Colour degree matrices of graphs with at most one cycle

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Abstract

Colour degree matrix problems, also known as edge-disjoint realisation and edge packing problems, have connections for example to discrete tomography. Necessary and sufficient conditions are known for a demand matrix to be the colour degree matrix of an edge-coloured forest. We will give necessary and sufficient conditions for a demand matrix to be realisable by a graph with at most one cycle, and a polynomial time algorithm to check these conditions.

Keywords:
colour degree matrix, degree constrained edge-partitioning, edge packing, discrete tomography, edge-disjoint realisation

1. Introduction

A *demand sequence* $(d_1, d_2, \ldots, d_n)$ of non-negative integers is called *graphical* if there is a (simple) graph G such that vertex v has degree d_v for each $v \in [n]$, where $[n] := \{1, \ldots, n\}$. Necessary and sufficient conditions are known for a sequence to be graphical [1], with corresponding polynomial time algorithms to test these conditions and to find a realisation if there is one [2, 3].

We consider an extension involving edges coloured with c colours. Let c and n be positive integers. An $n \times c$ *demand matrix* is a matrix $D = (d_{v,q} : v \in [n], q \in [c])$ of non-negative integers. We call D a *colour degree matrix* if there exists a c-edge coloured graph G on n vertices, which *realises* D; that is, for each $s \in [c]$, each vertex $v \in [n]$ has $d_{v,q}$ incident edges coloured q. Note that this colouring will not be proper if some $d_{v,q} \geq 2$. Finding a realisation of a demand matrix is also known as finding an *edge-disjoint realisation* [4], *edge packing* [5] or *degree constrained edge-partitioning* [6]. This problem is closely connected to *discrete tomography* [6] with many applications in industry [7].

Recently it was shown that for any fixed $c \geq 2$ deciding whether a demand matrix is a colour degree matrix is NP-hard [8, 9, 10]. However, we may be interested in realisations where the graph G has some specific structure. We will focus on colour degree matrices of graphs with at most one cycle, but first we consider briefly the already studied case of forests.

1.1. Colour degree matrices of forests

The following result was observed by Harary and Menon [11, 12].

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Proposition 1.1. *A non-zero sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers is the degree sequence of a forest if and only if the total demand $\sum_v d_v$ is even and at most $2s - 2$, where s is the number of v such that $d_v \neq 0$.*

The conditions are trivially necessary; and they are easily seen to be sufficient using an inductive argument, since if a non-zero sequence satisfies the conditions then there is a vertex v with $d_v = 1$.

When is a demand matrix the colour degree matrix of a forest? There are simple necessary conditions. Let c and n be positive integers, and consider an $n \times c$ demand matrix $D = (d_{v,q} : v \in [n], q \in [c])$. For each $I \subseteq [c]$, denote the demand of v in the set I of colours by $d_{v,I} = \sum_{q \in I} d_{v,q}$. Observe that $(d_{1,I}, \ldots, d_{n,I})$ is the demand sequence we obtain when we ignore colours not in I and identify colours in I. Clearly it is necessary that each such sequence $(d_{1,I}, \ldots, d_{n,I})$ is the degree sequence of a forest. These conditions were shown to be sufficient by Bentz et al. in [6] for the case $c = 2$, and by Carroll and Isaak in [13] for the general case.

Theorem 1.2. *[6, 13] The demand matrix $D = (d_{v,q} : v \in [n], q \in [c])$ is the colour degree matrix of a forest if and only if, for each $I \subseteq [c]$, the sequence $(d_{1,I}, \ldots, d_{n,I})$ is the degree sequence of a forest.*

We may restate this theorem in an apparently more quantitative way. For each $I \subseteq [c]$, let

$$S(I) = S_D(I) = \{v : d_{v,I} > 0\} = \{v : d_{v,q} > 0 \text{ for some } q \in I\}$$

be the *support* of I, and $s(I) = s_D(I) = |S_D(I)|$; and let

$$t(I) = t_D(I) = \sum_{v \in [n]} d_{v,I} = \sum_{q \in I} \sum_{v \in [n]} d_{v,q}$$

be the *total demand* of I. For a single element $q \in [c]$ we will write simply $S(q)$, $s(q)$ and $t(q)$.

Theorem 1.3. *[6, 13] The demand matrix $D = (d_{v,q} : v \in [n], q \in [c])$ is the colour degree matrix of a forest if and only if*

1. *$t(q)$ is even for each $q \in [c]$, and*
2. *$t(I) \leq 2s(I) - 2$ for each $I \subseteq [c]$ with $t(I) > 0$.*

Bentz et al. showed that these conditions can be checked in polynomial time and a realisation found if there is one for the case $c = 2$ [6]. See Section 4 for a polynomial time algorithm for the general case.

1.2. Colour degree matrices of graphs with at most one cycle

The following result was given by Harary and Boesch [14].

Proposition 1.4. *A sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers is the degree sequence of a graph with at most one cycle if and only if the total demand $\sum_v d_v$ is even and at most $2s$ (where s is the number of v such that $d_v \neq 0$), and if the total demand is $2s > 0$ then there are at least 3 vertices with demand at least 2.*

The conditions are clearly necessary; and they are easily seen to be sufficient by an inductive argument as for forests, since if a non-zero sequence satisfies the conditions then either each demand is 2 or there is a vertex v with $d_v = 1$.

When is a demand matrix D the colour degree matrix of a graph with at most one cycle? This is the question on which we focus in this paper. As with forests, there are simple necessary conditions.

Clearly, for each $I \subseteq [c]$, it is necessary that $(d_{1,I}, \ldots, d_{n,I})$ is the degree sequence of a graph with at most one cycle. Further, call $I \subseteq [c]$ *critical* if $t(I) = 2s(I) > 0$. If I is critical then in any realisation G there must be a cycle where each edge has a colour in I. Thus another necessary condition is that there are no two disjoint critical sets of colours. We shall see that these conditions are also sufficient.

Theorem 1.5. *The demand matrix $D = (d_{v,q} : v \in [n], q \in [c])$ is the colour degree matrix of a graph with at most one cycle if and only if (a) for each $I \subseteq [c]$ the sequence $(d_{1,I}, \ldots, d_{n,I})$ is the degree sequence of a graph with at most one cycle, and (b) there are no two disjoint critical sets.*

Let us restate Theorem 1.5, much as we restated Theorem 1.2 for forests. Theorems 1.5 and 1.6 together form our main result. We shall also consider algorithms briefly in Section 4.

Theorem 1.6. *The demand matrix $D = (d_{v,q} : v \in [n], q \in [c])$ is the colour degree matrix of a graph with at most one cycle if and only if*

(U1) $t(q)$ is even for each $q \in [c]$,

(U2) $t(I) \leq 2s(I)$ for all $I \subseteq [c]$,

(U3) each critical set I contains at least 3 vertices v with $d_{v,I} \geq 2$, and

(U4) there are no two disjoint critical sets.

This version is the one which we shall prove, and which fits better with algorithms. We call (U1) - (U4) the *unicycle conditions*. We have already seen that they are necessary. Let us end this section by noting a condition equivalent to condition (U3).

Given a demand matrix D, for each set I of colours and each $j = 1, \ldots, s(I)$, let $\Delta_{j,I}$ denote the jth largest demand $d_{v,I}$ in I. The new condition is:

(U5) $s(I) \geq 2$ and $\Delta_{1,I} + \Delta_{2,I} \leq s(I) + 1$ for each critical set $I \subseteq [c]$.

The fact that conditions (U5) and (U3) are equivalent follows immediately from:

Lemma 1.7. *Let I be a critical set. Then $s(I) \geq 2$ and $\Delta_{1,I} + \Delta_{2,I} \leq s(I) + 1$ if and only if there are at least 3 vertices v with $d_{v,I} \geq 2$.*

Proof. If at most 2 vertices v have $d_{v,I} \geq 2$ then

$$2s(I) = t(I) = \Delta_{1,I} + \Delta_{2,I} + s(I) - 2,$$

and so $\Delta_{1,I} + \Delta_{2,I} = s(I) + 2 > s(I) + 1$. Conversely, if at least 3 vertices v have $d_{v,I} \geq 2$, then $s(I) \geq 3$ and

$$2s(I) = t(I) \geq \Delta_{1,I} + \Delta_{2,I} + s(I) - 1,$$

and so $\Delta_{1,I} + \Delta_{2,I} \leq s(I) + 1$. □

Observe that if condition (U3) or (U5) holds, then

$$\Delta_{1,I} \leq s(I) - 1, \tag{1}$$

since $\Delta_{2,I} \geq 2$.

We introduce some preliminary results in Section 2. We show that the unicycle conditions are sufficient in Section 3, and we consider related algorithmic questions in Section 4.

2. Preliminary results

In this section we discuss some preliminary results for vertices with set degrees. Throughout the section we shall assume that $D = (d_{v,q} : v \in [n], q \in [c])$ is a demand matrix for which the unicycle conditions hold.

We denote the restriction of condition (Ui), $i \in [5]$, to a specific $I \subseteq [c]$ by $(Ui)_I$. E.g. for a fixed $I \subseteq [c]$, we denote $t(I) \leq 2s(I)$ by $(\mathrm{U2})_I$.

2.1. Forced edges

For distinct $v, w \in [n]$ and $q \in [c]$, we will call vw a *forced edge in colour* q if $t(q) = 2s(q) - 2$ and $d_{v,q} + d_{w,q} = s(q)$.

Lemma 2.1. *Let D satisfy the unicycle conditions. Suppose that $I \subseteq [c]$ is such that $t(I) \leq 2s(I) - 2$, and v and w are distinct and satisfy $d_{v,I} + d_{w,I} = s(I)$. Then there exists a colour $q \in I$, such that vw is a forced edge in colour q, and each $x \in S(q) \setminus \{v, w\}$ satisfies $d_{x,q} = 1$.*

Proof. $t(I) = 2s(I) - 2$ because by assumption

$$2s(I) - 2 \geq t(I) \geq d_{v,I} + d_{w,I} + s(I) - 2 = 2s(I) - 2.$$

Moreover, for every $x \in S := S(I) \setminus \{v, w\}$, $d_{x,I} = 1$, and any $I' \subseteq I$ has at most two vertices with degree at least 2. Thus by condition (U3), I has no critical subsets. Suppose that $d_{v,q} + d_{w,q} < s(q)$ for all $q \in I$. Condition (U1) gives $d_{v,q} + d_{w,q} \leq s(q) - 2$ if $v, w \in S(q)$ and $d_{v,q} + d_{w,q} \leq s(q) - 1$ otherwise; and so $d_{v,q} + d_{w,q} \leq |S(q) \setminus \{v, w\}|$. Hence

$$s(I) = \sum_{q \in I} (d_{v,q} + d_{w,q}) \leq \sum_{q \in I} |S(q) \setminus \{v, w\}| = |S(I) \setminus \{v, w\}| = s(I) - 2,$$

because each $x \in S$ appears in $S(q)$ for exactly one $q \in I$. This contradiction shows that there exists $q \in I$ such that $d_{v,q} + d_{w,q} \geq s(q)$. Now

$$t(q) = d_{v,q} + d_{w,q} + s(q) - 2 \geq 2s(q) - 2 \geq t(q)$$

(since $\{q\}$ is not critical), and it follows that $t(q) = 2s(q) - 2$ and $d_{v,q} + d_{w,q} = s(q)$, as required. $\square$

2.2. Forced triangles

Let D satisfy the unicycle conditions. Observe that for $I \subseteq [c]$ and distinct $x, y, z \in S(I)$, by condition (U2) we have

$$d_{x,I} + d_{y,I} + d_{z,I} + s(I) - 3 \leq t(I) \leq 2s(I). \tag{2}$$

Hence

$$d_{x,I} + d_{y,I} + d_{z,I} \leq s(I) + 3, \tag{3}$$

and if equality holds then I is critical and $d_{x,J} = 1$ for each $x \in S(J) \setminus \{u, v, w\}$.

For distinct $u, v, w \in [n]$ and $J \subseteq [c]$, we call uvw a *forced triangle with colours* J if $d_{u,J} + d_{v,J} + d_{w,J} = s(J) + 3$, and J is minimal with this property.

Lemma 2.2. *Let D satisfy the unicycle conditions. Suppose that uvw is a forced triangle with colours J, and let $\Delta = (d_{t,q} : t \in S(J), q \in J)$. Then Δ is the colour degree matrix of a unicyclic graph.*

To show this we will use the following lemma.

Lemma 2.3. *Let D satisfy the unicycle conditions. Let uvw be a forced triangle with colours J and let $B = \{u, v, w\}$ and $S = S(J) \setminus B$. Then for each $\emptyset \neq I \subseteq J$, $d_{B,I} \geq d_{S,I} + 2$.*

Proof. Observe that $d_{B,J} = d_{S,J} + 6 = s(J) + 3$. Suppose that $\emptyset \neq I \subset J$ satisfies $d_{B,I} \leq d_{S,I}$. Set $K = J \setminus I$ (so $\emptyset \neq K \subset J$). As $d_{S,I} = |S \cap S(I)| = s(I) - |S(I) \cap B|$ and $|S(I) \cap B| \geq |S(I) \cap S(K)|$,

$$\begin{aligned} d_{B,K} &= d_{B,J} + d_{S,J} \geq s(J) + 3 - d_{S,I} \\ &= s(J) - (s(I) - |S(I) \cap B|) + 3 \\ &\geq s(J) - s(I) + |S(I) \cap S(K)| + 3 = s(K) + 3 \end{aligned}$$

Thus by (3) $d_{B,K} = s(K) + 3$, contradicting minimality of J. Hence for each $\emptyset \neq I \subseteq J$, we have $d_{B,I} > d_{S,I}$ and since their sum is even $d_{B,I} \geq d_{S,I} + 2$. □

Proof of Lemma 2.2. We have $d_{u,J}, d_{v,J}, d_{w,J} \geq 2$ (since J is critical). Set $B = \{u, v, w\} = \{x, y, z\}$ and $S = S(J) \setminus B$. Lemma 2.3 shows that $|J| \leq 3$, since

$$6 = d_{B,J} - d_{S,J} = \sum_{q \in J} (d_{B,q} - d_{S,q}) \geq 2|J|.$$

Moreover $d_{S,J} = |S \cap S(J)|$, and the values $D_{S,q}$ for $q \in J$ determine the colours required on the triangle. If a triangle through u, v and w with the right colours can be formed, then connecting $x \in S$ to u, v or w appropriately gives the required realisation.

There are three possible cases:

1. $|J| = 1$, say $J = \{q\}$, and $d_{B,q} = d_{S,q} + 6$; Then the required (uni-coloured) triangle can be formed, as $d_{u,q}, d_{v,q}, d_{w,q} \geq 2$.

2. $|J| = 2$, say $J = \{q, r\}$, and $d_{B,q} = d_{S,q} + 4$ and $d_{B,r} = d_{S,r} + 2$; We are looking for a two-coloured triangle with two edges in q and one in r. If we have $d_{x,q} \geq 2$ and $d_{y,q}, d_{z,q} \geq 1$, and $d_{y,r}, d_{z,r} \geq 1$, for distinct $x, y, z \in B$, the required triangle can be formed. There must be $x \in B$, such that $d_{x,q} \geq 2$, as $d_{B,q} \geq 4$. Let it be u. Suppose there is $x \in B, x \neq u$, such that $d_{x,q} = 0$, then $d_{\{y,z\},q} = d_{S,q} + 4 \geq s(q) + 2$, contradicting condition (U5). Thus $d_{x,q} \geq 1$ for all $x \in B$. Suppose now that $d_{x,r} = 0$. Then

$$d_{\{y,z\},r} = d_{B,r} = d_{S,r} + 2 = s(r) + 2 - |B \cap S(r)| = s(r). \tag{4}$$

But $d_{y,r}, d_{z,r} < s(r)$ by Eq. 1, so $d_{y,r}, d_{z,r} \geq 1$. Thus we can form the required triangle, as any $x \in B$ for which $d_{x,r} = 0$, must have $d_{x,q} \geq 2$, as $d_{x,J} \geq 2, \forall x \in B$.

3. $|J| = 3$, say $J = \{q, r, s\}$ and $d_{B,q} = d_{S,q} + 2$, $d_{B,r} = d_{S,r} + 2$ and $d_{B,s} = d_{S,s} + 2$; We are looking for a three-coloured triangle with colours q, r, and s. We need for all $i \in \{q, r, s\}$ that there exists a distinct pair $x, y \in B$ such that $d_{x,i}, d_{y,i} \geq 1$. If there is $x \in B$ and $i \in J$ such that $d_{x,i} = 0$, then by Eq. 4, $d_{y,i}, d_{z,i} \geq 1$. If $d_{x,i} = d_{x,j} = 0$, for $i \neq j$ and $i, j \in J$, then $\{i, j\}$ is cycle critical

$$t(\{i, j\}) = s(i) + s(j) + d_{S,\{i,j\}} = s(i) + s(j) + s(\{i, j\}) - 2 = 2s(\{i, j\}).$$

But then by condition (U3), $u, v, w \in S(\{i, j\})$ contradicting $d_{x,i} = d_{x,j} = 0$ for some $x \in B$; and so every $x \in B$ misses at most one colour. Moreover these colours are distinct. Consider the bipartite graph with parts B and J (each of size 3) and an edge xj whenever $d_{x,j} \geq 1$. The edges missing form a matching, so the edges present must contain a 6-cycle, which specifies a coloured triangle as required.

□

2.3. Sufficiency when each total degree is two

Lemma 2.4. *Let $D = (d_{v,q} : q \in [c], v \in [n])$ satisfy the unicycle conditions, and suppose that $d_{v,[c]} = 2, \forall v \in [n]$. Then D can be realised by a single cycle through all vertices.*

Proof. Let each demand for a colour at a vertex $v \in [n]$ correspond to a half-edge of this colour. We will try to find a realisation by pairing the half-edges. There is an even number $t(q)$ of half-edges of each colour q by condition (U1). Arbitrarily pairing half-edges gives a (not necessarily simple) realisation of D, which consists of a disjoint union of cycles. Consider a realisation G with a minimal number of cycles and suppose there are at least two cycles. If two cycles have a colour in common on edges x_1y_1, x_2y_2 say, then switching these edges to x_1x_2, y_1y_2 gives a realisation of D with fewer cycles. If a cycle C has no common colours with $G \setminus C$, then the colours on C and the remaining colours are disjoint critical sets, contradicting condition (U4). Thus G is a single (simple) cycle through all vertices. □

3. The unicycle conditions are sufficient

We will consider a counterexample $D = (d_{v,q} : v \in [n], q \in [c])$, which is minimal with respect to n. In a series of claims we will show more and more properties of D, and eventually obtain a contradiction.

The main steps are as follows. First we show that the full colour set $[c]$ is the unique critical set. D. Lemma 2.2 then implies that there is no forced triangle. By Lemma 2.4 there is a leaf $v_1 \in [n]$, which only has demand for colour 1 say. We fix v_1: the rest of the proof involves looking for a suitable partner $w \in S(1)$ for v_1, so that we can make $v_1 w$ an edge with colour 1 in a unicyclic realisation of D. We consider $\tilde{D} = \tilde{D}(w)$, which is D with v_1 removed and $d_{w,1}$ reduced by one. If the unicycle conditions hold in $\tilde{D}$, then a unicyclic realisation of $\tilde{D}$ exists by minimality of D. This realisation will extend to a realisation of D, by appending v_1 to w with colour 1.

We see quickly that conditions (U1) and (U4) must hold in $\tilde{D}$. Further we see that, if w or I satisfy some extra conditions then $(\mathrm{U2})_I$ and $(\mathrm{U3})_I$ hold in $\tilde{D}$.

There are some sets $I \subsetneq [c]$, that we shall call dangerous, for which we need to be careful how we pick w. We see that $s(1) = 4$, and for each dangerous set I, we consider the set $W(I) = S(1) \cap S(I \setminus 1)$ of 'good' vertices. We finish the proof by showing that for any two dangerous sets I and J, we have $W(I) = W(J)$ (using the fact there is no forced triangle), and picking w from this set yields the desired contradiction.

Proof of Theorem 1.6. Suppose the conditions are not sufficient and consider a counterexample $D = (d_{v,q} : v \in [n], q \in [c])$, for which conditions (U1)-(U4) hold but there is no realisation as required, such that n is minimal.

There must exist a critical $I \subseteq [c]$, or D would be the degree sequence of a forest by Theorem 1.3. By condition (U3), $n \geq 3$.

Claim 1. *$[c]$ is the unique critical set in D.*

Proof. Suppose $I \subsetneq [c]$ is a critical set. We will show that a unicyclic realisation can be found.

Set $I^c = [c] \setminus I$. Note that $t(I^c) \geq 2$, $s(I^c) \geq 2$; and $s(I) \leq n-1$, since $s(I) = n$

$$2n = t([c]) = t(I) + t(I^c) \geq 2s(I) + s(I^c) \geq 2s(I) + 2.$$

Let $D_1 = (d_{v,q} : v \in S(I), q \in I)$ and let a be a new vertex with degrees $d_{a,q} = \sum_{v \in S(I)} d_{v,q}$, for each $q \in I^c$. Let $D_2 = (d_{v,q} : v \in ([n] \setminus S(I)) \cup \{a\}, q \in I^c)$.

By assumption conditions (U1) - (U4) hold for D_1, so by minimality of D there exists a unicyclic realisation G_1 of D_1. For all $J \subseteq I^c$, let $s_2(J) = s_{D_2}(J)$ and $t_2(J) = t_{D_2}(J)$. Observe that $t(J) = t_2(J)$. If for all $J \subseteq I^c$, $t_2(J) \leq 2s_2(J) - 2$, a forest realisation T_2 of D_2 exists by Theorem 1.3. A unicyclic realisation of D can then be found by replacing a by G_1.

Let $J \subseteq I^c$. If $S(J) \cap S(I) = \varnothing$, $t(J) = t_2(J)$ and $s(J) = s_2(J)$. By condition (U4), J is not critical because it is disjoint from I. If $S = S(J) \cap S(I) \neq \varnothing$, $s_2(J) = s(J) - |S| + 1$. Suppose $t_2(J) \geq 2s_2(J)$, then since $s(I \cup J) = s(I) + s(J) - |S|$, we have

$$t(I \cup J) = t(I) + t_2(J) \geq 2s(I) + 2s_2(J) = 2s(I) + 2s(J) - 2|S| + 2 = 2s(I \cup J) + 2.$$

Thus $t_2(J) \leq 2s_2(J) - 2$, as required. □

Claim 2. *There is no forced triangle.*

Proof. There can be no forced triangle in $I \subsetneq [c]$, as $[c]$ is the only critical set by Claim 1. By Lemma 2.2 a forced triangle in $[c]$ can be realised. □

Claim 3. *There is a leaf, that is, a vertex $v \in [n]$ with $d_{v,[c]} = 1$.*

Proof. If the minimal total degree is 2, then condition (U2) implies all total degrees are exactly 2 and D can be realised by a single cycle through all vertices by Lemma 2.4. □

Fix a leaf v_1. We may suppose without loss of generality $d_{v_1,1} = 1$ and $d_{v_1,q} = 0$ for all $q \neq 1$ and $S(1) \setminus \{v_1\} \neq \varnothing$. We aim to find a suitable partner w for v_1, such that we can make $v_1 w$ an edge with colour 1 in a unicyclic realisation of D.

Given $w \neq v_1$, $w \in S(1)$, let $\tilde{D} = \tilde{D}(w)$ be obtained from D by deleting v and decreasing $d_{w,1}$ by 1. Note that $\tilde{D}$ is a demand matrix on $\tilde{n} = n - 1$ vertices, so if the unicycle conditions hold for $\tilde{D}$, then a unicyclic realisation $\tilde{G}$ of $\tilde{D}$ can be found by minimality of D. This can be extended to a realisation of D by appending v_1 to w with colour 1.

Claim 4. *Let $w \neq v_1$ be in $S(1)$. Then (U1) and (U4) hold in $\tilde{D}$.*

Proof. It is clear that $(U1)$ holds. If $1 \notin I$ then $\tilde{t}(I) = t(I)$ and $\tilde{s}(I) = s(I)$, so if I is not critical in D then I is not critical in $\tilde{D}$. Thus any critical set in $\tilde{D}$ must contain 1, hence (U4) holds. □

Note also that $1 \notin I$ then $(\mathrm{U2})_I$ and $(\mathrm{U3})_I$ must hold, since (as noted above) $\tilde{t}(I) = t(I)$ and $\tilde{s}(I) = s(I)$. Therefore by Claim 4 it is enough to show that, for a suitable choice of $w \neq v_1$ in $S(1)$, for all $I \subseteq [c]$ with $1 \in I$,

$$(\mathrm{U2})_I \text{ and } (\mathrm{U3})_I \text{ hold in } \tilde{D}. \tag{5}$$

Claim 5. *Let $w \neq v_1$ be in $S(1)$, let $I \subseteq [c]$ contain 1, and suppose that $t(I) < 2s(I) - 2$. Then (5) holds.*

Proof. Since $t(I) \leq 2s(I) - 4$ by (U1), and $\tilde{s}(I) \geq s(I) - 2$,

$$\tilde{t}(I) = t(I) - 2 \leq 2s(I) - 4 - 2 \leq 2\tilde{s}(I) - 2.$$

Thus I is not critical for $\tilde{D}$, and (5) holds. □

Claim 6. *Let $w \neq v_1$ be in $S(1)$, let $I \subseteq [c]$ contain 1, and suppose that $w \in \tilde{S}(I)$. Then (5) holds.*

Proof. Consider first the case $I \subsetneq [c]$. Then I is not critical (since $[c]$ is the unique critical set), and $\tilde{s}(I) = s(I) - 1$. Hence

$$\tilde{t}(I) = t(I) - 2 \leq 2s(I) - 2 - 2 = 2\tilde{s}(I) - 2.$$

Thus I is not critical for $\tilde{D}$, and (5) holds.

Now consider the case $I = [c]$. Note that $s([c]) = n$ and $\tilde{s}([c]) = \tilde{n}$. Then

$$\tilde{t}([c]) = t([c]) - 2 = 2n - 2 = 2\tilde{n}.$$

Also, $\Delta_{1,[c]} + \Delta_{2,[c]} + \Delta_{3,[c]} \leq n + 2$, because there is no forced triangle by Claim 2, and and $\Delta_{3,[c]} \geq 2$ by (U3). Therefore

$$\tilde{\Delta}_{1,[c]} + \tilde{\Delta}_{2,[c]} \leq \Delta_{1,[c]} + \Delta_{2,[c]} \leq n = \tilde{n} + 1.$$

Thus $(\mathrm{U5})_I$ holds, and so also (5) holds. □

It now follows that $d_{x,1} = 1$ for each $x \in S(1)$: for otherwise we could pick w with $d_{w,1} \geq 2$ and then $w \in \tilde{S}(I)$ for each $I \subseteq [c]$ that contains 1. Also, observe from the last claim that, if we pick w from $S(1) \cap S([c] \setminus 1)$ and $I = [c]$, then (5) holds. We want to pick such a w.

Claim 7. *$S(1) \cap S([c] \setminus 1) \neq \varnothing$.*

Proof. If $S(1) \cap S([c] \setminus 1) = \varnothing$, then since $s(1) \geq 2$ and $[c] \setminus 1$ is not critical,

$$2n = t([c]) = t(1) + t([c] \setminus 1) \leq s(1) + 2s([c] \setminus 1) - 2 = 2n - s(1) - 2 \leq 2n - 4,$$

which is a contradiction. □

Claim 8. *$s(1) \geq 4$.*

Proof. Note that $s(1)$ is even. Suppose $s(1) = 2$. Then $t(1) = 2$, $s([c] \setminus 1) \leq s([c]) - 1 = n - 1$, and $2s([c] \setminus 1) \geq t([c] \setminus 1) = t([c]) - 2 = 2n - 2 \geq 2s([c] \setminus 1)$. So $[c] \setminus 1$ is critical, contradicting Claim 1. Thus $s(1) \geq 4$. □

We say that $I \subseteq [c]$ is *dangerous* if $1 \in I \subsetneq [c]$ and

$$\Delta_{1,I} + \Delta_{2,I} = s(I).$$

Claim 9. *Let $w \neq v_1$ be in $S(1)$, let $1 \in I \subsetneq [c]$, and suppose that I is not dangerous. Then (5) holds.*

Proof. By Claims 5 and 6 we may assume that $t(I) = 2s(I) - 2$ and $w \notin \tilde{S}(I)$. Thus

$$\tilde{t}(I) = t(I) - 2 = 2s(I) - 4 = \tilde{s}(I),$$

and so $(\mathrm{U2})_I$ holds in $\tilde{D}$. Further, $1 \in I \subsetneq [c]$ but I is not dangerous, so

$$\tilde{\Delta}_{1,I} + \tilde{\Delta}_{2,I} \leq \Delta_{1,I} + \Delta_{2,I} \leq s(I) - 1 = \tilde{s}(I) + 1.$$

Thus $(\mathrm{U5})_I$ holds, and so also (5) holds. □

We now see that there must be a dangerous set; since by Claim 7 we can choose $w \in S(1) \cap S([c] \setminus 1)$, and then (5) holds for each $I \subseteq [c]$ which is not dangerous. For each dangerous $I \subseteq [c]$, let $W(I) = S(1) \cap S(I \setminus 1)$ be the set of *good vertices* for I. Note that $v_1 \in S(1) \setminus S(I \setminus 1)$.

Claim 10. *$s(I) = 4$; and for each dangerous set I, $W(I)$ consists of the two vertices x with $d_{x,I} \geq 2$.*

Proof. Let I be dangerous and let x_1 and x_2 be distinct vertices with $d_{x_1,I} + d_{x_2,I} = \Delta_{1,I} + \Delta_{2,I} = s(I)$. Then $d_{y,I} = 1$ for each $y \in S(I) \setminus \{x_1, x_2\}$, and so $W(I) \subseteq \{x_1, x_2\}$. Let $I' = I \setminus 1$. Then I' is not critical and thus $d_{x_1,I'} + d_{x_2,I'} \leq s(I')$. Now

$$2s(I) - 2 = t(I) = d_{x_1,I} + d_{x_2,I} + s(I) - 2 \leq d_{x_1,1} + d_{x_2,1} + s(I') + s(I) - 2.$$

Hence

$$2 \geq d_{x_1,1} + d_{x_2,1} \geq s(I) - s(I') = s(1) - |S(1) \cap \{x_1, x_2\}| \geq s(1) - 2 \geq 2.$$

Thus $s(1) = 4$; and $d_{x_1,1} = d_{x_2,1} = 1$, so $W(I) = \{x_1, x_2\}$. □

Claim 11. *Let I and J be dangerous. Then $W(I) = W(J)$.*

Proof. Recall that $s(1) = 4$ and $v_1 \in S(1)$, and note that $v_1 \notin W(I) \cup W(J)$. Suppose that $W(I) \neq W(J)$. Then $W(I) \setminus W(J)$, $W(I) \cap W(J)$ and $W(J) \setminus W(I)$ are each singletons, say $\{x_1\}$, $\{x_2\}$ and $\{x_3\}$; and then $S(1) = \{x_1, x_2, x_3, v_1\}$. Since I is not critical, it follows from Lemma 2.1 that x_1x_2 is a forced edge in colour q, for some $q \in I$; and $q \neq 1$, since $d_{x_1,1} + d_{x_2,1} = 2 < 4 = s(1)$. Similarly, x_1x_3 is a forced edge in some colour $r \neq 1$, $r \in J$. Further, $x_1 \notin S(J \setminus \{1\})$, thus $q \neq r$. Let $K = \{1, q, r\}$, then

$$s(K) \leq |S(q) \setminus S(1)| + |S(r) \setminus S(1)| + s(1) = s(q) - 2 + s(r) - 2 + 4 = s(q) + s(r).$$

Also, since $d_{x_1,q} + d_{x_2,q} = s(q)$, $d_{x_2,r} + d_{x_3,r} = s(r)$ and $d_{x_1,1} + d_{x_2,1} + d_{x_3,1} = 3$, we have

$$d_{x_1,K} + d_{x_2,K} + d_{x_3,K} \geq s(q) + s(r) + 3 \geq s(K) + 3 :$$

but now we have a forced triangle, contradicting Claim 2. □

Let W^* be the common set $W(I)$ of good vertices for each dangerous set I. We may at last complete the proof of Theorem 1.6 by choosing $w \in W^*$. For now, $w \in \tilde{S}(I)$ for each dangerous set I. Hence (5) holds for each dangerous set I by Claim 6, and we have already seen that it holds for each non-dangerous set I. □

4. Checking the unicycle conditions

In this section we sketch a polynomial time algorithm to check the unicycle conditions of Theorem 1.6. Note that if c is a constant then this is easy to do in linear time, but we need to be careful for general c. Finally we sketch how a realisation can be found.

Suppose we are given a demand matrix $D = (d_{v,q} : v \in [n], q \in [c])$, where $d_{v,[c]} > 0$ for each $v \in [n]$ and $t(q) > 0$ for each $q \in [c]$. We may assume that $c \leq n$, as otherwise D would not satisfy condition (U2), and indeed we may assume that $t([c]) \leq 2n$. so $c = O(n)$. Clearly we can check condition (U1) quickly.

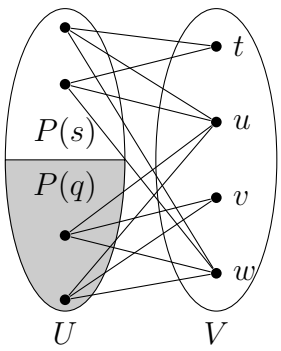
4.1. Checking condition (U2)

We shall construct a bipartite graph G with parts U and V, such that G has a matching covering U if and only if condition (U2) holds for D.

For each $q \in [c]$, let $P(q) = \{q_i : i = 1, \ldots, t(q)/2\}$ (so we are creating an element for every two times colour q is demanded). For each $I \subseteq [c]$, let $C(I) = \bigcup_{q \in I} P(q)$. Let $U = C([c])$ (so that $|U| = t([c])/2$) and let $V = [n]$, and let q_i and v be adjacent in G whenever $v \in S(q)$. See Fig. 1 for an example.

There is a matching in G covering U if and only if Hall's condition holds. Here Hall's condition is that for each $U' \subseteq U$, the total number of neighbours in V of the $q_i \in U'$ is at least $|U'|$. But it is not hard to see that we need to consider only sets U' made of complete sets $P(q)$, and so Hall's condition is equivalent to the condition that for each $I \subseteq [c]$ we have $s(I) \geq t(I)/2$ – in other words that condition (U2) holds.

Finally, note that G has at most $2n$ vertices, and so we can check condition (U2) in time $O(n^{2.5})$, by using the Hopcroft–Karp algorithm [15].



$[c] = \{s, q\}$
$[n] = \{t, u, v, w\}$

$D =$

	s	q
t	2	0
u	1	1
v	0	2
w	1	1

Figure 1: Example of a demand matrix D and corresponding graph G.

4.2. Checking conditions (U3) and (U4)

Suppose that (U2) holds. Let $\emptyset \neq I \subseteq [c]$. We want a subroutine that given $q \in I$, checks whether q appears in a critical subset of I. Form a bipartite graph G' by starting with the induced subgraph of G with parts $C(I)$ and V and adding a vertex q_0 to $C(I)$, adjacent to each $v \in S(q)$. Then Hall's condition fails for G' if and only if some $q \in K \subseteq I$ is critical. This subroutine takes $O(n^{2.5})$ time.

We use the subroutine as follows. First we use it for all $q \in [c]$ and $[c]$. Let

$$J = \{q : [c] \text{ contains a critical set containing } q\}.$$

If $J = \varnothing$, then all the conditions hold and we are done . Indeed, at this stage, our algorithm has checked whether the acyclic conditions in Theorem 1.3 hold, in time $O(n^{3.5})$.

So assume that there is a critical set. Now we run the subroutine on each of the sets $J \setminus \{q\}$ for each $r \in J \setminus q$, to determine the set

$$F = \{q \in [c] : [c] \setminus \{q\} \text{ contains no critical set}\} = \{q \in [c] : q \text{ is in each critical set}\}.$$

If (U4) fails, then clearly $F = \emptyset$. If (U4) holds, then there is a unique minimal critical set I^* (since the intersection of two critical sets is critical) and so $F = I^*$. Thus if $F = \emptyset$ we know that (U4) fails, and we are done. So assume that $F \neq \emptyset$, and then (U4) holds and $F = I^*$. But now we may easily check (U3), since we need consider only F.

We have now seen how to check all the unicyclic conditions, in total time $O(n^{4.5})$.

4.3. Finding a realisation

Suppose that D satisfies the unicyclic conditions. If each degree $d_{v,[c]} = 2$ then the proof of Lemma 2.4 indicates how to find a unicyclic realisation in time $O(n^2)$. Let v be a leaf, with say $d_{v,q} = d_{v,[c]} = 1$. We check the conditions for $\tilde{D}(w)$ as in the proof of Theorem 1.6, for all possible neighbours $w \in S(q)$ of v, and fix vw in colour q if they hold. There are at most $s(q) - 1 \leq n$ calls to check the conditions, so arguing crudely that gives an overall complexity of $O(n^{6.5})$.

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