Further result on acyclic chromatic index of planar graphs

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Abstract

An acyclic edge coloring of a graph G is a proper edge coloring such that every cycle is colored with at least three colors. The acyclic chromatic index $\chi'_a(G)$ of a graph G is the least number of colors in an acyclic edge coloring of G. It was conjectured that $\chi'_a(G) \leq \Delta(G) + 2$ for any simple graph G with maximum degree $\Delta(G)$. In this paper, we prove that every planar graph G admits an acyclic edge coloring with $\Delta(G) + 6$ colors.

Keywords: Acyclic edge coloring; Acyclic chromatic index; κ-deletion-minimal graph; κ-minimal graph; Acyclic edge coloring conjecture

1 Introduction

All graphs considered in this paper are finite, simple and undirected. An acyclic edge coloring of a graph G is a proper edge coloring such that every cycle is colored with at least three colors. The acyclic chromatic index $\chi'_a(G)$ of a graph G is the least number of colors in an acyclic edge coloring of G. It is obvious that $\chi'_a(G) \geq \chi'(G) \geq \Delta(G)$. Fiamčík [5] stated the following conjecture in 1978, which is well known as Acyclic Edge Coloring Conjecture, and Alon et al. [2] restated it in 2001.

Conjecture 1. For any graph G, $\chi'_a(G) \leq \Delta(G) + 2$.

Alon et al. [1] proved that $\chi'_a(G) \leq 64\Delta(G)$ for any graph G by using probabilistic method. Molloy and Reed [11] improved it to $\chi'_a(G) \leq 16\Delta(G)$. Recently, Ndreca et al. [12] improved the upper bound to $\lceil 9.62(\Delta(G)-1)\rceil$, and Esperet and Parreau [4] further improved it to $4\Delta(G) - 4$ by using the so-called entropy compression method. The best known general bound is $\lceil 3.74(\Delta(G) - 1)\rceil$ due to Giotis et al. [7]. Alon et al. [2] proved that there is a constant c such that $\chi'_a(G) \leq \Delta(G) + 2$ for a graph G whenever the girth is at least $c\Delta \log \Delta$.

Regarding general planar graph G, Fiedorowicz et al. [6] proved that $\chi'_a(G) \leq 2\Delta(G) + 29$; Hou et al. [10] proved that $\chi'_a(G) \leq \max\{2\Delta(G) - 2, \Delta(G) + 22\}$. Recently, Basavaraju et al. [3] showed that $\chi'_a(G) \leq \Delta(G) + 12$, and Guan et al. [8] improved it to $\chi'_a(G) \leq \Delta(G) + 10$, and Wang et al. [14] further improved it to $\chi'_a(G) \leq \Delta(G) + 7$.

In this paper, we improve the upper bound to $\Delta(G) + 6$ by the following theorem.

Theorem 1.1. If G is a planar graph, then $\chi'_a(G) \leq \Delta(G) + 6$.

2 Preliminary

Let $\mathbb{S}$ be a multiset and x be an element in $\mathbb{S}$. The *multiplicity* $\mathrm{mul}_{\mathbb{S}}(x)$ is the number of times x appears in $\mathbb{S}$. Let $\mathbb{S}$ and $\mathbb{T}$ be two multisets. The union of $\mathbb{S}$ and $\mathbb{T}$, denoted by $\mathbb{S} \uplus \mathbb{T}$, is a multiset with $\mathrm{mul}_{\mathbb{S}\uplus\mathbb{T}}(x) = \mathrm{mul}_{\mathbb{S}}(x) + \mathrm{mul}_{\mathbb{T}}(x)$. Throughout this paper, every coloring uses colors from $[\kappa] = \{1, 2, \dots, \kappa\}$.

We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices that are adjacent to v in G and

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$\deg_G(v)$ (or simple $\deg(v)$) to denote the degree of v in G. When G is a plane graph, we use $F(G)$ to denote its face set and $\deg_G(f)$ (or simple $\deg(f)$) to denote the degree of a face f in G. A k-, k^+-, k^--vertex (resp. face) is a vertex (resp. face) with degree k, at least k and at most k, respectively. A face $f = v_1v_2\dots v_k$ is a $(\deg(v_1), \deg(v_2), \dots, \deg(v_k))$-face.

A graph G with maximum degree at most κ is *κ-deletion-minimal* if $\chi'_a(G) > \kappa$ and $\chi'_a(H) \leq \kappa$ for every proper subgraph H of G. A graph property $\mathcal{P}$ is *deletion-closed* if $\mathcal{P}$ is closed under taking subgraphs. Analogously, we can define another type of minimal graphs by taking minors. A graph G with maximum degree at most κ is *κ-minimal* if $\chi'_a(G) > \kappa$ and $\chi'_a(H) \leq \kappa$ for every proper minor H with $\Delta(H) \leq \Delta(G)$. Obviously, every proper subgraph of a κ-minimal graph admits an acyclic edge coloring with at most κ colors, and then every κ-minimal graph is also a κ-deletion-minimal graph and all the properties of κ-deletion-minimal graphs are also true for κ-minimal graphs.

Let G be a graph and H be a subgraph of G. An acyclic edge coloring of H is a *partial acyclic edge coloring* of G. Let $\mathcal{U}_\phi(v)$ denote the set of colors which are assigned to the edges incident with v with respect to ϕ. Let $C_\phi(v) = [\kappa] \setminus \mathcal{U}_\phi(v)$ and $C_\phi(uv) = [\kappa] \setminus (\mathcal{U}_\phi(u) \cup \mathcal{U}_\phi(v))$. Let $\Upsilon_\phi(uv) = \mathcal{U}_\phi(v) \setminus \{\phi(uv)\}$ and $W_\phi(uv) = \{\, u_i \mid uu_i \in E(G) \text{ and } \phi(uu_i) \in \Upsilon_\phi(uv) \,\}$. Notice that $W_\phi(uv)$ may be not same with $W_\phi(vu)$. For simplicity, we will omit the subscripts if no confusion can arise.

An (α,β)-maximal dichromatic path with respect to ϕ is a maximal path whose edges are colored by α and β alternately. An (α,β,u,v)-critical path with respect to ϕ is an (α,β)-maximal dichromatic path which starts at u with color α and ends at v with color α. An (α,β,u,v)-alternating path with respect to ϕ is an (α,β)-dichromatic path starting at u with color α and ending at v with color β.

Let ϕ be a partial acyclic edge coloring of G. A color α is *candidate* for an edge e in G with respect to a partial edge coloring of G if none of the adjacent edges of e is colored with α. A candidate color α is *valid* for an edge e if assigning the color α to e does not result in any dichromatic cycle in G.

Fact 1 (Basavaraju et al. [3]). Given a partial acyclic edge coloring of G and two colors α,β, there exists at most one (α,β)-maximal dichromatic path containing a particular vertex v. □

Fact 2 (Basavaraju et al. [3]). Let G be a κ-deletion-minimal graph and uv be an edge of G. If ϕ is an acyclic edge coloring of $G-uv$, then no candidate color for uv is valid. Furthermore, if $\mathcal{U}(u)\cap\mathcal{U}(v) = \emptyset$, then $\deg(u)+\deg(v) = \kappa+2$; if $|\mathcal{U}(u) \cap \mathcal{U}(v)| = s$, then $\deg(u) + \deg(v) + \sum_{w\in W(uv)} \deg(w) \geq \kappa + 2s + 2$. □

We remind the readers that we will use these two facts frequently, so please keep these in mind and we will not refer it at every time.

3 Structural lemmas

Wang and Zhang [13] presented many structural results on κ-deletion-minimal graphs and κ-minimal graphs. In this section, we give more structural lemmas in order to prove our main result.

Lemma 1. If G is a κ-deletion-minimal graph, then G is 2-connected and $\delta(G) \geq 2$.

3.1 Local structure on the 2- or 3-vertices

Lemma 2 (Wang and Zhang [13]). Let G be a κ-minimal graph with $\kappa \geq \Delta(G) + 1$. If v_0 is a 2-vertex of G, then v_0 is contained in a triangle.

Lemma 3 (Wang and Zhang [13]). Let G be a κ-deletion-minimal graph. If v is adjacent to a 2-vertex v_0 and $N_G(v_0) = \{w, v\}$, then v is adjacent to at least $\kappa - \deg(w) + 1$ vertices with degree at least $\kappa - \deg(v) + 2$. Moreover,

- (A) if $\kappa \geq \deg(v) + 1$ and $wv \in E(G)$, then v is adjacent to at least $\kappa - \deg(w) + 2$ vertices with degree at least $\kappa - \deg(v) + 2$, and $\deg(v) \geq \kappa - \deg(w) + 3$;
- (B) if $\kappa \geq \Delta(G) + 2$ and v is adjacent to precisely $\kappa - \Delta(G) + 1$ vertices with degree at least $\kappa - \Delta(G) + 2$, then v is adjacent to at most $\deg(v) + \Delta(G) - \kappa - 3$ vertices with degree two and $\deg(v) \geq \kappa - \Delta(G) + 4$.

Lemma 4 (Wang and Zhang [13]). Let G be a κ-deletion-minimal graph with $\kappa \geq \Delta(G) + 2$. If v_0 is a 2-vertex, then every neighbor of v_0 has degree at least $\kappa - \Delta(G) + 4$.

Lemma 5 (Hou et al. [9]). Let G be a κ-deletion-minimal graph with $\kappa \geq \Delta(G) + 2$. If v is a 3-vertex, then every neighbor of v is a $(\kappa - \Delta(G) + 2)^{+}$-vertex.

Lemma 6 (Wang and Zhang [13]). Let G be a κ-minimal graph with $\kappa \geq \Delta(G) + 2$. If v is a 3-vertex in G, then every neighbor of v is a $(\kappa - \Delta(G) + 3)^{+}$-vertex.

Lemma 7 (Wang and Zhang [13]). Let G be a κ-deletion-minimal graph with $\kappa \geq \Delta(G) + 2$, and let w_0 be a 3-vertex with $N_G(w_0) = \{w, w_1, w_2\}$, and $\deg(w) = \kappa - \Delta(G) + 3$. If $ww_1, ww_2 \in E(G)$, then $\deg(w_1) = \deg(w_2) = \Delta(G)$ and w is adjacent to precisely one vertex (namely w_0) with degree less than $\Delta(G) - 1$.

Lemma 8. Let G be a κ-deletion-minimal graph with maximum degree Δ, and let w_0 be a 3-vertex with $N_G(w_0) = \{w, w_1, w_2\}$. If $\deg_G(w) = \kappa - \Delta + 4 = \ell$ with $8 \leq \ell \leq 10$ and $N_G(w) = \{w_0, w_1, w_2, \dots, w_{\ell-1}\}$, then there exists no 4-set $X^* \subseteq \{w_1, w_2, \dots, w_{\ell-1}\}$ satisfying the following four conditions: (1) every vertex in X^* is a 5^--vertex; (2) the degree-sum of vertices in X^* is at most $\kappa - \Delta + 9$; (3) the degree-sum of any two vertices in X^* is at most Δ; (4) X^* has at least two 4^--vertices.

Proof. Suppose to the contrary that there exists a 4-set X^* satisfying all the four conditions. Let X be the subscripts of vertices in X^*. Since G is κ-deletion-minimal, the graph $G - ww_0$ has an acyclic edge coloring ϕ with $\phi(ww_i) = i$ for $i \in \{1, \dots, \ell - 1\}$. The fact that $\deg_G(w) + \deg_G(w_0) \leq \Delta + 3 < \kappa + 2$ and Fact 2 imply that $\mathcal{U}(w) \cap \mathcal{U}(w_0) \neq \emptyset$.

Case 1. $|\mathcal{U}(w) \cap \mathcal{U}(w_0)| = 1$.

It follows that $|C(ww_0)| = \Delta - 4$.

Subcase 1.1. The common color is on ww_1 or ww_2.

Without loss of generality, we may assume that w_0w_1 is colored with ℓ and w_0w_2 is colored with 1. Note that there exists a $(1, \alpha, w, w_0)$-critical path for every $\alpha \in \{\ell+1, \dots, \kappa\}$, so we have that $\{\ell+1, \dots, \kappa\} \subseteq \mathcal{U}(w_1) \cap \mathcal{U}(w_2)$. Notice that the set $\{1, \dots, \ell\} \setminus (\mathcal{U}(w_1) \cup \mathcal{U}(w_2))$ is nonempty. Now, reassigning ℓ to ww_0 and a color in $\{1, \dots, \ell\} \setminus (\mathcal{U}(w_1) \cup \mathcal{U}(w_2))$ to w_0w_1 results in an acyclic edge coloring of G, a contradiction.

Subcase 1.2. The common color is not on ww_1 and ww_2.

Without loss of generality, we may assume that w_0w_1 is colored with ℓ and w_0w_2 is colored with 3. There exists a $(3, \alpha, w, w_0)$-critical path for $\alpha \in \{\ell + 1, \dots, \kappa\}$. It follows that $\{\ell + 1, \dots, \kappa\} \subseteq \Upsilon(ww_3) \cap \Upsilon(w_0w_2)$ and $\deg_G(w_3) \geq \Delta - 3 \geq 5$.

If $1 \notin \mathcal{U}(w_2)$, then reassigning 1 to w_0w_2 will take us back to Case 1.1. Hence, we have that $1 \in \Upsilon(w_0w_2)$ and $\deg_G(w_2) \geq \Delta - 1 \geq 7$. By Lemma 5, we have that $\deg_G(w_1) \geq \kappa - \Delta + 2 \geq 6$.

Note that w_1, w_2 and w_3 are 5^+-vertices, there exists a 4^--vertex w_x with $x \in X \setminus \mathcal{U}(w_2)$. If $\ell \notin \mathcal{U}(w_2)$, then reassigning the color x to w_0w_2 results in a new acyclic edge coloring σ of $G - ww_0$, and then $C_\sigma(ww_0) = \{\ell+1, \dots, \kappa\} \subseteq \Upsilon(ww_x)$ and $\deg_G(w_x) \geq \Delta - 3 \geq 5$, which contradicts that w_x is a 4^--vertex. Hence, $\Upsilon(w_0w_2) = \{1, 2\} \cup \{\ell, \dots, \kappa\}$ and $\deg_G(w_2) = \Delta$, which implies that $X \cap \Upsilon(w_0w_2) = \emptyset$.

Claim 1. There exists a $(3, \ell, w, w_2)$-alternating path.

Proof. Suppose to the contrary that there exists no $(3, \ell, w, w_2)$-alternating path. We can revise ϕ by assigning ℓ to ww_0 and erase the color from w_0w_1, and obtain an acyclic edge coloring of $G - w_0w_1$. If some color $\alpha \in \{\ell+1, \dots, \kappa\}$ is absent in $\mathcal{U}_\phi(w_1)$, then we can further assign α to w_0w_1, since there exists a $(3, \alpha, w, w_0)$-critical path with respect to ϕ. If some color $\alpha \in \{4, \dots, \ell - 1\}$ is absent in $\mathcal{U}_\phi(w_1)$, then we can further assign α to w_0w_1. Hence, $\mathcal{U}_\phi(w_1) \supseteq \{1\} \cup \{4, \dots, \kappa\}$ and $\deg_G(w_1) \geq \kappa - 2 > \Delta(G)$, a contradiction. □

Therefore, $\{\ell, \dots, \kappa\} \subseteq \Upsilon(ww_3)$ and $\deg_G(w_3) \geq \Delta - 2 \geq 6$, which implies that $X \cap \mathcal{U}(w_2) = \emptyset$.

There exists a (ℓ, m, w_0, w_2)-critical path for every $m \in X$; otherwise, reassigning m to w_0w_2 results in another new acyclic edge coloring ϕ_m of $G - ww_0$, by the above arguments, $\{\ell, \dots, \kappa\} \subseteq \Upsilon(ww_m)$ and $\deg_G(w_m) \geq \Delta - 2 \geq 6$, a contradiction. Thus, we have that $X \subseteq \Upsilon(w_0w_1)$. By symmetry, we may assume that $\{4, 5, 6, 7\} = X \subseteq \Upsilon(w_0w_1)$.

Suppose that $\{3, 8, 9, \dots, \ell - 1\} \nsubseteq \mathcal{U}(w_1)$, say λ is a such color. There exists a $(\lambda, \alpha, w, w_2)$-alternating path for $\ell + 1 \leq \alpha \leq \kappa$; otherwise, reassigning λ to w_0w_2 (if $\lambda = 3$ there is no change to w_0w_2) and α to ww_0 results in an acyclic edge coloring of G. Similar to Claim 1, there exists a (λ, ℓ, w, w_2)-alternating path. Reassigning λ to w_0w_1 and 4 to w_0w_2 results in a new acyclic edge coloring φ of $G - ww_0$. Since there is no $(\lambda, \alpha, w, w_0)$-critical path with respect to

φ, thus there exists a $(4, \alpha, w_0, w)$-critical path with respect to φ for $\alpha \in \{\ell, \dots, \kappa\}$, and then $\{\ell, \dots, \kappa\} \subseteq \Upsilon(ww_4)$, which contradicts the fact that w_4 is a 5^--vertex. Hence, we have that $\{1\} \cup \{3, 4, \dots, \ell\} \subseteq \mathcal{U}(w_1)$.

Let φ_m be obtained from ϕ by reassigning m to ww_0 and erasing the color on ww_m, where $m \in \{4, 5, 6, 7\}$. Note that φ_m is an acyclic edge coloring of $G - ww_m$ for $m \in \{4, 5, 6, 7\}$. By Fact 2, we have that $|\Upsilon(ww_m) \cap \{1, 2, \dots, \ell - 1\}| \geq 1$ for $m \in \{4, 5, 6, 7\}$.

Let α be an arbitrary color in $\{\ell, \dots, \kappa\} \setminus (\Upsilon(w_0w_1) \cup \Upsilon(ww_4) \cup \Upsilon(ww_5) \cup \Upsilon(ww_6) \cup \Upsilon(ww_7))$. Since there exists neither $(1, \alpha, w, w_x)$-critical path nor $(3, \alpha, w, w_x)$-critical path (with respect to φ_x) for every $x \in X$, thus there exists a $(\lambda_x, \alpha, w, w_x)$-critical path (with respect to φ_x), where $\lambda_x \in \{2, 8, 9, \dots, \ell - 1\}$. Moreover, there exists $(\lambda, \alpha, w, w_{x_1})$- and $(\lambda, \alpha, w, w_{x_2})$-critical path for some $\lambda \in \{2, 8, 9, \dots, \ell - 1\}$ since $|X| > |\{2, 8, 9, \dots, \ell - 1\}|$, but this contradicts Fact 1.

So we may assume that $\alpha \in \Upsilon(ww_4) \cup \Upsilon(ww_5) \cup \Upsilon(ww_6) \cup \Upsilon(ww_7)$ for every $\alpha \in \{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1)$.

$$
\begin{aligned}
\kappa - \Delta + 9 \geq & \deg_G(w_4) + \deg_G(w_5) + \deg_G(w_6) + \deg_G(w_7) \\
\geq & |\{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1)| + 4 + \sum_{t=4}^{7} |\Upsilon(ww_t) \cap \{1, \dots, \ell - 1\}| \\
\geq & (\kappa - \Delta) + 4 + (1 + 1 + 1 + 1) \\
= & \kappa - \Delta + 8.
\end{aligned}
$$

By symmetry, we may assume that $|\Upsilon(ww_4) \cap \{1, \dots, \ell-1\}| = |\Upsilon(ww_5) \cap \{1, \dots, \ell-1\}| = |\Upsilon(ww_6) \cap \{1, \dots, \ell-1\}| = 1$. Let $\Upsilon(ww_4) \cap \{1, \dots, \ell-1\} = \{\mu_1\}$, $\Upsilon(ww_5) \cap \{1, \dots, \ell-1\} = \{\mu_2\}$ and $\Upsilon(ww_6) \cap \{1, \dots, \ell-1\} = \{\mu_3\}$. If $\mu_1 = \mu_2 = \mu$, then there exists a (μ, α, w, w_4)- and (μ, α, w, w_5)-critical path, where $\alpha \in \{\ell, \dots, \kappa\} \setminus (\Upsilon(ww_4) \cup \Upsilon(ww_5))$, which contradicts Fact 1. Thus μ_1, μ_2, μ_3 are distinct.

If $\mu_1 \in \{4, 5, 6, 7\}$, then every color $\alpha \in \{\ell, \dots, \kappa\} \setminus (\Upsilon(ww_4) \cup \Upsilon(ww_{\mu_1}))$ is valid for ww_4 with respect to φ_4; note that $\{\ell, \dots, \kappa\} \setminus (\Upsilon(ww_4) \cup \Upsilon(ww_{\mu_1}))$ is a nonempty set. By symmetry, we may assume that $\{\mu_1, \mu_2, \mu_3\} \cap \{4, 5, 6, 7\} = \emptyset$.

Since μ_1, μ_2, μ_3 are distinct, we may assume that $\mu_1 \neq 2$. If $2 \notin \Upsilon(w_0w_1)$, then reassigning 2 to w_0w_1 and 4 to w_0w_2 results in a new acyclic edge coloring φ^* of $G - ww_0$. For every color $\beta \in \{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1)$, there exists no $(2, \beta, w, w_0)$-critical path with respect to φ^*, thus there exists a $(4, \beta, w, w_0)$-critical path with respect to φ^*, and then $\{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1) \subseteq \Upsilon(ww_4)$ and $\deg_G(w_4) \geq |\{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1)| + 2 \geq 6$, which contradicts the degree of w_4.

Hence, we have that $\{1, \dots, \ell - 1\} \subseteq \Upsilon(w_0w_1)$ and $|\{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1)| \geq \kappa - \Delta + 1$. By similar arguments as above, we can prove that $\Upsilon(ww_7) \cap \{1, \dots, \ell - 1\} = \{\mu_4\}$ and $\mu_1, \mu_2, \mu_3, \mu_4$ are distinct. Moreover, we can also conclude that $\{\mu_1, \mu_2, \mu_3, \mu_4\} \cap \{4, 5, 6, 7\} = \emptyset$.

Suppose that $\mu_1 = 3$. Since there exists no $(3, \alpha, w, w_4)$-critical path with respect to φ_4, where $\alpha \in \{\ell + 1, \dots, \kappa\}$, thus $\{\ell + 1, \dots, \kappa\} \subseteq \Upsilon(ww_4)$, a contradiction. So, by symmetry, we may assume that $\{\mu_1, \mu_2, \mu_3, \mu_4\} = \{1, 2, 8, 9\}$.

By symmetry, we assume that $\mu_1 = 1$. Note that there exists no $(1, \alpha, w, w_4)$-critical path (with respect to φ_4) for every $\alpha \in \{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1)$, thus $\{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1) \subseteq \Upsilon(ww_4)$; otherwise, reassigning 4 to ww_0 and a color α to ww_4 results in an acyclic edge coloring. Now, we have that $\deg_G(w_4) \geq 2 + |\{\ell, \dots, \kappa\} \setminus \Upsilon(w_0w_1)| \geq 6$, a contradiction.

Case 2. $\mathcal{U}(w) \cap \mathcal{U}(w_0) = \{\lambda_1, \lambda_2\}$, $\phi(w_0w_1) = \lambda_1$ and $\phi(w_0w_2) = \lambda_2$.

If follows that $|C(ww_0)| = \Delta - 3$. First of all, we show the following claim:

$(*)$ $C(ww_0) = \{\ell, \dots, \kappa\} \subseteq \mathcal{U}(w_1) \cap \mathcal{U}(w_2)$.

By contradiction and symmetry, assume that there exists a color ζ in $\{\ell, \dots, \kappa\} \setminus \mathcal{U}(w_1)$. Clearly, there exists a $(\lambda_2, \zeta, w_0, w)$-critical path, and then there exists no $(\lambda_2, \zeta, w_0, w_1)$-critical path. Now, reassigning ζ to w_0w_1 will take us back to Case 1. Hence, we have $\{\ell, \dots, \kappa\} \subseteq \mathcal{U}(w_1)$; similarly, we have $\{\ell, \dots, \kappa\} \subseteq \mathcal{U}(w_2)$. This completes the proof of $(*)$.

Note that w_1 and w_2 have degree at least $\Delta - 1 \geq 7$, this implies that $\{1, 2\} \cap X = \emptyset$ and $|X \cap \Upsilon(w_0w_1)| \leq 1$ and $|X \cap \Upsilon(w_0w_2)| \leq 1$. Let $\{p, q\} \subseteq X \setminus (\Upsilon(w_0w_1) \cup \Upsilon(w_0w_2))$. Reassigning p to w_0w_1 and q to w_0w_2 results in a new acyclic edge coloring ψ of $G - ww_0$. Hence, we have that $C_\psi(ww_0) \subseteq \Upsilon(ww_p) \cup \Upsilon(ww_q)$, and then $\deg_G(w_p) + \deg_G(w_q) \geq (\Delta - 3) + 2 + 2 \geq \Delta + 1$, which is a contradiction. □

3.2 Local structure on the 4-vertices

Lemma 9. Let G be a κ-deletion-minimal graph with maximum degree Δ and $\kappa \geq \Delta + 2$, and let w_0 be a 4-vertex with $N_G(w_0) = \{w, v_1, v_2, v_3\}$.

(a) If $\deg_G(w) \leq \kappa - \Delta$, then
$$\sum_{x \in N_G(w_0)} \deg_G(x) \geq 2\kappa - \deg_G(w_0) + 8 = 2\kappa + 4. \tag{1}$$

(b) If $\deg_G(w) \leq \kappa - \Delta + 1$ and ww_0 is contained in two triangles ww_1w_0 and ww_2w_0, then
$$\sum_{x \in N_G(w_0)} \deg_G(x) \geq 2\kappa - \deg_G(w_0) + 9 = 2\kappa + 5. \tag{2}$$

Furthermore, if the equality holds in (2), then all the other neighbors of w are 6^+-vertices.

Proof. We may assume that

($*$) The graph $G - ww_0$ admits an acyclic edge coloring ϕ such that the number of common colors at w and w_0 is minimum.

Here, (a) and (b) will be proved together, so we may assume that $\deg_G(w) \leq \kappa - \Delta + 1$. Since $\deg_G(w) + \deg_G(w_0) \leq \kappa - \Delta + 5 < \kappa + 2$, we have that $|\Upsilon(ww_0) \cap \Upsilon(w_0w)| = m \geq 1$. It follows that $|C(ww_0)| = \kappa - (\deg_G(w) + \deg_G(w_0) - m - 2) \geq \Delta - 2$. Without loss of generality, let $N_G(w) = \{w_0, w_1, w_2, \dots\}$ and $\phi(ww_i) = i$ for $1 \leq i \leq \deg_G(w) - 1$. Let $\mathbb{S} = \Upsilon(w_0v_1) \uplus \Upsilon(w_0v_2) \uplus \Upsilon(w_0v_3)$.

Claim 1. For every color θ in $C(ww_0)$, there exists a $(\lambda, \theta, w_0, w)$-critical path for some $\lambda \in \Upsilon(ww_0) \cap \Upsilon(w_0w)$. Consequently, every color in $C(ww_0)$ appears in $\mathbb{S}$.

Case 1. $\mathcal{U}(w) \cap \mathcal{U}(w_0) = \{\lambda\}$.

It follows that $|C(ww_0)| = \kappa - (\deg_G(w) + \deg_G(w_0) - 3)$.

(a) Suppose that $\deg_G(w) + \deg_G(w_0) \leq \kappa - \Delta + 4$. It follows that $|C(ww_0)| \geq \Delta - 1$. Without loss of generality, let $\phi(w_0v_1) = 1, \phi(w_0v_2) = \kappa - \Delta$ and $\phi(w_0v_3) = \kappa - \Delta + 1$. By Claim 1, there exists a $(1, \theta, w_0, w)$-critical path for every θ in $C(ww_0)$. Hence, we have that $\deg_G(w) = \kappa - \Delta$ and $\deg_G(v_1) = \deg_G(w_1) = \Delta$ and $\Upsilon(w_0v_1) = \Upsilon(ww_1) = \{\kappa - \Delta + 2, \dots, \kappa\}$. Notice that $\deg_G(w) = \kappa - \Delta \geq 3$ results from Lemma 4. Reassigning $\kappa - \Delta$, 1 and 2 to ww_1, ww_0 and w_0v_1 respectively, and we obtain an acyclic edge coloring of G, a contradiction.

(b) Suppose that $\deg_G(w) + \deg_G(w_0) = \kappa - \Delta + 5$ and ww_0 is contained in two triangles ww_1w_0 and ww_2w_0 ($w_1 = v_1$ and $w_2 = v_2$).

Subcase 1.1. The common color λ does not appear on w_0v_3, but it appears on ww_1 or ww_2.

By symmetry, assume that $\phi(w_0w_1) = 2$, $\phi(w_0v_2) = \kappa - \Delta + 1$, $\phi(w_0v_3) = \kappa - \Delta + 2$. By Claim 1, we have that $\{\kappa - \Delta + 3, \dots, \kappa\} \subseteq \Upsilon(w_0w_1) \cap \Upsilon(ww_2)$ and $\deg_G(w_1) = \deg_G(w_2) = \Delta$. Now, reassigning $\kappa - \Delta + 1$ to w_0w and reassigning 3 to w_0w_2 results in an acyclic edge coloring of G, a contradiction.

Subcase 1.2. The common color λ does not appear on w_0v_3 and it does not appear on ww_1 or ww_2 either.

By symmetry, assume that $\phi(w_0w_1) = 3$, $\phi(w_0w_2) = \kappa - \Delta + 1$, $\phi(w_0v_3) = \kappa - \Delta + 2$. By Claim 1, we have that $\{\kappa - \Delta + 3, \dots, \kappa\} \subseteq \Upsilon(w_0w_1) \cap \Upsilon(ww_3)$, $\deg_G(w_1) = \Delta$ and $\deg_G(w_3) \geq \Delta - 1$. Reassigning 2 to w_0w_1 will take us back to Subcase 1.1.

Subcase 1.3. The common color λ appears on w_0v_3 and it also appears on ww_1 or ww_2.

By symmetry, assume that $\phi(w_0w_1) = \kappa - \Delta + 1$, $\phi(w_0w_2) = \kappa - \Delta + 2$, $\phi(w_0v_3) = 2$. By Claim 1, we have that $\{\kappa - \Delta + 3, \dots, \kappa\} \subseteq \Upsilon(ww_2) \cap \Upsilon(w_0v_3)$, $\deg_G(w_2) = \Delta$ and $\deg_G(v_3) \geq \Delta - 1$. Now, reassigning $\kappa - \Delta + 1$ to ww_2 will take us back to Subcase 1.1.

Subcase 1.4. The common color λ appears on w_0v_3, but it does not appear on ww_1 or ww_2.

By symmetry, assume that $\phi(w_0w_1) = \kappa - \Delta + 1$, $\phi(w_0w_2) = \kappa - \Delta + 2$, $\phi(w_0v_3) = 3$. By Claim 1, we have that $\{\kappa - \Delta + 3, \dots, \kappa\} \subseteq \Upsilon(ww_3) \cap \Upsilon(w_0v_3)$, $\deg_G(w_3) \geq \Delta - 1$ and $\deg_G(v_3) \geq \Delta - 1$. If $\{2, \kappa - \Delta + 1\} \cap \Upsilon(w_0v_3) = \emptyset$, then reassigning 2 to w_0v_3 will take us back to Subcase 1.3. So we may assume that $\{2, \kappa - \Delta + 1\} \cap \Upsilon(w_0v_3) \neq \emptyset$. But we can still reassign 1 to w_0v_3 and go back to Subcase 1.3.

Case 2. $\mathcal{U}(w) \cap \mathcal{U}(w_0) = \{\lambda_1, \dots, \lambda_m\}$ and $m \geq 2$.

Let $\mathcal{A}(v_1) = C(ww_0) \setminus \Upsilon(w_0v_1) = \{\alpha_1, \alpha_2, \dots\}$, $\mathcal{A}(v_2) = C(ww_0) \setminus \Upsilon(w_0v_2) = \{\beta_1, \beta_2, \dots\}$ and $\mathcal{A}(v_3) = C(ww_0) \setminus \Upsilon(w_0v_3)$.

Claim 2. $\mathcal{A}(v_1), \mathcal{A}(v_2), \mathcal{A}(v_3) \neq \emptyset$.

Proof. Suppose to the contrary that $\mathcal{A}(v_*) = \emptyset$. It follows that $\Delta - 1 \geq |\Upsilon(w_0v_*)| \geq |C(ww_0)| = \kappa - (\deg_G(w) + \deg_G(w_0) - m - 2) \geq \kappa - (\kappa - \Delta + 5 - 2 - 2) = \Delta - 1$, thus $\deg_G(w) + \deg_G(w_0) = \kappa - \Delta + 5$, $m = 2$ and $\Upsilon(w_0v_*) = C(ww_0)$ with $|\Upsilon(w_0v_*)| = \Delta - 1$. This implies that the graph G satisfies the condition (b) with $v_* = v_3$ (assume that $w_1 = v_1$ and $w_2 = v_2$). We may assume that $\mathcal{U}(w_0) = \{\lambda_1, \lambda_2, \kappa - \Delta + 1\}$.

If the color on w_0w_1 is λ_1 and the color on w_0w_2 is λ_2, then reassigning α_1, β_1 and λ_2 to ww_0, w_0w_2 and w_0v_3, respectively, yields an acyclic edge coloring of G.

But if the color on w_0w_1 is $\kappa - \Delta + 1$ and the color on w_0w_2 is λ_2, then reassigning 2 to w_0v_3 and β_1 to ww_0 results in an acyclic edge coloring of G. □

Claim 3. Every color in $C(ww_0)$ appears at least twice in $\mathbb{S}$.

Proof. Suppose that there exists a color α in $C(ww_0)$ appearing only once in $\mathbb{S}$, say $\alpha \in \Upsilon(w_0v_1)$. Without loss of generality, assume that $\phi(w_0v_1) = \lambda_1$ and $\phi(w_0v_2) = \lambda_2$. By Claim 1, there exists a $(\lambda_1, \alpha, w_0, w)$-critical path. Reassigning α to w_0v_2 results in a new acyclic edge coloring ϕ^* of $G - ww_0$ with $|\mathcal{U}_{\phi^*}(w) \cap \mathcal{U}_{\phi^*}(w_0)| < |\mathcal{U}(w) \cap \mathcal{U}(w_0)|$, which contradicts the assumption $(*)$. □

Let $X = \{\, \alpha \mid \alpha \in C(ww_0) \text{ and } \mathrm{mul}_{\mathbb{S}}(\alpha) = 3 \,\}$.

$$
\begin{aligned}
&\sum_{x \in N_G(w_0)} \deg_G(x) \\
&= \deg_G(w_0) + \deg_G(w) - 1 + \sum_{\alpha \in [\kappa]} \mathrm{mul}_{\mathbb{S}}(\alpha) \\
&= \deg_G(w_0) + \deg_G(w) - 1 + \sum_{\alpha \in C(ww_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + \sum_{\alpha \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) \\
&= \deg_G(w_0) + \deg_G(w) - 1 + 2|C(ww_0)| + |X| + \sum_{\alpha \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) \\
&= \deg_G(w_0) + \deg_G(w) - 1 + 2(\kappa - (\deg_G(w) + \deg_G(w_0) - 2 - m)) + |X| + \sum_{\alpha \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) \\
&= 2\kappa - \deg_G(w_0) - \deg_G(w) + 2m + 3 + |X| + \sum_{\alpha \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha)
\end{aligned}
$$

It is sufficient to prove that

$$
\sum_{\alpha \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + |X| \geq \begin{cases} \deg_G(w) - 2m + 5, & \text{if } \deg_G(w) \leq \kappa - \Delta; \quad (3) \\ \deg_G(w) - 2m + 6, & \text{if } \deg_G(w) \leq \kappa - \Delta + 1 \text{ and } ww_0 \text{ is contained in two triangles.} \quad (4) \end{cases}
$$

Subcase 2.1. $\mathcal{U}(w) \cap \mathcal{U}(w_0) = \{\lambda_1, \lambda_2\}$.

Claim 4. Every color in $\mathcal{U}(w)$ is in $\mathbb{S}$.

Proof. Assume that w_0v_1 is colored with λ_1 and w_0v_2 is colored with λ_2. Notice that $C(ww_0) \subseteq \Upsilon(w_0v_1) \cup \Upsilon(w_0v_2)$ and $\mathcal{A}(v_1) \cap \mathcal{A}(v_2) = \emptyset$. By Claim 2, we have that $\mathcal{A}(v_1), \mathcal{A}(v_2), \mathcal{A}(v_3) \neq \emptyset$. If $\lambda_1 \notin \mathbb{S}$, then reassigning β_1, α_1 and λ_1 to w_0w, w_0v_1 and w_0v_3 respectively, results in an acyclic edge coloring of G, a contradiction. Thus, we have that $\lambda_1 \in \mathbb{S}$. Similarly, we can prove that $\lambda_2 \in \mathbb{S}$. Let τ be an arbitrary color in $\mathcal{U}(w) \setminus (\mathbb{S} \cup \{\lambda_1, \lambda_2\})$. Let σ be obtained from ϕ by reassigning τ to w_0v_1. It is obvious that σ is an acyclic edge coloring of $G - ww_0$. So we can obtain a similar contradiction by replacing ϕ with σ. □

Claim 5. The color in $\mathcal{U}(w_0) \setminus \{\lambda_1, \lambda_2\}$ appears at least twice in $\mathbb{S}$.

Proof. Suppose that λ_1, λ_2 and λ^* are on the edges w_0v_1, w_0v_2 and w_0v_3, respectively. There exists a $(\lambda^*, \alpha_1, w_0, v_1)$-critical path; otherwise, reassigning α_1 to w_0v_1 will take us back to Case 1. Hence, we have $\lambda^* \in \Upsilon(w_0v_1)$. Similarly, there exists a $(\lambda^*, \beta_1, w_0, v_2)$-critical path and $\lambda^* \in \Upsilon(w_0v_2)$. Therefore, the color λ^* appears exactly twice in $\mathbb{S}$. □

Now, we have

$$\sum_{\alpha \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + |X| \geq |\mathcal{U}(w)| + 2 + |X| = \deg_G(w) + 1 + |X|.$$

So conclusion (a) holds. Now, suppose that $\deg_G(w) + \deg_G(w_0) \leq \kappa - \Delta + 5$ and ww_0 is contained in two triangles ww_0w_1 and ww_0w_2 ($w_1 = v_1$ and $w_2 = v_2$).

Subcase 2.1.1. The two common colors λ_1 and λ_2 are on w_1w and w_1w_0.

There exists a $(\lambda_1, \alpha, w_0, w)$- or $(\lambda_2, \alpha, w_0, w)$-critical path for $\alpha \in C(ww_0)$. Hence, we have that $C(ww_0) \subseteq \mathcal{U}(w_1)$, and thus $\deg_G(w_1) \geq |C(ww_0)| + |\{\lambda_1, \lambda_2\}| \geq \Delta + 1$, a contradiction.

Subcase 2.1.2. The two common colors λ_1 and λ_2 are on w_2w and w_2w_0.

This is similar with Subcase 2.1.1.

Subcase 2.1.3. $\{\lambda_1, \lambda_2\} \cap \{1, 2\} = \{\lambda_1\}$ and λ_1 appears on w_0w_1 or w_0w_2.

Without loss of generality, assume that $\phi(w_0w_1) = \kappa - \Delta + 1$, $\phi(w_0w_2) = 1$, $\phi(w_0v_3) = 3$. If $2 \notin \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$, then reassigning 2 to w_0v_3 will take us back to Subcase 2.1.2. Hence, $2 \in \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$ and 2 appears at least twice in $\mathbb{S}$. Therefore, we have

$$\sum_{\alpha \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + |X| \geq |\mathcal{U}(w)| + 2 + |X| + |\{2\}| \geq \deg_G(w) + 2.$$

Suppose that

$$\sum_{x \in N_G(w_0)} \deg_G(x) = 2\kappa - \deg_G(w_0) + 9.$$

It follows that

$$\sum_{\alpha \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + |X| = |\mathcal{U}(w)| + 2 + |X| + |\{2\}| = \deg_G(w) + 2,$$

and every color in $\mathcal{U}(w) \setminus \{2\}$ appears only once in $\mathbb{S}$.

There exists a $(3, \kappa - \Delta + 1, w_0, w)$-critical path, otherwise, reassigning $\kappa - \Delta + 1$ to w_0w and α_1 to w_0w_1 results in an acyclic edge coloring of G, a contradiction. By Claim 5, we have that $\kappa - \Delta + 1 \in \Upsilon(w_0w_2) \cap \Upsilon(w_0v_3)$. And by Claim 4, we have that $3 \in \Upsilon(w_0w_1) \cup \Upsilon(w_0w_2)$. Since $|C(ww_0)| \geq \Delta - 1$ and $\{1, 2, 3, \kappa - \Delta + 1\} \subseteq \Upsilon(w_0w_1) \cup \Upsilon(w_0w_2)$, this implies that $|\mathcal{A}(w_1)| + |\mathcal{A}(w_2)| \geq 4$. There exists no $(1, \alpha, w, w_0)$-critical path for every $\alpha \in \mathcal{A}(w_1) \cup \mathcal{A}(w_2)$, thus there exists a $(3, \alpha, w, w_0)$-critical path, and then $\mathcal{A}(w_1) \cup \mathcal{A}(w_2) \subseteq \Upsilon(ww_3)$. Hence, $\deg_G(w_3) \geq |\mathcal{A}(w_1)| + |\mathcal{A}(w_2)| + |\{3, \kappa - \Delta + 1\}| \geq 6$.

Suppose that $4 \notin \Upsilon(w_0v_3)$ and there exists no $(\kappa - \Delta + 1, 4, w_0, v_3)$-critical path. Reassigning 4 to w_0v_3 results in a new acyclic edge coloring ϱ_1 of $G - ww_0$. Similarly, we can prove $\deg_G(w_4) \geq 6$ by replacing ϕ with ϱ_1.

Suppose that $4 \in \Upsilon(w_0v_3)$. This implies that $\{1, 2, 4, \kappa - \Delta + 1\} \subseteq \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$ and $|\mathcal{A}(w_1)| + |\mathcal{A}(v_3)| \geq 4$. Reassigning 4 to w_0w_2 and reassigning 1 to w_0v_3 results in another acyclic edge coloring π of $G - ww_0$. Hence, there exists a $(4, \alpha, w_0, w)$-critical path with respect to π for $\alpha \in \mathcal{A}(w_1) \cup \mathcal{A}(v_3)$, and then $\mathcal{A}(w_1) \cup \mathcal{A}(v_3) \subseteq \Upsilon(ww_4)$. Similarly as above, there exists a $(4, \kappa - \Delta + 1, w_0, w)$-critical path with respect to π. Hence, $\deg_G(w_4) \geq |\mathcal{A}(w_1)| + |\mathcal{A}(v_3)| + |\{4, \kappa - \Delta + 1\}| \geq 6$.

Suppose that there exists a $(\kappa - \Delta + 1, 4, w_0, v_3)$-critical path and $4 \in \Upsilon(w_0w_1)$. This implies that $\{1, 2, 4, \kappa - \Delta + 1\} \subseteq \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$ and $|\mathcal{A}(w_1)| + |\mathcal{A}(v_3)| \geq 4$. Reassigning 4 to w_0w_2 and reassigning 1 to w_0v_3 results in another acyclic edge coloring ϱ_2 of $G - ww_0$. Similarly as above, we can prove that $\deg_G(w_4) \geq 6$.

In one word, the degree of w_4 is at least six. By symmetry, we have that $\deg_G(w_i) \geq 6$ for $4 \leq i \leq \deg_G(w) - 1$.

Subcase 2.1.4. $\{\lambda_1, \lambda_2\} \cap \{1, 2\} = \{\lambda_1\}$ and λ_1 appears on w_0v_3.

Without loss of generality, assume that $\phi(w_0w_1) = \kappa - \Delta + 1$, $\phi(w_0w_2) = 3$, $\phi(w_0v_3) = 1$. If $2 \notin \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$, then reassigning 2 to w_0w_1 and reassigning β_1 to w_0w_2 will take us back to Subcase 2.1.1. Hence, $2 \in \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$ and 2 appears at least twice in $\mathbb{S}$. Therefore, we have

$$\sum_{\alpha\in\mathcal{U}(w)\cup\mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + |X| \geq |\mathcal{U}(w)| + 2 + |X| + |\{2\}| \geq \deg_G(w) + 2.$$

Suppose that

$$\sum_{x\in N_G(w_0)} \deg_G(x) = 2\kappa - \deg_G(w_0) + 9.$$

It follows that

$$\sum_{\alpha\in\mathcal{U}(w)\cup\mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + |X| = |\mathcal{U}(w)| + 2 + |X| + |\{2\}| = \deg_G(w) + 2,$$

and every color in $\mathcal{U}(w) \setminus \{2\}$ appears only once in $\mathbb{S}$.

There exists a $(3, \kappa-\Delta+1, w_0, w)$-critical path, otherwise, reassigning $\kappa-\Delta+1$ to w_0w and α_1 to w_0w_1 results in an acyclic edge coloring of G, a contradiction. Since $|C(ww_0)| \geq \Delta - 1$ and $\{1, 2, 3, \kappa-\Delta+1\} \subseteq \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$, this implies that $|\mathcal{A}(w_1)|+|\mathcal{A}(v_3)| \geq 4$. There exists no $(1, \alpha, w, w_0)$-critical path for every $\alpha \in \mathcal{A}(w_1)\cup\mathcal{A}(v_3)$, thus there exists a $(3, \alpha, w, w_0)$-critical path, and then $\mathcal{A}(w_1)\cup\mathcal{A}(v_3) \subseteq \Upsilon(ww_3)$. Hence, $\deg_G(w_3) \geq |\mathcal{A}(w_1)|+|\mathcal{A}(v_3)|+|\{3, \kappa-\Delta+1\}| \geq 6$.

Suppose that $4 \notin \Upsilon(w_0w_2)$ and there exists no $(\kappa-\Delta+1, 4, w_0, w_2)$-critical path. Reassigning 4 to w_0w_2 results in a new acyclic edge coloring ϱ_3 of $G - ww_0$. Similarly, we can prove $\deg_G(w_4) \geq 6$ by replacing ϕ with ϱ_3.

If $4 \in \Upsilon(w_0w_2)$, then reassigning 1 to w_0w_2 and reassigning 4 to w_0v_3 will take us back to Subcase 2.1.3. If there exists a $(\kappa-\Delta+1, 4, w_0, w_2)$-critical path and $4 \in \Upsilon(w_0w_1)$, then reassigning 1 to w_0w_2 and 4 to w_0v_3 will take us back to Subcase 2.1.3 again.

Hence, we have that $\deg_G(w_4) \geq 6$. By symmetry, we also have that $\deg_G(w_i) \geq 6$ for $4 \leq i \leq \deg_G(w) - 1$.

Subcase 2.1.5. $\{\lambda_1, \lambda_2\} \cap \{1, 2\} = \emptyset$ and the color on w_0v_3 is a common color.

Without loss of generality, assume that $\phi(w_0w_1) = \kappa-\Delta+1$, $\phi(w_0w_2) = 3$, $\phi(w_0v_3) = 4$. If $1 \notin \Upsilon(w_0w_2) \cup \Upsilon(w_0v_3)$, then reassigning 1 to w_0w_2 will take us back to Subcase 2.1.3. Hence, $1 \in \Upsilon(w_0w_2) \cup \Upsilon(w_0v_3)$ and 1 appears at least twice in $\mathbb{S}$. If $2 \notin \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$, then reassigning 2 to w_0w_1 and β_1 to w_0w_2 will take us back to Subcase 2.1.3. Therefore, we have

$$\sum_{\alpha\in\mathcal{U}(w)\cup\mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + |X| \geq |\mathcal{U}(w)| + 2 + |X| + |\{1, 2\}| \geq \deg_G(w) + 3.$$

Subcase 2.1.6. $\{\lambda_1, \lambda_2\} \cap \{1, 2\} = \emptyset$ and the color on w_0v_3 is not a common color.

Without loss of generality, assume that $\phi(w_0w_1) = 3$, $\phi(w_0w_2) = 4$, $\phi(w_0v_3) = \kappa-\Delta+1$.

Suppose that $1 \notin \Upsilon(w_0w_2) \cup \Upsilon(w_0v_3)$. Thus, there exists a $(3, 1, w_0, w_2)$-critical path; otherwise, reassigning 1 to w_0w_2 and α_1 to ww_0 results in an acyclic edge coloring of G. But reassigning α_1, β_1 and 1 to ww_0, w_0w_2 and w_0v_3 respectively, yields an acyclic edge coloring of G. Hence, $1 \in \Upsilon(w_0w_2) \cup \Upsilon(w_0v_3)$ and 1 appears at least twice in $\mathbb{S}$. Similarly, we have that $2 \in \Upsilon(w_0w_1) \cup \Upsilon(w_0v_3)$. Therefore, we have

$$\sum_{\alpha\in\mathcal{U}(w)\cup\mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\alpha) + |X| \geq |\mathcal{U}(w)| + 2 + |X| + |\{1, 2\}| \geq \deg_G(w) + 3.$$

Subcase 2.2. $|\mathcal{U}(w) \cap \mathcal{U}(w_0)| = 3$.

Claim 6. Every color in $\mathcal{U}(w)$ is in $\mathbb{S}$.

Proof. Assume that w_0v_1, w_0v_2 and w_0v_3 are colored with λ_1, λ_2 and λ_3, respectively. Suppose that $\lambda_1 \notin \mathbb{S}$. If there is no $(\lambda_2, \alpha_1, w_0, v_1)$-critical path, then reassigning α_1 and λ_1 to w_0v_1 and w_0v_3 respectively, results in a new acyclic edge coloring of $G - ww_0$, which contradicts $(*)$. Hence, there exists a $(\lambda_2, \alpha_1, w_0, v_1)$-critical path, and hence there exists a $(\lambda_3, \alpha_1, w_0, w)$-critical path. But reassigning α_1 and λ_1 to w_0v_1 and w_0v_2, yields another acyclic edge coloring of $G - ww_0$, which contradicts $(*)$.

Hence, we have that $\lambda_1 \in \mathbb{S}$. By symmetry, we have that $\{\lambda_1, \lambda_2, \lambda_3\} \subseteq \mathbb{S}$. Let τ be an arbitrary color in $\mathcal{U}(w) \setminus (\mathbb{S} \cup \{\lambda_1, \lambda_2, \lambda_3\})$. Let σ be obtained from ϕ by reassigning τ to w_0v_1. It is obvious that σ is an acyclic edge coloring of $G - ww_0$. So we can obtain a similar contradiction by replacing ϕ with σ. So we conclude that $\mathcal{U}(w) \subseteq \mathbb{S}$. $\square$

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\mathcal{U}(w)| = \deg_G(w) - 1,$$

In the following discussion, suppose that $\deg_G(w) + \deg_G(w_0) \leq \kappa - \Delta + 5$ and ww_0 is contained in two triangles ww_0w_1 and ww_0w_2 ($w_1 = v_1$ and $w_2 = v_2$).

Subcase 2.2.1. $\mathcal{U}(w_0) \cap \{1, 2\} = \{1, 2\}$.

By symmetry, assume that $\phi(w_0w_1) = 3, \phi(w_0w_2) = 1, \phi(w_0v_3) = 2$. Since $\alpha_1 \notin \mathcal{U}(w_1)$, it follows that there exists a $(2, \alpha_1, w_0, w)$-critical path. Reassigning α_1 to w_0w_1 will take us back to Subcase 2.1.2.

Subcase 2.2.2. $\mathcal{U}(w_0) \cap \{1, 2\} = \{\lambda^*\}$ and λ^* is not on w_0v_3.

By symmetry, assume that $\phi(w_0w_1) = 3, \phi(w_0w_2) = 1, \phi(w_0v_3) = 5$. Since $\alpha_1 \notin \mathcal{U}(w_1)$, it follows that there exists a $(5, \alpha_1, w_0, w)$-critical path. Reassigning α_1 to w_0w_1 will take us back to Subcase 2.1.3.

Subcase 2.2.3. $\mathcal{U}(w_0) \cap \{1, 2\} = \{\lambda^*\}$ and λ^* is on w_0v_3.

By symmetry, assume that $\phi(w_0w_1) = 3, \phi(w_0w_2) = 4, \phi(w_0v_3) = 1$. Since $\alpha_1 \notin \mathcal{U}(w_1)$, it follows that there exists a $(4, \alpha_1, w_0, w)$-critical path. Reassigning α_1 to w_0w_1 will take us back to Subcase 2.1.4.

Subcase 2.2.4. $\mathcal{U}(w_0) \cap \{1, 2\} = \emptyset$.

By symmetry, assume that $\phi(w_0w_1) = 3, \phi(w_0w_2) = 4, \phi(w_0v_3) = 5$. Suppose that 1 only appears once in $\mathbb{S}$. Reassigning 1 to w_0w_2 will create a (3, 1)-dichromatic cycle containing w_0w_2, for otherwise, we go back to Subcase 2.2.2. But Reassigning 1 to w_0v_3 will take us back to Subcase 2.2.3. Hence, the color 1 appears at least twice in $\mathbb{S}$. Similarly, the color 2 appears at least twice in $\mathbb{S}$. Hence, we have

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(w_0)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq \deg_G(w) - 1 + |\{1, 2\}| = \deg_G(w) + 1. \qquad \square$$

3.3 Local structure on 5-vertices

Lemma 10. Let G be a κ-deletion-minimal graph with $\kappa \geq \Delta + 5$ and let u be a 5-vertex.

(a) If u is contained in a triangle wuw_1w with $\deg_G(w) \leq \kappa - \Delta$ and $\deg_G(w_1) \leq 6$, then

$$\sum_{x \in N_G(u)} \deg_G(x) \geq 2\kappa - \deg_G(u) + 12 = 2\kappa + 7. \tag{5}$$

(b) If u is contained in a triangle wuw_1w with $\deg_G(w) \leq \kappa - \Delta - 1$ and $\deg_G(w_1) \leq 7$, then

$$\sum_{x \in N_G(u)} \deg_G(x) \geq 2\kappa - \deg_G(u) + 12 = 2\kappa + 7. \tag{6}$$

Proof. We may assume that

(*) The graph $G - wu$ admits an acyclic edge coloring ϕ such that the number of common colors at w and u is minimum.

Here, (a) and (b) will be proved together, so we may assume that $\deg_G(w) \leq \kappa - \Delta$. Since $\deg_G(w) + \deg_G(u) \leq \kappa - \Delta + 5 < \kappa + 2$, we have that $|\Upsilon(wu) \cap \Upsilon(uw)| = m \geq 1$. It follows that $|C(wu)| = \kappa - (\deg_G(w) + \deg_G(u) - m - 2) \geq \Delta - 2$. Without loss of generality, let $N_G(w) = \{u, w_1, w_2, \dots\}$ and $\phi(ww_i) = i$ for $1 \leq i \leq \deg_G(w) - 1$. Let $N_G(u) = \{w, u_1, u_2, u_3, u_4\}$ and $\mathbb{S} = \Upsilon(uu_1) \uplus \Upsilon(uu_2) \uplus \Upsilon(uu_3) \uplus \Upsilon(uu_4)$.

Let $\mathcal{A}(u_1) = C(wu) \setminus \mathcal{U}(u_1) = \{\alpha_1, \alpha_2, \dots\}$, $\mathcal{A}(u_2) = C(wu) \setminus \mathcal{U}(u_2) = \{\beta_1, \beta_2, \dots\}$, $\mathcal{A}(u_3) = C(wu) \setminus \mathcal{U}(u_3) = \{\xi_1, \xi_2, \dots\}$, $\mathcal{A}(u_4) = C(wu) \setminus \mathcal{U}(u_4) = \{\zeta_1, \zeta_2, \dots\}$ and $\mathcal{A}(w_2) = C(wu) \setminus \mathcal{U}(w_2) = \{\zeta_1^*, \zeta_2^*, \dots\}$.

Claim 1. For every color θ in $C(wu)$, there exists an (λ, θ, u, w)-critical path for some $\lambda \in \mathcal{U}(w) \cap \mathcal{U}(u)$. Consequently, $\mathrm{mul}_{\mathbb{S}}(\theta) \geq 1$.

Case 1. $\mathcal{U}(w) \cap \mathcal{U}(u) = \{\lambda\}$. By symmetry, we may assume that $w_1 = u_1$.

It follows that $|C(wu)| = \kappa - (\deg_G(w) + \deg_G(u) - 3) \geq \Delta - 2$.

Subcase 1.1. The edge ww_1 is colored with λ. By symmetry, assume that $\phi(uw_1) = \kappa-\Delta+2$, $\phi(uu_2) = 1$, $\phi(uu_3) = \kappa-\Delta$, $\phi(uu_4) = \kappa - \Delta + 1$.

By Claim 1, we have that $\{\kappa-\Delta+3, \dots, \kappa\} \subseteq \Upsilon(ww_1) \cap \Upsilon(uu_2)$. Moreover, $\mathcal{U}(w_1) = \{1, \kappa-\Delta+2\} \cup \{\kappa-\Delta+3, \dots, \kappa\}$, $\deg_G(w_1) = \Delta$ and $\deg_G(u_2) \geq \Delta - 1$. Notice that $|\Upsilon(uu_2) \cap \{2, 3, \dots, \kappa - \Delta - 1\}| \leq 1$, thus there exists a color ζ which is in $\{2, 3, \dots, \kappa - \Delta - 1\} \setminus \Upsilon(uu_2)$ (note that this set is nonempty). But assigning $\kappa - \Delta + 2$ to uw and ζ to uw_1 results in an acyclic edge coloring of G, a contradiction.

Subcase 1.2. The edge uw_1 is colored with λ. By symmetry, assume that $\phi(ww_1) = 2$, $\phi(uu_2) = \kappa-\Delta$, $\phi(uu_3) = \kappa-\Delta+1$, $\phi(uu_4) = \kappa - \Delta + 2$.

By Claim 1, we have that $\{\kappa - \Delta + 3, \dots, \kappa\} \subseteq \Upsilon(uw_1) \cap \Upsilon(ww_2)$ and $\deg_G(w_1) = \Delta$ and $\deg_G(w_2) \geq \Delta - 1$. Modify ϕ by reassigning 1 to wu and reassigning a color in $\{\kappa - \Delta, \kappa - \Delta + 1, \kappa - \Delta + 2\} \setminus \mathcal{U}(w_2)$ to ww_1, we obtain an acyclic edge coloring of G, a contradiction.

Subcase 1.3. Neither w_1w nor w_1u is colored with λ. By symmetry, assume that $\phi(uw_1) = \kappa - \Delta$, $\phi(uu_2) = 2$, $\phi(uu_3) = \kappa - \Delta + 1$, $\phi(uu_4) = \kappa - \Delta + 2$.

By Claim 1, we have that $C(wu) \subseteq \Upsilon(uu_2) \cap \Upsilon(ww_2)$ and $\deg_G(w_2) \geq \Delta - 1$ and $\deg_G(u_2) \geq \Delta - 1$. Notice that $\{1, \kappa - \Delta\} \nsubseteq \mathcal{U}(w_2)$.

If $\deg_G(w) \leq \kappa - \Delta - 1$, then $\mathcal{U}(w) = \{1, 2, \dots, \kappa - \Delta - 2\}$ and $\deg(w_2) = \deg(u_2) = \Delta$, but reassigning 1 to uu_2 will take us back to Subcase 1.1.

So we may assume that $\deg(w) = \kappa - \Delta$, $C(wu) = \{\kappa - \Delta + 3, \dots, \kappa\}$ and $\deg(w_1) \leq 6$.

Suppose that $C(wu) \subseteq \mathcal{U}(w_1)$. Thus $\mathcal{U}(w_1) = \{1, \kappa - \Delta\} \cup C(wu)$. If $1 \notin \mathcal{U}(w_2)$, then reassigning $1, \kappa - \Delta$ and 3 to wu, ww_1 and w_1u respectively results in an acyclic edge coloring of G, a contradiction. So we may assume that $1 \in \mathcal{U}(w_2)$ and $\Upsilon(ww_2) = \{1\} \cup \{\kappa - \Delta + 3, \dots, \kappa\}$. But reassigning $\kappa - \Delta$ to wu and 3 to w_1u results in an acyclic edge coloring of G, a contradiction. Hence, we have that $C(wu) \nsubseteq \mathcal{U}(w_1)$.

We further suppose that $1 \in \mathcal{U}(u_2)$ and $\Upsilon(uu_2) = \{1\} \cup \{\kappa - \Delta + 3, \dots, \kappa\}$. If there is a $(2, 1, u, w)$-critical path, then $\deg_G(w_2) = \Delta(G)$ and $\Upsilon(ww_2) = \{1\} \cup \{\kappa - \Delta + 3, \dots, \kappa\}$, but reassigning $\kappa - \Delta$ to ww_2 will take us back to Subcase 1.2. So we may assume that there is no $(2, 1, u, w)$-critical path. There exists a $(\tau^*, \alpha_1, w, w_1)$-critical path with some $\tau^* \in \mathcal{U}(w) \setminus \{1, 2\}$, otherwise reassigning α_1 to ww_1 and 1 to uw will result in an acyclic edge coloring of G. By symmetry, assume that $\tau^* = 3$ and there exists a $(3, \alpha_1, w, w_1)$-critical path. But reassigning 3 to uu_2 and α_1 to wu results in an acyclic edge coloring of G.

So we may assume that $1 \notin \mathcal{U}(u_2)$. There exists a $(\kappa-\Delta+1, 1, u, u_2)$- or $(\kappa-\Delta+2, 1, u, u_2)$-critical path; otherwise, reassigning 1 to uu_2 will take us back to Subcase 1.1. By symmetry, assume that there exists a $(\kappa-\Delta+2, 1, u, u_2)$-critical path and $1 \in \Upsilon(uu_4)$, thus $\deg_G(u_2) = \Delta(G)$ and $\Upsilon(uu_2) = \{\kappa - \Delta + 2, \kappa - \Delta + 3, \dots, \kappa\}$.

There exists a $(\kappa-\Delta+1, \alpha_1, u, w_1)$- or $(\kappa-\Delta+2, \alpha_1, u, w_1)$-critical path; otherwise, reassigning α_1 to uw_1 and $\kappa-\Delta$ to uw will result in an acyclic edge coloring of G. Hence, $\{\kappa - \Delta + 1, \kappa - \Delta + 2\} \cap \mathcal{U}(w_1) \neq \emptyset$. Similarly, there exists a (τ, α_1, w, w_1)-critical path with some $\tau \in \mathcal{U}(w) \setminus \{1, 2\}$. By symmetry, assume that $\tau = 3$ and there exists a $(3, \alpha_1, w, w_1)$-critical path. Hence, $|\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))| \geq 4$ and $|\mathcal{U}(w_1) \cap C(wu)| \leq 2$, and then $|C(wu) \setminus \mathcal{U}(w_1)| \geq \Delta - 4$.

Suppose that $\mathcal{A}(u_4) \cap \mathcal{A}(w_1) \neq \emptyset$, say $\zeta \in \mathcal{A}(u_4) \cap \mathcal{A}(w_1)$. Thus there exists a $(\kappa - \Delta + 1, \zeta, u, w_1)$-critical path; otherwise, reassigning ζ to uw_1 and $\kappa - \Delta$ to uw will result in an acyclic edge coloring of G. There exists a $(2, \kappa - \Delta + 2, u, w)$-critical path, otherwise reassigning ζ to uu_4 and $\kappa - \Delta + 2$ to uw will result in an acyclic edge coloring of G. Hence, we have that $\Upsilon(ww_2) = \Upsilon(uu_2) = \{\kappa - \Delta + 2, \kappa - \Delta + 3, \dots, \kappa\}$. But reassigning $\kappa - \Delta$ to ww_2 will take us back to Subcase 1.2. So we have that $\mathcal{A}(u_4) \cap \mathcal{A}(w_1) = \emptyset$.

There exists a $(\kappa - \Delta + 2, 3, u, u_2)$-critical path, for otherwise reassigning 3 to uu_2 and α_1 to uw will result in an acyclic edge coloring of G. It follows that $\{1, 3\} \cup \mathcal{A}(w_1) \subseteq \Upsilon(uu_4)$. If $2 \notin \mathcal{U}(u_4)$ and there exists no $(\kappa-\Delta+1, 2, u, u_4)$-critical path, then reassigning 2, 1 and α_1 to uu_4, uu_2 and uw, respectively, will result in an acyclic edge coloring of G. Hence, $\mathcal{U}(u_4) = \{1, 2, 3, \kappa-\Delta+2\} \cup \mathcal{A}(w_1)$ or $\mathcal{U}(u_4) = \{1, 3, \kappa-\Delta+1, \kappa-\Delta+2\} \cup \mathcal{A}(w_1)$. Consequently, we have that $|\mathcal{A}(w_1)| = \Delta - 4$, and then $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, 3, \kappa - \Delta, \kappa - \Delta + 1\}$ or $\{1, 3, \kappa - \Delta, \kappa - \Delta + 2\}$.

There exists a $(\kappa - \Delta + 1, 2, u, w_1)$- or $(\kappa - \Delta + 2, 2, u, w_1)$-critical path, for otherwise we reassign $\alpha_1, 2$ and $\kappa - \Delta$ to uw, uw_1 and uu_2. Thus, $\mathcal{U}(u_4) = \{1, 2, 3, \kappa - \Delta + 2\} \cup \mathcal{A}(w_1)$. If $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, 3, \kappa - \Delta, \kappa - \Delta + 2\}$, then we reassign $\alpha_1, 2, 3$ and $\kappa - \Delta$ to uw, uw_1, uu_2 and uu_4. Therefore, $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, 3, \kappa-\Delta, \kappa-\Delta+1\}$, but reassigning $\kappa - \Delta + 2$, 1 and 4 to uw, uu_2 and uu_4 results in an acyclic edge coloring of G.

Case 2. $\mathcal{U}(w) \cap \mathcal{U}(u) = \{\lambda_1, \dots, \lambda_m\}$ and $m \geq 2$.

We can relabel the vertices in $\{u_1, u_2, u_3, u_4\}$ as $\{v_1, v_2, v_3, v_4\}$. By symmetry, we may assume that $\phi(uv_i) = \lambda_i$ for $i \in \{1, \dots, m\}$.

Claim 2. The sets $\mathcal{A}(v_1), \mathcal{A}(v_2), \dots, \mathcal{A}(v_m)$ are pairwise disjoint.

Proof. Suppose, to the contrary, that $\alpha \in \mathcal{A}(v_1) \cap \mathcal{A}(v_2)$. By Claim 1 and the symmetry, we may assume that there exists a $(\lambda_3, \alpha, u, w)$-critical path and $m \geq 3$, which implies that there exists no $(\lambda_3, \alpha, u, v_2)$-critical path. Consequently, there exists a $(\phi(uv_4), \alpha, u, v_2)$-critical path; otherwise, reassigning α to uv_2 to obtain a new acyclic edge coloring of $G - wu$, which contradicts the minimality of m. Now, reassigning α to uv_1 to obtain an acyclic edge coloring π of $G - wu$, but $|\mathcal{U}_\pi(u) \cap \mathcal{U}_\pi(w)| < |\mathcal{U}(u) \cap \mathcal{U}(w)|$, which is a contradiction. $\square$

Claim 3. Every color in $C(wu)$ appears at least twice in $\mathbb{S}$.

Proof. Suppose that there exists a color α in $C(wu)$ such that $\mathrm{mul}_{\mathbb{S}}(\alpha) = 1$. By Claim 1 and symmetry, we may assume that there exists a $(\lambda_1, \alpha, u, w)$-critical path and $\alpha \in \mathcal{U}(v_1)$. But reassigning α to uv_2 results in a new acyclic edge coloring of $G - wu$, which contradicts the assumption $(*)$. $\square$

Let $X = \{\, \theta \mid \theta \in C(wu) \text{ and } \mathrm{mul}_{\mathbb{S}}(\theta) \geq 3 \,\}$.

$$\begin{aligned}
&\sum_{x \in N_G(u)} \deg_G(x) \\
&= \deg_G(u) + \deg_G(w) - 1 + \sum_{\theta \in [\kappa]} \mathrm{mul}_{\mathbb{S}}(\theta) \\
&= \deg_G(u) + \deg_G(w) - 1 + \sum_{\theta \in C(wu)} \mathrm{mul}_{\mathbb{S}}(\theta) + \sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) \\
&\geq \deg_G(u) + \deg_G(w) - 1 + 2|C(wu)| + |X| + \sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) \\
&= \deg_G(u) + \deg_G(w) - 1 + 2(\kappa - (\deg_G(w) + \deg_G(u) - 2 - m)) + |X| + \sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) \\
&= 2\kappa - \deg_G(u) - \deg_G(w) + 2m + 3 + |X| + \sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta)
\end{aligned}$$

It is sufficient to prove that

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq \deg_G(w) - 2m + 9. \tag{7}$$

Subcase 2.1. $\mathcal{U}(w) \cap \mathcal{U}(u) = \{\lambda_1, \lambda_2\}$ and $w_1 = u_1$. Note that $\mathcal{A}(w_1) \neq \emptyset$.

Subcase 2.1.1. The two colors on the edges $w_1 w$ and $w_1 u$ are all common colors.

Without loss of generality, assume that $\phi(uw_1) = 2$, $\phi(uu_2) = 1$, $\phi(uu_3) = \kappa - \Delta$ and $\phi(uu_4) = \kappa - \Delta + 1$. Consequently, we conclude that $\{\kappa - \Delta + 2, \dots, \kappa\} \subseteq \mathcal{U}(w_1)$ and $\deg_G(w_1) \geq \Delta + 1$, a contradiction.

Subcase 2.1.2. The color on $w_1 w$ is a common color and the color on $w_1 u$ is not a common color.

Without loss of generality, assume that $\phi(uw_1) = \kappa - \Delta$, $\phi(uu_2) = 1$, $\phi(uu_3) = 2$ and $\phi(uu_4) = \kappa - \Delta + 1$.

For every color $\alpha_i \in \mathcal{A}(w_1)$, there exists a $(\theta_i, \alpha_i, w, w_1)$-critical path with some $\theta_i \in \mathcal{U}(w) \setminus \{1, 2\}$; otherwise, reassigning α_i to ww_1 will take us back to Case 1. By symmetry, we may assume that there exists a $(3, \alpha^*, w, w_1)$-critical path with some α^*, and then $3 \in \mathcal{U}(w_1)$. If $\Upsilon(ww_2) \subseteq C(wu)$, then reassigning $\kappa - \Delta$ to ww_2 will take us back to Subcase 2.1.1. So we have that $\Upsilon(ww_2) \nsubseteq C(wu)$ and $\mathcal{A}(w_2) \neq \emptyset$. Consequently, for every color $\zeta_i^* \in \mathcal{A}(w_2)$, there exists a $(\mu_i, \zeta_i^*, w, w_2)$-critical path with some $\mu_i \in \mathcal{U}(w) \setminus \{1, 2\}$; otherwise, reassigning ζ_i^* to ww_2 will take us back to Case 1. Hence, $\{1, 3, \kappa - \Delta\} \subseteq \mathcal{U}(w_1)$ and $\{2, \mu_1\} \subseteq \mathcal{U}(w_2)$, and then $|\mathcal{A}(w_1)| \geq 2$ and $|\mathcal{A}(w_2)| \geq 1$.

If $\Upsilon(uu_2) \subseteq C(wu)$, then reassigning μ_1 to uu_2 and ζ_1^* to wu results in an acyclic edge coloring of G, a contradiction. Thus, we have that $\Upsilon(uu_2) \nsubseteq C(wu)$ and $\mathcal{A}(u_2) \neq \emptyset$. For every color $\beta_i \in \mathcal{A}(u_2)$, there exists an $(\varepsilon_i, \beta_i, u, u_2)$-critical path with some $\varepsilon_i \in \{\kappa - \Delta, \kappa - \Delta + 1\}$; otherwise, reassigning β_i to uu_2 will take us back to Case 1.

If $\Upsilon(uu_3) \subseteq C(wu)$, then reassigning 3 to uu_3 and α^* to wu results in an acyclic edge coloring of G, a contradiction. Thus, we have that $\Upsilon(uu_3) \nsubseteq C(wu)$ and $\mathcal{A}(u_3) \neq \emptyset$. Consequently, for every color $\xi_i \in \mathcal{A}(u_3)$, there exists a (m_i, ξ_i, u, u_3)-critical path with some $m_i \in \{\kappa - \Delta, \kappa - \Delta + 1\}$; otherwise, reassigning ξ_i to uu_3 will take us back to Case 1.

Claim 4. $\mathcal{A}(u_2) \cap \mathcal{A}(u_4) = \emptyset$.

Proof of Claim 4. Suppose that $\beta_1 \in \mathcal{A}(u_2) \cap \mathcal{A}(u_4)$. It follows that there exists a $(\kappa - \Delta, \beta_1, u, u_2)$-critical path, $\kappa - \Delta \in \Upsilon(uu_2)$ and $\beta_1 \in \Upsilon(uw_1)$. Also, there exists a $(1, \kappa - \Delta + 1, w, u)$- or $(2, \kappa - \Delta + 1, w, u)$-critical path; otherwise, reassigning β_1 to uu_4 and $\kappa - \Delta + 1$ to wu results in an acyclic edge coloring of G. Suppose that there exists a $(1, \kappa - \Delta + 1, w, u)$-critical path and $\kappa - \Delta + 1 \in \Upsilon(ww_1) \cap \Upsilon(uu_2)$. It follows that $\{1, \kappa - \Delta, \kappa - \Delta + 1\} \subseteq \mathcal{U}(u_2)$ and $|\mathcal{A}(u_2)| \geq 2$. Furthermore, we can conclude that $\{1, 3, \kappa-\Delta, \kappa-\Delta+1, \beta_1\} \cup \mathcal{A}(w_2) \subseteq \mathcal{U}(w_1)$. Note that $\mathcal{A}(w_2) \cap \mathcal{A}(u_2) = \emptyset$, thus $\mathcal{U}(w_1) = \{1, 3, \kappa - \Delta, \kappa - \Delta + 1, \beta_1\} \cup \mathcal{A}(w_2)$ and $\mathcal{U}(w_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{2, \mu_1\}$. Recall that $\beta_2 \notin \mathcal{A}(w_2)$, thus $\varepsilon_2 = \kappa - \Delta + 1$ and there exists a $(\kappa - \Delta + 1, \beta_2, u, u_2)$-critical path. Now, reassigning β_2 to uw_1 and $\kappa - \Delta$ to wu results in an acyclic edge coloring of G.

So, we may assume that there exists a $(2, \kappa - \Delta + 1, w, u)$-critical path. Hence, $\{1, \kappa - \Delta, 3, \beta_1\} \cup \mathcal{A}(w_2) \subseteq \mathcal{U}(w_1)$ and $\{2, \mu_1, \kappa-\Delta+1\} \subseteq \Upsilon(ww_2)$. It follows that $\mathcal{U}(w_1) = \{1, \kappa-\Delta, 3, \beta_1\} \cup \mathcal{A}(w_2)$ and $\mathcal{U}(w_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{2, \mu_1, \kappa-\Delta+1\}$. Now, reassigning α_1 to uw_1 and $\kappa - \Delta$ to wu results in an acyclic edge coloring of G. This completes the proof of Claim 4. □

Claim 5. $\mathcal{A}(u_2) \cap \mathcal{A}(w_1) = \emptyset$.

Proof of Claim 5. By contradiction, assume that $\alpha_1 = \beta_1$. It follows that there exists a $(\kappa - \Delta + 1, \beta_1, u, u_2)$-critical path and $\kappa - \Delta + 1 \in \mathcal{U}(u_2)$. There exists a $(2, \kappa - \Delta, w, u)$-critical path; otherwise, reassigning β_1 to uw_1 and $\kappa - \Delta$ to uw results in an acyclic edge coloring of G, a contradiction. So we have that $\kappa - \Delta \in \Upsilon(ww_2) \cap \Upsilon(uu_3)$.

Note that $\{1, \kappa-\Delta, 3\} \subseteq \mathcal{U}(w_1)$ and $\{2, \mu_1, \kappa-\Delta\} \subseteq \mathcal{U}(w_2)$. If $\deg_G(w) \leq \kappa-\Delta$ and $\deg_G(w_1) \leq 6$, then $\mathcal{A}(w_2) \subseteq \mathcal{U}(w_1)$ with $|\mathcal{A}(w_2)| \geq 2$, and then $|\mathcal{U}(w_1) \cap \mathcal{U}(w)| \leq 3$; similarly, if $\deg_G(w) \leq \kappa-\Delta-1$ and $\deg_G(w_1) \leq 7$, then $\mathcal{A}(w_2) \subseteq \mathcal{U}(w_1)$ with $|\mathcal{A}(w_2)| \geq 3$, and then $|\mathcal{U}(w_1) \cap \mathcal{U}(w)| \leq 3$. If $\mathcal{U}(w_1) \cap \mathcal{U}(w) = \{1, 3\}$, then $\{2, \kappa - \Delta\} \subsetneq \mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))$; otherwise, reassigning α_1, α_2 and 3 to uw_1, uw and uu_3 respectively results in an acyclic edge coloring of G. Suppose that $\mathcal{U}(w_1) \cap \mathcal{U}(w) = \{1, 3, s\}$. Since $|\mathcal{A}(w_1)| \geq 3$, thus there exists a $\tau \in \{3, s\}$ and α_i, α_j such that both (τ, α_i, w, w_1)- and (τ, α_j, w, w_1)-critical path exist, and thus $\{2, \kappa - \Delta\} \subsetneq \mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))$; otherwise, reassigning α_i, α_j and τ to uw_1, uw and uu_3 respectively. Anyway, we have that $|\mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))| \geq 3$.

If $\kappa - \Delta$ only appears only once (at u_3) in $\mathbb{S}$, then reassigning $\kappa - \Delta$ to uu_2 and β_1 to uw_1 will take us back to Case 1. So we conclude that the color $\kappa - \Delta$ appears at least twice in $\mathbb{S}$.

If $1 \notin \mathbb{S} \setminus \mathcal{U}(w_1)$, then reassigning $1, \beta_1$ and ξ_1 to uu_4, uu_2 and wu respectively, results in an acyclic edge coloring of G, a contradiction. Therefore, the color 1 appears at least twice in $\mathbb{S}$.

Suppose that $4 \notin \mathbb{S}$. Thus there exists a $(4, \xi_1, w, u_2)$-alternating path; otherwise, reassigning 4 to uu_2 and ξ_1 to wu results in an acyclic edge coloring of G, a contradiction. Now, reassigning $4, \beta_1$ and ξ_1 to uu_4, uu_2 and wu respectively, results in an acyclic edge coloring G, a contradiction. So we conclude that $4 \in \mathbb{S}$. By symmetry, we can also obtain that every color in $\mathcal{U}(w) \setminus \{1, 2, 3\}$ appears in $\mathbb{S}$.

Suppose that every color in $\mathcal{U}(w) \setminus \{1, 2\}$ appears exactly once in $\mathbb{S}$. Suppose that $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \setminus \{1, 2\}) = \{3, s\}$. Thus, $\mathcal{U}(w_1) = \{1, \kappa - \Delta, 3, s\} \cup \mathcal{A}(w_2)$ and $\kappa - \Delta + 1 \notin \mathcal{U}(w_1)$. Since $|\mathcal{A}(w_1)| \geq 3$, thus there exists a $\tau \in \{3, s\}$ and α_i, α_j such that both (τ, α_i, w, w_1)- and (τ, α_j, w, w_1)-critical path exist. Reassigning τ, α_i and α_j to uu_3, uw_1 and wu respectively, results in an acyclic edge coloring of G, a contradiction. So we may assume that $|\mathcal{U}(w_1) \cap (\mathcal{U}(w) \setminus \{1, 2\})| = 1$, that is $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \setminus \{1, 2\}) = \{3\}$. Reassigning $3, \alpha_1$ and α_2 to uu_3, uw_1 and wu respectively, results in an acyclic edge coloring of G, a contradiction. Hence, we may assume that the color 3 appears at least twice in $\mathbb{S}$.

Suppose that $\xi_1 \in \mathcal{A}(u_4)$. Thus, there exists a $(\kappa - \Delta, \xi_1, u, u_3)$-critical path; otherwise, reassigning ξ_1 to uu_3 will take us back to Case 1. Furthermore, $\kappa - \Delta + 1 \in \Upsilon(ww_1) \cup \Upsilon(uu_3)$; otherwise, reassigning ξ_1 to uu_4 and $\kappa - \Delta + 1$ to wu results in an acyclic edge coloring of G. If $2 \notin \mathbb{S}$, then reassigning $\alpha_1, 2$ and ξ_1 to wu, uw_1 uu_3 respectively, results in an acyclic edge coloring of G. So we have that $2 \in \mathbb{S}$. Hence,

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{4, \ldots, \deg_G(w) - 1\}| + 2|\{1, 3, \kappa - \Delta, \kappa - \Delta + 1\}| + |\{2\}| = \deg_G(w) + 5.$$

So we may assume that $\mathcal{A}(u_3) \cap \mathcal{A}(u_4) = \emptyset$. It is obvious that $\mathcal{A}(u_3) \subseteq X$. Hence,

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{4, \ldots, \deg_G(w) - 1\}| + 2|\{1, 3, \kappa - \Delta\}| + |\{\kappa - \Delta + 1\}| + |\mathcal{A}(u_3)| \geq \deg_G(w) + 4.$$

The equality holds only if $\kappa - \Delta + 1$ appears only once in $\mathbb{S}$ and 2 does not appear in $\mathbb{S}$; but reassigning $\alpha_1, 2$ and ξ_1 to wu, uw_1 and uu_3 respectively, results in an acyclic edge coloring. Therefore, inequality (7) holds, we are done. This completes the proof Claim 5. □

By Claim 5, the three sets $\mathcal{A}(w_1), \mathcal{A}(u_2)$ and $\mathcal{A}(u_3)$ are pairwise disjoint.

(1) Suppose that there exists no $(2, \kappa-\Delta, w, u)$-critical path. This implies that there exists a $(\kappa-\Delta+1, \alpha_i, u, w_1)$-critical path; otherwise, reassigning α_i to uw_1 and $\kappa-\Delta$ to wu results in an acyclic edge coloring of G. Thus, $\{1, 3, \kappa-\Delta, \kappa-\Delta+1\} \subseteq \mathcal{U}(w_1)$ and $\mathcal{A}(w_1) \subseteq \mathcal{U}(u_4)$. Note that $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \subseteq \mathcal{U}(w_1)$, thus $|\mathcal{A}(u_2)| = |\mathcal{A}(u_3)| = 1$ and $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, 3, \kappa-\Delta, \kappa-\Delta+1\}$. Similarly, we know that $\mathcal{A}(u_2) \cup \mathcal{A}(w_2) \subseteq \mathcal{U}(w_1)$, which implies that $|\mathcal{A}(w_2)| = |\mathcal{A}(u_3)| = 1$ and $\mathcal{A}(w_2) = \mathcal{A}(u_3)$. Hence, $\mathcal{U}(w_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{2, \mu_1\}$. By Claim 4, we conclude that $\mathcal{U}(u_4) \supseteq \mathcal{A}(w_1) \cup \mathcal{A}(u_2) \cup \{\kappa-\Delta+1\}$. If $\Upsilon(uu_4) \subseteq C(uw)$, then reassigning $3, \alpha^*$ and $\kappa-\Delta$ to uu_4, uw_1 and wu respectively results in an acyclic edge coloring of G. Note that $|\mathcal{A}(w_1)| + |\mathcal{A}(u_2)| = \Delta - 2$, so we may assume that $|\Upsilon(uu_4) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))| = 1$. In addition, $\Upsilon(uu_4) \cap C(uw) = \mathcal{A}(w_1) \cup \mathcal{A}(u_2)$ and $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, \varepsilon_1\}$. Recall that $\mathcal{A}(w_1), \mathcal{A}(u_2)$ and $\mathcal{A}(u_3)$ are pairwise disjoint, thus $\mathcal{A}(u_3) \cap \mathcal{U}(u_4) = \emptyset$, and then there exists a $(\kappa-\Delta, \xi_1, u, u_3)$-critical path and $\kappa-\Delta \in \Upsilon(uu_3)$. There exists a $(1, \kappa-\Delta+1, u, w)$-critical path; otherwise, reassigning ξ_1 and $\kappa-\Delta+1$ to uu_4 and uw results in an acyclic edge coloring of G. Hence, $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, \varepsilon_1\} = \{1, \kappa-\Delta+1\}$. There exists a $(\kappa-\Delta+1, \mu_1, u, u_2)$-critical path, otherwise, reassigning μ_1 to uu_2 and ζ_1^* to uw results in an acyclic edge coloring of G. Hence, $\Upsilon(uu_4) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{\mu_1, \kappa-\Delta+1\}$. Now, reassigning $\zeta_1^*, \alpha_1, \mu_1$ and $\kappa-\Delta$ to uw, uw_1, uu_2 and uu_4 respectively, yields an acyclic edge coloring of G.

(2) Now, we may assume that there exists a $(2, \kappa-\Delta, w, u)$-critical path and $\kappa-\Delta \in \Upsilon(uu_3) \cap \Upsilon(ww_2)$. Clearly, $\mathcal{U}(w_1) \supseteq \mathcal{A}(u_2) \cup \mathcal{A}(w_2) \cup \{1, 3, \kappa-\Delta\}$. If $\deg_G(w) \le \kappa-\Delta-1$, then $\deg_G(w_1) \ge 2+3+3 = 8$, a contradiction. Thus, $\deg_G(w) = \kappa-\Delta$, which implies that $\mathcal{U}(w_1) = \mathcal{A}(u_2) \cup \mathcal{A}(w_2) \cup \{1, 3, \kappa-\Delta\}$, $|\mathcal{A}(u_2)| = 1$ and $|\mathcal{A}(w_2)| = 2$. It is easy to see that $\mathcal{U}(w_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{2, \kappa-\Delta, \mu_1\}$ and $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, \varepsilon_1\}$. If $3 \notin \mathcal{U}(u_3)$, then there exists a $(\kappa-\Delta+1, 3, u, u_3)$-critical path; otherwise, reassigning α_1, α_2 and 3 to uw_1, uw and uu_3 respectively results in an acyclic edge coloring of G. Hence, we have that $\{3, \kappa-\Delta+1\} \cap \mathcal{U}(u_3) \neq \emptyset$ and $|\mathcal{A}(u_3)| \ge 2$. Recall that $\mathcal{A}(w_1), \mathcal{A}(u_2)$ and $\mathcal{A}(u_3)$ are disjoint, thus $\mathcal{U}(w_1) \supseteq \mathcal{A}(u_2) \cup \mathcal{A}(u_3) \cup \{1, 3, \kappa-\Delta\}$. Moreover, $\mathcal{U}(w_1) = \mathcal{A}(u_2) \cup \mathcal{A}(u_3) \cup \{1, 3, \kappa-\Delta\}$, $\mathcal{A}(u_3) = \mathcal{A}(w_2)$. If there exists a $\xi_i \notin \mathcal{U}(u_4)$, then there exists a $(\kappa-\Delta, \xi_i, u, u_3)$-critical path, and then reassigning ξ_i to uu_4 and $\kappa-\Delta+1$ to uw results in an acyclic edge coloring of G. So we have that $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \subseteq \mathcal{U}(u_4)$.

There exists a $(\kappa-\Delta, \mu_1, u, u_2)$- or $(\kappa-\Delta+1, \mu_1, u, u_2)$-critical path; otherwise, reassigning μ_1 to uu_2 and ζ_1^* to uw results in an acyclic edge coloring of G. If there exists a $(\kappa-\Delta, \mu_1, u, u_2)$-critical path, then $\mu_1 = 3$ and $\varepsilon_1 = \kappa-\Delta$; but reassigning μ_1, α^* and ζ_1^* to uu_2, uw_1 and uw results in an acyclic edge coloring. So there exists a $(\kappa-\Delta+1, \mu_1, u, u_2)$-critical path, thus $\varepsilon_1 = \kappa-\Delta+1$ and $\mathcal{U}(u_4) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{\mu_1, \kappa-\Delta+1\}$. Now, reassigning $\kappa-\Delta, \mu_1, \alpha^*$ and ζ_1^* to uu_4, uu_2, uw_1 and uw respectively, yields an acyclic edge coloring of G.

Subcase 2.1.3. The color on w_1w is not a common color and the color on w_1u is a common color.

Without loss of generality, assume that $\phi(uw_1) = 3$, $\phi(uu_2) = 2$, $\phi(uu_3) = \kappa-\Delta$ and $\phi(uu_4) = \kappa-\Delta+1$.

For every color $\alpha_i \in \mathcal{A}(w_1)$, there exists a $(\theta_i, \alpha_i, u, w_1)$-critical path with some $\theta_i \in \{\kappa-\Delta, \kappa-\Delta+1\}$; otherwise, reassigning α_i to uw_1 will take us back to Case 1. If $\Upsilon(uu_2) \subseteq C(wu)$, then reassigning 1 to uu_2 will take us back to Subcase 2.1.1. So we have that $\Upsilon(uu_2) \nsubseteq C(wu)$ and $\mathcal{A}(u_2) \neq \emptyset$. Consequently, for every color $\beta_i \in \mathcal{A}(u_2)$, there exists a $(\varepsilon_i, \beta_i, u, u_2)$-critical path with some $\varepsilon_i \in \{\kappa-\Delta, \kappa-\Delta+1\}$; otherwise, reassigning β_i to uu_2 will take us back to Case 1. Hence, we have $\{\kappa-\Delta, \kappa-\Delta+1\} \cap \Upsilon(uu_2) \neq \emptyset$.

Subcase 2.1.3.1. Suppose that $\{\kappa-\Delta, \kappa-\Delta+1\} \subseteq \mathcal{U}(w_1)$.

If $\{\kappa-\Delta, \kappa-\Delta+1\} \subseteq \mathcal{U}(u_2)$, then $\mathcal{U}(w_1) = \{1, 3, \kappa-\Delta, \kappa-\Delta+1\} \cup \mathcal{A}(u_2)$ and $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{\kappa-\Delta, \kappa-\Delta+1, 2\}$; but reassigning α_1 to ww_1 and 1 to uw results in an acyclic edge coloring of G. This implies that $|\{\kappa-\Delta, \kappa-\Delta+1\} \cap \mathcal{U}(u_2)| = 1$, say $\kappa-\Delta \in \mathcal{U}(u_2)$. Hence, we have $\varepsilon_i = \kappa-\Delta$ and $\mathcal{A}(u_2) \subseteq \mathcal{U}(u_3)$.

Suppose that there exists no $(2, 1, u, w)$-critical path. Thus, there exists a $(\mu_i, \alpha_i, w, w_1)$-critical path with $\mu_i \in \mathcal{U}(w) \setminus \{1, 2, 3\}$; otherwise, reassigning α_i to ww_1 and 1 to wu will result in an acyclic edge coloring of G. Note that $\mathcal{U}(w_1) \supseteq \{1, 3, \kappa-\Delta, \kappa-\Delta+1, \mu_1\} \cup \mathcal{A}(u_2)$, it follows that $\mathcal{U}(u_2) \cap (\mathcal{U}(u) \cup \mathcal{U}(w)) = \{2, \kappa-\Delta\}$ and $|\mathcal{U}(u_2) \cap C(wu)| = \Delta-2$. Moreover, $\mathcal{U}(w_1) = \{1, 3, \kappa-\Delta, \kappa-\Delta+1, \mu_1\} \cup \mathcal{A}(u_2)$ and $|\mathcal{A}(u_2)| = 1$, say $\mu_1 = 4$. Thus, there exists a $(\kappa-\Delta, 1, u, u_2)$-critical path; otherwise, reassigning 1 to uu_2 will take us back to Subcase 2.1.1. So, we have $1 \in \mathcal{U}(u_3)$. Furthermore, there exists a $(\kappa-\Delta, 4, u, u_2)$-critical path; otherwise, reassigning 4 to uu_2 and α_1 to wu results in an acyclic edge coloring of G. Hence, $\{1, 4, \kappa-\Delta\} \subseteq \mathcal{U}(u_3)$. Recall that $|\mathcal{A}(u_3)| \ge 2$ and $|\mathcal{A}(u_2)| = 1$, it follows that $\mathcal{A}(w_1) \cap \mathcal{A}(u_3) \neq \emptyset$, say $\alpha_1 \notin \mathcal{U}(u_3)$. If $1 \notin \mathcal{U}(u_4)$, then reassigning 1 to uu_4 and α_1 to uw_1 will take us back to Subcase 2.1.2. Thus, we have $1 \in \mathcal{U}(u_4)$. If $2 \notin \mathbb{S}$, then reassigning $2, \beta_1$ and α_1 to uu_3, uu_2 and uw respectively results in an acyclic edge coloring of G. Thus $2 \in \mathbb{S}$. If $3 \notin \mathbb{S}$, then reassigning $3, \alpha_1$ and β_1 to uu_4, uw_1 and uw respectively results in an acyclic edge coloring of G. Thus $3 \in \mathbb{S}$. If $5 \notin \mathbb{S}$, then there exists a $(5, \alpha_1, w, u_2)$-alternating path, otherwise, reassigning 5 to

uu_2 and α_1 to uw results in an acyclic edge coloring of G; but reassigning $5, \beta_1$ and α_1 to uu_3, uu_2 and uw results in an acyclic edge coloring of G. Thus $5 \in \mathbb{S}$. Similarly, $\{5, 6, \dots, \deg_G(w) - 1\} \subseteq \mathbb{S}$. Therefore, we have

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq 3|\{1\}| + 2|\{4, \kappa - \Delta\}| + |\{2, 3, \kappa - \Delta + 1, 5, 6, \dots, \deg_G(w) - 1\}| = \deg_G(w) + 5.$$

Suppose that there exists a $(2, 1, w, u)$-critical path. It follows that $\{1, 2, \kappa - \Delta\} \subseteq \mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))$. It is obvious that $\mathcal{A}(u_2) \subseteq \mathcal{U}(w_1)$, thus $\mathcal{U}(w_1) = \{1, 3, \kappa - \Delta, \kappa - \Delta + 1\} \cup \mathcal{A}(u_2)$, $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, 2, \kappa - \Delta\}$ and $|\mathcal{U}(u_2) \cap C(wu)| = \Delta - 3$. If $\mathcal{A}(w_1) \subseteq \mathcal{U}(u_3)$, then $\mathcal{U}(u_3) = \mathcal{A}(w_1) \cup \mathcal{A}(u_2) \cup \{\kappa - \Delta\} = C(wu) \cup \{\kappa - \Delta\}$; but reassigning $2, \beta_1$ and α_1 to uu_3, uu_2 and uw respectively results in an acyclic edge coloring of G. So we may assume that $\mathcal{A}(w_1) \nsubseteq \mathcal{U}(u_3)$ and $\alpha_1 \notin \mathcal{U}(u_3)$. If $3 \notin \mathbb{S}$, then reassigning $3, \alpha_1$ and β_1 to uu_4, uw_1 and uw respectively results in an acyclic edge coloring of G. Thus, we have $3 \in \mathbb{S}$. For every color θ in $\mathcal{U}(w) \setminus \{3\}$, we have that $\theta \in \mathbb{S}$; otherwise, reassigning θ, β_1 and α_1 to uu_3, uu_2 and uw respectively results in an acyclic edge coloring of G. If $1 \notin \mathcal{U}(u_3) \cup \mathcal{U}(u_4)$, then reassigning 1 to uu_4 and α_1 to uw_1 will take us back to Subcase 2.1.1. Hence, the color 1 appears exactly three times in $\mathbb{S}$. If $\kappa - \Delta + 1$ appears at least twice in $\mathbb{S}$ or $|X| \geq 1$, then

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq \deg_G(w) + 5.$$

So we may assume that $\kappa - \Delta + 1$ appears precisely once (at w_1) and $X = \emptyset$. Note that $\beta_1 \notin \mathcal{U}(u_4)$. But reassigning β_1 to uu_4 and $\kappa - \Delta + 1$ to uu_2 will take us back to Case 1.

Subcase 2.1.3.2. Now, we may assume that $\{\kappa - \Delta, \kappa - \Delta + 1\} \nsubseteq \mathcal{U}(w_1)$ and $\kappa - \Delta + 1 \notin \mathcal{U}(w_1)$.

Thus, there exists a $(\kappa - \Delta, \alpha_i, u, w_1)$-critical path for every α_i; otherwise, reassigning α_i to uw_1 will take us back to Case 1. It follows that $\kappa - \Delta \in \mathcal{U}(w_1)$ and $\mathcal{A}(w_1) \subseteq \mathcal{U}(u_3) \cap \mathcal{U}(u_2)$. If $\Upsilon(uu_3) \subseteq C(uw)$, then reassigning α_1 to uw_1 and 1 to uu_3 will take us back to Subcase 2.1.2. So we may assume that $\Upsilon(uu_3) \nsubseteq C(wu)$ and $C(wu) \nsubseteq \Upsilon(uu_3)$.

(1) Suppose that $\mathcal{A}(u_2) \cap \mathcal{A}(u_3) = \emptyset$. It follows that $\mathcal{A}(w_1), \mathcal{A}(u_2)$ and $\mathcal{A}(u_3)$ are pairwise disjoint. Suppose that there exists no $(2, 1, u, w)$-critical path. Thus, there exists a (τ, α_1, w, w_1)-critical path, where $\tau \in \mathcal{U}(w) \setminus \{1, 2, 3\}$; otherwise, reassigning 1 to uw and α_1 to ww_1 results in an acyclic edge coloring of G. Since $\mathcal{U}(u_3) \supseteq \mathcal{A}(w_1) \cup \mathcal{A}(u_2)$ and $C(wu) \nsubseteq \mathcal{U}(u_3)$, it follows that $|\mathcal{A}(w_1)| = \Delta - 3$, $|\mathcal{A}(u_2)| = 1$ and $|\mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))| = 2$. If $1 \notin \mathcal{U}(u_3)$, then there exists a $(\kappa - \Delta + 1, 1, u, u_3)$-critical path; otherwise, reassigning 1 to uu_3 and α_1 to uw_1 will take us back to Subcase 2.1.2. Thus, $\Upsilon(uu_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1\}$ or $\{\kappa - \Delta + 1\}$. If there exists no $(\kappa - \Delta + 1, 3, u, u_3)$-critical path, then reassigning $3, \alpha_1$ and β_1 to uu_3, uw_1 and uw results in an acyclic edge coloring of G. Hence, there exists a $(\kappa - \Delta + 1, 3, u, u_3)$-critical path and $\Upsilon(uu_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{\kappa - \Delta + 1\}$. But reassigning 1 to uu_2 will take us back to Subcase 2.1.1.

Now, we consider the other subcase: suppose that there exists a $(2, 1, u, w)$-critical path and $1 \in \mathcal{U}(u_2)$. Since $\mathcal{U}(u_3) \supseteq \mathcal{A}(w_1) \cup \mathcal{A}(u_2)$ and $C(wu) \nsubseteq \mathcal{U}(u_3)$, so we have that $|\mathcal{A}(w_1)| = \Delta - 4$, $|\mathcal{A}(u_2)| = 2$ and $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, 3, \kappa - \Delta\}$, $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, 2, \varepsilon_1\}$ and $|\mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))| = 2$. If $1 \in \mathcal{U}(u_3)$, then $\mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, \kappa - \Delta\}$, and then reassigning β_1, α_1 and 3 to uw, uw_1 and uu_3 results in an cyclic edge coloring. Thus, $1 \notin \mathcal{U}(u_3)$. There exists a $(\kappa - \Delta + 1, 1, u, u_3)$-critical path; otherwise, reassigning 1 to uu_3 and α_1 to uw_1 will take us back to Subcase 2.1.2. This implies that $\mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{\kappa - \Delta, \kappa - \Delta + 1\}$ and 1 appears three times in $\mathbb{S}$. There exists a $(\kappa - \Delta + 1, 3, u, u_3)$-critical path, otherwise, reassigning β_1, α_1 and 3 to uw, uw_1 and uu_3 results in an acyclic edge coloring of G. Now, we have $\{1, 3\} \subseteq \Upsilon(uu_4)$. If $\beta \in \mathcal{A}(u_2) \cap \mathcal{A}(u_4)$, then $\varepsilon_1 = \kappa - \Delta$ and there exists a $(\kappa - \Delta, \beta, u, u_2)$-critical path; but reassigning β to uu_4 and $\kappa - \Delta + 1$ to uw results in an acyclic edge coloring of G. This implies that $\mathcal{A}(u_2) \subseteq \mathcal{U}(u_4)$ and $\mathcal{A}(u_2) \subseteq X$. Suppose that $\{4, 5, \dots, \deg_G(w) - 1\} \nsubseteq \mathbb{S}$. So, by symmetry, we may assume that $4 \notin \mathbb{S}$. There exists a $(4, \beta_1, w, w_1)$-alternating path; otherwise, reassigning 4 to uw_1 and β_1 to uw results in an acyclic edge coloring of G. But reassigning $4, \alpha_1$ and β_1 to uu_3, uw_1 and uw results in an acyclic edge coloring of G. Hence, $\{3, 4, \dots, \deg_G(w) - 1\} \subseteq \mathbb{S}$.

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{3, 4, \dots, \deg_G(w) - 1\}| + 3|\{1\}| + |\{\varepsilon_1\}| + |\{\kappa - \Delta\}| + |\{\kappa - \Delta + 1\}| + |\mathcal{A}(u_2)| \geq \deg_G(w) + 5.$$

(2) So we may assume that $\mathcal{A}(u_2) \cap \mathcal{A}(u_3) \neq \emptyset$, say $\beta_1 \in \mathcal{A}(u_2) \cap \mathcal{A}(u_3)$. Thus, there exists a $(\kappa - \Delta + 1, \beta_1, u, u_2)$-critical path; otherwise, reassigning β_1 to uu_2 will take us back to Case 1. So, we have $\kappa - \Delta + 1 \in \mathcal{U}(u_2)$.

Suppose that the color 1 only appears once (at w_1) in $\mathbb{S}$. If there exists no $(3, 1, u, u_2)$-critical path, then reassigning 1 to uu_2 will take us back to Subcase 2.1.1. But if there exists a $(3, 1, u, u_2)$-critical path, then reassigning 1 to uu_4 and β_1 to uu_2 will take us back to Subcase 2.1.1 again. Hence, the color 1 appears at least twice in $\mathbb{S}$.

If $2 \notin \mathbb{S}$, then reassigning $2, \beta_1$ and α_1 to uu_4, uu_2 and uw respectively results in an acyclic edge coloring of G. If $3 \notin \mathbb{S}$, then reassigning $3, \alpha_1$ and β_1 to uu_3, uw_1 and uw respectively, results in an acyclic edge coloring of G. Suppose that $4 \notin \mathbb{S}$. There exists a $(4, \beta_1, w, w_1)$-alternating path; otherwise, reassigning 4 to uw_1 and β_1 to uw results in an acyclic edge coloring of G. Now, reassigning $4, \alpha_1$ and β_1 to uu_3, uw_1 and uw respectively results in an acyclic edge coloring of G. Thus, $\{2, 3, 4\} \subseteq \mathbb{S}$. By symmetry, we have that $\mathcal{U}(w) \setminus \{1, 2, 3\} \subseteq \mathbb{S}$.

Suppose that $\kappa - \Delta$ appears only once (at w_1) in $\mathbb{S}$. Thus, there exists a $(3, \kappa - \Delta, u, w)$-critical path; otherwise, reassigning $\kappa - \Delta$ to uw and β_1 to uu_3 results in an acyclic edge coloring of G. But reassigning β_1 to uu_3 and $\kappa - \Delta$ to uu_2 will take us back to Case 1. Hence, the color $\kappa - \Delta$ appears at least twice in $\mathbb{S}$.

Note that $|\mathcal{A}(w_1)| \geq 2$. If $\mathcal{A}(w_1) \subseteq \mathcal{U}(u_4)$, then $\mathcal{A}(w_1) \subseteq X$, and then

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{\kappa - \Delta + 1, 2, 3, \ldots, \deg_G(w) - 1\}| + 2|\{1, \kappa - \Delta\}| + |\mathcal{A}(w_1)| \geq \deg_G(w) + 5.$$

So we may assume that $\mathcal{A}(w_1) \nsubseteq \mathcal{U}(u_4)$, say $\alpha_1 \notin \mathcal{U}(u_4)$. There exists a $(2, \kappa - \Delta + 1, w, u)$-critical path; otherwise, reassigning α_1 to uu_4 and $\kappa - \Delta + 1$ to uw results in an acyclic edge coloring of G. Consequently, there exists a $(\kappa - \Delta, \kappa - \Delta + 1, u, w_1)$-critical path and $\kappa - \Delta + 1 \in \mathcal{U}(u_3)$; otherwise, reassigning α_1 to uu_4 and $\kappa - \Delta + 1$ to uw_1 will take us back to Case 1. Hence, the color $\kappa - \Delta + 1$ appears exactly twice in $\mathbb{S}$.

Suppose that there exists no $(2, 1, u, w)$-critical path. Thus, there exists a (τ, α_1, w, w_1)-critical path with $\tau \in \mathcal{U}(w) \setminus \{1, 2, 3\}$; otherwise, reassigning 1 to uw and α_1 to ww_1 results in an acyclic edge coloring of G. If τ only appears once (at w_1) in $\mathbb{S}$, then reassigning τ, α_1 and β_1 to uu_3, uw_1 and uw respectively results in an acyclic edge coloring of G. Hence, the color τ appears at least twice in $\mathbb{S}$. Hence,

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{2, 3, \ldots, \deg_G(w) - 1\}| + 2|\{1, \kappa - \Delta, \kappa - \Delta + 1\}| + |\{\tau\}| = \deg_G(w) + 5.$$

Suppose there exists a $(2, 1, u, w)$-critical path and $1 \in \mathcal{U}(u_2)$. If $1 \notin \mathcal{U}(u_3) \cup \mathcal{U}(u_4)$, then reassigning 1 to uu_3 and α_1 to uw_1 will take us back to Subcase 2.1.1. Hence, the color 1 appears at least three times in $\mathbb{S}$,

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{2, 3, \ldots, \deg_G(w) - 1\}| + 3|\{1\}| + 2|\{\kappa - \Delta, \kappa - \Delta + 1\}| = \deg_G(w) + 5.$$

Subcase 2.1.4. Neither the color on $w_1 w$ nor the color on $w_1 u$ is a common color.

By symmetry, assume that $\phi(uw_1) = \kappa - \Delta$, $\phi(uu_2) = 2$, $\phi(uu_3) = 3$ and $\phi(uu_4) = \kappa - \Delta + 1$.

If $\Upsilon(uu_2) \subseteq C(wu)$, then reassigning 1 to uu_2 will take us back to Subcase 2.1.2. This implies that $\Upsilon(uu_2) \nsubseteq C(wu)$ and $\mathcal{A}(u_2) \neq \emptyset$. Thus, there exists a $(\varepsilon_i, \beta_i, u, u_2)$-critical path with $\varepsilon_i \in \{\kappa - \Delta, \kappa - \Delta + 1\}$; otherwise, reassigning β_i to uu_2 will take us back to Case 1. Similarly, we have that $\Upsilon(uu_3) \nsubseteq C(wu)$ and $\mathcal{A}(u_3) \neq \emptyset$, and thus there exists a (m_i, ξ_i, u, u_3)-critical path with $m_i \in \{\kappa - \Delta, \kappa - \Delta + 1\}$. If $1 \notin \mathcal{U}(u_2) \cup \mathcal{U}(u_3)$, then reassigning 1 to uu_2 will create a $(1, \kappa - \Delta + 1)$-dichromatic cycle containing uu_2, otherwise, it will take us back to Subcase 2.1.2; but reassigning 1 to uu_3 will take us back to Subcase 2.1.2 again. It follows that $1 \in \mathcal{U}(u_2) \cup \mathcal{U}(u_3)$ and 1 appears at least twice in $\mathbb{S}$.

Subcase 2.1.4.1. Suppose that $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \nsubseteq \mathcal{U}(w_1)$ and $\beta_1 = \alpha_1 \notin \mathcal{U}(w_1)$.

Hence, there exists a $(3, \beta_1, u, w)$-critical path and $(\kappa - \Delta + 1, \beta_1, u, u_2)$-critical path, thus $\kappa - \Delta + 1 \in \mathcal{U}(u_2)$.

There exists a $(2, \kappa - \Delta, u, w)$- or $(3, \kappa - \Delta, u, w)$-critical path; otherwise, reassigning β_1 to uw_1 and $\kappa - \Delta$ to uw results in an acyclic edge coloring of G. It follows that $\kappa - \Delta \in \mathcal{U}(u_2) \cup \mathcal{U}(u_3)$. Moreover, $\kappa - \Delta$ appears at least twice in $\mathbb{S}$; otherwise, assume that $\kappa - \Delta$ only appears at u_2, thus reassigning $\kappa - \Delta$ to uu_3 and β_1 to uw_1 will take us back to Case 1.

If $2 \notin \mathbb{S}$, then reassigning $\beta_1, 2$ and ξ_1 to uu_2, uu_4 and uw respectively results in an acyclic edge coloring of G. Thus $2 \in \mathbb{S}$.

Suppose that $4 \notin \mathbb{S}$. There exists a $(4, \xi_1, w, u_2)$-alternating path for every $\xi_i \in \mathcal{A}(u_3)$; otherwise, reassigning 4 to uu_2 and ξ_1 to uw results in an acyclic edge coloring of G. Now, reassigning $\beta_1, 4$ and ξ_1 to uu_2, uu_4 and uw respectively results in an acyclic edge coloring of G again. Hence, the color 4 appears in $\mathbb{S}$. Similarly, we can prove that $\mathcal{U}(w) \setminus \{1, 2, 3\} \subseteq \mathbb{S}$.

Suppose that $3 \notin \mathbb{S}$. If there exists no $(\kappa - \Delta + 1, \xi_i, u, u_3)$-critical path, then reassigning 3 to uw_1 and ξ_i to uu_3 will take us back to Subcase 2.1.3. Hence, there exists a $(\kappa - \Delta + 1, \xi_i, u, u_3)$-critical path for every $\xi_i \in \mathcal{A}(u_3)$, and then $\kappa-\Delta+1 \in \mathcal{U}(u_3)$ and $\mathcal{A}(u_3) \subseteq \mathcal{U}(u_4)$. If there exists no $(\kappa-\Delta, \xi_i, u, u_3)$-critical path, then reassigning $3, \xi_i$ and β_1 to uu_4, uu_3 and uw respectively, results in an acyclic edge coloring of G. Hence, both $(\kappa-\Delta, \xi_i, u, u_3)$- and $(\kappa-\Delta+1, \xi_i, u, u_3)$-critical path exist for every $\xi_i \in \mathcal{A}(u_3)$, and then $\{\kappa-\Delta, \kappa-\Delta+1\} \subseteq \mathcal{U}(u_3)$ and $\mathcal{A}(u_3) \subseteq \mathcal{U}(w_1) \cap \mathcal{U}(u_4)$. Clearly, every color in $\mathcal{A}(u_3)$ appears precisely three times in $\mathbb{S}$. Therefore,

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{2\} \cup \{4, \ldots, \deg_G(w)-1\}| + 2|\{1, \kappa-\Delta, \kappa-\Delta+1\}| + |\mathcal{A}(u_3)| \geq \deg_G(w) + 5.$$

So, in the following, we may assume that $3 \in \mathbb{S}$.

If there exists a $(2, 1, u, w)$-critical path (or $(3, 1, u, w)$-critical path) and 1 appears only twice in $\mathbb{S}$, then reassigning 1 to uu_3 (to uu_2) will take us back to Subcase 2.1.2. In other words, if there exists a $(2, 1, u, w)$-critical path or $(3, 1, u, w)$-critical path, then the color 1 appears at least three times in $\mathbb{S}$.

Suppose that neither $(2, 1, u, w)$-critical path nor $(3, 1, u, w)$-critical path exists. If there exists no (τ, β_1, w, w_1)-critical path with some $\tau \in \mathcal{U}(w) \setminus \{1, 3\}$, then reassigning 1 to uw and β_1 to ww_1 results in an acyclic edge coloring of G. Hence, there exists a (τ, β_1, w, w_1)-critical path with some $\tau \in \mathcal{U}(w) \setminus \{1, 3\}$. Suppose that there exists a $(2, \beta_1, w, w_1)$-critical path and 2 appears only once in $\mathbb{S}$. This implies that there exists a $(\kappa-\Delta, 2, u, u_4)$-critical path; otherwise reassigning $2, \beta_1$ and ξ_1 to uu_4, uu_2 and uw respectively results in an acyclic edge coloring of G. But reassigning $2, \beta_1, \xi_1$ and ζ^* ($\zeta^* = \beta_2$ if $|\mathcal{A}(u_2)| \geq 2$, otherwise, $\zeta^* = \kappa-\Delta$) to uu_4, uw_1, uw and uu_2 respectively, and we obtain an acyclic edge coloring of G. Thus, if there exists a $(2, \beta_1, w, w_1)$-critical path, then the color 2 appears at least twice in $\mathbb{S}$. Suppose that there exists a $(4, \beta_1, w, w_1)$-critical path and 4 only appears once in $\mathbb{S}$. Hence, there is a $(\kappa-\Delta, 4, u, u_3)$-critical path, otherwise, reassigning 4 to uu_3 and β_1 to uw results in an acyclic edge coloring of G. Now, reassigning $4, \beta_1$ and ξ_1 to uu_4, uu_2 and uw will create a $(4, \xi_1)$-dichromatic cycle containing uw; otherwise, the resulting coloring is an acyclic edge coloring of G. But reassigning 4 to uu_2 and ξ_1 to uw results in an acyclic edge coloring of G. Thus, if there exists a $(4, \beta_1, w, w_1)$-critical path, then the color 4 appears at least twice in $\mathbb{S}$. Similarly, if there exists a (τ, β_1, w, w_1)-critical path with $\tau \geq 4$, then the color τ appears at least twice in $\mathbb{S}$. Therefore, the color τ appears at least twice in $\mathbb{S}$.

By the above arguments, regardless of the existence of $(2, 1, u, w)$-critical path or $(3, 1, u, w)$-critical path, if $\kappa-\Delta+1$ appears at least twice or $|X| \geq 1$, then

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq \deg_G(w) + 5.$$

So we may assume that the color $\kappa-\Delta+1$ appears only once (at u_2) in $\mathbb{S}$ and $X = \emptyset$. If $\mathcal{A}(u_3) \nsubseteq \mathcal{U}(w_1)$, say $\xi_1 \notin \mathcal{U}(w_1)$, then there exists a $(2, \xi_1, u, w)$-critical path and $(\kappa-\Delta+1, \xi_1, u, u_3)$-critical path, and then $\kappa-\Delta+1 \in \mathcal{U}(u_3)$, a contradiction. So we may assume that $\mathcal{A}(u_3) \subseteq \mathcal{U}(w_1)$ and $\mathcal{A}(u_3) \cap \mathcal{U}(u_4) = \emptyset$.

Clearly, there exists a $(2, \xi_1, u, w)$-critical path. Thus, there exists a $(\kappa-\Delta, \xi_1, u, u_3)$-critical path; otherwise, reassigning ξ_1 to uu_3 will take us back to Case 1. Hence, there exists a $(2, \kappa-\Delta+1, u, w)$-critical path; otherwise, reassigning ξ_1 to uu_4 and $\kappa-\Delta+1$ to uw will result in an acyclic edge coloring of G. Now, reassigning ξ_1 to uu_4 and $\kappa-\Delta+1$ to uu_3 will take us back to Case 1.

Subcase 2.1.4.2. $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \subseteq \mathcal{U}(w_1)$.

Firstly, suppose that $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \nsubseteq \mathcal{U}(u_4)$ and $\beta_1 = \zeta_1 \notin \mathcal{U}(u_4)$. Hence, there exists a $(3, \beta_1, u, w)$-critical path and a $(\kappa-\Delta, \beta_1, u, u_2)$-critical path, and then $\kappa-\Delta \in \mathcal{U}(u_2)$.

If $\{2, 3, \kappa-\Delta+1\} \cap \mathcal{U}(w_1) = \emptyset$, then reassigning β_1 to uu_2 and 2 to uw_1 will take us back to Subcase 2.1.3. Hence, $\{2, 3, \kappa-\Delta+1\} \cap \mathcal{U}(w_1) \neq \emptyset$. Recall that $1 \in \mathcal{U}(u_2) \cup \mathcal{U}(u_3)$. Since $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \subseteq \mathcal{U}(w_1)$, it follows that $\deg_G(w_1) = 6$, $\deg_G(w) = \kappa-\Delta$, and $|\mathcal{A}(u_2)| + |\mathcal{A}(u_3)| = 3$. Furthermore, we have that $\{\kappa-\Delta, \kappa-\Delta+1\} \cap \mathcal{U}(u_2) = \{\kappa-\Delta\}$ and $|\{\kappa-\Delta, \kappa-\Delta+1\} \cap \mathcal{U}(u_3)| = 1$. Thus, there exists a $(3, \kappa-\Delta+1, w, u)$-critical path; otherwise reassigning β_1 to uu_4 and $\kappa-\Delta+1$ to uw will result in an acyclic edge coloring of G. Hence, $\{\kappa-\Delta, \kappa-\Delta+1\} \cap \mathcal{U}(u_3) = \{\kappa-\Delta+1\}$.

If $1 \notin \mathcal{U}(u_2)$, then $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{2, \kappa-\Delta\}$, but reassigning 1 to uu_2 will take us back to Subcase 2.1.2. This implies that $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, 2, \kappa-\Delta\}$ and $\mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{3, \kappa-\Delta+1\}$. Now, there is a $(\kappa-\Delta+1, 1, u, u_3)$-critical path; otherwise, reassigning 1 to uu_3 will take us back to Subcase 2.1.2. Thus, there exists a $(\kappa-\Delta, \kappa-\Delta+1, u, u_2)$-critical path; otherwise, reassigning β_1 to uu_4 and $\kappa-\Delta+1$ to uu_2 will take us back to Case 1. Hence, $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, \kappa-\Delta, \kappa-\Delta+1\}$. Moreover, there exists a $(\kappa-\Delta+1, 2, u, w_1)$-critical path; otherwise, reassigning $\beta_1, 2$ and $\kappa-\Delta$ to uu_2, uw_1 and uw respectively, results in an acyclic edge coloring of G. It is obvious that $2 \in \mathcal{U}(u_4)$. If $\kappa-\Delta \notin \mathcal{U}(u_4)$, then reassigning $\kappa-\Delta, \beta_1, 2$ and ξ_1 to uu_4, uu_2, uw_1 and uw respectively, results in an acyclic edge coloring of G. Thus, $\kappa-\Delta \in \mathcal{U}(u_4)$. Recall that $\{1,2\} \subseteq \mathcal{U}(u_4)$. If there exists a color τ in $\mathcal{U}(w) \setminus \mathcal{U}(u_4)$, then reassigning τ, ξ_1 and β_1 to uu_4, uu_3 and uw respectively, will result in an acyclic edge coloring of G. Hence, $\mathcal{U}(w) \subseteq \mathcal{U}(u_4)$. Then

$$\sum_{\theta\in\mathcal{U}(w)\cup\mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\mathcal{U}(w)| + 2|\{1, \kappa-\Delta, \kappa-\Delta+1\}| = \deg_G(w) + 5.$$

Secondly, suppose that $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \subseteq \mathcal{U}(u_4)$. Thus, every color in $\mathcal{A}(u_2) \cup \mathcal{A}(u_3)$ appears three times in $\mathbb{S}$, and then $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \subseteq X$. Recall that $1 \in \mathcal{U}(u_2) \cup \mathcal{U}(u_3)$. Since $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \subseteq \mathcal{U}(w_1)$, it follows that $|\{\kappa-\Delta, \kappa-\Delta+1\} \cap \mathcal{U}(u_2)| = 1$ or $|\{\kappa-\Delta, \kappa-\Delta+1\} \cap \mathcal{U}(u_3)| = 1$.

Suppose that $1 \in \Upsilon(uu_2) \cap \Upsilon(uu_3)$. It follows that $\deg_G(w) = 6$ and $\mathcal{U}(w_1) = \{1, \kappa-\Delta\} \cup \mathcal{A}(u_2) \cup \mathcal{A}(u_3)$. Moreover, we have that $|\{\kappa-\Delta, \kappa-\Delta+1\} \cap \Upsilon(uu_2)| = 1$ and $|\{\kappa-\Delta, \kappa-\Delta+1\} \cap \Upsilon(uu_3)| = 1$. If $\kappa-\Delta+1 \notin \Upsilon(uu_2)$, then reassigning $\beta_1, 2$ and ξ_1 to uu_2, uw_1 and wu respectively results in an acyclic edge coloring of G. So we have that $\kappa-\Delta+1 \in \Upsilon(uu_2)$. Similarly, we have that $\kappa-\Delta+1 \in \Upsilon(uu_3)$. But reassigning α_1 to uw_1 and $\kappa-\Delta$ to wu results in an acyclic edge coloring of G.

So we may assume that $1 \notin \Upsilon(uu_2) \cap \Upsilon(uu_3)$ and $1 \in \Upsilon(uu_2)$. If there is a $(2, 1, u, u_3)$-critical path, then $\mathcal{U}(w_1) = \{1, \kappa-\Delta\} \cup \mathcal{A}(u_2) \cup \mathcal{A}(u_3)$ and $2 \in \mathcal{U}(u_3)$, but reassigning α_1 to ww_1 and 1 to wu results in an acyclic edge coloring of G. Hence, there exists no $(2, 1, u, u_3)$-critical path. Thus, there exists a $(\kappa-\Delta+1, 1, u, u_3)$-critical path, otherwise, reassigning 1 to uu_3 will take us back to Subcase 2.1.2. This implies that the color 1 appears at least three times in $\mathbb{S}$.

Suppose that $3 \notin \mathbb{S}$. Thus, there exists a $(2, \kappa-\Delta+1, w, u)$-critical path; otherwise, reassigning 3, 1 and $\kappa-\Delta+1$ to uu_4, uu_3 and uw respectively, results in an acyclic edge coloring of G. Hence, $\kappa-\Delta+1 \in \mathcal{U}(u_2) \cap \mathcal{U}(u_3)$. If $\kappa-\Delta \notin \mathcal{U}(u_3)$, then reassigning $3, \xi_1$ and β_1 to uu_4, uu_3 and uw respectively, results in an acyclic edge coloring of G. So we may assume that $\kappa-\Delta \in \mathcal{U}(u_3)$. Hence, $|\mathcal{A}(u_2)| = |\mathcal{A}(u_3)| = 2$ and $\mathcal{U}(w_1) = \{1, \kappa-\Delta\} \cup \mathcal{A}(u_2) \cup \mathcal{U}(u_3)$. Now, reassigning $3, 1, \beta_1$ and α_1 to uu_4, uu_3, uw and ww_1, results in an acyclic edge coloring of G. Therefore, we can conclude that $3 \in \mathbb{S}$.

Suppose that $4 \notin \mathbb{S}$. Thus, there is a $(4, \beta_1, w, u_3)$-alternating path; otherwise, reassigning 4 to uu_3 and β_1 to uw will result in an acyclic edge coloring of G. Similarly, there exists a $(4, \xi_1, w, u_2)$-alternating path. Moreover, there exists a $(\kappa-\Delta, \xi_1, u, u_3)$-critical path; otherwise, reassigning 4, ξ_1 and β_1 to uu_4, uu_3 and uw respectively, results in an acyclic edge coloring of G. Thus, $\mathcal{U}(u_3) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{3, \kappa-\Delta, \kappa-\Delta+1\}$, $|\mathcal{A}(u_3)| = 2$ and $|\mathcal{A}(u_2)| = 2$. Hence, $|\mathcal{U}(u_2) \cap \{\kappa-\Delta, \kappa-\Delta+1\}| = 1$. If $\kappa-\Delta \notin \mathcal{U}(u_2)$, then reassigning $4, \beta_1$ and ξ_1 to uu_4, uu_2 and uw respectively, results in an acyclic edge coloring of G. Hence, we have that $\mathcal{U}(u_2) \cap (\mathcal{U}(w) \cup (u)) = \{1, 2, \kappa-\Delta\}$. But reassigning $\xi_1, 4$ and β_1 to uw, uw_1 and uu_2 respectively, results in an acyclic edge coloring of G. So, $4 \in \mathbb{S}$. Similarly, we have that $\mathcal{U}(w) \setminus \{1, 2, 3\} \subseteq \mathbb{S}$.

Recall that $|\mathcal{A}(u_2) \cup \mathcal{A}(u_3)| \geq 3$, $|\{\kappa-\Delta, \kappa-\Delta+1\} \cap \mathcal{U}(u_2)| \geq 1$ and $|\{\kappa-\Delta, \kappa-\Delta+1\} \cap \mathcal{U}(u_3)| \geq 1$. Hence,

$$\sum_{\theta\in\mathcal{U}(w)\cup\mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{3, 4, \ldots, \deg_G(w)-1\}| + 3|\{1\}| + 1 + 1 + |\mathcal{A}(u_2) \cup \mathcal{A}(u_3)| \geq \deg_G(w) + 5.$$

Subcase 2.2. $\mathcal{U}(w) \cap \mathcal{U}(u) = \{\lambda_1, \lambda_2, \lambda_3\}$ and $w_1 = u_1$. Note that $|C(wu)| \geq \Delta$.

Subcase 2.2.1. The color on uw_1 is a common color.

By symmetry, assume that $\phi(uw_1) = \lambda_1$, $\phi(uu_2) = \lambda_2$, $\phi(uu_3) = \lambda_3$ and $\phi(uu_4) = \kappa-\Delta$.

If $\Upsilon(uu_2) \subseteq C(wu)$, then reassigning β_1 to uu_2 will take us back to Subcase 2.1. So we have that $\Upsilon(uu_2) \nsubseteq C(wu)$ and $|\mathcal{U}(u_2) \cap C(wu)| \leq \Delta-2$; similarly, we also have that $|\mathcal{U}(u_3) \cap C(wu)| \leq \Delta-2$. If $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, \lambda_1\}$, then reassigning α_1 to uw_1 will take us back to Subcase 2.1 again. Hence, $|\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))| \geq 3$. By Claim 2, we have $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \subseteq \mathcal{U}(w_1)$ and $\mathcal{A}(u_2) \cap \mathcal{A}(u_3) = \emptyset$. Further, we have that $|\mathcal{U}(w_1)| \geq 3 + |\mathcal{A}(u_2)| + |\mathcal{A}(u_3)| > \deg_G(w_1)$, which is a contradiction.

Subcase 2.2.2. The color on uw_1 is not a common color, but the color on ww_1 is a common color.

By symmetry, assume that $\phi(uw_1) = \kappa-\Delta$, $\phi(uu_2) = 2$, $\phi(uu_3) = 3$ and $\phi(uu_4) = 1$.

If $\Upsilon(uu_2) \subseteq C(wu)$, then reassigning β_1 to uu_2 will take us back to Subcase 2.1. So we have that $\Upsilon(uu_2) \nsubseteq C(wu)$ and $|\mathcal{U}(u_2) \cap C(wu)| \leq \Delta-2$; similarly, we also have that $|\mathcal{U}(u_3) \cap C(wu)| \leq \Delta-2$ and $|\mathcal{U}(u_4) \cap C(wu)| \leq \Delta-2$.

If $\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u)) = \{1, \kappa-\Delta\}$, then reassigning α_1 to ww_1 will take us back to Subcase 2.1 again. Hence, $|\mathcal{U}(w_1) \cap (\mathcal{U}(w) \cup \mathcal{U}(u))| \geq 3$.

Furthermore, we have that $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \nsubseteq \mathcal{U}(w_1)$. Otherwise, if $\deg_G(w) \leq \kappa - \Delta$, then $|\mathcal{U}(w_1)| \geq 3 + |\mathcal{A}(u_2)| + |\mathcal{A}(u_3)| \geq 3 + 2 + 2 > 6$; and if $\deg_G(w) \leq \kappa - \Delta - 1$, then $|\mathcal{U}(w_1)| \geq 3 + |\mathcal{A}(u_2)| + |\mathcal{A}(u_3)| \geq 3 + 3 + 3 > 7$. Without loss of generality, assume that $\beta_1 \notin \mathcal{U}(w_1)$. Since $\beta_1 \notin \mathcal{U}(w_1) \cup \mathcal{U}(u_2)$, it follows that there exists a $(3, \beta_1, u, w)$-critical path. There exists a $(1, \beta_1, u, u_2)$-critical path; otherwise, reassigning β_1 to uu_2 will take us back to Subcase 2.1. Hence, $1 \in \mathcal{U}(u_2)$ and 1 appears at least twice in $\mathbb{S}$.

There exists a $(2, \kappa - \Delta, u, w)$- or $(3, \kappa - \Delta, u, w)$-critical path; otherwise, reassigning $\kappa - \Delta$ to uw and β_1 to uw_1 will result in an acyclic edge coloring of G. If $\kappa - \Delta$ appears only once in $\mathbb{S}$, then reassigning β_1 to uw_1 and $\kappa - \Delta$ to uu_4 will take us back to Subcase 2.1. Hence, the color $\kappa - \Delta$ appears at least twice in $\mathbb{S}$.

Let $t \in \mathcal{U}(w) \setminus \{1, 3\}$. If $t \notin \mathbb{S}$, then reassigning β_1 to uu_2 and t to uu_4 will take us back to Subcase 2.1. Hence, we have that $\mathcal{U}(w) \setminus \{1, 3\} \subseteq \mathbb{S}$.

If $\mathcal{A}(u_3) \subseteq \mathcal{U}(w_1)$, then every color in $\mathcal{A}(u_3)$ appears precisely three times in $\mathbb{S}$, and then

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{2, 4, 5, \ldots, \deg_G(w) - 1\}| + 2|\{1, \kappa - \Delta\}| + |\mathcal{A}(u_3)| \geq \deg_G(w) + 3.$$

So we may assume that $\mathcal{A}(u_3) \nsubseteq \mathcal{U}(w_1)$ and $\xi_1 \notin \mathcal{U}(w_1) \cup \mathcal{U}(u_3)$. Similar to above, we can prove that there exists a $(2, \xi_1, w, u)$- and $(1, \xi_1, u, u_3)$-critical path, and then 1 appears precisely three times in $\mathbb{S}$. If $3 \notin \mathbb{S}$, then reassigning 3 to uu_4 and ξ_1 to uu_3 will take us back to Subcase 2.1. Thus, the color 3 appears at least once in $\mathbb{S}$. Therefore, we have

$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq |\{2, 3, \ldots, \deg_G(w) - 1\}| + 2|\{\kappa - \Delta\}| + 3|\{1\}| = \deg_G(w) + 3.$$

Subcase 2.2.3. Neither the color on $w_1 w$ nor the color on $w_1 u$ is a common color.

By symmetry, assume that $\phi(uw_1) = \kappa - \Delta$, $\phi(uu_2) = 2$, $\phi(uu_3) = 3$ and $\phi(uu_4) = 4$.

If $\Upsilon(uu_2) \subseteq C(wu)$, then reassigning β_1 to uu_2 will take us back to Subcase 2.1. So we have that $\Upsilon(uu_2) \nsubseteq C(wu)$ and $|\mathcal{U}(u_2) \cap C(wu)| \leq \Delta - 2$; similarly, we also have that $|\mathcal{U}(u_3) \cap C(wu)| \leq \Delta - 2$ and $|\mathcal{U}(u_4) \cap C(wu)| \leq \Delta - 2$.

Suppose that the color 1 appears at most twice in $\mathbb{S}$; by symmetry, assume that $1 \notin \mathcal{U}(u_3) \cup \mathcal{U}(u_4)$. Thus there exists a $(2, 1, u, u_4)$-critical path; otherwise, reassigning 1 to uu_4 will take us back to Subcase 2.2.2. But reassigning 1 to uu_3 will take us back to Subcase 2.2.2 again. Hence, the color 1 appears at least three times in $\mathbb{S}$.

Furthermore, $\mathcal{A}(u_2) \cup \mathcal{A}(u_3) \cup \mathcal{A}(u_4) \nsubseteq \mathcal{U}(w_1)$; otherwise, we have $|\mathcal{U}(w_1)| \geq 2 + |\mathcal{A}(u_2)| + |\mathcal{A}(u_3)| + |\mathcal{A}(u_4)| > \deg_G(w_1)$, which is a contradiction. Without loss of generality, assume that $\beta_1 \notin \mathcal{U}(w_1)$. Clearly, there exists a $(3, \beta_1, u, w)$- or $(4, \beta_1, u, w)$-critical path. By symmetry, assume that there exists a $(3, \beta_1, u, w)$-critical path. There exists a $(4, \beta_1, u, u_2)$-critical path; otherwise, reassigning β_1 to uu_2 will take us back to Subcase 2.1. It follows that $4 \in \mathcal{U}(u_2)$.

If $2 \notin \mathbb{S}$, then reassigning 2 to uu_4 and β_1 to uu_2 will take us back to Subcase 2.1. So we have $2 \in \mathbb{S}$; similarly, we can obtain that $\mathcal{U}(w) \setminus \{1, 3, 4\} \subseteq \mathbb{S}$.

If $3 \notin \mathbb{S}$, then $4 \in \mathcal{U}(w_1) \cup \mathcal{U}(u_3)$; otherwise, reassigning $3, 4$ and β_1 to uu_4, uu_3 and uu_2 respectively, and then we go back to Subcase 2.1. Anyway, we have that $\mathrm{mul}_{\mathbb{S}}(3) + \mathrm{mul}_{\mathbb{S}}(4) \geq 2$.

There exists a $(2, \kappa - \Delta, u, w)$- or $(3, \kappa - \Delta, u, w)$- or $(4, \kappa - \Delta, u, w)$-critical path; otherwise, reassigning β_1 to uw_1 and $\kappa - \Delta$ to uw results in an acyclic edge coloring of G. If $\kappa - \Delta \notin \mathcal{U}(u_3) \cup \mathcal{U}(u_4)$, then reassigning $\kappa - \Delta$ to uu_3 and β_1 to uw_1 will take us back to Case 2.1. This implies that $\kappa - \Delta \in \mathcal{U}(u_3) \cup \mathcal{U}(u_4)$; similarly, we can prove that $\kappa - \Delta \in \mathcal{U}(u_2) \cup \mathcal{U}(u_4)$ and $\kappa - \Delta \in \mathcal{U}(u_2) \cup \mathcal{U}(u_3)$. Hence, the color $\kappa - \Delta$ appears at least twice in $\mathbb{S}$. Therefore, we have

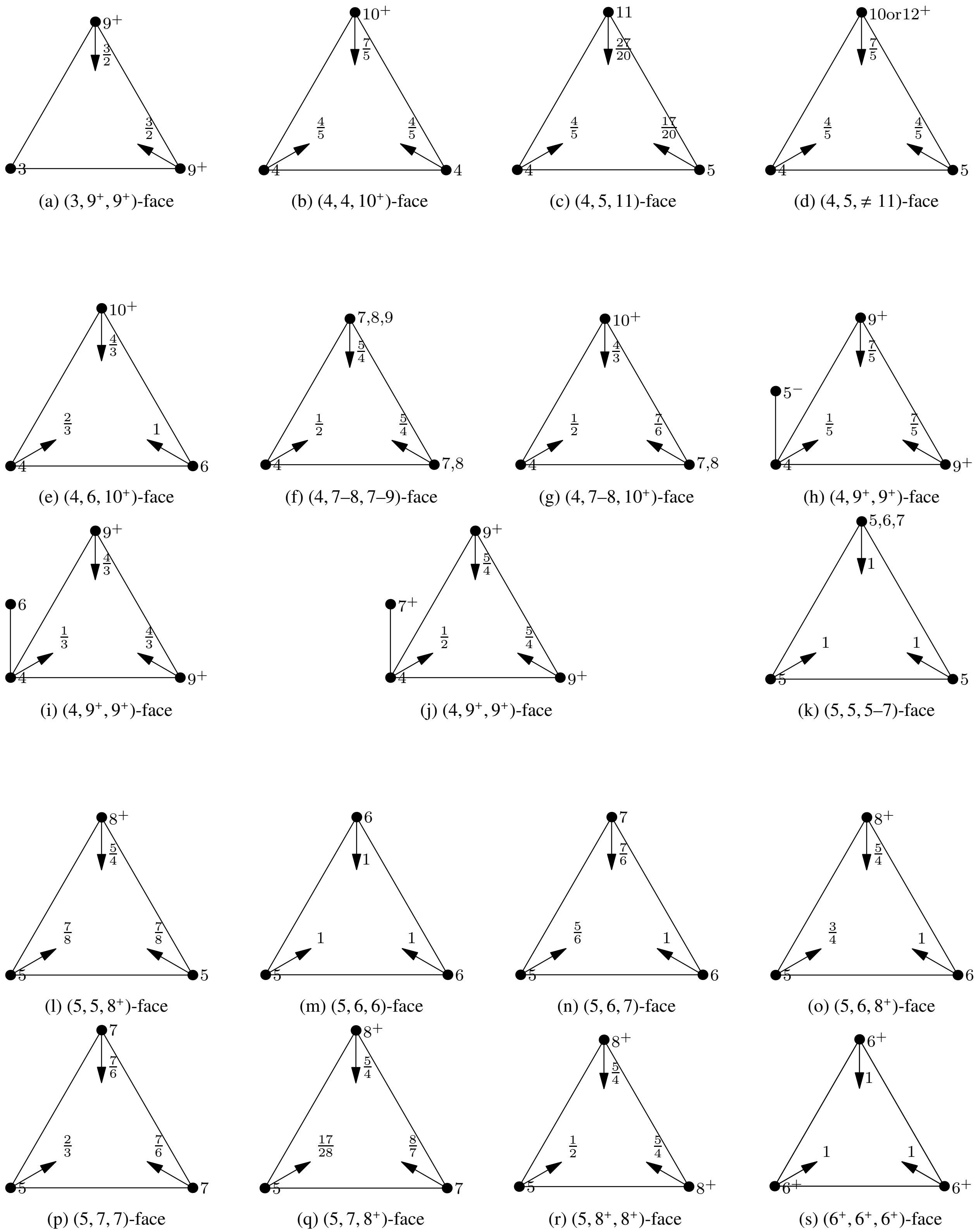
$$\sum_{\theta \in \mathcal{U}(w) \cup \mathcal{U}(u)} \mathrm{mul}_{\mathbb{S}}(\theta) + |X| \geq 3|\{1\}| + 2|\{\kappa - \Delta\}| + \sum_{\theta \in \mathcal{U}(w) \setminus \{1\}} \mathrm{mul}_{\mathbb{S}}(\theta) \geq \deg_G(w) + 3.$$

Subcase 2.3. $|\mathcal{U}(w) \cap \mathcal{U}(u)| = 4$.

In other words, $\mathcal{U}(u) \subseteq \mathcal{U}(w)$. It follows that $|C(wu)| = \kappa - \deg_G(w) + 1 \geq \Delta + 1$ and $|\mathcal{A}(u_i)| \geq 2$ for $i = 2, 3, 4$. By Claim 2, we have that $\mathcal{A}(u_2), \mathcal{A}(u_3)$ and $\mathcal{A}(u_4)$ are pairwise disjoint and $\mathcal{U}(w_1) \supseteq \mathcal{A}(u_2) \cup \mathcal{A}(u_3) \cup \mathcal{A}(u_4)$, which implies that $|\mathcal{U}(w_1)| \geq 2 + |\mathcal{A}(u_2)| + |\mathcal{A}(u_3)| + |\mathcal{A}(u_4)| > \deg_G(w_1)$, a contradiction. □

4 The main result

Now, we are ready to prove the main result, Theorem 1.1.



(a) $(3, 9^+, 9^+)$-face
(b) $(4, 4, 10^+)$-face
(c) $(4, 5, 11)$-face
(d) $(4, 5, \neq 11)$-face
(e) $(4, 6, 10^+)$-face
(f) $(4, 7–8, 7–9)$-face
(g) $(4, 7–8, 10^+)$-face
(h) $(4, 9^+, 9^+)$-face
(i) $(4, 9^+, 9^+)$-face
(j) $(4, 9^+, 9^+)$-face
(k) $(5, 5, 5–7)$-face
(l) $(5, 5, 8^+)$-face
(m) $(5, 6, 6)$-face
(n) $(5, 6, 7)$-face
(o) $(5, 6, 8^+)$-face
(p) $(5, 7, 7)$-face
(q) $(5, 7, 8^+)$-face
(r) $(5, 8^+, 8^+)$-face
(s) $(6^+, 6^+, 6^+)$-face

Fig. 1: Discharging rules

Proof of Theorem 1.1. Suppose that G is a counterexample with $|V| + |E|$ is minimum, and fix $\kappa = \Delta(G) + 6$. Since the hypothesis is minor-closed, it follows that G is a κ-minimal graph. Let G^* be obtained from G by removing all the 2-vertices. By Lemma 1 and Lemma 3, the minimum degree of G^* is at least three. Take a component H of G^* and embed it in the plane. In the following, we will do arguments on the graph H to obtain a contradiction.

By Lemma 3 (A), we have the following claims.

Claim 1. If $\deg_H(v) < \deg_G(v)$, then $\deg_H(v) \geq 8 + m$, where m is the number of adjacent 7^--vertices in H.

Claim 2. If $\deg_H(v) \leq 7$, then $\deg_G(v) = \deg_H(v)$.

From the Euler's formula, we have the following equality:

$$\sum_{v \in V(H)} (2\deg_H(v) - 6) + \sum_{f \in F(H)} (\deg_H(f) - 6) = -12 \tag{8}$$

Assign the initial charge of every vertex v to be $2\deg_H(v) - 6$ and the initial charge of every face f to be $\deg_H(f) - 6$. Clearly, the sum of the initial charge of vertices and faces is -12. We design appropriate discharging rules and redistribute charge among the vertices and faces, such that the final charge of every vertex and every face is nonnegative, which derive a contradiction.

Discharging Rules:

(R1) If w is a 4-vertex adjacent to a 5^--vertex u, then w sends $\frac{4}{5}$ to each face incident with wu, and sends $\frac{1}{5}$ to each other face.

(R2) If w is a 4-vertex adjacent to a 6-vertex u, then w sends $\frac{2}{3}$ to each face incident with wu, and sends $\frac{1}{3}$ to each other face.

(R3) If w is a 4-vertex which is not adjacent to 6^--vertices, then w sends $\frac{1}{2}$ to each incident face.

(R4) All the rules regarding 3-faces are in the Fig (a)–(s).

(R5) Every 9^+-vertex sends 1 to each incident 4^+-face.

(R6) Every vertex with degree $5, 6, 7$ or 8 sends $\frac{1}{2}$ to each incident 4^+-face.

Computing the final charge of faces.

Let $f = w_1w_2w_3$ be a 3-face with $\deg_H(w_1) \leq \deg_H(w_2) \leq \deg_H(w_3)$.

If w_1 is a 3-vertex, then Lemma 6 implies that both w_2 and w_3 are 9^+-vertices in G, and they also are 9^+-vertices in H by Claim 1, thus f is a $(3, 9^+, 9^+)$-face in H and the final charge is $-3 + 2 \times \frac{3}{2} = 0$.

If w_1w_2 is a $(4, 4)$-edge, then Lemma 9 implies that w_3 is a 12^+-vertex in G, and it is a 10^+-vertex in H by Claim 1, thus f is a $(4, 4, 10^+)$-face and the final charge is $-3 + 2 \times \frac{4}{5} + \frac{7}{5} = 0$.

If w_1w_2 is a $(4, 5)$-edge, then Lemma 9 implies that w_3 is a 11^+-vertex in G, and it is a 10^+-vertex in H by Claim 1, thus the final charge of f is $-3 + \frac{4}{5} + \frac{17}{20} + \frac{27}{20} = 0$ if $\deg_H(w_3) = 11$, or $-3 + 2 \times \frac{4}{5} + \frac{7}{5} = 0$ if w_3 is a 10- or 12^+-vertex in H.

If w_1w_2 is a $(4, 6)$-edge, then Lemma 9 implies that w_3 is a 10^+-vertex in G, and it is a 10^+-vertex in H by Claim 1, and then the final charge is $-3 + \frac{2}{3} + 1 + \frac{4}{3} = 0$.

If $\deg_H(w_1) = 4$, $\deg_H(w_2) \in \{7, 8\}$ and $\deg_H(w_3) \in \{7, 8, 9\}$, then the final charge of f is $-3 + \frac{1}{2} + 2 \times \frac{5}{4} = 0$.

If $\deg_H(w_1) = 4$, $\deg_H(w_2) \in \{7, 8\}$ and $\deg_H(w_3) \geq 10$, then the final charge of f is $-3 + \frac{1}{2} + \frac{7}{6} + \frac{4}{3} = 0$.

Suppose that f is a $(4, 9^+, 9^+)$-face. If w_1 is adjacent to a 5^--vertex u, then w_1 sends $\frac{1}{5}$ to f, and then the final charge of f is $-3 + \frac{1}{5} + 2 \times \frac{7}{5} = 0$; if w_1 is adjacent to a 6-vertex u, then w_1 sends $\frac{1}{3}$ to f, and then the final charge of f is $-3 + \frac{1}{3} + 2 \times \frac{4}{3} = 0$; if w_1 is not adjacent to 6^--vertices, then w_1 sends $\frac{1}{2}$ to f, and then the final charge of f is $-3 + \frac{1}{2} + 2 \times \frac{5}{4} = 0$.

If $\deg_H(w_1) = \deg_H(w_2) = 5$ and $\deg_H(w_3) \in \{5, 6, 7\}$, then the final charge of f is $-3 + 3 \times 1 = 0$.

If f is a $(5, 5, 8^+)$-face, then the final charge is $-3 + 2 \times \frac{7}{8} + \frac{5}{4} = 0$.

If f is a $(5, 6, 6)$-face, then the final charge is $-3 + 3 \times 1 = 0$.

If f is a $(5, 6, 7)$-face, then the final charge is $-3 + \frac{5}{6} + 1 + \frac{7}{6} = 0$.

If f is a $(5,6,8^+)$-face, then the final charge is $-3+\frac{3}{4}+1+\frac{5}{4}=0$.

If f is a $(5,7,7)$-face, then the final charge is $-3+\frac{2}{3}+2\times\frac{7}{6}=0$.

If f is a $(5,7,8^+)$-face, then the final charge is $-3+\frac{17}{28}+\frac{8}{7}+\frac{5}{4}=0$.

If f is a $(5,8^+,8^+)$-face, then the final charge is $-3+\frac{1}{2}+2\times\frac{5}{4}=0$.

If f is a $(6^+,6^+,6^+)$-face, then the final charge is $-3+3\times 1=0$.

Next, we compute the final charge of 4-faces. Let $w_1w_2w_3w_4$ be a 4-face with w_2 having the minimum degree on the boundary. If $\deg_H(w_2)\geq 5$, then the final charge of f is at least $-2+4\times\frac{1}{2}=0$. If $\deg_H(w_1),\deg_H(w_3)\geq 9$, then the final charge is at least $-2+2\times 1=0$. So we may assume that $\deg_H(w_2)\in\{3,4\}$ and $\deg_H(w_1)\leq 8$. By Lemma 6 and Claim 1, we have that $\deg_H(w_2)=4$ and $\deg_G(w_1)=\deg_H(w_1)\leq 8$. By Lemma 9 and discharging rules, the face f receives at least $\frac{1}{2}$ from each incident vertex, so the final charge of f is at least $-2+4\times\frac{1}{2}=0$.

Suppose that f is a 5-face. If f is incident with a 9^+-vertex, then the final charge is at least $-1+1=0$. So we may assume that f is incident with five 8^--vertices. It is obvious that f is incident with at least two 5^+-vertices, and then the final charge is at least $-1+2\times\frac{1}{2}=0$.

If f is a 6^+-face, then the final charge is at least $\deg_H(f)-6\geq 0$.

Computing the final charge of vertices.

Let v be a 3-vertex. Clearly, the final charge is zero.

Let v be a 4-vertex. If v is adjacent to a 5^--vertex, then Lemma 9 and Claim 1 implies that v is adjacent to three 9^+-vertices, and then the final charge is $2-2\times\frac{4}{5}-2\times\frac{1}{5}=0$. If v is adjacent to a 6-vertex, then Lemma 9 and Claim 1 implies that v is adjacent to three 9^+-vertices, and then the final charge is $2-2\times\frac{2}{3}-2\times\frac{1}{3}=0$. If v is not adjacent to 6^--vertices, then the final charge is $2-4\times\frac{1}{2}=0$.

Let v be a 5-vertex with neighbors $v_1,v_2,\dots,v_5$ in anticlockwise order. If v sends at most $\frac{4}{5}$ to each incident face, then the final charge is at least $4-5\times\frac{4}{5}=0$. So we may assume that v sends more than $\frac{4}{5}$ to some face f.

If f is a $(5,5,5)$-face, then Lemma 10 and Claim 1 implies that the other three vertices adjacent to v are 9^+-vertices, and then the final charge of v is at least $4-1-2\times\frac{7}{8}-2\times\frac{1}{2}>0$.

If f is a $(5,5,6)$-face, then Lemma 10 and Claim 1 implies that the other three vertices adjacent to v are 8^+-vertices, and then the final charge of v is at least $4-1-\frac{7}{8}-\frac{3}{4}-2\times\frac{1}{2}>0$.

If f is a $(5,5,7)$-face, then Lemma 10 and Claim 1 implies that the other three vertices adjacent to v are 7^+-vertices, and then the final charge of v is at least $4-2\times 1-3\times\frac{2}{3}=0$.

If f is a $(5,6,6)$-face, then Lemma 10 and Claim 1 implies that the other three vertices adjacent to v are 7^+-vertices, and then the final charge of v is at least $4-1-2\times\frac{5}{6}-2\times\frac{2}{3}=0$.

If v sends at most $\frac{1}{2}$ to an incident face, then the final charge of v is at least $4-4\times\frac{7}{8}-\frac{1}{2}=0$. So we may assume that the 5-vertex v sends more than $\frac{1}{2}$ to each incident face, thus v is incident with five 3-faces.

Suppose that $f=vv_1v_2$ is a 3-face with $\deg_H(v_1)=5$ and $\deg_H(v_2)\geq 8$. By the excluded cases in the above, the vertex v_5 is an 8^+-vertex. Since v sends more than $\frac{1}{2}$ to the 3-face vv_2v_3, the vertex v_3 is a 7^--vertex. Similarly, the vertex v_4 is also a 7^--vertex. Now, the 3-face vv_3v_4 is a $(5,7^-,7^-)$-face. By the excluded cases, we only have to consider the edge v_3v_4 is a $(6,7)$- or $(7,6)$- or $(7,7)$-edge. If v_3v_4 is a $(7,7)$-edge, then the final charge of v is at least $4-2\times\frac{7}{8}-2\times\frac{17}{28}-\frac{2}{3}>0$. If v_3v_4 is $(6,7)$- or $(7,6)$-edge, then the final charge of v is at least $4-2\times\frac{7}{8}-\frac{3}{4}-\frac{5}{6}-\frac{17}{28}>0$.

Suppose that $f=vv_1v_2$ is a $(5,6,7)$-face with $\deg_H(v_1)=6$ and $\deg_H(v_2)=7$. By the excluded cases, the vertex v_3 is a 6^+-vertex and the vertex v_5 is a 7^+-vertex. By Lemma 6 and Claim 1, the vertex v_4 is a 4^+-vertex. If $\deg_H(v_4)=4$, then Lemma 9 and Claim 1 implies that both v_3 and v_5 are 11^+-vertices, thus the final charge of v is at least $4-\frac{5}{6}-\frac{3}{4}-\frac{17}{28}-2\times\frac{17}{20}>0$. By the excluded cases, the vertex v_4 cannot be a 5-vertex. If $\deg_H(v_4)=6$, then $\deg_H(v_3)\geq 7$, and then the final charge of v is at least $4-\frac{2}{3}-4\times\frac{5}{6}=0$. If $\deg_H(v_4)\geq 7$, then the final charge is at least $4-\frac{2}{3}-4\times\frac{5}{6}=0$.

Suppose that $f=vv_1v_2$ is a $(5,4,11)$-face. By Lemma 9 and Claim 1, the vertex v_5 is a 10^+-vertex. If one of v_3 and v_4 is a 8^+-vertex, then v sends $\frac{1}{2}$ to an incident 3-face, a contradiction. So we may assume that $\deg_H(v_3),\deg_H(v_4)\leq 7$. By the excluded cases, the edge v_3v_4 is a $(7,7)$-edge, and then the final charge of v is at least $4-2\times\frac{17}{28}-\frac{2}{3}-2\times\frac{17}{20}>0$.

Let v be a 6-vertex. The final charge is at least $6-6\times 1=0$.

Let v be a 7-vertex. If v sends at most $\frac{1}{2}$ to an incident face, then the final charge is at least $8 - 6 \times \frac{5}{4} - \frac{1}{2} = 0$. So we may assume that v sends more than $\frac{1}{2}$ to each incident face, thus v is incident with seven 3-faces. By Lemma 9 (b) and Claim 1, the vertex v is not incident with $(4, 7, 9^-)$-faces. Now, the vertex v sends at most $\frac{7}{6}$ to each incident face. If v is incident with a $(5^-, 5^-, 7)$- or $(6^+, 6^+, 7)$-face, then the final charge is at least $8 - 6 \times \frac{7}{6} - 1 = 0$. So every face incident with v is a $(5^-, 6^+, 7)$-face, but the vertex v is a 7-vertex and the number 7 is odd, a contradiction.

Let v be an 8-vertex. Every 8-vertex sends at most $\frac{5}{4}$ to each incident face, thus the final charge is at least $10 - 8 \times \frac{5}{4} = 0$.

Let v be a 9-vertex. If $\deg_G(v) > 9$, then Claim 1 implies that v is adjacent to at most one 7^--vertex in H, and then the final charge of v is at least $12 - 7 \times 1 - 2 \times \frac{3}{2} > 0$. So we may assume that $\deg_G(v) = \deg_H(v) = 9$.

Suppose that $(3, 9)$-edge uv is incident with two 3-faces. By Lemma 7, the vertex v is adjacent to eight 8^+-vertices, and then the final charge is at least $12 - 7 \times 1 - 2 \times \frac{3}{2} > 0$. So every $(3, 9)$-edge uv is incident with at most one 3-face.

Let τ be the number of incident 4^+-faces. If $\tau \geq 4$, then the final charge is at least $12 - 5 \times \frac{3}{2} - 4 \times 1 > 0$. Since $\deg_G(v) = \deg_H(v) = 9$, Lemma 9 implies that v is not incident with face (h) or (i). If $\tau \leq 3$, then the final charge is at least $12 - \tau - 2\tau \times \frac{3}{2} - (9 - 3\tau) \times \frac{5}{4} \geq 0$.

Let v be a 10-vertex. If $\deg_G(v) > 10$, then Claim 1 implies that v is adjacent to at most two 7^--vertices, and then the final charge is at least $14 - 4 \times \frac{3}{2} - 6 \times 1 > 0$. So we may assume that $\deg_G(v) = \deg_H(v) = 10$. Hence, the vertex v is not incident with face (b), (d) or (h), and thus v sends $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ or 1 to each incident face.

If v is incident with at least two 4^+-faces, then the final charge is at least $14 - 8 \times \frac{3}{2} - 2 \times 1 = 0$. Hence, the vertex v is incident with at most one 4^+-face. Lemma 9 implies that v is adjacent to at most five 4^--vertices. Let s be the number of incident $(10, 3, 9^+)$-faces, and let s^* be the number of incident $(10, 4, 6^+)$-faces.

If $s \leq 4$, then the final charge is at least $14 - s \times \frac{3}{2} - (10 - s) \times \frac{4}{3} = \frac{2}{3} - \frac{s}{6} \geq 0$. So we may assume that $s \geq 5$, and then the number of adjacent 3-vertices is at least three.

(1) $s \in \{5, 6\}$.

If $s^* = 0$, then the final charge is at least $14 - 6 \times \frac{3}{2} - 4 \times \frac{5}{4} = 0$. If v is incident with exactly one 4^+-face, then the final charge is at least $14 - 6 \times \frac{3}{2} - 1 - 3 \times \frac{4}{3} = 0$. So we may assume that $s^* \geq 1$ and v is not incident with any 4^+-face. Clearly, the vertex v is incident with exactly six $(10, 3, 9^+)$-faces and $s = 6$. It is obvious that v is adjacent to at least one 4-vertex. Lemma 8 implies that the vertex v is adjacent to exactly three 3-vertices, one 4-vertex and six 6^+-vertices. Hence, it is incident with exactly two $(10, 6^+, 6^+)$-faces, and then the final charge is at least $14 - 6 \times \frac{3}{2} - 2 \times \frac{4}{3} - 2 \times 1 > 0$.

(2) $s \geq 7$.

Clearly, the vertex v is adjacent to at least four 3-vertices. Lemma 8 implies that the vertex v is adjacent to exactly four 3-vertices and six 6^+-vertices. Hence, the vertex v is incident with two $(10, 6^+, 6^+)$-faces, or one $(10, 6^+, 6^+)$-face and one 4^+-face, thus the final charge is at least $14 - 8 \times \frac{3}{2} - 2 \times 1 = 0$.

Let v be an 11-vertex. If $\deg_G(v) > 11$, then v is adjacent to at most three 7^--vertices in H, and then the final charge is at least $16 - 6 \times \frac{3}{2} - 5 \times 1 > 0$. So we may assume that $\deg_G(v) = \deg_H(v) = 11$.

If v sends at most 1 to an incident face, then the final charge is at least $16 - 10 \times \frac{3}{2} - 1 = 0$. So we may assume that v is not incident with 4^+-faces and is not incident with $(11, 6^+, 6^+)$-faces. Since the degree of v is odd, the vertex v cannot be incident with eleven $(11, 5^-, 6^+)$-faces. So v is incident with a $(11, 5^-, 5^-)$-face f. Lemma 6 and Lemma 9 implies that the face f is a $(4, 5, 11)$-face or $(5, 5, 11)$-face. Hence, the vertex v is adjacent to at most four 3-vertices. If v is adjacent to at most three 3-vertices, then the final charge is at least $16 - 6 \times \frac{3}{2} - 5 \times \frac{7}{5} = 0$. Hence, the vertex v is adjacent to exactly four 3-vertices, see Fig. 2. If f is a $(5, 5, 11)$-face, then the final charge of v is $16 - 8 \times \frac{3}{2} - 3 \times \frac{5}{4} > 0$. If f is a $(4, 5, 11)$-face, then the final charge is $16 - 8 \times \frac{3}{2} - \frac{5}{4} - \frac{27}{20} - \frac{7}{5} = 0$.

Let v be a 12^+-vertex. The final charge is at least $2 \deg_H(v) - 6 - \deg_H(v) \times \frac{3}{2} = \frac{1}{2} \deg_H(v) - 6 \geq 0$. □

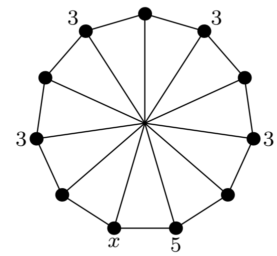


Fig. 2: The vertex x is a 4- or 5-vertex.

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