13], where it has been proved that there are 350 minimal forbidden induced subgraphs for (1,2)-split graphs. When $G$ is a perfect graph, being a $(1,2)$-split graph is equivalent to being a $(2,1)$-graph: observe that a perfect graph $G$ contains a clique triangle-transversal if and only if $G$ contains a clique that intersects all of its odd cycles. In [4] and [9], respectively, characterizations by forbidden induced subgraphs of cographs and chordal graphs which are ( 1,2 )-split graphs are presented.
Deciding whether a distance-hereditary graph admits a cct can be done in linear-time using the clique-width approach, since the existence of a cct can be represented by a Monadic Second Order Logic (MSOL) formula using only predicates over vertex sets [6. 10]. However, no structural characterization for distance-hereditary graphs admitting a cct was known. In order to fill this gap, in this note we describe a characterization of distance-hereditary graphs with cct in terms of forbidden induced subgraphs.

## 2 Background

In this work, all graphs are finite, simple and undirected. Given a graph $G=(V(G), E(G))$, we denote by $\bar{G}$ the complement of $G$. For $V^{\prime} \subseteq V(G), G\left[V^{\prime}\right]$ denotes the subgraph of $G$ induced by $V^{\prime}$. Let $X=\left(V_{X}, E_{X}\right)$ and $Y=\left(V_{Y}, E_{Y}\right)$ be two graphs such that $V_{X} \cap V_{Y}=\emptyset$. The operations " + " and " $\cup$ " are defined as follows: the disjoint union $X \cup Y$, sometimes referred simply as graph union, is the graph with vertex set $V_{X} \cup V_{Y}$ and edge set $E_{X} \cup E_{Y}$; the join $X+Y$ is the graph with vertex set $V_{X} \cup V_{Y}$ and edge set
$E_{X} \cup E_{Y} \cup\left\{x y \mid x \in V_{X}, y \in V_{Y}\right\}$.
Let $N(x)=\{y \mid y \neq x$ and $x y \in E\}$ denote the open neighborhood of $x$ and let $N[x]=$ $\{x\} \cup N(x)$ denote the closed neighborhood of $x$. A cut-vertex is a vertex $x$ such that $G[V \backslash\{x\}$ has more connected components than $G$. A block (or 2-connected component) of $G$ is a maximal induced subgraph of $G$ having no cut-vertex. A block is nontrivial if it contains a cycle; otherwise it is trivial.

For a set $\mathcal{F}$ of graphs, $G$ is $\mathcal{F}$-free if no induced subgraph of $G$ is in $\mathcal{F}$.
Vertices $x$ and $y$ are true twins (false twins, respectively) in $G$ if $N[x]=N[y](N(x)=N(y)$, respectively).
Adding a true twin (false twin, pendant vertex, respectively) $y$ to vertex $x$ in graph $G$ means that for $G$ and $y \notin V(G)$, a new graph $G^{\prime}$ is constructed with $V\left(G^{\prime}\right)=V(G) \cup\{y\}$ and $E\left(G^{\prime}\right)=E(G) \cup\{x y\} \cup\{u y \mid u \in N(x)\}\left(E\left(G^{\prime}\right)=E(G) \cup\{u y \mid u \in N(x)\}\right.$, $E\left(G^{\prime}\right)=E(G) \cup\{x y\}$, respectively).

The complete (resp. edgeless) graph with $n$ vertices is denoted by $K_{n}$ (respectively $I_{n}$ ). The graphs $K_{1}$ and $K_{3}$ are called trivial graph and triangle, respectively. The chordless cycle (chordless path, respectively) with $n$ vertices is denoted by $C_{n}$ ( $P_{n}$, respectively). The graph $C_{n}\left(\overline{C_{n}}\right.$, respectively) for $n \geq 5$ is a hole (anti-hole, respectively).

The house is the graph with vertices $a, b, c, d, e$ and edges $a b, b c, c d, a d, a e, b e$. The $g e m$ is the graph with vertices $a, b, c, d, e$ and edges $a b, b c, c d, a e, b e, c e, d e$. The domino is the graph with vertices $a, b, c, d, e, h$ and edges $a b, b c, c d, a d, b e, e h, c h$.


Figure 1: House, hole, domino, and gem.
If $H$ is an induced subgraph of $G$ then we say that $G$ contains $H$, otherwise $G$ is $H$-free. A clique (resp. stable or independent set) is a subset of vertices inducing a complete (resp. edgeless) subgraph. A universal vertex is a vertex adjacent to all the other vertices of the graph. A split graph is a graph whose vertex set can be partitioned into a stable set and a clique. It is well known that $G$ is a split graph if an only if $G$ is $\left(2 K_{2}, C_{4}, C_{5}\right)$-free.
A star is a graph whose vertex set can be partitioned into a stable set and a universal vertex. A bipartite graph is a graph whose vertex set can be partitioned into two stable sets. A cograph is a graph containing no $P_{4}$. A chordal graph is a graph containing no $C_{k}$, for $k \geq 4$. A distance-hereditary graph is a graph in which the distances in any connected induced subgraph are the same as they are in the original graph. A threshold graph is a graph that can be constructed from a one-vertex graph by repeated applications of the
following two operations: (a) addition of a single isolated vertex to the graph; (b) addition of a single universal vertex to the graph. It is well known that $G$ is a threshold graph if an only if $G$ is $\left(2 K_{2}, C_{4}, P_{4}\right)$-free.
If $T \cap V(C) \neq \emptyset$ for cycle $C$, we say that $T$ covers $C$.

## 3 The forbidden subgraph characterization

The following well-known characterization of distance-hereditary graphs, also called $H H D G$ free graphs, will be fundamental for our result:

Theorem 1 [1] The following are equivalent for any graph $G$ :
(i) $G$ is a distance-hereditary graph.
(ii) $G$ can be generated from a single vertex by repeatedly adding a pendant vertex, a false twin, or a true twin, respectively.
(iii) $G$ is (house, hole, domino, gem)-free. (See Figure 1.)

Let $G=(V, E)$ be a graph, and for a vertex $x \in V$, let $N^{k}(x)=\left\{y \in V \mid \operatorname{dist}_{G}(x, y)=k\right\}$ for $k \geq 0$ denote the distance levels in $G$ with respect to $x$. For $k=2$, let $R=V \backslash(N[x] \cup$ $\left.N^{2}(x)\right)$. The following are useful properties of distance-hereditary graphs:

Proposition 1 Let $G$ be distance hereditary and $u, v \in N^{2}(x)$.
(i) If $u v \in E(G)$ or $u v \notin E(G)$ but connected by a path in $N^{2}(x) \cup R$ then $N(u) \cap N(x)=$ $N(v) \cap N(x)$.
(ii) If $N(x)$ is a stable set then for $u, v \in N^{2}(x)$ with $u v \notin E, N(u) \cap N(x)$ and $N(v) \cap$ $N(x)$ do not overlap.

Proof. ( $i$ ): Since $G$ is (house,hole)-free, $u, v$ cannot have incomparable neighborhoods in $N(x)$. Moreover, since $G$ is (house, hole, domino, gem)-free, the neighborhoods of $u$ and $v$ in $N(x)$ cannot properly contain one another, which shows Proposition 1 .
(ii): Let us suppose, by contradiction, that $u$ and $v$ overlap. In this case, let $a, b, c$ be vertices in $N(x)$ such that $a u \in E(G)$, av $\notin E(G)$, bu, bv $\in E(G)$, cu $\notin E(G)$ and $c v \in E(G)$. Then $G[x, a, b, c, u, v]$ induces a domino, which is a contradiction.

Theorem 2 Let $G$ be a distance-hereditary graph. Then $G$ admits a clique cycle transversal if and only if $G$ is $\left(G_{1}, \ldots, G_{12}\right)$-free.

$\mathrm{G}_{3}$

$\mathrm{G}_{6}$

$\mathrm{G}_{7}$



Figure 2: Forbidden subgraphs for distance-hereditary graphs with cct.

Proof. It is easy to see that $G_{1}, \ldots, G_{12}$ from Figure 2 have no cct. For the converse direction, let $G^{\prime}$ be a distance-hereditary $\left(G_{1}, \ldots, G_{12}\right)$-free graph. By Theorem 1, $G^{\prime}$ results, starting with a single vertex, by repeatedly applying one of the three operations in Theorem 1 (ii). Since adding a pendant vertex $y$ to a vertex $x$ in $G$ does not create cycles with $y$, we can restrict ourselves to the following two cases: $G^{\prime}$ results from $G$ by either adding a true twin or a false twin $y$ to vertex $x$ in $G$, and in both cases, we have to show that $G^{\prime}$ has a cct.

We can inductively assume that $G$ has a cct $Q$. The vertex set $V(G)$ can be partitioned into $\{x\} \cup N(x) \cup N^{2}(x) \cup R$. Let $Q_{1}=Q \cap N(x), Q_{2}=Q \cap N^{2}(x)$ and $N_{1}(x)=N(x) \backslash Q_{1}$.

### 3.1 Case 1: $y$ is a true twin to $x$.

Let $G^{\prime}$ result from $G$ by adding a true twin $y$ to $x$ in $G$. In this case, the possible cycles with $y$ in $G^{\prime}$ are triangles $x y a$ for $a \in N(x)$, triangles $y a b$ for $a, b \in N(x), a b \in E(G)$, and $C_{4}$ 's $y a b c$ for $a, b \in N(x), a b \notin E(G), c \in N^{2}(x)$. If $x \in Q$ or, more generally, $Q \subseteq N[x]$, then $Q \cup\{y\}$ is a cct of $G^{\prime}$. Thus we have to consider the case $x \notin Q$. Since for a triangle $y a b$ also $x a b$ is a triangle which is covered by $Q$, the triangle yab is covered by $Q$, and similarly for the $C_{4} y a b c$ where $x a b c$ is a $C_{4}$ in $G$ covered by $Q$. Thus, we only have to
deal with triangles $x y a$.

Claim 1 If $x \notin Q$ then $G[N(x)]$ is a split graph with partition $\left(N_{1}(x), Q_{1}\right)$.

Proof of Claim 1. For each edge $a b \in G[N(x)], x a b$ is a triangle. Hence, $a$ or $b$ is in $Q$ and $N_{1}(x)$ is a stable set. Since $Q_{1}$ is a clique, the claim follows.
Since $G^{\prime}$ is $\left(G_{1}, G_{2}, G_{3}\right)$-free, $R$ induces a cycle-free subgraph in $G$.
Case 1.1: $G\left[N^{2}(x) \cup R\right]$ is cycle-free.
Claim 2 If $x \notin Q$ then $G^{\prime}$ has a cct.
Proof of Claim 2. Since $Q$ is a clique and $G\left[N^{2}(x) \cup R\right]$ is cycle-free, $Q_{2}$ contains at most two vertices. If $Q_{2}=\emptyset$ then $Q \cup\{y\}$ is a cct of $G^{\prime}$. If a vertex $u \in Q_{2}$ has no neighbors in $N_{1}(x)$ then every cycle containing $u$ also contains a vertex of $Q_{1}$, i.e., $Q \backslash\{u\}$ is still a cct of $G$. Thus, assume without loss of generality that every vertex in $Q_{2}$ has a neighbor in $N_{1}(x)$.
If $Q_{2}=\{u, v\}$, the neighborhood of $Q_{2}$ in $N_{1}(x)$ cannot contain two vertices $a$ and $b$, otherwise by Claim 1 and Proposition 1 vertices $x, y, a, b, u, v$ induce $G_{5}$. Hence $u$ and $v$ have precisely one neighbor $a \in N_{1}(x)$ which must be adjacent to all vertices of $Q_{1}$, otherwise if $a$ misses a vertex $b \in Q_{1}$ then vertices $x, y, a, b, u, v$ induce $G_{5}$. Therefore $(Q \backslash\{u, v\}) \cup\{a, y\}$ is a cct of $G^{\prime}$.
If $Q_{2}=\{u\}$, we consider two subcases:
(i) Vertex $u$ has a neighbor $a \in N_{1}(x)$ which misses some vertex $b \in Q_{1}$. Then every cycle $C$ in $G$ containing $a, u$ but no vertex of $Q_{1}$ must also contain $x$. This is shown as follows: If $C$ does not contain $x$ then, by Proposition 1, $C$ is either a triangle auv with $v \in N^{2}(x)$ or a $C_{4}$ aucv with $c \in N_{1}(x), v \in N^{2}(x)$. In the former case by using Proposition 1, vertices $x, y, a, b, u, v$ induce $G_{5}$. The latter case cannot occur since the existence of cycle aucv implies the existence of cycle axcv in $G$, not covered by $Q$. This implies that $(Q \backslash\{u\}) \cup\{x, y\}$ is a cct of $G^{\prime}$.
(ii) Every neighbor $a \in N_{1}(x)$ of $u$ sees all vertices in $Q_{1}$. Then $Q \cup\{a\}$ is a cct of $G$ for some $a \in N_{1}(x)$ and, since by Claim $1 N_{1}(x)$ is a stable set, every other neighbor $a^{\prime} \in N_{1}(x)$ of $u$ misses some vertex in $Q_{1} \cup\{a\}$. By applying a similar argument as in (i), every cycle $C$ in $G$ containing $a^{\prime}, u$ but no vertex of $Q_{1} \cup\{a\}$ must also contain $x$. We conclude that $(Q \backslash\{u\}) \cup\{x, y, a\}$ is a cct of $G^{\prime}$. This completes the proof of Claim 2 .

Case 1.2: $G\left[N^{2}(x) \cup R\right]$ is not cycle-free.
We now assume that $G\left[N^{2}(x) \cup R\right]$ contains a cycle $C$. This cycle can be one of the following types (see Figure 3):
$\left(A_{1}\right) C$ has exactly one vertex $u$ in $N^{2}(x)$, and $C$ is a $C_{4}$.
$\left(A_{2}\right) C$ has exactly one vertex $u$ in $N^{2}(x)$, and $C$ is a $C_{3}$.
$\left(B_{1}\right) C$ has exactly two vertices $u, v$ in $N^{2}(x), u v \in E(G)$, and $C$ is a $C_{4}$.
$\left(B_{2}\right) C$ has exactly two vertices $u, v$ in $N^{2}(x), u v \in E(G)$, and $C$ is a $C_{3}$.
$\left(B_{3}\right) C$ has exactly two vertices $u, v$ in $N^{2}(x), u v \notin E(G)$, and $C$ is a $C_{4}$.
$\left(C_{1}\right) C$ is a $C_{4}$ with exactly three vertices $u, v, w$ in $N^{2}(x)$ (which form a $P_{3}$ in $N^{2}(x)$ ).
$\left(D_{1}\right) C$ is a $C_{3}$ in $N^{2}(x)$.
$\left(D_{2}\right) C$ is a $C_{4}$ in $N^{2}(x)$.
a)


b)


c)

d)



Figure 3: Cycles in $G\left[N^{2}(x) \cup R\right]$.

Claim $3 N(x)$ is a clique.

Proof of Claim 3. Suppose to the contrary that there are $a, b \in N(x)$ with $a b \notin E(G)$. Since $G^{\prime}$ is $\left(G_{1}, G_{2}\right)$-free, $a$ and $b$ must see each cycle in $G\left[N^{2}(x) \cup R\right]$. If $C$ is a cycle of type $\left(A_{1}\right)$ or $\left(A_{2}\right)$ in $G\left[N^{2}(x) \cup R\right]$, i.e., with exactly one vertex $u$ in $N^{2}(x)$ then $a$ and $b$ see $u$ and we obtain $G_{7}$ or $G_{8}$ - contradiction. If $C$ is of type $\left(B_{3}\right)$ with $u, v, \in N^{2}(x)$, $u v \notin E(G)$, then both $a$ and $b$ have to see $C$, and if not both $a$ and $b$ see both $u$ and $v$ then there is either a hole or domino or $G_{8}$. Thus $a$ and $b$ see both $u$ and $v$, i.e., there is $G_{11}$ - contradiction.
We analyze the remaining cases by considering the following situation: If $a$ sees vertex $u$ and $b$ sees vertex $v \neq u$ in $N^{2}(x)$ such that there exists a path linking $u$ and $v$ in $N^{2}(x)$ then by Proposition 1, $a$ and $b$ see a common edge $u^{\prime} v^{\prime}$; but then $G^{\prime}$ contains $G_{5}$ with $x, y, a, b, u^{\prime}, v^{\prime}$ - contradiction. This shows Claim 3.

Let $N_{C}^{2}(x)$ denote the set of all vertices in $N^{2}(x)$ which are contained in cycles of subgraph $G\left[N^{2}(x) \cup R\right]$. Since $G$ is $\left(G_{1}, G_{2}, G_{3}\right)$-free, there is only one connected component in $G\left[N_{C}^{2}(x) \cup R\right]$. In addition, if $a \in N(x)$ then there is a triangle $x y a$, and $a$ must see every cycle in $G\left[N_{C}^{2}(x) \cup R\right]$.

Claim 4 Every vertex in $N(x)$ sees every vertex in $Q_{2}$.
Proof of Claim 4. Since $G^{\prime}$ is $\left(G_{1}, G_{2}, G_{3}\right)$-free, any vertex $a \in N(x)$ sees at least one vertex $u$ in each cycle of $G\left[N^{2}(x) \cup R\right]$. Since by Claim 3, $N(x)$ is a clique, and by Proposition 1, all vertices in $Q_{2}$ have the same neighborhood in $N(x)$, and vertex $u$ sees $a$, all vertices in $Q_{2}$ see all vertices in $N(x)$ which shows Claim 4 .
We conclude that if there is a cycle in $G\left[N^{2}(x) \cup R\right]$ and $x \notin Q$ then $N(x) \cup Q_{2}$ is a cct of $G^{\prime}$, which finishes the proof in Case 1.

### 3.2 Case 2: $y$ is a false twin to $x$.

Let $G^{\prime}$ result from $G$ by adding a false twin $y$ to $x$ in $G$. We again inductively suppose that $G$ has a cct $Q$. The possible cycles with $y$ in $G^{\prime}$ are triangles yab for $a, b \in N(x), a b \in E(G)$, $C_{4}$ 's $y a b c$ for $a, b \in N(x), a b \notin E(G), c \in N^{2}(x)$, and $C_{4}$ 's $x y a b$ for $a, b \in N(x)$.
If $|N(x)|=1$ then $Q$ is also a cct of $G^{\prime}$. Now assume that $|N(x)| \geq 2$.
Recall that $V(G)$ is partitioned into $\{x\} \cup N(x) \cup N^{2}(x) \cup R$, and since $G^{\prime}$ is $\left(G_{1}, G_{2}, G_{3}\right)$ free, $R$ induces a cycle-free subgraph in $G^{\prime}$.
The fact below strengthens Claim 1.

Claim $5 G^{\prime}[N(x)]$ is a threshold graph.

Proof of Claim 5. Since $G^{\prime}$ is distance hereditary, $N(x)$ is $P_{4}$-free, and since $G^{\prime}$ is $G_{5^{-}}$and $G_{6}$-free, $N(x)$ is $2 K_{2^{-}}$and $C_{4}$-free, i.e., $G^{\prime}[N(x)]$ is a threshold graph which shows Claim 5 . $\diamond$

Case 2.1: $G\left[N^{2}(x) \cup R\right]$ is cycle-free.
We are going to show that also in this case, $G^{\prime}$ has a cct.
Recall that $G$ has a cct $Q$, and let $Q_{1}=Q \cap N(x), Q_{2}=Q \cap N^{2}(x), N_{1}(x)=N(x) \backslash Q_{1}$. As in Claim 2, $Q_{2}$ can contain at most two vertices, and we can assume that every vertex in $Q_{2}$ has a neighbor in $N_{1}(x)$. Moreover, if $Q_{2} \neq \emptyset$ then $x \notin Q$.
Case 2.1.1: $\left|Q_{2}\right|=2$.
Let $Q_{2}=\{u, v\}$; recall that by Proposition 1, $u$ and $v$ have the same neighborhood in $N_{1}(x)$. We distinguish between three subcases:
(i) If $u, v$ have three neighbors $a, b, c$ in (the stable set) $N_{1}(x)$ then vertices $x, y, u, a, b, c$ (vertices $x, y, v, a, b, c$, respectively) induce $G_{4}$ in $G^{\prime}$, which is impossible.
(ii) If $u, v$ have exactly two neighbors $a, b$ in $N_{1}(x)$ then there is no other vertex $c \in N_{1}(x)$, otherwise $x, y, a, b, c, u, v$ induce $G_{11}$ in $G^{\prime}$. In addition, since $G^{\prime}$ is $G_{4}$-free and by Claim 5 , $G^{\prime}[N(x)]$ is a threshold graph, either $a$ or $b$ sees all vertices of $Q_{1}$, otherwise if $a$ misses $a^{\prime}$ and $b$ misses $b^{\prime}$ in $Q_{1}$, respectively, then either $G^{\prime}$ contains $G_{4}$ (if we can choose $a^{\prime}=b^{\prime}$ ) or there is a $P_{4} a b^{\prime} a^{\prime} b$ in $N(x)$. Suppose that $a$ sees all vertices of $Q_{1}$. Then every cycle in $G^{\prime}$ containing $y$ also contains some vertex in $Q \cup\{a\}$, showing that $Q \cup\{a\}$ is a cct of $G^{\prime}$.
(iii) If $u, v$ have exactly one neighbor $a$ in $N_{1}(x)$, we analyze the neighborhood of $a$. If $a$ misses some vertex $b \in Q_{1}$ then there is no other vertex $c \in N_{1}(x)$ (otherwise $x, y, a, b, c, u, v$ induce $G_{11}$ if $b c \notin E(G)$ or $c, b, x, u, a$ induce house if $b c \in E(G)$ ), and hence every cycle in $G^{\prime}$ containing $y$ also contains some vertex in $Q$, i.e., $Q$ is still a cct of $G^{\prime}$. If $a$ sees all vertices in $Q_{1}$, every cycle containing $u$ or $v$ also contains some vertex in $Q_{1} \cup\{a\}$, and hence $(Q \backslash\{u, v\}) \cup\{a, y\}$ is a cct of $G^{\prime}$.
Case 2.1.2: $\left|Q_{2}\right|=1$.
Let $Q_{2}=\{u\}$. Since $G^{\prime}$ is $G_{4}$-free, $u$ has at most two neighbors in $N_{1}(x)$. If $u$ has two neighbors $a, b$ in $N_{1}(x)$ then, by Claim 5 and since $G^{\prime}$ is $G_{4}$-free, one of them, say $a$, must see all vertices in $Q_{1}$, and this means that every cycle containing $u$ also contains some vertex in $Q_{1} \cup\{a\}$ (recall that $G\left[N^{2}(x) \cup R\right]$ is cycle-free), i.e., $(Q \backslash\{u\}) \cup\{a, y\}$ is a cct of $G^{\prime}$. If $u$ has precisely one neighbor $a$ in $N_{1}(x)$ and $a$ sees all vertices in $Q_{1}$, again $(Q \backslash\{u\}) \cup\{a, y\}$ is a cct of $G^{\prime}$; otherwise, $a$ misses a vertex $b$ in $Q_{1}$, and the analysis is as follows:
$\left(i^{\prime}\right)$ If $N_{1}(x)$ consists only of vertex $a$ then every cycle in $G^{\prime}$ containing $y$ also contains a vertex of $Q_{1}$, and hence $Q$ is a cct of $G^{\prime}$.
(ii') If $N_{1}(x)$ contains a vertex $c \neq a$, we must have $b c \notin E(G)$ (otherwise $x, a, b, c, u$ induce a house). We show that there is no cycle $C$ in $G$ containing $a, u$ but no vertex of $Q_{1}$. If there is such a cycle $C$ then by Proposition 1, it must be a triangle auv with $v \in N^{2}(x)$, and then vertices $x, y, a, b, c, u, v$ induce graph $G_{11}$, or it is a $C_{4}$ auvw with $u, v \in N^{2}(x)$ and $w \in R$ but then there is a $G_{12}$ or domino in $G^{\prime}$. We conclude that $(Q \backslash\{u\}) \cup\{y\}$ is a cct of $G^{\prime}$.

Case 2.1.3: $Q_{2}=\emptyset$.
In this case, $Q \subseteq N[x]$. If there is a cct $Q$ of $G$ with $x \notin Q$ then $Q \cup\{y\}$ is a cct of $G^{\prime}$ and we are done. So we have to show that in Case 2.1.3, $G$ has a cct $Q$ with $Q \subseteq N(x)$.
In $G$, there are two types of cycles containing $x$ : Triangles $x a b$ with $a, b \in N(x)$ and $C_{4}$ 's $x a b c$ with $a, b \in N(x)$ and $c \in N^{2}(x)$. Recall that by Claim 5, $N(x)$ induces a threshold graph and in particular is partitioned into a clique $Q_{1}$ and a stable set $N_{1}(x)$. Then $Q_{1}$ (and in general, every maximal clique in $N(x)$ ) covers every triangle $x a b$ since $a b \in E$.
The case of $C_{4}$ with $x$ in $G$ is more involved. Assume that there is a $C_{4} x a b c$ with
$a, b \in N(x)$ and $c \in N^{2}(x)$ which is not covered by $Q_{1}$. If vertex $a$ (vertex $b$, respectively) sees all vertices in $Q_{1}$ then the clique $Q_{1} \cup\{a\}\left(Q_{1} \cup\{b\}\right.$, respectively) covers xabc as well. Otherwise both $a$ and $b$ have non-neighbors in $Q_{1}$. Since by Claim 5, $N(x)$ is $P_{4}$ free, $a$ and $b$ have a common non-neighbor, say $d$, in $Q_{1}$. It follows that $c d \notin E$ (otherwise $x, y, c, a, b, d$ induce $\left.G_{4}\right)$. If $a$ and $b$ miss another vertex $d^{\prime} \in Q_{1}$ then $x, y, d, d^{\prime}, a, b, c$ induce $G_{11}$, a contradiction. Thus one of $a$ and $b$, say $a$, has at most one non-neighbor, say $d$, in $Q_{1}$, and since $N(x)$ is a threshold graph, without loss of generality, the neighborhood of $b$ in $Q_{1}$ is contained in the neighborhood of $a$ in $Q_{1}$; in particular, $b$ misses $d$, and, as above, $c$ misses $d$. Let $e$ be a neighbor of $d$ in $N^{2}(x)$.

Claim 6 Let $e$ be a neighbor of $d$ in $N^{2}(x)$. Then $e$ misses $a, b$ and $c$. (with $a, b, c$ and $d$ as described above).

## Proof of Claim 6.

We begin by observing that if $c e \in E$, by Proposition 1 (i), $c$ and $e$ have the same neighbors in $N(x)$ which is impossible since $e$ sees $d$ and $c$ misses $d$. Therefore $c e \notin E(G)$. In this case, by Proposition 1 (ii), if $e$ sees one of $a$ and $b$, it must see both of them but now, $x, y, e, a, b, d$ induce $G_{4}$ - a contradiciton. Then $e$ must also miss both $a$ and $b$. $\diamond$
Suppose that $Q_{1}^{\prime}:=\left(Q_{1} \backslash\{d\}\right) \cup\{a\}$ is not a cct of $G$. Note that $N(x) \backslash Q_{1}^{\prime}$ is stable. Then there is a cycle in $G$ whose only vertex from $Q_{1}$ is $d$. Obviously, if $C$ is a cycle containing $d$ and an edge in $N(x)$ then $Q_{1}^{\prime}$ covers $C$ since $N(x) \backslash Q_{1}^{\prime}$ is stable. Thus, we have to consider cycles without an edge in $N(x)$.
First consider a $C_{3} d u v$ with $u, v \in N^{2}(x)$. Then by Claim 6, $u$ and $v$ miss $a, b$ and $c$, and now, together with $y, G^{\prime}$ contains $G_{9}$, a contradiction. If $d$ is in a $C_{4} d u v w$ with $u, v \in N^{2}(x)$ and $w \in R$ then very similarly, together with $y, G^{\prime}$ contains $G_{10}$, a contradiction. Thus, $d$ is not contained in any of such cycles.
If $C$ is a $C_{4}$ with $d, z \in N(x)$ and $u, v \in N^{2}(x)$ then, again by Claim 6, $u$ and $v$ miss $a, b$ and $c$. Then $z$ must see $a$ and $b$, otherwise there is a house or $G_{12}$ in $G^{\prime}$, together with $y$, but then $x a d z u$ induce a house, a contradiction.
This also happens when $d$ is in a $C_{4}$ with $x$ and no vertex from $Q_{1}^{\prime}$. This final contradiction shows that in Case 2.1.3, there is a cct $Q$ of $G$ without $x$, and thus, there is a cct $Q \cup\{y\}$ in $G^{\prime}$.
Case 2.2: $G\left[N^{2}(x) \cup R\right]$ is not cycle-free.
As in Case 1, we now assume that $G\left[N^{2}(x) \cup R\right]$ contains a cycle $C$ (which implies that in this case, $x \notin Q$ holds).

Claim $7 N(x)$ is $I_{3}$-free.
Proof of Claim 7. Assume that $G\left[N^{2}(x) \cup R\right]$ is not cycle-free and $N(x)$ contains a stable set of three vertices $a_{1}, a_{2}, a_{3}$. Let $S$ be a maximal stable set in $N(x)$ containing $a_{1}, a_{2}, a_{3}$.

Recall that by Proposition 1, for vertices $u, v \in N^{2}(x)$ in the same connected component of $G\left[N^{2}(x) \cup R\right]$, their neighborhoods in $S$ are equal. In addition, no vertex $u \in N^{2}(x)$ sees at least three vertices in $S$, otherwise $G_{4}$ is contained in $G^{\prime}$. Thus, for every pair of vertices $u, v \in N^{2}(x), N(u) \cap S=N(v) \cap S \subseteq\left\{a_{1}, a_{2}\right\}$ holds.
If $u \in N^{2}(x)$ is in a cycle of type $\left(A_{1}\right)$ or $\left(A_{2}\right)$ then $x, y, a_{1}, a_{2}, a_{3}, u$ and the vertices of the remaining cycle induce $G_{2}, G_{3}, G_{9}$ or $G_{10}$. If the cycle with $u, v \in N^{2}(x)$ is of type $\left(B_{1}\right)$, there is a house; if it is of type $\left(B_{2}\right)$ or $\left(D_{1}\right)$, there is $G_{2}$ or $G_{11}$; if of type $\left(B_{3}\right)$, there is $G_{3}$ or $G_{12}$; and finally, if of type $\left(C_{1}\right)$ or $\left(D_{2}\right)$, there is $G_{3}$ or $G_{11}$. This shows Claim 7 . $\diamond$
We conclude that if $y$ is a false twin to $x$ and $G\left[N^{2}(x) \cup R\right]$ is not cycle-free then by Claim 7, $N_{1}(x)$ contains at most two vertices. If $\left|N_{1}(x)\right| \leq 1$ then $Q$ is a cct of $G^{\prime}$. If $N_{1}(x)=\{a, b\}$ then by Claims 5 and 7 , one of them, say $a$, sees all vertices in $Q_{1}$. By Proposition 1, either $Q_{2} \cup\{a\}$ is a clique, and then $Q \cup\{a\}$ is a cct of $G^{\prime}$, or $a$ sees no vertex of $Q_{2}$, and then let $C$ be a cycle in $G\left[N^{2}(x) \cup R\right]$ and let $u \in V(C) \cap Q_{2}$. Since $x y a b$ is a $C_{4}$ and $G$ is $\left(G_{2}, G_{3}\right)$-free, there is an edge linking xyab and $C$. By Proposition 11, we conclude that $b$ sees all vertices in $C \cap N^{2}(x)$ and, therefore, in $Q_{2}$. Now if there is some $a^{\prime} \in Q_{1}$ then $x, a, a^{\prime}, b, u$ induce either a house or a gem - a contradiction. Therefore $Q_{1}=\emptyset$ and $Q_{2}^{\prime} \cup\{b\}$ is a cct of $G^{\prime}$. This finishes the proof in Case 2 and thus also the proof of Theorem 2 .

As a direct consequence of Theorem 2, we obtain another proof of a result in [3):

Corollary 1 If $G$ is a cograph then $G$ admits a clique cycle-transversal if and only if $G$ is $\left(G_{1}, \ldots, G_{6}\right)$-free.

Proof. Graphs $G_{1}$ to $G_{6}$ admit no cct. Conversely, $G$ is also $\left(G_{7}, \ldots, G_{12}\right)$-free (because all of them contain $P_{4}$ ). Since every cograph is a distance-hereditary graph, by Theorem 2 the corollary follows.

Corollary 2 Let $G$ be a distance-hereditary graph. Then $G$ is a (2,1)-graph if and only if $G$ is $\left(G_{1}, G_{5}, G_{6}, G_{7}\right)$-free.

Proof. Graphs $G_{1}, G_{5}, G_{6}$ and $G_{7}$ are not (2,1)-graphs. Conversely, assume that $G$ is $\left(G_{1}, G_{5}, G_{6}, G_{7}\right)$-free and $G$ is not a (2,1)-graph. Let $G^{\prime}$ be a minimal induced subgraph of $G$ which is not a $(2,1)$-graph. Note that being a $(2,1)$-graph is equivalent to admitting a clique that intersects every odd cycle. Thus $G^{\prime}$ does not admit a cct. By Theorem 2, $G^{\prime}$ is isomorphic to one of the graphs $G_{1}, G_{2}, \ldots, G_{12}$. Since $G_{2}, G_{3}, G_{4}, G_{8}, G_{9}, G_{10}, G_{11}$, and $G_{12}$ are (2,1)-graphs, it follows that $G$ contains $G_{1}, G_{5}, G_{6}$, or $G_{7}$ as an induced subgraph.

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