# A permutation code preserving a double Eulerian bistatistic 

Jean-Luc Baril and Vincent Vajnovszki<br>LE2I, Université de Bourgogne Franche-Comté<br>BP 47870, 21078 Dijon Cedex, France<br>\{barjl\}\{vvajnov\}@u-bourgogne.fr

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#### Abstract

Visontai conjectured in 2013 that the joint distribution of ascent and distinct nonzero value numbers on the set of subexcedant sequences is the same as that of descent and inverse descent numbers on the set of permutations. This conjecture has been proved by Aas in 2014, and the generating function of the corresponding bistatistics is the double Eulerian polynomial. Among the techniques used by Aas are the Möbius inversion formula and isomorphism of labeled rooted trees. In this paper we define a permutation code (that is, a bijection between permutations and subexcedant sequences) and show the more general result that two 5 -tuples of set-valued statistics on the set of permutations and on the set of subexcedant sequences, respectively, are equidistributed. In particular, these results give a bijective proof of Visontai's conjecture.


## 1 Introduction

In enumerative combinatorics it is a classical result that the descent number des and the inverse descent number ides (defined as ides $\pi=\operatorname{des} \pi^{-1}$ ) on permutations are Eulerian statistics, and their distributions on the set $\mathfrak{S}_{n}$ of length- $n$ permutations are given by the $n$th Eulerian polynomial $A_{n}$, that is

$$
A_{n}(u)=\sum_{\pi \in \mathfrak{S}_{n}} u^{\operatorname{des} \pi+1}=\sum_{\pi \in \mathfrak{S}_{n}} u^{\text {ides } \pi+1}
$$

and the joint distribution of des and ides is given by the $n$th double Eulerian polynomial,

$$
A_{n}(u, v)=\sum_{\pi \in \mathfrak{S}_{n}} u^{\operatorname{des} \pi+1} v^{\operatorname{ides} \pi+1}
$$

see for instance [2, 7].
An alternative way to represent a permutation is its Lehmer code [5], which is a subexcedant sequence. The ascent number asc on the set $S_{n}$ of subexcedant sequences is still an Eulerian statistic (see for example [8), and in 6] the statistic that counts the number of distinct nonzero symbols in a subexcedant sequence (that following [1] we denote by row) is proved to be still Eulerian; this result is credited to Dumont by the authors of 6]. In terms of generating functions we have

$$
A_{n}(u)=\sum_{s \in S_{n}} u^{\mathrm{asc} s+1}=\sum_{s \in S_{n}} u^{\mathrm{row} s+1}
$$

Moreover, in [11] Visontai conjectured that the joint distribution of des and ides on the set of permutations is the same as that of asc and row on the set of subexcedant sequences, that is

$$
A_{n}(u, v)=\sum_{\pi \in \mathfrak{S}_{n}} u^{\operatorname{des} \pi+1} v^{\text {ides } \pi+1}=\sum_{s \in S_{n}} u^{\operatorname{asc} s+1} v^{\text {row } s+1}
$$

In 2014 Aas [1] proved Visontai's conjecture, and among the techniques he used are the Möbius inversion formula and isomorphism of labeled rooted trees. In the present paper we define a bijection between permutations and subexcedant sequences (i.e., a permutation code) and show that the tuple of set-valued statistics (Des, Ides, Lrmax, Lrmin, RImax) on the set of permutations has the same distribution as (Asc, Row, Posz, Max, RImax) on the set of subexcedant sequences (each of the occurring statistics is defined below). In particular, our bijection gives a constructive proof of Visontai's conjecture.

## 2 Notation and definitions

A length-n word $w$ over the alphabet $A$ is a sequence $w_{1} w_{2} \ldots w_{n}$ of symbols in $A$, and we will consider only finite alphabets $A \subset \mathbb{N}$.

## Statistics

A statistic on a set $X$ of words is simply a function from $X$ to $\mathbb{N}$; a set-valued statistic is a function from $X$ to $2^{\mathbb{N}}$; and a multistatistic is a tuple of statistics.
Let $w=w_{1} w_{2} \ldots w_{n}$ be a length- $n$ word. A descent in $w$ is a position $i$ in $w, 1 \leq i<n$, with $w_{i}>w_{i+1}$, and the descent set of $w$ is

$$
\text { Des } w=\left\{i: 1 \leq i<n \text { with } w_{i}>w_{i+1}\right\} .
$$

A left-to-right maximum in $w$ is a position $i$ in $w, 1 \leq i \leq n$, with $w_{j}<w_{i}$ for all $j<i$, and the set of left-to-right maxima is

$$
\operatorname{Lrmax} w=\left\{i: 1 \leq i \leq n \text { with } w_{j}<w_{i} \text { for all } j<i\right\} .
$$

Clearly $1 \in \operatorname{Lrmax} w$, and Des and Lrmax are classical examples of set-valued statistics on words. We define similarly the sets Asc $w$ of ascents, Lrmin $w$ of left-to-right minima, RImax $w$ of right-to-left maxima and RImin $w$ of right-to-left minima in $w$.

To each set-valued statistic St corresponds an (integer-valued) statistic st defined as st $w=$ card St $w$, for example des $w$ and $\operatorname{Irmax} w$ counts, respectively, the number of descents and the number of left-to-right maxima in $w$.

Let $X$ and $X^{\prime}$ be two sets of words, and st and st' be two statistics defined on $X$ and $X^{\prime}$, respectively. We say that st on $X$ has the same distribution as st ${ }^{\prime}$ on $X^{\prime}$ (or equivalently, st and st' are equidistributed) if, for any integer $u$,

$$
\operatorname{card}\{w \in X: \text { st } w=u\}=\operatorname{card}\left\{w \in X^{\prime}: \text { st'}^{\prime} w=u\right\},
$$

and the multistatistic ( $\mathrm{st}_{1}, \mathrm{st}_{2}, \ldots, \mathrm{st}_{p}$ ) defined on $X$ has the same distribution as the multistatistic $\left(\mathrm{st}_{1}^{\prime}, \mathrm{st}_{2}^{\prime}, \ldots, \mathrm{st}_{p}^{\prime}\right.$ ) defined on $X^{\prime}$ (or the two multistatistics are equidistributed) if, for any integer $p$-tuple $u=\left(u_{1}, u_{2}, \ldots, u_{p}\right)$,

$$
\operatorname{card}\left\{w \in X:\left(\operatorname{st}_{1}, \mathrm{st}_{2}, \ldots, \mathrm{st}_{p}\right) w=u\right\}=\operatorname{card}\left\{w \in X^{\prime}:\left(\operatorname{st}_{1}^{\prime}, \mathrm{st}_{2}^{\prime}, \ldots, \mathrm{st}_{p}^{\prime}\right) w=u\right\}
$$

The notion of equidistribution of (multi)statistics can naturally be extended to set-valued (multi)statistics.

## Permutations, subexcedant sequences and codes

This paper deals with two particular classes of words: permutations and subexcedant sequences. A permutation is a length- $n$ word over $\{1,2, \ldots, n\}$ with distinct symbols. Alternatively, a permutation is an element of the symmetric group on $\{1,2, \ldots, n\}$ written in one line notation, and $\mathfrak{S}_{n}$ denotes the set of length- $n$ permutations. If two permutations $\pi=\pi_{1} \pi_{2} \ldots \pi_{n}$ and $\sigma=\sigma_{1} \sigma_{2} \ldots \sigma_{n}$ are such that $\sigma_{\pi_{1}} \sigma_{\pi_{2}} \cdots \sigma_{\pi_{n}}=12 \cdots n$ (i.e., the identity in $\mathfrak{S}_{n}$ ), then $\sigma$ is the inverse of $\pi$, which is denoted by $\pi^{-1}$.

A length-n subexcedant sequenc 1 is a word $s=s_{1} s_{2} \ldots s_{n}$ over $\{0,1, \ldots, n-1\}$ with $0 \leq s_{i} \leq i-1$ for $1 \leq i \leq n$, and $S_{n}$ denotes the set of length- $n$ subexcedant sequences; and we have $S_{n}=\{0\} \times\{0,1\} \times \cdots \times\{0,1, \ldots, n-1\}$.

Some statistics are consistently defined only on particular classes of words, e.g. permutations or subexcedant sequences.

For a permutation $\pi \in \mathfrak{S}_{n}$, an inverse descent (ides for short) in $\pi$ is a position $i$ for which $\pi_{i}+1$ appears to the left of $\pi_{i}$ in $\pi$. Equivalently, $i$ is an ides in $\pi$ if $j=\pi_{i}$ is a descent in $\pi^{-1}$. The ides set is defined as

$$
\text { Ides } \pi=\left\{i: 1<i \leq n \text { with } \pi_{i}+1 \text { appears in } \pi \text { to the left of } \pi_{i}\right\},
$$

and ides $\pi=\operatorname{des} \pi^{-1}$, but in general $\operatorname{Ides} \pi$ is not equal to Des $\pi^{-1}$.
Let $s=s_{1} s_{2} \ldots s_{n}$ be a subexcedant sequence in $S_{n}$. The Posz statistic gives the positions of 0 s in $s$,

$$
\text { Posz } s=\left\{i: 1 \leq i \leq n, s_{i}=0\right\},
$$

and obviously $1 \in \operatorname{Posz} s$. The Max statistic is defined as

$$
\operatorname{Max} s=\left\{i: 1 \leq i \leq n, s_{i}=i-1\right\},
$$

and as above, $1 \in \operatorname{Max} s$.
A last-value position in $s$ is a position $i$ in $s$ such that $s_{i} \neq 0$ and $s_{i}$ does not occur in the suffix $s_{i+1} s_{i+2} \ldots s_{n}$ of $s$. The last-value position set, denoted by Row, is defined as

$$
\text { Row } s=\left\{i: s_{i} \neq 0 \text { and } s_{i} \text { does not occur in the suffix } s_{i+1} s_{i+2} \ldots s_{n}\right\} \text {. }
$$

Clearly $1 \notin \operatorname{Row} s$, and row $s=$ card Row $s$ counts the number of distinct nonzero symbols in $s$.
Example 1. If $\pi=62587314 \in \mathfrak{S}_{8}$ and $s=01102363 \in S_{8}$, then
Des $\pi=$ Asc $s=\{1,4,5,6\}$,
Ides $\pi=$ Row $s=\{3,5,7,8\}$,
$\operatorname{Lrmax} \pi=\operatorname{Posz} s=\{1,4\}$,
$\operatorname{Lrmin} \pi=\operatorname{Max} s=\{1,2,7\}$,

[^0]$\operatorname{RImax} \pi=\operatorname{RImin} s=\{4,5,8\}$.
An inversion in a permutation $\pi=\pi_{1} \pi_{2} \ldots \pi_{n} \in \mathfrak{S}_{n}$ is a pair $(i, j)$ with $i<j$ and $\pi_{i}>\pi_{j}$. The set $\mathfrak{S}_{n}$ is in bijection with $S_{n}$, and any such bijection is called permutation code. The Lehmer code $L$ defined in [5] is a classical example of permutation code; it maps each permutation $\pi=\pi_{1} \pi_{2} \ldots \pi_{n}$ to a subexcedant sequence $s_{1} s_{2} \ldots s_{n}$ where, for all $j, 1 \leq j \leq n, s_{j}$ is the number of inversions $(i, j)$ in $\pi$ (or equivalently, the number of entries in $\pi$ larger than $\pi_{j}$ and on its left). For example $L(62587314)=01101464$. See also [10] for a family of permutation codes in the context of Mahonian statistics on permutations.

In [3] is showed that dmc statistic which counts the number of distinct nonzero symbols in the Lehmer code of a permutation $\pi$ (the statistic $\pi \mapsto$ row $L(\pi)$ with the above notations) is Eulerian, and so has the same distribution as des, asc or ides on $\mathfrak{S}_{n}$. See also [9] where Dumont's statistic dmc is extended to words.

Although the following properties are folklore, they are easy to check.
Property 1. If $\pi \in \mathfrak{S}_{n}$ and $L(\pi) \in S_{n}$ is its Lehmer code, then Des $\pi=\operatorname{Asc} L(\pi)$, $\operatorname{Lrmax} \pi=$ $\operatorname{Posz} L(\pi)$, $\operatorname{Lrmin} \pi=\operatorname{Max} L(\pi)$, and $\mathrm{RImax} \pi=\mathrm{R} \operatorname{lmin} L(\pi)$.

## 3 The permutation code $b$

We define a mapping $b: \mathfrak{S}_{n} \rightarrow S_{n}$ and Theorem 1 shows that $b$ is a bijection, that is, a permutation code, and it is the main tool in proving that (Des, Ides, Lrmax, Lrmin, Rlmax) on $\mathfrak{S}_{n}$ has the same distribution as (Asc, Row, Posz, Max, RImax) on $S_{n}$ (see Theorem 21).

A position $i$ in $\pi=\pi_{1} \pi_{2} \ldots \pi_{n} \in \mathfrak{S}_{n}, 1 \leq i \leq n$, can satisfy the following properties:
P1: $\pi_{i}+1$ occurs in $\pi$ at the right of $\pi_{i}$,
P2: $\pi_{i}-1$ occurs in $\pi 0$ at the right of $\pi_{i}$,
where $\pi 0$ is the permutation of $\{0,1, \ldots, n\}$ obtained by adding a 0 at the end of $\pi$. And to each position $i$ in $\pi$ we associate an integer $\lambda_{i}(\pi) \in\{0,1,2,3\}$ according to $i$ satisfies both, one, or none of these properties:

$$
\lambda_{i}(\pi)=\left\{\begin{array}{l}
0, \text { if } i \text { satisfies both } \mathrm{P} 1 \text { and } \mathrm{P} 2 \\
1, \text { if } i \text { satisfies } \mathrm{P} 2 \text { but not } \mathrm{P} 1 \\
2, \text { if } i \text { satisfies P1 but not P2 } \\
3, \text { if } i \text { satisfies neither P1 nor } \mathrm{P} 2
\end{array}\right.
$$

and we denote it simply by $\lambda_{i}$ when there is no ambiguity.
Alternatively, using the Iverson bracket notation ( $[P]=1$ iff the statement $P$ is true), we have the more concise expression: $\lambda_{i}=\left[\pi_{i}=n\right.$ or $\pi_{i}+1$ occurs at the left of $\left.\pi_{i}\right]+2 \cdot\left[\pi_{i}-\right.$ 1 occurs at the left of $\pi_{i}$ ].

For example, for any $\pi \in \mathfrak{S}_{n}$ we have $\lambda_{1}=0$ except $\lambda_{1}=1$ if $\pi_{1}=n$; and when $n>1$, then $\lambda_{n}=3$ except $\lambda_{n}=2$ if $\pi_{n}=1$. Each $\lambda_{i}$ is uniquely determined by $\pi$, for instance if $\pi=62587314$, then $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{8}=0,0,1,1,3,2,1,3$, see Figure 2.

An interval $I=[a, b], a \leq b$, is the set of integers $\{x: a \leq x \leq b\}$; and a labeled interval is a pair $(I, \ell)$ where $I$ is an interval and $\ell$ and integer. In order to give the construction of the mapping $b$ we define below the slices of a permutation, and some of their properties are given in Remark [1]

Definition 1. For a permutation $\pi=\pi_{1} \pi_{2} \ldots \pi_{n} \in \mathfrak{S}_{n}$ and an $i, 0 \leq i<n$, the $i$ th slice of $\pi$ is the sequence of labeled intervals $U_{i}(\pi)=\left(I_{1}, \ell_{1}\right),\left(I_{2}, \ell_{2}\right), \ldots,\left(I_{k}, \ell_{k}\right)$, defined by following process (see Figure (1).

- $U_{0}(\pi)=([0, n], 0)$.
- For $i \geq 1$, let $U_{i-1}(\pi)=\left(I_{1}, \ell_{1}\right),\left(I_{2}, \ell_{2}\right), \ldots,\left(I_{k}, \ell_{k}\right)$ be the $(i-1)$ th slice of $\pi$ and $v$, $1 \leq v \leq k$, be the integer such that $\pi_{i} \in I_{v}$. The $i$ th slice $U_{i}(\pi)$ of $\pi$ is defined according to $\lambda_{i}$ :
- If $\lambda_{i}=0$ (or equivalently, $\min I_{v}<\pi_{i}<\max I_{v}$ ), then

$$
U_{i}(\pi)=\left(I_{1}, \ell_{1}\right), \ldots,\left(I_{v-1}, \ell_{v-1}\right),\left(H, \ell_{v}\right),\left(J, \ell_{v+1}\right),\left(I_{v+1}, \ell_{v+2}\right), \ldots,\left(I_{k-1}, \ell_{k}\right),\left(I_{k}, \ell_{k}+1\right)
$$

where $H=\left[\pi_{i}+1, \max I_{v}\right]$ and $J=\left[\min I_{v}, \pi_{i}-1\right]$;

- If $\lambda_{i}=1$ (or equivalently, $\min I_{v}<\max I_{v}=\pi_{i}$ ), then

$$
U_{i}(\pi)=\left(I_{1}, \ell_{1}\right), \ldots,\left(I_{v-1}, \ell_{v-1}\right),\left(J, \ell_{v+1}\right),\left(I_{v+1}, \ell_{v+2}\right), \ldots,\left(I_{k-1}, \ell_{k}\right),\left(I_{k}, \ell_{k}+1\right)
$$

where $J=\left[\min I_{v}, \pi_{i}-1\right]$;

- If $\lambda_{i}=2$ (or equivalently, $\min I_{v}=\pi_{i}<\max I_{v}$ ), then

$$
U_{i}(\pi)=\left(I_{1}, \ell_{1}\right), \ldots,\left(I_{v-1}, \ell_{v-1}\right),\left(J, \ell_{v}\right),\left(I_{v+1}, \ell_{v+1}\right), \ldots,\left(I_{k-1}, \ell_{k-1}\right),\left(I_{k}, \ell_{k}+1\right),
$$

where $J=\left[\pi_{i}+1, \max I_{v}\right]$;

- If $\lambda_{i}=3$ (or equivalently, $\min I_{v}=\max I_{v}=\pi_{i}$ ), then

$$
U_{i}(\pi)=\left(I_{1}, \ell_{1}\right), \ldots,\left(I_{v-1}, \ell_{v-1}\right),\left(I_{v+1}, \ell_{v+1}\right), \ldots,\left(I_{k-1}, \ell_{k-1}\right),\left(I_{k}, \ell_{k}+1\right) .
$$

Example 2. For the permutation $\pi=62587314$ in Figure 2, $\lambda_{1}(\pi), \lambda_{2}(\pi), \ldots, \lambda_{8}(\pi)=$ $0,0,1,1,3,2,1,3$, and the process described in Definition $\mathbb{1}$ gives the slices below.

$$
\begin{aligned}
& U_{0}(\pi)=([0,8], 0) ; \\
& U_{1}(\pi)=([7,8], 0),([0,5], 1) ; \\
& U_{2}(\pi)=([7,8], 0),([3,5], 1),([0,1], 2) ; \\
& U_{3}(\pi)=([7,8], 0),([3,4], 2),([0,1], 3) ; \\
& \left.U_{4}(\pi)=([7,7], 2),([3,4], 3),,[0,1], 4\right) ; \\
& U_{5}(\pi)=([3,4], 3),([0,1], 5) ; \\
& U_{6}(\pi)=([4,4], 3),([0,1], 6) ; \\
& U_{7}(\pi)=([4,4], 3),([0,0], 7) .
\end{aligned}
$$



Remark 1. Let $U_{i}(\pi)=\left(I_{1}, \ell_{1}\right),\left(I_{2}, \ell_{2}\right), \ldots,\left(I_{k}, \ell_{k}\right)$ be the $i$ th slice of $\pi, 0 \leq i<n$. Then, the following properties can be easily checked:

- the intervals $I_{1}, I_{2}, \ldots, I_{k}$ are in decreasing order, that is $\max I_{j+1}<\min I_{j}$ for any $j$, $1 \leq j<k$;
- the sequence $\ell_{1}, \ell_{2}, \ldots, \ell_{k}$ is increasing, that is $\ell_{j}<\ell_{j+1}$ for any $j, 1 \leq j<k$;


Figure 1: The four cases in Definition 1.


Figure 2: The permutation $\pi=62587314$ with $b(\pi)=01102363$ and $\lambda_{1}(\pi), \lambda_{2}(\pi), \ldots, \lambda_{8}(\pi)=0,0,1,1,3,2,1,3$.
$-\left\{\ell_{1}, \ell_{2}, \ldots, \ell_{k}\right\} \subseteq[0, i]$, and $\ell_{k}=i$;
$-0 \in I_{k}$;
$-\cup_{j=1}^{k} I_{j}=\left\{\pi_{i+1}, \pi_{i+2}, \ldots, \pi_{n}\right\} \cup\{0\} ;$

- the $(i+1)$ th entry of the Lehmer code of $\pi$ is given by the number of entries $\pi_{j}>\pi_{i+1}$, with $j<i+1$, that is the cardinality of $\left[\pi_{i+1}, n\right] \backslash \cup_{j=1}^{k} I_{j}$.
A byproduct of Definition 1 is the construction of $b: \mathfrak{S}_{n} \rightarrow S_{n}$ defined below.
Definition 2. Let $\pi=\pi_{1} \pi_{2} \ldots \pi_{n} \in \mathfrak{S}_{n}$. For each $i, 1 \leq i \leq n$, we define $b_{i}=\ell_{v}$, where $v$ is such that $\left(I_{v}, \ell_{v}\right)$ is a labeled interval in the $(i-1)$ th slice of $\pi$ with $\pi_{i} \in I_{v}$, and we denote by $b(\pi)$ the sequence $b_{1} b_{2} \ldots b_{n}$.

From Remark 1 it follows that $b(\pi)$ is a subexcedant sequence, see for instance Example 2 and Figures 2 and 3

Proposition 1. Let $\pi=\pi_{1} \pi_{2} \ldots \pi_{n} \in \mathfrak{S}_{n}, b(\pi)=b_{1} b_{2} \ldots b_{n}$, and let an $i, 1 \leq i \leq n$.

1. $i$ is a descent in $\pi$ iff $i$ is an ascent in $b(\pi)$;
2. $i$ is an ides in $\pi$ iff $b_{i}$ does not occur in $b_{i+1} b_{i+2} \ldots b_{n}$;
3. $i$ is a left-to-right maximum in $\pi$ iff $b_{i}=0$;
4. $i$ is a left-to-right minimum in $\pi$ iff $b_{i}=i-1$;
5. $i$ is a right-to-left maximum in $\pi$ iff $i$ is right-to-left minimum in $b$.

Proof. Points 1 and 2 obviously follow from the definition of $b$.
Point 3. Let $j$ be such that $\pi_{j}=n ; b_{i}=0$ iff $i \leq j$ and $\pi_{i}$ lies in the first interval of the ( $i-1$ )th slice of $\pi$, which in turn is equivalent to $i$ is a left-to-right maximum in $\pi$.
Point 4. Similarly, let $j$ be such that $\pi_{j}=1 ; b_{i}=i-1$ iff $i \leq j$ and $\pi_{i}$ lies in the last interval of the $(i-1)$ th slice of $\pi$, which in turn is equivalent to $i$ is a left-to-right minimum in $\pi$.
Point 5. By the construction of $b, i$ is a right-to-left maximum in $\pi$ iff $\pi_{i}$ is the largest element of the first interval of the $(i-1)$ th slice of $\pi$, which in turn is equivalent to $b_{i}$ is smaller than any of $b_{i+1}, b_{i+2}, \ldots, b_{n}$.

See for instance Example 1, where $s=b(\pi)$.
For a length $-n$ subexcedant sequence $b=b_{1} b_{2} \ldots b_{n}$ let consider the following properties that a position $i, 1 \leq i \leq n$, can satisfy:

R1: $b_{i}$ occurs in the suffix $b_{i+1} b_{i+2} \ldots b_{n}$ of $b$,
R2: $i-1$ occurs in $b$.
The next proposition shows that each $\lambda_{i}(\pi)$ can be obtained solely from $b(\pi)$.
Proposition 2. Let $\pi \in \mathfrak{S}_{n}$ and $b(\pi)=b_{1} b_{2} \ldots b_{n}$. Then for any $i, 1 \leq i \leq n$, we have:

$$
\lambda_{i}(\pi)=\left\{\begin{array}{l}
0, \text { if } i \text { satisfies both R1 and R2, } \\
1, \text { if } i \text { satisfies } R 2 \text { but not } R 1, \\
2, \text { if i satisfies R1 but not } R 2, \\
3, \text { if } i \text { satisfies neither R1 nor } R 2 .
\end{array}\right.
$$

Proof. By the construction given in Definition 2 for $b(\pi)$ from the slices of $\pi$, it follows that the position $i$ in $\pi$ satisfies property P1 (resp. P2) if and only if the position $i$ in $b(\pi)$ satisfies property R1 (resp. R2), and the statement holds.

Proposition 3. Let $\pi, \sigma \in \mathfrak{S}_{n}$ with $b(\pi)=b(\sigma)$. Then

1. $\lambda_{i}(\pi)=\lambda_{i}(\sigma)$ for any $i, 1 \leq i \leq n$.
2. If $\left(I_{1}, \ell_{1}\right),\left(I_{2}, \ell_{2}\right), \ldots,\left(I_{k}, \ell_{k}\right)$ is the $i$ th slice of $\pi$, and $\left(J_{1}, m_{1}\right),\left(J_{2}, m_{2}\right), \ldots,\left(J_{p}, m_{p}\right)$ that of $\sigma$, for some $i, 1 \leq i<n$, then $k=p$ and $\ell_{j}=m_{j}$, for $1 \leq j \leq k$.

Proof. The first point is a consequence of Proposition 2.
The second point follows by the next considerations. The $i$ th, slice of $\pi, i \leq 1<n$ has the same number of intervals as its $(i-1)$ th slice, except in two cases: $\lambda_{i}(\pi)=0$ (when an interval is split into two intervals); and when $\lambda_{i}(\pi)=3$ (when a one-element interval is removed). The result follows by considering the first point and by induction on $i$.

The sequence $b(\pi)=b_{1} b_{2} \ldots b_{n} \in S_{n}$ was defined by means of the slices of $\pi$, but in proving the bijectivity of $b$ we need rather the complement of these slices. Let $\pi \in \mathfrak{S}_{n}$ and $U_{i}(\pi)=\left(I_{1}, \ell_{1}\right),\left(I_{2}, \ell_{2}\right), \ldots,\left(I_{k}, \ell_{k}\right)$ be the $i$ th slice of $\pi$ for an $i, 1 \leq i<n$. The $i$ th profile of $\pi$ is the sequence $X_{1}, X_{2}, \ldots, X_{p}$ of decreasing nonempty maximal intervals (that is, max $X_{j+1}<$ $\min X_{j}$, and none of them has the form $X_{j} \cup X_{j+1}$ ) with $\cup_{j=1}^{p} X_{j}=\{1,2, \ldots, n\} \backslash \cup_{j=1}^{k} I_{j}$. And clearly, $\cup_{j=1}^{p} X_{j}$ is the set of entries in $\pi$ to the left of $\pi_{i+1}$, and $\sum_{j=1}^{p}$ card $X_{j}=i$.
Example 3. The vertical grey regions on the right side of Example 2 correspond to the profiles of $\pi=62587314$ in Figure 22. These profiles are: $[6,6] ;[6,6],[2,2] ;[5,6],[2,2]$; $[8,8],[5,6],[2,2] ;[5,8],[2,2] ;[5,8],[2,3] ;$ and $[5,8],[1,3]$.

In the proof of Theorem 1 we need the next result.
Proposition 4. Let $\pi, \sigma \in \mathfrak{S}_{n}$ with $b(\pi)=b(\sigma)$, and let an $i, 1 \leq i<n$. If $X_{1}, X_{2}, \ldots, X_{p}$ and $Y_{1}, Y_{2}, \ldots, Y_{m}$ are the $i$ th profiles of $\pi$ and of $\sigma$, then
$-n \in X_{1}$ if and only if $n \in Y_{1}$,
$-p=m$, and
$-\operatorname{card} X_{j}=\operatorname{card} Y_{j}$ for any $j, 1 \leq j \leq p$.
Proof. It is easy to see that $n \in X_{1}$ iff 0 does not appear in $b_{i+1}(\pi) \ldots b_{n}(\pi)=b_{i+1}(\sigma) \ldots b_{n}(\sigma)$, that is, iff $n \in Y_{1}$. And if $i=1$, then the first profile of $\pi$ and of $\sigma$ are one-element intervals, and the statement holds.

From the first point of Proposition 3 we have $\lambda_{i}(\pi)=\lambda_{i}(\sigma)$. Let suppose that the statement is true for $i-1$, and we will prove it for $i$.

In passing from the $(i-1)$ th profiles of $\pi$ and of $\sigma$ to their $i$ th profiles, the following cases can occur (we refer the reader to Definition 1 and Figure 1).

- If $\lambda_{i}(\pi)=\lambda_{i}(\sigma)=0$, or $\lambda_{i}(\pi)=\lambda_{i}(\sigma)=1$ and $b_{i}(\pi)=b_{i}(\sigma)=0$, then a new oneelement interval is added to the $i$ th profile of $\pi$ and of $\sigma$. Moreover, since $b(\pi)=b(\sigma)$, by the second point of Proposition 3, it follows that these intervals are both, for some $k$, the $k$ th intervals in the $i$ th profile of $\pi$ and $\sigma$.
- If $\lambda_{i}(\pi)=\lambda_{i}(\sigma)=1$ and $b_{i}(\pi)=b_{i}(\sigma) \neq 0$, then for some $k$, a new element is added to the $k$ th interval of both $i$ th profiles of $\pi$ and $\sigma$; this element is the smallest one in the obtained intervals.
- If $\lambda_{i}(\pi)=\lambda_{i}(\sigma)=2$, or $\lambda_{i}(\pi)=\lambda_{i}(\sigma)=3$ and $b_{i}(\pi)=b_{i}(\sigma)=0$, then for some $k$, a new element is added to the $k$ th interval of both $i$ th profiles of $\pi$ and $\sigma$; this element is the largest one in the obtained intervals.
- If $\lambda_{i}(\pi)=\lambda_{i}(\sigma)=3$ and $b_{i}(\pi)=b_{i}(\sigma) \neq 0$, then two consecutive intervals are merged in the $i$ th profiles of $\pi$ and of $\sigma$ : the $k$ th and $(k+1)$ th ones, for some $k$.

Now we explain how the Lehmer code $c_{1} c_{2} \ldots c_{n}$ is linked to the profiles of a permutation. By definition, $c_{1}=0$ and $c_{i}, i>1$, is the number of entries in $\pi$ at the left of $\pi_{i}$ and larger than $\pi_{i}$. If $X_{1}, X_{2}, \ldots, X_{p}$ is the $(i-1)$ th profile of $\pi$, it follows that $c_{i}=\sum_{j=1}^{u} \operatorname{card} X_{j}$, where $u$ is such that $\cup_{j=1}^{u} X_{j}$ is the set of entries in $\pi$ at the left of $\pi_{i}$ and larger than $\pi_{i}$, and so $c_{i}=\operatorname{card} \cup_{j=1}^{u} X_{j}$.
Theorem 1. The mapping b: $\mathfrak{S}_{n} \rightarrow S_{n}$ is a bijection.
Proof. Let $\pi, \sigma \in \mathfrak{S}_{n}$ with $b(\pi)=b(\sigma)$, and $c_{1} c_{2} \ldots c_{n}$ and $d_{1} d_{2} \ldots d_{n}$ be the Lehmer codes of $\pi$ and $\sigma$. Let also $i$ be an integer, $1<i \leq n$, and $\left(I_{1}, \ell_{1}\right),\left(I_{2}, \ell_{2}\right), \ldots,\left(I_{k}, \ell_{k}\right)$ be the $(i-1)$ th slice of $\pi$, and $v$ such that $\pi_{i} \in I_{v}$ (see Definition (2). If $X_{1}, X_{2}, \ldots, X_{p}$ is the ( $i-1$ )th profile of $\pi$, then

> if $n \in X_{1}$, it follows that $c_{i}=\sum_{j=1}^{v} \operatorname{card} X_{j}$, and
> if $n \notin X_{1}$, it follows that $c_{i}=\sum_{j=1}^{v-1} \operatorname{card} X_{j}$.

Since $b(\pi)=b(\sigma)$, combining Proposition 4 and the second point of Proposition 3, we have that $c_{i}=d_{i}$. It follows that the Lehmer code of $\pi$ and of $\sigma$ are equal, and so are $\pi$ and $\sigma$, and thus $b$ is injective. And by cardinality reasons it follows that $b$ is bijective.

It is straightforward to see that the 4-tuple of statistics (Des, Lrmax, Lrmin, RImax) on $\mathfrak{S}_{n}$ has the same distribution as (Asc, Posz, Max, Rlmin) on $S_{n}$. Indeed, for the Lehmer code $L(\pi)$ of a permutation $\pi$ we have (Des, Lrmax, Lrmin, RImax) $\pi=$ (Asc, Posz, Max, RImin) $L(\pi)$, see Property 1. But, generally, Ides $\pi$ is different from Row $L(\pi)$. For example, if $\pi=62587314$, then $L(\pi)=01101464$, Ides $\pi=\{3,5,7,8\}$ and Row $L(\pi)=\{5,7,8\}$.

Combining Theorem 1 and Proposition it follows that $b$ not only behaves as the Lehmer code for the above 4 -tuples of statistics, but also it transforms Ides $\pi$ into Row $b(\pi)$. Formally, we have the next theorem, which subsequently gives Row as a set-valued partner for Asc, thereby answering to an open question stated in [1].
Theorem 2. For any $\pi \in \mathfrak{S}_{n}$,

$$
\text { (Des, Ides, Lrmax, Lrmin, RImax) } \pi=(\text { Asc, Row, Posz, Max, RImin }) b(\pi),
$$

and so the multistatistic (Des, Ides, Lrmax, Lrmin, RImax) on $\mathfrak{S}_{n}$ has the same distribution as (Asc, Row, Posz, Max, RImin) on $S_{n}$.

The next corollaries are consequences of Theorem 2. The first of them is Visontai's conjecture 11 and says that (asc, row) on subexcedant sequences is a double Eulerian bistatistic.

Corollary 1. The bistatistics (asc, row) on the set of subexcedant sequences has the same distribution as (des, ides) on the set of permutations.

Corollary 2. The bistatistics (Asc, Row) and (Row, Asc) are equidistributed on the set of subexcedant sequences.

Proof. Let $s \in S_{n}$ and let define $t=b(\sigma)$ where $\sigma=\pi^{-1}$ with $\pi=b^{-1}(s)$. It is clear that (Asc, Row) $s=($ Des, Ides) $\pi=($ Ides, Des $) \sigma=($ Row, Asc $) t$.


Figure 3: The length-20 permutation $\pi=1115741817514610312013819216912$ with $b(\pi)=00230153751011091221341211$ and $\lambda_{1}(\pi), \lambda_{2}(\pi), \ldots, \lambda_{20}(\pi)=$ $0,0,0,0,0,1,2,1,3,1,1,0,1,1,2,3,3,3,3,3$.

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[^0]:    ${ }^{1}$ known in literature also as inversion sequence, inversion table or subexceedant function

