Nordhaus-Gaddum-type results for the generalized edge-connectivity of graphs*

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Abstract

Let G be a graph, S be a set of vertices of G, and $\lambda(S)$ be the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \cdots, T_\ell$ in G such that $S \subseteq V(T_i)$ for every $1 \leq i \leq \ell$. The generalized k-edge-connectivity $\lambda_k(G)$ of G is defined as $\lambda_k(G) = min\{\lambda(S)|S \subseteq V(G)\ and\ |S| = k\}$. Thus $\lambda_2(G) = \lambda(G)$. In this paper, we consider the Nordhaus-Gaddum-type results for the parameter $\lambda_k(G)$. We determine sharp upper and lower bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$ for a graph G of order n, as well as for a graph of order n and size m. Some graph classes attaining these bounds are also given.

Keywords: edge-connectivity; Steiner tree; edge-disjoint trees; generalized edge-connectivity; complementary graph.

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1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [4] for graph theoretical notation and terminology not described here. For a graph $G(V, E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *an Steiner tree connecting S* (Shortly, *a Steiner tree*) is a subgraph $T(V', E')$ of G which is a tree such that $S \subseteq V'$. Two Steiner trees T and T' connecting S are *edge-disjoint* if $E(T) \cap E(T') = \varnothing$. The *Steiner Tree Packing Problem* for a given graph $G(V, E)$ and $S \subseteq V(G)$ asks to find a set of maximum number of edge-disjoint S-Steiner trees in G. This problem has obtained wide attention and many results have been worked out, see [18, 19, 20]. The problem for $S = V(G)$ is called the *Spanning Tree Packing Problem*. For any graph G of order n, the *spanning tree packing number* or *STP number*, is the maximum number of edge-disjoint spanning trees contained in G. For the STP number, Palmer gave a good survey, see [17].

Recently, we introduced the concept of generalized edge-connectivity of a graph G in [13]. For $S \subseteq V(G)$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint trees in G connecting S. Then the *generalized k-edge-connectivity* $\lambda_k(G)$

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of G is defined as $\lambda_k(G) = min\{\lambda(S) : S \subseteq V(G) \ and \ |S| = k\}$. Thus $\lambda_2(G) = \lambda(G)$. Set $\lambda_k(G) = 0$ when G is disconnected. We call it the generalized k-edge-connectivity since Chartrand et al. in [5] introduced the concept of generalized (vertex) connectivity in 1984. There have been many results on the generalized connectivity, see [10, 11, 12, 13].

One can see that the Steiner Tree Packing Problem studies local properties of graphs, but the generalized edge-connectivity focuses on global properties of graphs. Actually, the STP number of a graph G is just $\lambda_n(G)$.

In addition to being natural combinatorial measures, the Steiner Tree Packing Problem and the generalized edge-connectivity can be motivated by their interesting interpretation in practice as well as theoretical consideration. For the practical backgrounds, we refer to [7, 8, 15].

From a theoretical perspective, both extremes of this problem are fundamental theorems in combinatorics. One extreme of the problem is when we have two terminals. In this case internally (edge-)disjoint trees are just internally (edge-)disjoint paths between the two terminals, and so the problem becomes the well-known Menger theorem. The other extreme is when all the vertices are terminals. In this case internally disjoint trees and edge-disjoint trees are just spanning trees of the graph, and so the problem becomes the classical Nash-Williams-Tutte theorem.

Theorem 1. *(Nash-Williams [14], Tutte [16]) A multigraph G contains a system of ℓ edge-disjoint spanning trees if and only if*

$$\|G/\mathscr{P}\| \geq \ell(|\mathscr{P}| - 1)$$

holds for every partition $\mathscr{P}$ of $V(G)$, where $\|G/\mathscr{P}\|$ denotes the number of crossing edges in G, i.e., edges between distinct parts of $\mathscr{P}$.

Corollary 1. *Every 2ℓ-edge-connected graph contains a system of ℓ edge-disjoint spanning trees.*

Let $\mathcal{G}(n)$ denote the class of simple graphs of order n and $\mathcal{G}(n, m)$ the subclass of $\mathcal{G}(n)$ having m edges. Give a graph theoretic parameter $f(G)$ and a positive integer n, the *Nordhaus-Gaddum(**N-G**) Problem* is to determine sharp bounds for: (1) $f(G) + f(\overline{G})$ and (2) $f(G) \cdot f(\overline{G})$, as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide investigations. Recently, Aouchiche and Hansen published a survey paper on this subject, see [3].

In this paper, we study $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$ for the parameter $\lambda_k(G)$ where $G \in \mathcal{G}(n)$ and $G \in \mathcal{G}(n, m)$.

2 Nordhaus-Gaddum-type results in $\mathcal{G}(n)$

The following observation is easily seen.

Observation 1. (1) *If G is a connected graph, then $1 \leq \lambda_k(G) \leq \lambda(G) \leq \delta(G)$;*

(2) *If H is a spanning subgraph of G, then $\lambda_k(H) \leq \lambda_k(G)$.*

(3) *Let G be a connected graph with minimum degree δ. If G has two adjacent vertices of degree δ, then $\lambda_k(G) \leq \delta - 1$.*

Alavi and Mitchem in [2] considered Nordhaus-Gaddum-type results for the connectivity and edge-connectivity parameters. In [13] we were concerned with analogous inequalities involving the generalized k-connectivity and generalized k-edge-connectivity. We showed that $1 \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$, but this is just a starting result and now we will further study the Nordhaus-Guddum type relations.

To start with, let us recall the Harary graph $H_{n,d}$ on n vertices, which is constructed by arranging the n vertices in circular order and spreading the d edges around the boundary in a nice way, keeping the chords as short as possible. They have the maximum connectivity for their size and $\kappa(H_{n,d}) = \lambda(H_{n,d}) = \delta(H_{n,d}) = d$. Palmer [17] gave the STP number of some special graph classes.

Lemma 1. *[17] (1) The STP number of a complete bipartite graph $K_{a,b}$ is $\lfloor \frac{ab}{a+b-1} \rfloor$.*

(2) *The STP number of a Harary graph $H_{n,d}$ is $\lfloor d/2 \rfloor$.*

Corresponding to (1) of Observation 1, we can obtain a sharp lower bound for the generalized k-edge-connectivity by Corollary 1. Actually, a connected graph G contains $\lfloor \frac{1}{2}\lambda(G) \rfloor$ spanning trees. Each of them is also a Steiner tree connecting S. So the following proposition is immediate.

Proposition 1. *For a connected graph G of order n and $3 \leq k \leq n$, $\lambda_k(G) \geq \lfloor \frac{1}{2}\lambda(G) \rfloor$. Moreover, the lower bound is sharp.*

In order to show the sharpness of this lower bound for $k = n$, we consider the Harary graph $H_{n,2r}$. Clearly, $\lambda(G) = 2r$. From (2) of Lemma 1, $H_{n,2r}$ contains r spanning trees, that is, $\lambda_n(H_{n,2r}) = r$. So $\lambda_n(H_{n,2r}) = \lfloor \frac{1}{2}\lambda(G) \rfloor$. For general k ($3 \leq k \leq n$), one can check that the cycle C_n can attain the lower bound since $\frac{1}{2}\lambda(C_n) = 1 = \lambda_k(C_n)$.

The following proposition indicates that the monotone properties of λ_k, that is, $\lambda_n \leq \lambda_{n-1} \leq \cdots \lambda_4 \leq \lambda_3 \leq \lambda$, is true for $2 \leq k \leq n$.

Proposition 2. *For two integers k and n with $2 \leq k \leq n-1$, and a connected graph G, $\lambda_{k+1}(G) \leq \lambda_k(G)$.*

Proof. Assume $3 \leq k \leq n-1$. Set $\lambda_{k+1}(G) = \ell$. For each $S \subseteq V(G)$ with $|S| = k$, we let $S' = S \cup \{u\}$, where $u \notin S$. Since $\lambda_{k+1}(G) = \ell$, there exist ℓ edge-disjoint trees connecting S'. These trees are also ℓ edge-disjoint trees connecting S. So $\lambda_k(G) \geq \ell$ and $\lambda_{k+1}(G) \leq \lambda_k(G)$. Combining this with (1) of Observation 1, we get that $\lambda_{k+1}(G) \leq \lambda_k(G)$ for $2 \leq k \leq n-1$. $\square$

Now we give the lower bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$.

Lemma 2. *Let $G \in \mathcal{G}(n)$. Then*

(1) $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$;

(2) $\lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0$.

Moreover, the two lower bounds are sharp.

Proof. (1) If $\lambda_k(G) + \lambda_k(\overline{G}) = 0$, then $\lambda_k(G) = \lambda_k(\overline{G}) = 0$, that is, G and $\overline{G}$ are all disconnected, which is impossible, and so $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$.

(2) By definition, $\lambda_k(G) \geq 0$ and $\lambda_k(\overline{G}) \geq 0$, and so $\lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0$. □

The following observation indicates the graphs attaining the lower bound of (1) in Lemma 2.

Observation 2. $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$ *if and only if* G *or* $\overline{G}$ *is disconnected.*

In [13] we obtained the exact value of the generalized k-edge-connectivity of a complete graph K_n.

Lemma 3. *[13] For two integers* n *and* k *with* $2 \leq k \leq n$, $\lambda_k(K_n) = n - \lceil k/2 \rceil$.

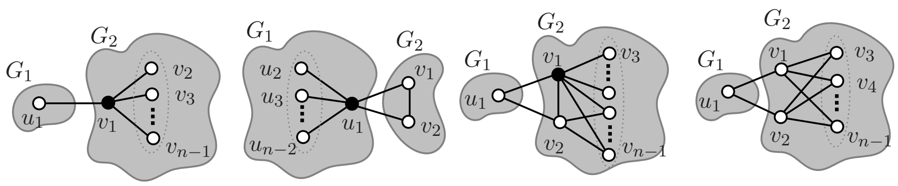
For a connected graph G of order n, we know that $1 \leq \lambda_k(G) \leq \lambda_k(K_n) = n - \lceil k/2 \rceil$. In [13] we characterized the graphs attaining the upper bound.

Lemma 4. *[13] For a connected graph* G *of order* n *with* $3 \leq k \leq n$, $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ *if and only if* $G = K_n$ *for* k *even;* $G = K_n \setminus M$ *for* k *odd, where* M *is an edge set such that* $0 \leq |M| \leq \frac{k-1}{2}$.

As we know, it is difficult to characterize the graphs with $\lambda_k(G) = 1$, even with $\lambda_3(G) = 1$. So we want to add some conditions to attack such a problem. Motivated by such an idea, we hope to characterize the graphs with $\lambda_k(G) + \lambda_k(\overline{G}) = 1$. Actually, the Norhaus-Gaddum-type problems also need to characterize the extremal graphs attaining the bounds.

Before studying the lower bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$, we give some graph classes (Every element of each graph class has order n), which will be used later.

For $n \geq 5$, $\mathcal{G}_n^1$ is a graph class as shown in Figure 1 (a) such that $\lambda(G) = 1$ and $d_G(v_1) = n - 1$ for $G \in \mathcal{G}_n^1$, where $v_1 \in V(G)$; $\mathcal{G}_n^2$ is a graph class as shown in Figure 1 (b) such that $\lambda(G) = 2$ and $d_G(u_1) = n - 1$ for $G \in \mathcal{G}_n^2$, where $u_1 \in V(G)$; $\mathcal{G}_n^3$ is a graph class as shown in Figure 1 (c) such that $\lambda(G) = 2$ and $d_G(v_1) = n - 1$ for $G \in \mathcal{G}_n^3$, where $v_1 \in V(G)$; $\mathcal{G}_n^4$ is a graph class as shown in Figure 1 (d) such that $\lambda(G) = 2$.



(a) $\mathcal{G}_n^1$ (b) $\mathcal{G}_n^2$ (c) $\mathcal{G}_n^3$ (d) $\mathcal{G}_n^4$

Figure 1. Graphs for Proposition 3 (The degree of a black vertex is $n - 1$).

The following observation and lemma are some preparations for Proposition 3.

For $n \geq 5$, let $K_{2,n-2}^{+}$ and $K_{2,n-2}^{++}$ be two graphs obtained from the complete bipartite graph $K_{2,n-2}$ by adding one and two edges on the part having $n - 2$ vertices, respectively.

Observation 3. (1) $\lambda_n(K^{++}_{2,n-2}) \geq 2$; (2) $\lambda_{n-1}(K^{+}_{2,n-2}) \geq 2$, $\lambda_n(K^{+}_{2,n-2}) = 1$; (3) $\lambda_{n-2}(K_{2,n-2}) \geq 2$, $\lambda_n(K_{2,n-2}) = \lambda_{n-1}(K_{2,n-2}) = 1$.

Proof. (1) As shown in Figure 2 (a), $\lambda_n(K^{++}_{2,n-2}) \geq 2$.

(2) As shown in Figure 2 (b), we have $\lambda_{n-1}(K^{+}_{2,n-2}) \geq 2$. Since $|E(K^{+}_{2,n-2})| = 2(n-2)+1$, $\lambda_n(K^{+}_{2,n-2}) \leq \lfloor \frac{2(n-2)+1}{n-1} \rfloor$, which implies that $\lambda_n(K^{+}_{2,n-2}) \leq 1$. Since $K^{+}_{2,n-2}$ is connected, $\lambda_n(K^{+}_{2,n-2}) = 1$.

(3) As shown in Figure 2 (c), it follows that $\lambda_{n-2}(K_{2,n-2}) \geq 2$. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \cdots, w_{n-2}\}$ be two parts of the complete bipartite graph $K_{2,n-2}$. Choose $S = \{u_1, u_2, w_1, w_2, \cdots, w_{n-3}\}$. If there exists an S-tree containing vertex w_{n-2}, then this tree will use $n-1$ edges of $E(K_{2,n-2})$, which implies that $\lambda_{n-1}(K_{2,n-2}) \leq 1$ since $|E(K_{2,n-2})| = 2(n-2)$. Suppose that there is no S-tree containing vertex w_2. Pick up a such tree, say T. Then there exists a vertex of degree 2 in T, which implies that there is no other S-tree in $K_{2,n-2}$. So $\lambda_{n-1}(K_{2,n-2}) \leq 1$. Since $K_{2,n-2}$ is connected, $\lambda_{n-1}(K_{2,n-2}) = 1$. From Proposition 2, $\lambda_n(K_{2,n-2}) = 1$. $\square$

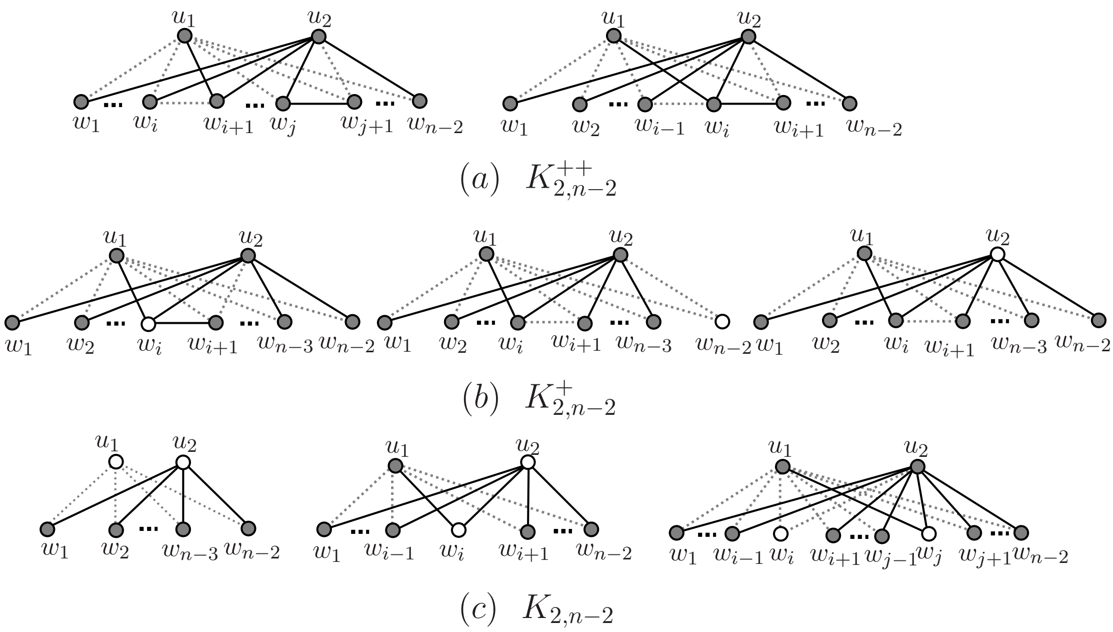


Figure 2. Graphs for Observation 2.

Lemma 5. *Let G be a connected graph. If $\lambda(G) = 3$ and there exists a vertex $u \in V(G)$ such that $d_G(u) = n-1$, then $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$.*

Proof. Let $G_1, \cdots, G_r$ be the connected components of $G \setminus u$. Since $\lambda(G) = 3$, it follows that $\delta(G_i) \geq 2$ $(1 \leq i \leq r)$. Let $|V(G_i)| = n_i$ $(1 \leq i \leq r)$ and $V(G_i) = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$. Then there exists an edge, without loss of generality, say $e_i = v_{i1}v_{i2} \in E(G_i)$ such that $G_i \setminus e_i$ is connected for $1 \leq i \leq r$. Thus $G_i \setminus e_i$ contains a spanning tree, say T_i $(1 \leq i \leq r)$. The trees $T = uv_{11} \cup T_1 \cup uv_{21} \cup T_2 \cup \cdots \cup uv_{r1} \cup T_r$ and $T' = v_{11}v_{12} \cup uv_{12} \cup \cdots \cup uv_{1n_1} \cup v_{21}v_{22} \cup uv_{22} \cup \cdots \cup uv_{2n_2} \cup \cdots \cup v_{r1}v_{r2} \cup uv_{r2} \cup \cdots \cup uv_{rn_r}$ are two spanning trees of G, that is, $\lambda_n(G) \geq 2$. Combining this with Proposition 2, $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$. $\square$

Proposition 3. *$\lambda_k(G) + \lambda_k(\overline{G}) = 1$ if and only if G (symmetrically, $\overline{G}$) satisfies one of the following conditions:*

(1) $G \in \mathcal{G}_n^1$ or $G \in \mathcal{G}_n^2$;

(2) $G \in \mathcal{G}_n^3$ and there exists a component G_i of $G \setminus v_1$ such that G_i is a tree and $|V(G_i)| < k$;

(3) $G \in \{K_{2,n-2}^+, K_{2,n-2}\}$ for $k = n$ and $n \geq 5$, or $G \in \{P_3, C_3\}$ for $k = n = 3$, or $G \in \{C_4, K_4 \setminus e\}$ for $k = n = 4$, or $G = K_{3,3}$ for $k = n = 6$, or $G = K_{2,n-2}$ for $k = n-1$ and $n \geq 5$, or $G = C_4$ for $k = n-1 = 3$.

Proof. Necessity. Let G be a graph satisfying one of the conditions of (1), (2) and (3). One can see that G is connected and its complement $\overline{G}$ is disconnected. Thus $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(G)$ and $\lambda_k(G) \geq 1$. We only need to show that $\lambda_k(G) \leq 1$ for each graph G satisfying one of the conditions of (1), (2) and (3). For $G \in \mathcal{G}_n^1$, since $\delta(G) = 1$ we have $\lambda_k(G) \leq 1$ by (1) of Observation 1. For $G \in \mathcal{G}_n^2$, it follows that $\lambda_k(G) \leq \delta(G) - 1 = 1$ by (3) of Observation 1 since $d_G(v_1) = d_G(v_2) = \delta(G) = 2$. Suppose $G \in \mathcal{G}_n^3$ and there exists a connected component G_i of $G \setminus v_1$ such that G_i is a tree and $|V(G_i)| < k$. Set $V(G_i) = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$. We choose $S \subseteq V(G)$ such that $V(G_i) \cup \{v_1\} = S' \subseteq S$. Then $|E(G[S'])| = 2n_i - 1$. Since every spanning tree of $G[S']$ uses $n_i - 1$ edges of $E(G[S'])$, there exists at most one spanning tree of $G[S']$, which implies that there is at most one tree connecting S in G. So $\lambda_k(G) \leq 1$. For $G = K_{2,n-2}^+$, $\lambda_n(G) = 1$ by (2) of Observation 3. For $G = K_{2,n-2}$, by (3) of Observation 3, we have $\lambda_n(K_{2,n-2}) = \lambda_{n-1}(K_{2,n-2}) = 1$. For $G = K_{3,3}$, $\lambda_n(G) \leq \lfloor \frac{|E(G)|}{n-1} \rfloor = \lfloor \frac{9}{5} \rfloor = 1$. For $G \in \{P_3, C_3, C_4, K_4 \setminus e\}$, one can check that $\lambda_k(G) \leq 1$ for $k = n$ or $k = n-1$. From these together with $\lambda_k(G) \geq 1$, we have $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(G) = 1$.

Sufficiency. Suppose $\lambda_k(G) + \lambda_k(\overline{G}) = 1$. Then $\lambda_k(G) = 1$ and $\lambda_k(\overline{G}) = 0$, or $\lambda_k(\overline{G}) = 1$ and $\lambda_k(G) = 0$. By symmetry, without loss of generality, we let $\lambda_k(G) = 1$ and $\lambda_k(\overline{G}) = 0$. From these together with Proposition 1, $\lambda(\overline{G}) = 0$ and $1 \leq \lambda(G) \leq 3$. So we have the following three cases to consider.

Case 1. $\lambda(G) = 1$.

For $n = 3$, one can check that $G = P_3$ satisfies $\lambda(G) = 1$ but $\lambda(\overline{G}) = 0$. Now we assume $n \geq 4$. Since $\lambda(G) = 1$, there exists at least one cut edge in G, say $e = u_1v_1$. Let G_1 and G_2 be two connected components of $G \setminus e$ such that $u_1 \in V(G_1)$ and $v_1 \in V(G_2)$. Set $V(G_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_{n_2}\}$, where $n_1 + n_2 = n$. Suppose $n_i \geq 2$ $(i = 1, 2)$. For any $u_i, u_j \in V(G_1)$, u_i and u_j are connected in $\overline{G}$ since there exists a path $u_iv_2u_j$ in $\overline{G}$; for any $v_i, v_j \in V(G_2)$, v_i and v_j are connected in $\overline{G}$ since there exists a path $v_iu_2v_j$ in $\overline{G}$; for any $u_i \in V(G_1)$ and $v_j \in V(G_2)$ $(i \neq 1$ or $j \neq 1)$, $v_iv_j \in E(\overline{G})$. Clearly, the path $u_1v_2u_2v_1$ connects u_1 and v_1 in $\overline{G}$. So $\overline{G}$ is connected, a contradiction. Thus $n_1 = 1$ or $n_2 = 1$. Without loss of generality, let $n_1 = 1$. Then $V(G_1) = \{u_1\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_{n-1}\}$. Clearly, G is a graph obtained from G_2 by attaching the edge $e = u_1v_1$. Since $u_1v_j \notin E(G)$ $(1 \leq j \leq n-1)$, $u_1v_j \in E(\overline{G})$. If $d_G(v_1) \leq n-2$, then there exists one vertex v_j such that $v_1v_j \in E(\overline{G})$, which results in $\lambda(\overline{G}) \geq 1$, a contradiction. So $d_G(v_1) = n-1$ and $G \in \mathcal{G}_n^1$ (See Figure 1 (a)).

Case 2. $\lambda(G) = 2$.

For $n = 3, 4$, the graph $G \in \{C_3, C_4, K_4 \backslash e\}$ satisfies that $\lambda(G) = 2$ and $\lambda(\overline{G}) = 0$. Since $\lambda_3(C_3) = 1$, $\lambda_3(C_4) = 1$, $\lambda_4(C_4) = 1$, $\lambda_3(K_4 \setminus e) = 2$ and $\lambda_4(K_4 \setminus e) = 1$, we have $G = C_3$ for $k = n = 3$; $G \in \{C_4, K_4 \setminus e\}$ for $k = n = 4$; $G = C_4$ for $k = n - 1 = 3$. Now we assume $n \geq 5$. Since $\lambda(G) = 2$, there exists an edge cut M such that $|M| = 2$. Let G_1 and G_2 be two connected components of $G \setminus M$, $V(G_1) = \{u_1, \cdots, u_{n_1}\}$ and $V(G_2) = \{v_1, \cdots, v_{n_2}\}$, where $n_1 + n_2 = n$. Clearly, $G[M] = 2K_2$ or $G[M] = P_3$.

At first, we consider the case $G[M] = 2K_2$. Without loss of generality, let $M = \{u_1v_1, u_2v_2\}$. Since $n \geq 5$, $n_1 \geq 3$ or $n_2 \geq 3$. Without loss of generality, let $n_1 \geq 3$. Clearly, any two vertices $v_i, v_j \in V(G_2)$ are connected in $\overline{G}$ since there exists a path $v_iu_3v_j$ in $\overline{G}$. Furthermore, for any $u_i \in V(G_1)$, $u_iv_1 \in E(\overline{G})$ or $u_iv_2 \in E(\overline{G})$. So $\overline{G}$ is connected and $\lambda(\overline{G}) \geq 1$, a contradiction.

Next, we consider the case $G[M] = P_3$. Without loss of generality, let $P = v_1u_1v_2$ be the path of order 3. Since $n \geq 5$, there exist at least two vertices in $G \setminus \{u_1, v_1, v_2\}$. If $n_1 \geq 2$ and $n_2 \geq 3$, then we can check that $\overline{G}$ is connected, a contradiction. So we assume that $n_1 = 1$ or $n_2 = 2$, that is, $V(G_2) = \{v_1, v_2\}$ or $V(G_1) = \{u_1\}$.

For the former, $V(G_1) = \{u_1, u_2, \cdots, u_{n-2}\}$. Since $\lambda(G) = 2$, $v_1v_2 \in E(G)$. Clearly, $v_1u_j, v_2u_j \notin E(G)$ $(2 \leq j \leq n-2)$, which implies that $v_1u_j, v_2u_j \in E(\overline{G})$. Therefore, $u_1u_j \notin E(\overline{G})$ $(2 \leq j \leq n-2)$ since $\overline{G}$ is disconnected. Thus $u_1u_j \in E(G)$ for each j $(2 \leq j \leq n-2)$. So $d_G(u_1) = n - 1$ and $G \in \mathcal{G}_n^2$ (See Figure 1 (b)).

For the latter, let $V(G_2) = \{v_1, v_2, \cdots, v_{n-1}\}$. First we consider the case $v_1v_2 \in E(G)$. Since $u_1v_j \notin E(G)$ $(3 \leq j \leq n-1)$, we have $u_1v_j \in E(\overline{G})$. If $3 \leq d_G(v_1) \leq n-2$ and $3 \leq d_G(v_2) \leq n-2$, then there exist two vertices v_i and v_j such that $v_1v_i, v_2v_j \in E(\overline{G})$ $(3 \leq i, j \leq n-1)$, which implies that $\overline{G}$ is connected, a contradiction. So $d_G(v_1) = n-1$ or $d_G(v_2) = n-1$. Without loss of generality, let $d_G(v_1) = n-1$. Thus $G \in \mathcal{G}_n^3$ (See Figure 1 (c)). Now we focus on the graph $G \setminus v_1$. Let $G_1, G_2, \cdots, G_r$ be the connected components of $G \setminus v_1$ and $V(G_i) = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ $(1 \leq i \leq r)$, where $\sum_{i=1}^{r} n_i = n - 1$. If there exists some connected component G_i such that $G_i = K_2$, then $G \in \mathcal{G}_n^2$ (See Figure 1 (b)). So we assume $n_i \geq 3$. Then we prove the following claim and get a contradiction.

Claim 1. For each connected component G_i of $G \setminus v_1$, if $n_i \geq k$, or $n_i \leq k-1$ and $|E(G_i)| \geq n_i$, then $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$.

Proof of Claim 1. For an arbitrary $S \subseteq V(G)$ with $|S| = k$, we only prove $\lambda(S) \geq 2$ for $v_1 \notin S$. The case for $v_1 \in S$ can be proved similarly. If there exists some connected component G_i such that $S = V(G_i)$, then $n_i = k$ and G_i has a spanning tree, say T_i. It is also a Steiner tree connecting S. Since $T_i' = v_1v_{i1} \cup v_1v_{i2} \cdots \cup v_1v_{in_i}$ is another Steiner tree connecting S and T_i, T_i' are two edge-disjoint trees, we have $\lambda(S) \geq 2$. Let us assume now $S \neq V(G_i)$ for $n_i \geq k$ $(1 \leq i \leq r)$. Let $S_i = S \cap V(G_i)$ $(1 \leq i \leq r)$ and $|S_i| = k_i$. Clearly, $\bigcup_{i=1}^{r} S_i = S$ and $\sum_{i=1}^{r} k_i = k$. Thus $S_i \subset V(G_i)$ for each connected component G_i such that $n_i \geq k$, and $S_j \subseteq V(G_j)$ for each connected component G_j such that $n_j \leq k-1$ and $|E(G_j)| \geq n_j$. We will show that there are two edge-disjoint Steiner trees connecting $S_i \cup \{v_1\}$ in $G[S_i \cup \{v_1\}]$ for each i $(1 \leq i \leq r)$ so that we can combine these trees to form two edge-disjoint Steiner trees connecting S in G. Suppose that G_i is a connected component such that $n_i \geq k$. Note that $V(G_i) = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$. Since $S_i \subset V(G_i)$, there exists a vertex, without loss of generality, say v_{i1}, such that $v_{i1} \notin S_i$. Clearly, G_i contains a spanning tree, say T'_{i1}. Thus $T_{i1} = v_1v_{i1} \cup T'_{i1}$ is a Steiner tree connecting $S_i \cup \{v_1\}$ in $G[G_i \cup \{v_1\}]$. Since $T_{i2} = v_1v_{i2} \cup v_1v_{i3} \cup \cdots \cup v_1v_{in_i}$ is another Steiner tree connecting $S_i \cup \{v_1\}$. Clearly, T_{i1} and T_{i2} are edge-disjoint. Assume that G_j is a connected component such that $n_j \leq k-1$ and $|E(G_j)| \geq n_j$. Note that $V(G_j) = \{v_{j1}, v_{j2}, \cdots, v_{jn_j}\}$. Then there exists an edge, without loss of generality, say $e_j = v_{j1}v_{j2} \in E(G_j)$ such that $G_j \backslash e_j$ contains a spanning tree of G_j, say T'_{j1}. Thus $T_{j1} = v_1v_{j1} \cup T'_{j1}$ and $T_{j2} = v_{j1}v_{j2} \cup v_1v_{j2} \cup \cdots \cup v_1v_{jn_j}$ are two edge-disjoint Steiner trees connecting $S_j \cup \{v_1\}$. Now we combine these small trees connecting $S_i \cup \{v_1\}$ $(1 \leq i \leq r)$ by the vertex v_1 to form two big trees connecting S. Clearly, $T_1 = T_{11} \cup T_{21} \cup \cdots \cup T_{r1}$ and $T_2 = T_{12} \cup T_{22} \cup \cdots \cup T_{r2}$ are our desired trees, that is, $\lambda(S) \geq 2$. From the arbitrariness of S, we have $\lambda_k(G) \geq 2$. $\square$

By Claim 1, we know that $G \in \mathcal{G}_n^3$ and there exists a connected component G_i of $G \setminus \{v_1\}$ such that $n_i \leq k-1$ and G_i is a tree.

We next consider the case $v_1v_2 \notin E(G)$ (See Figure 1 (d)). Thus $v_1v_2 \in E(\overline{G})$. Since $u_1v_j \notin E(G)$ $(3 \leq j \leq n-1)$, $u_1v_j \in E(\overline{G})$, which results in $v_1v_j, v_2v_j \notin E(\overline{G})$ since $\overline{G}$ is disconnected. Thus $v_1v_j, v_2v_j \in E(G)$ for each j $(3 \leq j \leq n-1)$. Let $R = \{v_j | 3 \leq j \leq n-1\}$. If $|E(G[R])| \geq 2$, then G contains a subgraph $K_{2,n-2}^{++}$, which implies that $\lambda_n(G) \geq 2$ by (1) of Observation 3. Combining this with Proposition 2, $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$, a contradiction. If $|E(G[R])| < 2$, then $G = K_{2,n-2}$ and $K_{2,n-2}^{+}$. From Observation 3 and Proposition 2, we have $\lambda_k(K_{2,n-2}^{+}) \geq 2$ for $3 \leq k \leq n-1$ and $\lambda_k(K_{2,n-2}) \geq 2$ for $3 \leq k \leq n-2$, a contradiction. So $G = K_{2,n-2}^{+}$ for $k = n$, or $G = K_{2,n-2}$ for $k = n$, or $G = K_{2,n-2}$ for $k = n-1$.

Case 3. $\lambda(G) = 3$.

For $n = 4$, $G = K_4$, $\lambda_3(G) = \lambda_4(G) = 2$ by Lemma 3. Then $\lambda_k(G) \geq 2$, a contradiction. Assume $n \geq 5$. Since $\lambda(G) = 3$, there exists an edge cut M such that $|M| = 3$. Let G_1

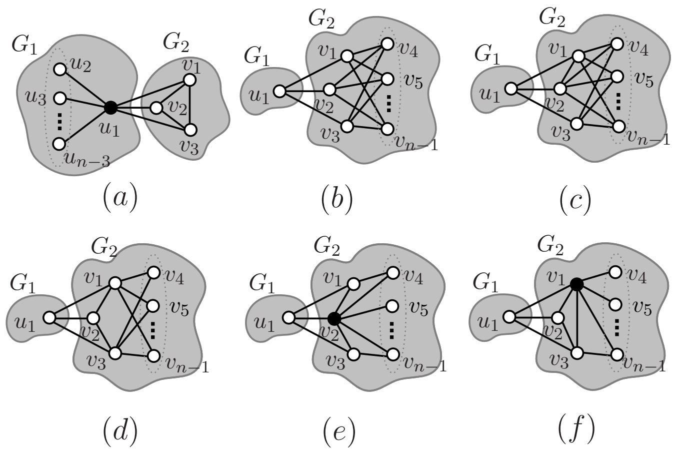


Figure 3. Graphs for Case 3 of Proposition 3.

and G_2 be two connected components of $G \setminus M$, $V(G_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_{n_2}\}$, where $n_1 + n_2 = n$. Clearly, $G[M] = P_4$ or $G[M] = P_3 \cup K_2$ or $G[M] = 3K_2$ or $G[M] = K_{1,n-3}$. For the former three cases, $n_i \geq 3$ $(i = 1, 2)$ and $n \geq 6$ since $\lambda(G) = 3$. To shorten the discussion, we only prove $\lambda(\overline{G}) \geq 1$ for $G[M] = P_4$ and get a contradiction among the former three cases. Without loss of generality, let

$G[M] = P_4 = u_1v_1u_2v_2$. For any $u_i, u_j \in V(G_1)$ $(1 \leq i \leq n_1)$, u_i and u_j are connected in $\overline{G}$ since there exists a path $u_iv_3u_j$ in $\overline{G}$; for any $v_i, v_j \in V(G_2)$ $(1 \leq i \leq n_2)$, v_i and v_j are connected in $\overline{G}$ since there exists a path $v_iu_3v_j$ in $\overline{G}$; for any $u_i \in V(G_1)$ and $v_j \in V(G_2)(i \neq 3$ and $j \neq 3)$, u_i and u_j are connected in $\overline{G}$ since there exists a path $u_iv_3u_3v_j$ in $\overline{G}$. Since $u_3v_j \in E(\overline{G})$ $(1 \leq j \leq n_2)$ and $v_3u_i \in E(\overline{G})$ $(1 \leq i \leq n_1)$, $\overline{G}$ is connected, a contradiction.

Now we consider the graph G such that $G[M] = K_{1,n-3}$. Assume $n_1 \geq 2$. If $n_2 \geq 4$, then we can check that $\overline{G}$ is connected and get a contradiction. Therefore, $n_2 = 3$, $V(G_2) = \{v_1, v_2, v_3\}$ and $V(G_1) = \{u_1, u_2 \cdots, u_{n-3}\}$. Since $\lambda(G) = 3$, it follows that $v_1v_2, v_2v_3, v_1v_3 \in E(G)$. Since $v_iu_j \notin E(G)$ $(1 \leq i \leq 3,\ 2 \leq j \leq n-3)$, we have $v_iu_j \in E(\overline{G})$. If there exists some vertex u_j $(2 \leq j \leq n-3)$ such that $u_1u_j \in E(\overline{G})$, then $\overline{G}$ is connected, a contradiction. So $u_1u_j \in E(G)$ for $2 \leq j \leq n-3$. Thus $d_G(u_1) = n-1$ (See Figure 3 (a)). From Lemma 5, $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ since $\lambda(G) = 3$, a contradiction.

Let us now assume $n_1 = 1$. Then $V(G_1) = \{u_1\}$ and $V(G_2) = \{v_1, v_2 \cdots, v_{n-1}\}$. If $G[\{v_1, v_2, v_3\}] = 3K_1$ or $G[\{v_1, v_2, v_3\}] = 2K_1 \cup K_2$, then we have $u_1v_j \in E(\overline{G})$ since $u_1v_j \notin E(G)$ $(4 \leq j \leq n-1)$. From this together with the fact that $\overline{G}$ is disconnected and $v_1v_3, v_2v_3 \in E(\overline{G})$, $v_iv_j \notin E(\overline{G})$ $(1 \leq i \leq 3,\ 4 \leq j \leq n-1)$, we have that $v_iv_j \in E(G)$ $(1 \leq i \leq 3,\ 4 \leq j \leq n-1)$. Thus G contains a complete bipartite graph $K_{3,n-3}$ as its subgraph (See Figure 3 (b) and (c)). From (1) of Lemma 1, $\lambda_n(G) = \lfloor \frac{3(n-3)}{n-1} \rfloor \geq 2$ for $n \geq 7$, which implies $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ and $n \geq 7$. Since $\lambda(G) = 3$, $n \geq 6$. So we only need to consider the case $n = 6$. Thus $G = H_i$ $(1 \leq i \leq 4)$ (See Figure 4). If $G = H_i$ $(2 \leq i \leq 4)$, then $\lambda_n(G) \geq 2$ for $k = n = 6$ (See Figure 4 $(b), (c), (d)$). Therefore $\lambda_k(G) \geq 2$ for $3 \leq k \leq 6$. If $G = H_1$, then $\lambda_n(G) \leq \lfloor \frac{|E(G)|}{n-1} \rfloor = \lfloor \frac{9}{5} \rfloor = 1$ for $k = n = 6$. For $k = 5$, we can check that $\lambda_3(G) \geq \lambda_4(G) \geq \lambda_5(G) \geq 2$ (See Figure 4 (e)). So $G = K_{3,3}$ for $k = n = 6$.

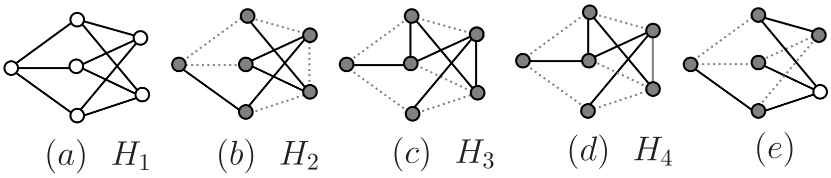


Figure 4. Graphs for Case 3 of Proposition 3.

Suppose $G[\{v_1, v_2, v_3\}] = P_3$. Without loss of generality, let $v_1v_2, v_2v_3 \in E(G)$. If $3 \leq d_G(v_2) \leq n-2$ (See Figure 3 (d)), then there exists at least one vertex v_j such that $v_2v_j \in E(\overline{G})$, which results in $v_1v_j, v_3v_j \notin E(\overline{G})$ $(4 \leq j \leq n-1)$ since $u_1v_j \in E(\overline{G})$ $(4 \leq j \leq n-1)$, $v_1v_3 \in E(\overline{G})$ and $\overline{G}$ is disconnected. Thus $v_1v_j, v_3v_j \in E(G)$ for each j $(4 \leq j \leq n-1)$. Since $d(v_4) \geq \delta(G) \geq \lambda(G) = 3$, we have $v_4v_2 \in E(G)$ or there exists some vertex v_j $(5 \leq j \leq n-1)$ such that $v_4v_j \in E(G)$, which implies that G contains a subgraph $K_{2,n-2}^{++}$ and so $\lambda_n(G) \geq 2$ by (1) of Observation 3. From Proposition 2, $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$, a contradiction. If $d_G(v_2) = n-1$ (See Figure 3 (e)), then $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ by Lemma 5 since $\lambda(G) = 3$, a contradiction.

Suppose $G[\{v_1, v_2, v_3\}] = K_3$. Without loss of generality, let $v_1v_2, v_1v_3, v_2v_3 \in E(G)$.

If $d_G(v_1) = n-1$ or $d_G(v_2) = n-1$ or $d_G(v_3) = n-1$ (See Figure 3 (f)), then by Lemma 5 $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ since $\lambda(G) = 3$, a contradiction. If $3 \leq d_G(v_i) \leq n-2 (1 \leq i \leq 3)$, then $\overline{G}$ is connected, a contradiction. $\square$

We now investigate the upper bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$.

Lemma 6. *Let $G \in \mathcal{G}(n)$. Then*

(1) $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$*;*

(2) $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq \left[\frac{n-\lceil k/2 \rceil}{2}\right]^2$.

Moreover, the two upper bounds are sharp.

Proof. (1) Since $G \cup \overline{G} = K_n$, $\lambda_k(G) + \lambda_k(\overline{G}) \leq \lambda_k(K_n)$. Combining this with Lemma 3, $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil \frac{k}{2} \rceil$.

(2) The conclusion holds by (1). $\square$

Let us focus on (1) of Lemma 6. If one of G and $\overline{G}$ is disconnected, we can characterize the graphs attaining the upper bound by Lemma 4.

Proposition 4. *For any graph G of order n, if G is disconnected, then $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$ if and only if $\overline{G} = K_n$ for k even; $\overline{G} = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

If both G and $\overline{G}$ are all connected, we can obtain a structural property of the graphs attaining the upper bound although it seems too difficult to characterize them.

Proposition 5. *If $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$, then $\Delta(G) - \delta(G) \leq \lceil \frac{k}{2} \rceil - 1$.*

Proof. Assume that $\Delta(G) - \delta(G) \geq \lceil \frac{k}{2} \rceil$. Since $\lambda_k(\overline{G}) \leq \delta(\overline{G}) = n - 1 - \Delta(G)$, $\lambda_k(G) + \lambda_k(\overline{G}) \leq \delta(G) + n - 1 - \Delta(G) \leq n - 1 - \lceil \frac{k}{2} \rceil$, a contradiction. $\square$

One can see that the graphs with $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$ must have a uniform degree distribution. Actually, we can construct a graph class to show that the two upper bounds of Lemma 6 are tight for $k = n$.

Example 2. Let n, r be two positive integers such that $n = 4r + 1$. From (1) of Lemma 1, we know that the STP number of the complete bipartite graph $K_{2r,2r+1}$ is $\lfloor \frac{2r(2r+1)}{2r+(2r+1)-1} \rfloor = r$, that is, $\lambda_n(K_{2r,2r+1}) = r$. Let $\mathcal{E}$ be the set of the edges of these r spanning trees in $K_{2r,2r+1}$. Then there exist $2r(2r+1) - 4r^2 = 2r$ remaining edges in $K_{2r,2r+1}$ except the edges in $\mathcal{E}$. Let M be the set of these $2r$ edges. Set $G = K_{2r,2r+1} \setminus M$. Then $\lambda_n(G) = r$, $M \subseteq E(\overline{G})$ and $\overline{G}$ is a graph obtained from two cliques K_{2r} and K_{2r+1} by adding $2r$ edges in M between them, that is, one endpoint of each edge belongs to K_{2r} and the other endpoint belongs to K_{2r+1}. Note that $E(\overline{G}) = E(K_{2r}) \cup M \cup E(K_{2r+1})$. Now we show that $\lambda_n(\overline{G}) \geq r$. As we know, K_{2r} contains r Hamiltonian paths, say $P_1, P_2, \cdots, P_r$, and so does K_{2r+1}, say $P_1', P_2', \cdots, P_r'$. Pick up r edges from M, say $e_1, e_2, \cdots, e_r$, let $T_i = P_i \cup P_i' \cup e_i (1 \leq i \leq r)$. Then $T_1, T_2, \cdots, T_r$ are r spanning trees in $\overline{G}$, namely, $\lambda_n(\overline{G}) \geq r$. Since $|E(\overline{G})| = \binom{2r}{2} + \binom{2r+1}{2} + 2r = 4r^2 + 2r$ and each spanning tree uses $4r$

edges, these edges can form at most $\lfloor \frac{4r^2+2r}{4r} \rfloor = r$ spanning trees, that is, $\lambda_n(\overline{G}) \leq r$. So $\lambda_n(\overline{G}) = r$.

Clearly, $\lambda_n(G) + \lambda_n(\overline{G}) = 2r = \frac{n-1}{2} = n - \lceil \frac{n}{2} \rceil$ and $\lambda_n(\overline{G}) \cdot \lambda_n(\overline{G}) = r^2 = \left[\frac{n-\lceil n/2 \rceil}{2}\right]^2$, which implies that the upper bound of Lemma 6 is sharp.

Combining Lemmas 2 and 6, we give our main result.

Theorem 2. *Let $G \in \mathcal{G}(n)$. Then*

(1) $1 \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$;

(2) $0 \leq \lambda_k(G) \cdot \lambda_k(\overline{G}) \leq \left[\frac{n-\lceil k/2 \rceil}{2}\right]^2$.

Moreover, the upper and lower bounds are sharp.

3 Nordhaus-Gaddum-type results in $\mathcal{G}(n, m)$

Achthan et. al. [1] restricted their attention to the subclass of $\mathcal{G}(n, m)$ consisting of graphs with exactly m edges. They investigated the edge-connectivity, diameter and chromatic number parameters. For edge-connectivity $\lambda(G)$, they showed that $\lambda(G) + \lambda(\overline{G}) \geq max\{1, n-1-m\}$. In this section, we consider a similar problem on the generalized edge-connectivity.

Lemma 7. *If M is an edge set of the complete graph K_n such that $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$ where $|M| = m$, then $G = K_n \setminus M$ contains ℓ edge-disjoint spanning trees, where $\ell = min\{n - 2m - 1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\}$.*

Proof. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(G)$ with $|V_i| = n_i$ $(1 \leq i \leq p)$, and $\mathcal{E}_p$ be the set of edges between distinct parts of $\mathscr{P}$ in G. It suffices to show that $|\mathcal{E}_p| \geq \ell(|\mathscr{P}| - 1)$ so that we can use Nash-Williams-Tutte Theorem.

The case $p = 1$ is trivial, thus we assume $2 \leq p \leq n$. Then $|\mathcal{E}_p| \geq \binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - |M| \geq \binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - m$. We will show that $\binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - m \geq \ell(p-1)$, that is, $\frac{n(n-1)}{2} - m - \ell(p-1) \geq \sum_{i=1}^{p} \binom{n_i}{2}$. We only need to prove that $\frac{n(n-1)}{2} - m - \ell(p-1) \geq max\{\sum_{i=1}^{p} \binom{n_i}{2}\}$. Since $f(n_1, n_2, \cdots, n_p) = \sum_{i=1}^{p} \binom{n_i}{2}$ achieves its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n - p + 1$, we need the inequality $\frac{n(n-1)}{2} - m - \ell(p - 1) \geq \binom{1}{2}(p-1) + \binom{n-p+1}{2}$, that is, $\frac{n(n-1)}{2} - m - \frac{(n-p+1)(n-p)}{2} \geq \ell(p-1)$. Actually, $\ell \leq \frac{n(n-1)-(n-p+1)(n-p)-2m}{2(p-1)}$ is our required inequality, namely, $\ell \leq n - \frac{1}{2} - (\frac{p-1}{2} + \frac{2m}{p-1})$. Since $f(x) = \frac{x}{2} + \frac{2m}{x}$ achieves its maximum value $max\{2m + \frac{1}{2}, \frac{n-1}{2} + \frac{2m}{n-1}\}$ when $1 \leq x \leq n-1$, we need $\ell \leq min\{n - 2m - 1, \frac{n}{2} - \frac{2m}{n-1}\}$. Since this inequality holds for $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$, we have $|\mathcal{E}_p| \geq \binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - |M| \geq \ell(p-1)$. From Theorem 1, we know that G has ℓ edge-disjoint spanning trees. $\square$

Lemma 8. *Let $G \in \mathcal{G}(n, m)$. For $n \geq 6$, we have*

(1) $\lambda_k(G) + \lambda_k(\overline{G}) \geq L(n, m)$, where

$$L(n,m) = \begin{cases} max\{1, \lfloor \frac{1}{2}(n-2-m) \rfloor\} & if\ \lfloor \frac{n}{3} \rfloor + 1 \leq m \leq \binom{n}{2}, \\ min\{n-2m-1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\} & if\ 0 \leq m \leq \lfloor \frac{n}{3} \rfloor. \end{cases}$$

(2) $\lambda_k(G)\cdot\lambda_k(\overline{G})\geq 0$.

Proof. (1) Since at least one of G and $\overline{G}$ must be connected, we have $\lambda_k(G)+\lambda_k(\overline{G})\geq 1$. For $m<n-1$, $\lambda_k(G)+\lambda_k(\overline{G})\geq \lfloor\frac{1}{2}\lambda(G)\rfloor+\lfloor\frac{1}{2}\lambda(\overline{G})\rfloor\geq\lfloor\frac{1}{2}(\lambda(G)+\lambda(\overline{G})-1)\rfloor\geq\lfloor\frac{1}{2}(max\{1,n-1-m\}-1)\rfloor\geq\lfloor\frac{1}{2}(n-2-m)\rfloor$ by Proposition 1. So $\lambda_k(G)+\lambda_k(\overline{G})\geq max\{1,\lfloor\frac{1}{2}(n-2-m)\rfloor\}$. In particular, for $0\leq m\leq\lfloor\frac{n}{3}\rfloor$, we can give a better lower bound of $\lambda_k(G)+\lambda_k(\overline{G})$ by Lemma 7, that is, $\lambda_k(G)+\lambda_k(\overline{G})=\lambda_k(\overline{G})\geq\lambda_n(\overline{G})\geq min\{n-2m-1,\lfloor\frac{n}{2}-\frac{2m}{n-1}\rfloor\}$.

To show the sharpness of the above lower bound for $\lfloor\frac{n}{3}\rfloor+1\leq m\leq\binom{n}{2}$, we consider the graph $G=K_{1,n-2}\cup K_1$. Then $m=n-2$ and $\overline{G}$ is a graph obtained from a complete graph K_{n-1} by attaching a pendant edge. Clearly, $\lambda_k(G)=0$ and $\lambda_k(\overline{G})=1$. So $\lambda_k(G)+\lambda_k(\overline{G})=1=max\{1,\lfloor\frac{1}{2}(n-2-m)\rfloor\}$. To show the sharpness of the above lower bound for $0\leq m\leq\lfloor\frac{n}{3}\rfloor$, we consider the graph $G=nK_1$. Thus $m=0$ and $\overline{G}=K_n$. Since $\lambda_n(G)+\lambda_n(\overline{G})=0+\lfloor\frac{n}{2}\rfloor=min\{n-2\cdot 0-1,\lfloor\frac{n}{2}-\frac{2\cdot 0}{n-1}\rfloor\}$, that is, the lower bound is sharp for $k=n$.

(2) The inequality follows from Theorem 2. $\square$

It was pointed out by Harary [9] that given the number of vertices and edges of a graph, the largest connectivity possible can also be read out of the inequality $\kappa(G)\leq\lambda(G)\leq\delta(G)$.

Theorem 3. *[9] For each n,m with $0\leq n-1\leq m\leq\binom{n}{2}$,*

$$\kappa(G)\leq\lambda(G)\leq\left\lfloor\frac{2m}{n}\right\rfloor,$$

where the maximum are taken over all graphs $G\in\mathcal{G}(n,m)$.

Now we will study a similar problem for the generalized edge-connectivity, which will be used in (2) of Lemma 9.

Corollary 2. *For any graph $G\in\mathcal{G}(n,m)$ and $3\leq k\leq n$, $\lambda_k(G)=0$ for $m<n-1$; $\lambda_k(G)\leq\lfloor\frac{2m}{n}\rfloor$ for $m\geq n-1$.*

Proof. Let $G\in\mathcal{G}(n,m)$. When $0\leq m<n-1$, G must be disconnected and hence $\lambda_k(G)=0$. If $m\geq n-1$, $\lambda_k(G)\leq\lambda(G)\leq\lfloor\frac{2m}{n}\rfloor$ by (1) of Observation 1 and Theorem 3. $\square$

Although the above bound of $\lambda_k(G)$ is the same as $\lambda(G)$, the graphs attaining the upper bound seems to be very rare. Actually, we can obtain some structural properties of these graphs.

Proposition 6. *For any $G\in\mathcal{G}(n,m)$ and $3\leq k\leq n$, if $\lambda_k(G)=\lfloor\frac{2m}{n}\rfloor$ for $m\geq n-1$, then*

(1) *$\frac{2m}{n}$ is not an integer;*

(2) *$\delta(G)=\lfloor\frac{2m}{n}\rfloor$;*

(3) *for $u,v\in V(G)$ such that $d_G(u)=d_G(v)=\lfloor\frac{2m}{n}\rfloor$, $uv\notin E(G)$.*

Proof. One can check that the conclusion holds for the case $m = n-1$. Assume $m \geq n$. We claim that $\frac{2m}{n}$ is not an integer. Otherwise, let $r = \frac{2m}{n}$ be an integer. We will show that $\lambda_k(G) \leq r-1 = \frac{2m}{n} - 1$ and get a contradiction. If G has at least one vertex v_i such that $d(v_i) > r$, then, since the average degree of G is exactly r, there must be a vertex v_j whose degree $d(v_j) < r$. From (1) of Observation 1, we have $\lambda_k(G) \leq \delta(G) \leq d(v_j) < r$, that is, $\lambda_k(G) \leq r-1$. If, on the other hand, G is a regular graph, then by (3) of Observation 1, $\lambda_k(G) \leq \delta(G) - 1 = r - 1$. So (1) holds.

For a graph G such that $\frac{2m}{n}$ is not an integer, $\lfloor \frac{2m}{n} \rfloor = \lambda_k(G) \leq \delta(G) \leq \lfloor \frac{2m}{n} \rfloor$, that is, $\delta(G) = \lfloor \frac{2m}{n} \rfloor$. So (2) holds.

For $u, v \in V(G)$ such that $d_G(u) = d_G(v) = \lfloor \frac{2m}{n} \rfloor$, we claim that $uv \notin E(G)$. Otherwise, $uv \in E(G)$. Since $d_G(u) = d_G(v) = \delta(G) = \lfloor \frac{2m}{n} \rfloor$, $\lambda_k(G) \leq \delta(G) - 1 = \lfloor \frac{2m}{n} \rfloor - 1$ by (3) of Observation 1, a contradiction. So (3) holds. □

Corollary 3. *For any graph G of order n and size m, if $\frac{2m}{n}$ is an integer, then $\lambda_k(G) \leq \frac{2m}{n} - 1$.*

Lemma 9. *Let $G \in \mathcal{G}(n,m)$. Then*

(1) $\lambda_k(G) + \lambda_k(\overline{G}) \leq M(n,m)$, *where*

$$M(n,m) = \begin{cases} n - \lceil \frac{k}{2} \rceil & \text{if } m \geq n-1, \\ & \quad \text{or } k \text{ is even and } m = 0, \\ & \quad \text{or } k \text{ is odd and } 0 \leq m \leq \frac{k-1}{2}; \\ n - \lceil \frac{k}{2} \rceil - 1 & \text{if } k \text{ is even and } 1 \leq m < n-1, \\ & \quad \text{or } k \text{ is odd and } \frac{k+1}{2} \leq m < n-1. \end{cases}$$

(2) $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq N(n,m)$, *where*

$$N(n,m) = \begin{cases} 0 & \text{if } 0 \leq m \leq n-2 \qquad\qquad , \\ (\frac{2m}{n} - 1)(n - 2 - \frac{2m}{n}) & \text{if } m \geq n-1 \text{ and } 2m \equiv 0 (mod\ n), \\ \lfloor \frac{2m}{n} \rfloor (n - 2 - \lfloor \frac{2m}{n} \rfloor) & \text{otherwise.} \qquad\qquad . \end{cases}$$

Moreover, these upper bounds are sharp.

Proof. From Theorem 2, (1) holds for $m \geq n-1$. We have given a graph class to show that the upper bound is sharp. From Proposition 4, $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$ for k even and $m = 0$, or k odd and $0 \leq m \leq \frac{k-1}{2}$. So for k even and $1 \leq m < n-1$, or k odd and $\frac{k+1}{2} \leq m < n-1$, $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil \frac{k}{2} \rceil - 1$.

To prove the sharpness of the bound for k odd and $\frac{k+1}{2} \leq m < n-1$, we consider the graph $G = K_{1,\frac{k+1}{2}} \cup (n - \frac{k+3}{2})K_1$. Now $\overline{G}$ is a graph obtained from the complete graph K_n by deleting all the edges of a star $K_{1,\frac{k+1}{2}}$. On one hand, by Lemma 4, $\lambda_k(\overline{G}) \leq n - \frac{k+1}{2} - 1$. On the other hand, by Lemma 4, we have $\lambda_k(\overline{G} + e) = n - \frac{k+1}{2}$ for any $e \notin E(\overline{G})$, which implies that $\lambda_k(\overline{G}) \geq n - \frac{k+1}{2} - 1$ (Note that $\lambda_k(H \setminus e) \geq \lambda_k(H) - 1$ for a connected graph H, where $e \in E(H)$). So $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) = n - \frac{k+1}{2} - 1$. By the same reason, for

k even and $1 \le m < n-1$ one can check that the graph $G = K_2 \cup (n-2)K_1$ satisfies that $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) \ge n - \frac{k}{2} - 1$.

(2) First, if $0 \le m \le n-2$, then $G \in \mathcal{G}(n,m)$ is disconnected. So $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$. Next if $\frac{2m}{n} = r$ is an integer, then $\frac{2e(\overline{G})}{n} = n-1-r$ is also an integer. From Corollary 3, we have $\lambda_k(G) \le r-1$ and $\lambda_k(\overline{G}) \le n-2-r$. So $\lambda_k(G) \cdot \lambda_k(\overline{G}) \le (r-1)(n-2-r) = (\frac{2m}{n}-1)(n-2-\frac{2m}{n})$. Finally, if $2m = nr + \ell$ where $1 \le \ell \le n-1$, then $\Delta(G) \ge r+1$. By (1) of Observation 1, $\lambda_k(\overline{G}) \le \delta(\overline{G}) = n-1-\Delta(G) \le n-2-r$. So $\lambda_k(G) \cdot \lambda_k(\overline{G}) \le r(n-2-r) = \lfloor \frac{2m}{n} \rfloor (n-2-\lfloor \frac{2m}{n} \rfloor)$. □

To show the sharpness of the upper bound for $m \ge n-1$ and $2m \equiv 0\ (mod\ n)$, we consider the following example.

Example 3. Let G be a cycle $C_n = w_1 w_2 \cdots w_n w_1 (n \ge 9)$. Since $\frac{2m}{n} = 2$ is an integer, $\lambda_3(G) = \frac{2m}{n} - 1 = 1$. It suffices to prove that $\lambda_3(\overline{G}) = n - 2 - \frac{2m}{n} = n-4$.

Choose $S = \{x, y, z\} \subseteq V(C_n) = V(G)$. We will show that $\lambda(S) \ge n-4$. If $d_{C_n}(x,y) = 1$ and $d_{C_n}(y,z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, y\}$ and $N_{C_n}(z) = \{y, z_2\}$, then the trees $T_i = xw_i \cup yw_i \cup zw_i$ together with $T_1 = xz \cup zx_1 \cup x_1 y$ form $n-4$ edge-disjoint S-trees (See Figure 5 (a)), namely, $\lambda(S) \ge n-4$, where $\{w_1, w_2, \cdots, w_{n-5}\} = V(G) \setminus \{x, y, z, x_1, z_2\}$.

If $d_{C_n}(x,y) = 2$ and $d_{C_n}(y,z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, y_1\}$ and $N_{C_n}(z) = \{y_1, z\}$ and $N_{C_n}(z) = \{y, z_2\}$, then the trees $T_i = xw_i \cup yw_i \cup zw_i$ together with $T_1 = xy \cup xz$ and $T_2 = z_2 x \cup z_2 y \cup z_2 y_1 \cup y_1 z$ form $n-4$ edge-disjoint S-trees (See Figure 5 (b)), namely, $\lambda(S) \ge n-4$, where $\{w_1, w_2, \cdots, w_{n-6}\} = V(G) \setminus \{x, y, z, x_1, y_1, z_2\}$.

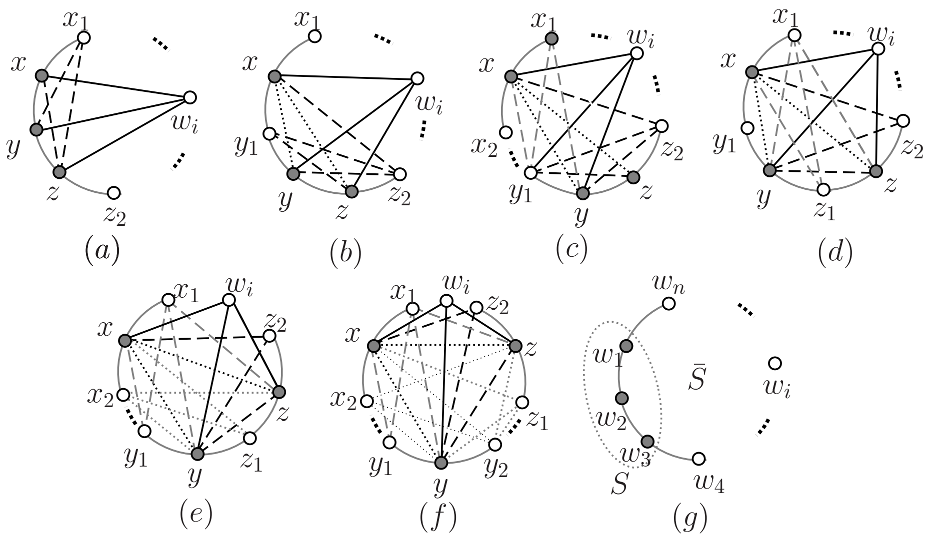


Figure 5. Graphs for Example 3.

If $d_{C_n}(x,y) \ge 3$ and $d_{C_n}(y,z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(z) = \{y_1, z\}$ and $N_{C_n}(z) = \{y, z_2\}$, then the trees $T_i = xw_i \cup yw_i \cup zw_i$ together with $T_1 = xy \cup xz$ and $T_2 = z_2 x \cup z_2 y \cup z_2 y_1 \cup y_1 z$ and $T_3 = xy_1 \cup y_1 x_1 \cup x_1 y \cup x_1 z$ form $n-4$ edge-disjoint S-trees (See Figure 5 (c)), namely, $\lambda(S) \ge n-4$, where $\{w_1, w_2, \cdots, w_{n-7}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, z_2\}$.

If $d_{C_n}(x,y) = 2$ and $d_{C_n}(y,z) = 2$, without loss of generality, let $N_{C_n}(x) = \{x_1, y_1\}$ and $N_{C_n}(z) = \{y_1, z_1\}$ and $N_{C_n}(z) = \{z_1, z_2\}$, then the trees $T_i = xw_i \cup yw_i \cup zw_i$ together with $T_1 = xz \cup xy$ and $T_2 = xz_2 \cup yz_2 \cup yz$ and $T_3 = x_1y \cup x_1z \cup x_1z_1 \cup xz_1$ form $n-4$ edge-disjoint S-trees (See Figure 5 (d)), namely, $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-7}\} = V(G) \setminus \{x, y, z, x_1, y_1, z_1, z_2\}$.

If $d_{C_n}(x,y) \geq 3$ and $d_{C_n}(y,z) = 2$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(z) = \{y_1, z_1\}$ and $N_{C_n}(z) = \{z_1, z_2\}$, then the trees $T_i = xw_i \cup yw_i \cup zw_i$ together with $T_1 = xz \cup xy$ and $T_2 = xz_2 \cup z_2y \cup yz$ and $T_3 = x_1y \cup x_1z \cup x_1y_1 \cup xy_1$ and $T_4 = x_2y \cup x_2z \cup x_2z_1 \cup z_1x$ form $n-4$ edge-disjoint S-trees (See Figure 5 (e)), namely, $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-8}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, y_2, z_2\}$.

Suppose that $d_{C_n}(x,y) \geq 3$ and $d_{C_n}(y,z) \geq 3$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(z) = \{y_1, y_2\}$ and $N_{C_n}(z) = \{z_1, z_2\}$. Then the trees $T_i = xw_i \cup yw_i \cup zw_i$ together with $T_1 = xz \cup xy$ and $T_2 = xz_2 \cup yz_2 \cup yz$ and $T_3 = xz_1 \cup yz_1 \cup y_2z_1 \cup y_2z$ and $T_4 = x_1y \cup x_1z \cup x_1y_1 \cup y_1x$ and $T_5 = x_2y \cup x_2z \cup x_2y_2 \cup y_2x$ form $n-4$ edge-disjoint S-trees (See Figure 5 (f)), namely, $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-9}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, y_2, z_1, z_2\}$.

From the arbitrariness of S, we know that $\lambda_3(\overline{G}) \geq n-4$ by definition. Now we show that $\lambda_3(\overline{G}) \leq n-4$ for $\overline{G} = \overline{C_n}$. Choose $S = \{w_1, w_2, w_3\} \subseteq V(G) = V(C_n)$. Then $w_1w_n \in E(C_n)$ and $w_3w_4 \in E(C_n)$. Thus $|E(\overline{G}[S])| = 1$ and $|E_{\overline{G}}[S, \bar{S}]| = 3(n-3) - 2$, which implies that $|E(\overline{G}[S]) \cup E_{\overline{G}}[S, \bar{S}]| = 3(n-3) - 1$ (See Figure 5 (g)). One can see that each tree connecting S in $\overline{G}$ uses at least 3 edges from $E(\overline{G}[S]) \cup E_{\overline{G}}[S, \bar{S}]$. Therefore $\lambda_3(\overline{G}) \leq \frac{3(n-3)-1}{3} = n-3-\frac{1}{3}$, which results in $\lambda_3(\overline{G}) \leq n-4$ since $\lambda_3(\overline{G})$ is an integer. So $\lambda_3(\overline{G}) = n-4$ and $\lambda_3(G) \cdot \lambda_3(\overline{G}) = \lambda_3(C_n) \cdot \lambda_3(\overline{C_n}) = 1 \cdot (n-4) = (\frac{2m}{n} - 1)(n - 2 - \frac{2m}{n})$. The upper bound is sharp.

For $m \geq n-1$ and $\frac{2m}{n} = r + \ell (1 \leq \ell \leq n-1)$, let $G = P_4$. Then $\lambda_3(G) = 1 = \lfloor \frac{6}{4} \rfloor = \lfloor \frac{2m}{n} \rfloor$ and $\lambda_3(\overline{G}) = \lambda_3(P_4) = 1 = 4-2-\lfloor \frac{6}{4} \rfloor = n-2-\lfloor \frac{2m}{n} \rfloor$. So $\lambda_3(G) \cdot \lambda_3(\overline{G}) = \lfloor \frac{2m}{n} \rfloor (n-2-\lfloor \frac{2m}{n} \rfloor)$.

Combining with Lemmas 8 and 9, we can obtain the following result.

Theorem 4. *Let $G \in \mathcal{G}(n,m)$. For $n \geq 6$, we have*

(1) $L(n,m) \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq M(n,m)$;

(2) $0 \leq \lambda_k(G) \cdot \lambda_k(\overline{G}) \leq N(n,m)$,

where $L(n,m), M(n,m), N(n,m)$ are defined in Lemmas 8 and 9.

Moreover, the upper and lower bounds are sharp.

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