A permutation code preserving a double Eulerian bistatistic

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Abstract

Visontai conjectured in 2013 that the joint distribution of ascent and distinct nonzero value numbers on the set of subexcedant sequences is the same as that of descent and inverse descent numbers on the set of permutations. This conjecture has been proved by Aas in 2014, and the generating function of the corresponding bistatistics is the double Eulerian polynomial. Among the techniques used by Aas are the Möbius inversion formula and isomorphism of labeled rooted trees. In this paper we define a permutation code (that is, a bijection between permutations and subexcedant sequences) and show the more general result that two 5-tuples of set-valued statistics on the set of permutations and on the set of subexcedant sequences, respectively, are equidistributed. In particular, these results give a bijective proof of Visontai's conjecture.

1 Introduction

In enumerative combinatorics it is a classical result that the descent number des and the inverse descent number ides (defined as $ides \pi = des \pi^{-1}$) on permutations are Eulerian statistics, and their distributions on the set \mathfrak{S}_n of length-*n* permutations are given by the *n*th Eulerian polynomial A_n , that is

$$A_n(u) = \sum_{\pi \in \mathfrak{S}_n} u^{\operatorname{des} \pi + 1} = \sum_{\pi \in \mathfrak{S}_n} u^{\operatorname{ides} \pi + 1},$$

and the joint distribution of des and ides is given by the nth double Eulerian polynomial,

$$A_n(u,v) = \sum_{\pi \in \mathfrak{S}_n} u^{\operatorname{des} \pi + 1} v^{\operatorname{ides} \pi + 1},$$

see for instance [2, 7].

An alternative way to represent a permutation is its Lehmer code [5], which is a subexcedant sequence. The ascent number asc on the set S_n of subexcedant sequences is still an Eulerian statistic (see for example [8]), and in [6] the statistic that counts the number of distinct nonzero symbols in a subexcedant sequence (that following [1] we denote by row) is proved to be still Eulerian; this result is credited to Dumont by the authors of [6]. In terms of generating functions we have

$$A_n(u) = \sum_{s \in S_n} u^{\operatorname{asc} s + 1} = \sum_{s \in S_n} u^{\operatorname{row} s + 1}.$$

Moreover, in [11] Visontai conjectured that the joint distribution of des and ides on the set of permutations is the same as that of asc and row on the set of subexcedant sequences, that is

$$A_n(u,v) = \sum_{\pi \in \mathfrak{S}_n} u^{\operatorname{des} \pi + 1} v^{\operatorname{ides} \pi + 1} = \sum_{s \in S_n} u^{\operatorname{asc} s + 1} v^{\operatorname{row} s + 1}.$$

In 2014 Aas [1] proved Visontai's conjecture, and among the techniques he used are the Möbius inversion formula and isomorphism of labeled rooted trees. In the present paper we define a bijection between permutations and subexcedant sequences (i.e., a permutation code) and show that the tuple of set-valued statistics (Des, Ides, Lrmax, Lrmin, Rlmax) on the set of permutations has the same distribution as (Asc, Row, Posz, Max, Rlmax) on the set of subexcedant sequences (each of the occurring statistics is defined below). In particular, our bijection gives a constructive proof of Visontai's conjecture.

2 Notation and definitions

A length-*n* word *w* over the alphabet *A* is a sequence $w_1w_2...w_n$ of symbols in *A*, and we will consider only finite alphabets $A \subset \mathbb{N}$.

Statistics

A *statistic* on a set X of words is simply a function from X to \mathbb{N} ; a *set-valued statistic* is a function from X to $2^{\mathbb{N}}$; and a *multistatistic* is a tuple of statistics.

Let $w = w_1 w_2 \dots w_n$ be a length-*n* word. A *descent* in *w* is a position *i* in *w*, $1 \le i < n$, with $w_i > w_{i+1}$, and the *descent set* of *w* is

Des
$$w = \{i : 1 \le i < n \text{ with } w_i > w_{i+1}\}.$$

A left-to-right maximum in w is a position i in w, $1 \le i \le n$, with $w_j < w_i$ for all j < i, and the set of left-to-right maxima is

$$\operatorname{Lrmax} w = \{i : 1 \le i \le n \text{ with } w_j < w_i \text{ for all } j < i\}.$$

Clearly $1 \in \text{Lrmax} w$, and Des and Lrmax are classical examples of set-valued statistics on words. We define similarly the sets Asc w of ascents, Lrmin w of left-to-right minima, Rlmax w of right-to-left maxima and Rlmin w of right-to-left minima in w.

To each set-valued statistic St corresponds an (integer-valued) statistic st defined as st $w = \operatorname{card} \operatorname{St} w$, for example des w and Irmax w counts, respectively, the number of descents and the number of left-to-right maxima in w.

Let X and X' be two sets of words, and st and st' be two statistics defined on X and X', respectively. We say that st on X has the same distribution as st' on X' (or equivalently, st and st' are equidistributed) if, for any integer u,

$$\operatorname{card}\{w \in X : \operatorname{st} w = u\} = \operatorname{card}\{w \in X' : \operatorname{st}' w = u\},\$$

and the multistatistic $(\mathsf{st}_1, \mathsf{st}_2, \ldots, \mathsf{st}_p)$ defined on X has the same distribution as the multistatistic $(\mathsf{st}'_1, \mathsf{st}'_2, \ldots, \mathsf{st}'_p)$ defined on X' (or the two multistatistics are equidistributed) if, for any integer p-tuple $u = (u_1, u_2, \ldots, u_p)$,

$$\operatorname{card}\{w \in X : (\mathsf{st}_1, \mathsf{st}_2, \dots, \mathsf{st}_p) \, w = u\} = \operatorname{card}\{w \in X' : (\mathsf{st}_1', \mathsf{st}_2', \dots, \mathsf{st}_p') \, w = u\}.$$

The notion of equidistribution of (multi)statistics can naturally be extended to set-valued (multi)statistics.

Permutations, subexcedant sequences and codes

This paper deals with two particular classes of words: permutations and subexcedant sequences. A permutation is a length-*n* word over $\{1, 2, ..., n\}$ with distinct symbols. Alternatively, a permutation is an element of the symmetric group on $\{1, 2, ..., n\}$ written in one line notation, and \mathfrak{S}_n denotes the set of length-*n* permutations. If two permutations $\pi = \pi_1 \pi_2 ... \pi_n$ and $\sigma = \sigma_1 \sigma_2 ... \sigma_n$ are such that $\sigma_{\pi_1} \sigma_{\pi_2} \cdots \sigma_{\pi_n} = 12 \cdots n$ (i.e., the identity in \mathfrak{S}_n), then σ is the *inverse* of π , which is denoted by π^{-1} .

A length-*n* subexcedant sequence¹ is a word $s = s_1 s_2 \dots s_n$ over $\{0, 1, \dots, n-1\}$ with $0 \le s_i \le i-1$ for $1 \le i \le n$, and S_n denotes the set of length-*n* subexcedant sequences; and we have $S_n = \{0\} \times \{0, 1\} \times \dots \times \{0, 1, \dots, n-1\}$.

Some statistics are consistently defined only on particular classes of words, e.g. permutations or subexcedant sequences.

For a permutation $\pi \in \mathfrak{S}_n$, an *inverse descent* (*ides* for short) in π is a position *i* for which $\pi_i + 1$ appears to the left of π_i in π . Equivalently, *i* is an ides in π if $j = \pi_i$ is a descent in π^{-1} . The *ides set* is defined as

 $\mathsf{Ides}\,\pi = \{i : 1 < i \le n \text{ with } \pi_i + 1 \text{ appears in } \pi \text{ to the left of } \pi_i\},\$

and ides $\pi = \text{des } \pi^{-1}$, but in general Ides π is not equal to $\text{Des } \pi^{-1}$.

Let $s = s_1 s_2 \dots s_n$ be a subexcedant sequence in S_n . The Posz statistic gives the positions of 0s in s,

Posz
$$s = \{i : 1 \le i \le n, s_i = 0\},\$$

and obviously $1 \in \mathsf{Posz} s$. The Max statistic is defined as

$$Max s = \{i : 1 \le i \le n, s_i = i - 1\},\$$

and as above, $1 \in \mathsf{Max} s$.

A last-value position in s is a position i in s such that $s_i \neq 0$ and s_i does not occur in the suffix $s_{i+1}s_{i+2}\ldots s_n$ of s. The last-value position set, denoted by Row, is defined as

Row $s = \{i : s_i \neq 0 \text{ and } s_i \text{ does not occur in the suffix } s_{i+1}s_{i+2}\dots s_n\}.$

Clearly $1 \notin \text{Row } s$, and row s = card Row s counts the number of distinct nonzero symbols in s.

Example 1. If $\pi = 62587314 \in \mathfrak{S}_8$ and $s = 01102363 \in S_8$, then

Des
$$\pi$$
 = Asc s = {1, 4, 5, 6},
Ides π = Row s = {3, 5, 7, 8},
Lrmax π = Posz s = {1, 4},
Lrmin π = Max s = {1, 2, 7},

¹known in literature also as inversion sequence, inversion table or subexceedant function

 $\operatorname{RImax} \pi = \operatorname{RImin} s = \{4, 5, 8\}.$

An inversion in a permutation $\pi = \pi_1 \pi_2 \dots \pi_n \in \mathfrak{S}_n$ is a pair (i, j) with i < j and $\pi_i > \pi_j$. The set \mathfrak{S}_n is in bijection with S_n , and any such bijection is called *permutation code*. The Lehmer code L defined in [5] is a classical example of permutation code; it maps each permutation $\pi = \pi_1 \pi_2 \dots \pi_n$ to a subexcedant sequence $s_1 s_2 \dots s_n$ where, for all $j, 1 \leq j \leq n, s_j$ is the number of inversions (i, j) in π (or equivalently, the number of entries in π larger than π_j and on its left). For example L(62587314) = 01101464. See also [10] for a family of permutation codes in the context of Mahonian statistics on permutations.

In [3] is showed that dmc statistic which counts the number of distinct nonzero symbols in the Lehmer code of a permutation π (the statistic $\pi \mapsto \text{row } L(\pi)$ with the above notations) is Eulerian, and so has the same distribution as des, asc or ides on \mathfrak{S}_n . See also [9] where Dumont's statistic dmc is extended to words.

Although the following properties are folklore, they are easy to check.

Property 1. If $\pi \in \mathfrak{S}_n$ and $L(\pi) \in S_n$ is its Lehmer code, then $\mathsf{Des}\,\pi = \mathsf{Asc}\,L(\pi)$, $\mathsf{Lrmax}\,\pi = \mathsf{Posz}\,L(\pi)$, $\mathsf{Lrmin}\,\pi = \mathsf{Max}\,L(\pi)$, and $\mathsf{RImax}\,\pi = \mathsf{RImin}\,L(\pi)$.

3 The permutation code b

We define a mapping $b: \mathfrak{S}_n \to S_n$ and Theorem 1 shows that b is a bijection, that is, a permutation code, and it is the main tool in proving that (Des, Ides, Lrmax, Lrmin, Rlmax) on \mathfrak{S}_n has the same distribution as (Asc, Row, Posz, Max, Rlmax) on S_n (see Theorem 2).

A position *i* in $\pi = \pi_1 \pi_2 \dots \pi_n \in \mathfrak{S}_n$, $1 \leq i \leq n$, can satisfy the following properties:

P1: $\pi_i + 1$ occurs in π at the right of π_i ,

P2: $\pi_i - 1$ occurs in $\pi 0$ at the right of π_i ,

where $\pi 0$ is the permutation of $\{0, 1, ..., n\}$ obtained by adding a 0 at the end of π . And to each position *i* in π we associate an integer $\lambda_i(\pi) \in \{0, 1, 2, 3\}$ according to *i* satisfies both, one, or none of these properties:

$$\lambda_i(\pi) = \begin{cases} 0, \text{ if } i \text{ satisfies both P1 and P2,} \\ 1, \text{ if } i \text{ satisfies P2 but not P1,} \\ 2, \text{ if } i \text{ satisfies P1 but not P2,} \\ 3, \text{ if } i \text{ satisfies neither P1 nor P2.} \end{cases}$$

and we denote it simply by λ_i when there is no ambiguity.

Alternatively, using the Iverson bracket notation ([P] = 1 iff the statement P is true), we have the more concise expression: $\lambda_i = [\pi_i = n \text{ or } \pi_i + 1 \text{ occurs at the left of } \pi_i] + 2 \cdot [\pi_i - 1 \text{ occurs at the left of } \pi_i]$.

For example, for any $\pi \in \mathfrak{S}_n$ we have $\lambda_1 = 0$ except $\lambda_1 = 1$ if $\pi_1 = n$; and when n > 1, then $\lambda_n = 3$ except $\lambda_n = 2$ if $\pi_n = 1$. Each λ_i is uniquely determined by π , for instance if $\pi = 62587314$, then $\lambda_1, \lambda_2, \ldots, \lambda_8 = 0, 0, 1, 1, 3, 2, 1, 3$, see Figure 2.

An interval I = [a, b], $a \leq b$, is the set of integers $\{x : a \leq x \leq b\}$; and a labeled interval is a pair (I, ℓ) where I is an interval and ℓ and integer. In order to give the construction of the mapping b we define below the slices of a permutation, and some of their properties are given in Remark 1. **Definition 1.** For a permutation $\pi = \pi_1 \pi_2 \dots \pi_n \in \mathfrak{S}_n$ and an $i, 0 \leq i < n$, the *i*th slice of π is the sequence of labeled intervals $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$, defined by following process (see Figure 1).

• $U_0(\pi) = ([0, n], 0).$

• For $i \geq 1$, let $U_{i-1}(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$ be the (i-1)th slice of π and v, $1 \leq v \leq k$, be the integer such that $\pi_i \in I_v$. The *i*th slice $U_i(\pi)$ of π is defined according to λ_i :

- If $\lambda_i = 0$ (or equivalently, $\min I_v < \pi_i < \max I_v$), then

$$U_i(\pi) = (I_1, \ell_1), \dots, (I_{v-1}, \ell_{v-1}), (H, \ell_v), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \dots, (I_{k-1}, \ell_k), (I_k, \ell_k+1),$$

where $H = [\pi_i + 1, \max I_v]$ and $J = [\min I_v, \pi_i - 1];$

- If $\lambda_i = 1$ (or equivalently, min $I_v < \max I_v = \pi_i$), then

$$U_i(\pi) = (I_1, \ell_1), \dots, (I_{\nu-1}, \ell_{\nu-1}), (J, \ell_{\nu+1}), (I_{\nu+1}, \ell_{\nu+2}), \dots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1)$$

where $J = [\min I_v, \pi_i - 1];$

- If $\lambda_i = 2$ (or equivalently, $\min I_v = \pi_i < \max I_v$), then

$$U_i(\pi) = (I_1, \ell_1), \dots, (I_{v-1}, \ell_{v-1}), (J, \ell_v), (I_{v+1}, \ell_{v+1}), \dots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1),$$

where $J = [\pi_i + 1, \max I_v];$

- If $\lambda_i = 3$ (or equivalently, min $I_v = \max I_v = \pi_i$), then

$$U_i(\pi) = (I_1, \ell_1), \dots, (I_{\nu-1}, \ell_{\nu-1}), (I_{\nu+1}, \ell_{\nu+1}), \dots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1).$$

Example 2. For the permutation $\pi = 62587314$ in Figure 2, $\lambda_1(\pi), \lambda_2(\pi), \ldots, \lambda_8(\pi) = 0, 0, 1, 1, 3, 2, 1, 3$, and the process described in Definition 1 gives the slices below.

$U_0(\pi) = ([0,8],0);$	$U_0(\pi)$	$U_1(\pi)$	$U_2(\pi)$	$U_3(\pi)$	$U_4(\pi)$	$U_5(\pi)$	$U_{6}(\pi)$	$U_7(\pi)$
$U_1(\pi) = ([7, 8], 0), ([0, 5], 1);$	8	0	0	0	•			
$U_2(\pi) = ([7,8],0), ([3,5],1), ([0,1],2);$	6	•			2			
$U_3(\pi) = ([7,8],0), ([3,4],2), ([0,1],3);$	5			•				
$U_4(\pi) = ([7,7],2), ([3,4],3), ([0,1],4);$	4 0		1	2	3	3	3	3
$U_5(\pi) = ([3,4],3), ([0,1],5);$	3	1					•	
$U_6(\pi) = ([4,4],3), ([0,1],6);$	1		•					•
$U_7(\pi) = ([4,4],3), ([0,0],7).$	0		2	3	4	5	6	7

Remark 1. Let $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$ be the *i*th slice of $\pi, 0 \le i < n$. Then, the following properties can be easily checked:

- the intervals I_1, I_2, \ldots, I_k are in decreasing order, that is $\max I_{j+1} < \min I_j$ for any j, $1 \le j < k$;
- the sequence $\ell_1, \ell_2, \ldots, \ell_k$ is increasing, that is $\ell_j < \ell_{j+1}$ for any $j, 1 \le j < k$;



Figure 1: The four cases in Definition 1.

8	0	0	0	•				
7	0	0	0	2	•			
6	٠							
5	1	1	•					
4	1	1	2	3	3	3	3	•
3	1	1	2	3	3	•		
2	1	٠						
1	1	2	3	4	5	6	•	
	1	2	3	4	5	6	7	8

Figure 2: The permutation $\pi = 62587314$ with $b(\pi) = 01102363$ and $\lambda_1(\pi), \lambda_2(\pi), \dots, \lambda_8(\pi) = 0, 0, 1, 1, 3, 2, 1, 3.$

- $\{\ell_1, \ell_2, \dots, \ell_k\} \subseteq [0, i]$, and $\ell_k = i$;
- $0 \in I_k;$
- $\cup_{j=1}^{k} I_j = \{\pi_{i+1}, \pi_{i+2}, \dots, \pi_n\} \cup \{0\};$
- the (i+1)th entry of the Lehmer code of π is given by the number of entries $\pi_j > \pi_{i+1}$, with j < i+1, that is the cardinality of $[\pi_{i+1}, n] \setminus \bigcup_{j=1}^k I_j$.

A byproduct of Definition 1 is the construction of $b: \mathfrak{S}_n \to S_n$ defined below.

Definition 2. Let $\pi = \pi_1 \pi_2 \dots \pi_n \in \mathfrak{S}_n$. For each $i, 1 \leq i \leq n$, we define $b_i = \ell_v$, where v is such that (I_v, ℓ_v) is a labeled interval in the (i-1)th slice of π with $\pi_i \in I_v$, and we denote by $b(\pi)$ the sequence $b_1 b_2 \dots b_n$.

From Remark 1 it follows that $b(\pi)$ is a subexcedant sequence, see for instance Example 2 and Figures 2 and 3.

Proposition 1. Let $\pi = \pi_1 \pi_2 \dots \pi_n \in \mathfrak{S}_n$, $b(\pi) = b_1 b_2 \dots b_n$, and let $an i, 1 \leq i \leq n$.

- 1. *i* is a descent in π iff *i* is an ascent in $b(\pi)$;
- 2. *i* is an ides in π iff b_i does not occur in $b_{i+1}b_{i+2}\dots b_n$;
- 3. *i* is a left-to-right maximum in π iff $b_i = 0$;
- 4. *i* is a left-to-right minimum in π iff $b_i = i 1$;
- 5. *i* is a right-to-left maximum in π iff *i* is right-to-left minimum in *b*.

Proof. Points 1 and 2 obviously follow from the definition of b.

Point 3. Let j be such that $\pi_j = n$; $b_i = 0$ iff $i \leq j$ and π_i lies in the first interval of the (i-1)th slice of π , which in turn is equivalent to i is a left-to-right maximum in π .

Point 4. Similarly, let j be such that $\pi_j = 1$; $b_i = i - 1$ iff $i \leq j$ and π_i lies in the last interval of the (i - 1)th slice of π , which in turn is equivalent to i is a left-to-right minimum in π . Point 5. By the construction of b, i is a right-to-left maximum in π iff π_i is the largest element

of the first interval of the (i-1)th slice of π , which in turn is equivalent to b_i is smaller than any of $b_{i+1}, b_{i+2}, \ldots, b_n$.

See for instance Example 1, where $s = b(\pi)$.

For a length-*n* subexcedant sequence $b = b_1 b_2 \dots b_n$ let consider the following properties that a position $i, 1 \leq i \leq n$, can satisfy:

R1: b_i occurs in the suffix $b_{i+1}b_{i+2}\ldots b_n$ of b_i ,

R2: i - 1 occurs in b.

The next proposition shows that each $\lambda_i(\pi)$ can be obtained solely from $b(\pi)$.

Proposition 2. Let $\pi \in \mathfrak{S}_n$ and $b(\pi) = b_1 b_2 \dots b_n$. Then for any $i, 1 \leq i \leq n$, we have:

$$\lambda_i(\pi) = \begin{cases} 0, & \text{if } i \text{ satisfies both } R1 \text{ and } R2, \\ 1, & \text{if } i \text{ satisfies } R2 \text{ but not } R1, \\ 2, & \text{if } i \text{ satisfies } R1 \text{ but not } R2, \\ 3, & \text{if } i \text{ satisfies neither } R1 \text{ nor } R2. \end{cases}$$

Proof. By the construction given in Definition 2 for $b(\pi)$ from the slices of π , it follows that the position i in π satisfies property P1 (resp. P2) if and only if the position i in $b(\pi)$ satisfies property R1 (resp. R2), and the statement holds.

Proposition 3. Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$. Then

- 1. $\lambda_i(\pi) = \lambda_i(\sigma)$ for any $i, 1 \le i \le n$.
- 2. If $(I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$ is the *i*th slice of π , and $(J_1, m_1), (J_2, m_2), \dots, (J_p, m_p)$ that of σ , for some $i, 1 \leq i < n$, then k = p and $\ell_j = m_j$, for $1 \leq j \leq k$.

Proof. The first point is a consequence of Proposition 2.

The second point follows by the next considerations. The *i*th, slice of π , $i \leq 1 < n$ has the same number of intervals as its (i-1)th slice, except in two cases: $\lambda_i(\pi) = 0$ (when an interval is split into two intervals); and when $\lambda_i(\pi) = 3$ (when a one-element interval is removed). The result follows by considering the first point and by induction on *i*.

The sequence $b(\pi) = b_1 b_2 \dots b_n \in S_n$ was defined by means of the slices of π , but in proving the bijectivity of b we need rather the complement of these slices. Let $\pi \in \mathfrak{S}_n$ and $U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \dots, (I_k, \ell_k)$ be the *i*th slice of π for an $i, 1 \leq i < n$. The *i*th profile of π is the sequence X_1, X_2, \dots, X_p of decreasing nonempty maximal intervals (that is, max $X_{j+1} < \min X_j$, and none of them has the form $X_j \cup X_{j+1}$) with $\bigcup_{j=1}^p X_j = \{1, 2, \dots, n\} \setminus \bigcup_{j=1}^k I_j$. And clearly, $\bigcup_{j=1}^p X_j$ is the set of entries in π to the left of π_{i+1} , and $\sum_{j=1}^p \operatorname{card} X_j = i$.

Example 3. The vertical grey regions on the right side of Example 2 correspond to the profiles of $\pi = 62587314$ in Figure 2. These profiles are: [6,6]; [6,6], [2,2]; [5,6], [2,2]; [8,8], [5,6], [2,2]; [5,8], [2,2]; [5,8], [2,3]; and [5,8], [1,3].

In the proof of Theorem 1 we need the next result.

Proposition 4. Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$, and let an $i, 1 \leq i < n$. If X_1, X_2, \ldots, X_p and Y_1, Y_2, \ldots, Y_m are the *i*th profiles of π and of σ , then

- $-n \in X_1$ if and only if $n \in Y_1$,
- -p=m, and
- card X_j = card Y_j for any j, $1 \le j \le p$.

Proof. It is easy to see that $n \in X_1$ iff 0 does not appear in $b_{i+1}(\pi) \dots b_n(\pi) = b_{i+1}(\sigma) \dots b_n(\sigma)$, that is, iff $n \in Y_1$. And if i = 1, then the first profile of π and of σ are one-element intervals, and the statement holds.

From the first point of Proposition 3 we have $\lambda_i(\pi) = \lambda_i(\sigma)$. Let suppose that the statement is true for i - 1, and we will prove it for i.

In passing from the (i-1)th profiles of π and of σ to their *i*th profiles, the following cases can occur (we refer the reader to Definition 1 and Figure 1).

- If $\lambda_i(\pi) = \lambda_i(\sigma) = 0$, or $\lambda_i(\pi) = \lambda_i(\sigma) = 1$ and $b_i(\pi) = b_i(\sigma) = 0$, then a new oneelement interval is added to the *i*th profile of π and of σ . Moreover, since $b(\pi) = b(\sigma)$, by the second point of Proposition 3, it follows that these intervals are both, for some k, the *k*th intervals in the *i*th profile of π and σ .
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 1$ and $b_i(\pi) = b_i(\sigma) \neq 0$, then for some k, a new element is added to the kth interval of both *i*th profiles of π and σ ; this element is the smallest one in the obtained intervals.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 2$, or $\lambda_i(\pi) = \lambda_i(\sigma) = 3$ and $b_i(\pi) = b_i(\sigma) = 0$, then for some k, a new element is added to the kth interval of both *i*th profiles of π and σ ; this element is the largest one in the obtained intervals.
- If $\lambda_i(\pi) = \lambda_i(\sigma) = 3$ and $b_i(\pi) = b_i(\sigma) \neq 0$, then two consecutive intervals are merged in the *i*th profiles of π and of σ : the *k*th and (k+1)th ones, for some *k*.

Now we explain how the Lehmer code $c_1c_2...c_n$ is linked to the profiles of a permutation. By definition, $c_1 = 0$ and c_i , i > 1, is the number of entries in π at the left of π_i and larger

than π_i . If X_1, X_2, \ldots, X_p is the (i-1)th profile of π , it follows that $c_i = \sum_{j=1}^u \operatorname{card} X_j$, where u is such that $\bigcup_{j=1}^u X_j$ is the set of entries in π at the left of π_i and larger than π_i , and so $c_i = \operatorname{card} \bigcup_{j=1}^u X_j$.

Theorem 1. The mapping $b: \mathfrak{S}_n \to S_n$ is a bijection.

Proof. Let $\pi, \sigma \in \mathfrak{S}_n$ with $b(\pi) = b(\sigma)$, and $c_1c_2 \ldots c_n$ and $d_1d_2 \ldots d_n$ be the Lehmer codes of π and σ . Let also *i* be an integer, $1 < i \leq n$, and $(I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)$ be the (i - 1)th slice of π , and *v* such that $\pi_i \in I_v$ (see Definition 2). If X_1, X_2, \ldots, X_p is the (i - 1)th profile of π , then

if $n \in X_1$, it follows that $c_i = \sum_{j=1}^{v} \operatorname{card} X_j$, and

if $n \notin X_1$, it follows that $c_i = \sum_{j=1}^{v-1} \operatorname{card} X_j$.

Since $b(\pi) = b(\sigma)$, combining Proposition 4 and the second point of Proposition 3, we have that $c_i = d_i$. It follows that the Lehmer code of π and of σ are equal, and so are π and σ , and thus b is injective. And by cardinality reasons it follows that b is bijective.

It is straightforward to see that the 4-tuple of statistics (Des, Lrmax, Lrmin, Rlmax) on \mathfrak{S}_n has the same distribution as (Asc, Posz, Max, Rlmin) on S_n . Indeed, for the Lehmer code $L(\pi)$ of a permutation π we have (Des, Lrmax, Lrmin, Rlmax) $\pi = (Asc, Posz, Max, Rlmin) L(\pi)$, see Property 1. But, generally, Ides π is different from Row $L(\pi)$. For example, if $\pi = 62587314$, then $L(\pi) = 01101464$, Ides $\pi = \{3, 5, 7, 8\}$ and Row $L(\pi) = \{5, 7, 8\}$.

Combining Theorem 1 and Proposition 1 it follows that b not only behaves as the Lehmer code for the above 4-tuples of statistics, but also it transforms $\mathsf{Ides}\,\pi$ into $\mathsf{Row}\,b(\pi)$. Formally, we have the next theorem, which subsequently gives Row as a set-valued partner for Asc , thereby answering to an open question stated in [1].

Theorem 2. For any $\pi \in \mathfrak{S}_n$,

(Des, Ides, Lrmax, Lrmin, RImax) $\pi = (Asc, Row, Posz, Max, RImin) b(\pi)$,

and so the multistatistic (Des, Ides, Lrmax, Lrmin, Rlmax) on \mathfrak{S}_n has the same distribution as (Asc, Row, Posz, Max, Rlmin) on S_n .

The next corollaries are consequences of Theorem 2. The first of them is Visontai's conjecture [11] and says that (asc, row) on subexcedant sequences is a double Eulerian bistatistic.

Corollary 1. The bistatistics (asc, row) on the set of subexcedant sequences has the same distribution as (des, ides) on the set of permutations.

Corollary 2. The bistatistics (Asc, Row) and (Row, Asc) are equidistributed on the set of subexcedant sequences.

Proof. Let $s \in S_n$ and let define $t = b(\sigma)$ where $\sigma = \pi^{-1}$ with $\pi = b^{-1}(s)$. It is clear that $(Asc, Row) s = (Des, Ides) \pi = (Ides, Des) \sigma = (Row, Asc) t$.



Figure 3: The length-20 permutation $\pi = 1115741817514610312013819216912$ with $b(\pi) = 00230153751011091221341211$ and $\lambda_1(\pi), \lambda_2(\pi), \dots, \lambda_{20}(\pi) = 0, 0, 0, 0, 0, 1, 2, 1, 3, 1, 1, 0, 1, 1, 2, 3, 3, 3, 3.$

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