# Maximum cuts in edge-colored graphs ${ }^{\text {T }}$ 

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#### Abstract

The input of the Maximum Colored Cut problem consists of a graph $G=(V, E)$ with an edge-coloring $c: E \rightarrow\{1,2,3, \ldots, p\}$ and a positive integer $k$, and the question is whether $G$ has a nontrivial edge cut using at least $k$ colors. The Colorful Cut problem has the same input but asks for a nontrivial edge cut using all $p$ colors. Unlike what happens for the classical Maximum Cut problem, we prove that both problems are NP-complete even on complete, planar, or bounded treewidth graphs. Furthermore, we prove that Colorful Cut is NP-complete even when each color class induces a clique of size at most 3, but is trivially solvable when each color induces a $K_{2}$. On the positive side, we prove that Maximum Colored Cut is fixedparameter tractable when parameterized by either $k$ or $p$, by constructing a cubic kernel in both cases.


Keywords: colored cut; edge cut; maximum cut; planar graph; parameterized complexity; polynomial kernel.

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## 1. Introduction

Given an edge-colored graph $G$ and an edge-set property $\Pi$, in maximum (minimum) colored/labeled $\Pi$ problems we are asked to find a subset of edges satisfying property $\Pi$ with respect to $G$ that uses the maximum (minimum) number of colors/labels. These problems have a lot of applications and have been widely studied in recent years:

- when $\Pi$ is the property of being a spanning tree of the input graph $G$, the Maximum (Minimum) Colored Spanning Tree problems have been studied in [5, 6, 7, 9, 20, 21, 28];
- when $\Pi$ is the property of being a path between two designated vertices of the input graph $G$, the Maximum (Minimum) Colored Path problems have been studied in [6, 8, 20, 30];
- when $\Pi$ is the property of being a perfect matching of the input graph $G$, the Maximum (Minimum) Perfect Matching problems have been studied in [17, 22];
- when $\Pi$ is the property of being a Hamiltonian cycle of the input graph $G$, the Minimum Colored Hamiltonian Cycle problem has been studied in [17];
- when $\Pi$ is the property of being a edge dominating set of the input graph $G$, the Minimum Colored Edge Domination Set problem has been studied in [17];

In this work, we focus our studies on the complexity analysis of colored problems where $\Pi$ is the property of being an edge cut of the input graph $G$. More precisely, let $G=(V, E)$ be a simple graph with an edge coloring $c: E \rightarrow\{1,2, \ldots, p\}$, not necessarily proper. Given a proper subset $S \subset V$, we define the edge cut $\partial S$ as the subset of $E$ where the edges have one endpoint in $S$ and the other in $V \backslash S$. We represent by $c(\partial S)$ the set of colors that appear in $\partial S$, i.e., $c(\partial S)=\{c(e) \mid e \in \partial S\}$. The problem of finding a subset $S \subset V$ such that $|c(\partial S)| \leq|c(\partial T)|$ for every $T \subset V$ is called Minimum Colored Cut, and its decision version is stated as follows.

## Minimum Colored Cut

Instance: A graph $G=(V, E)$ with an edge coloring $c: E \rightarrow$ $\{1,2, \ldots, p\}$ and an integer $k>0$.
Question: Is there a proper subset $S \subset V$ such that $|c(\partial S)| \leq k$ ?
Associated with Minimum Colored Cut we have the Minimum Col$\operatorname{ORED}(s, t)$-CuT problem, in which we are asked to find an edge cut that separates a given pair $s, t$ of vertices using as few colors as possible.

Minimum Colored $(s, t)$-Cut
Instance: A graph $G=(V, E)$ with an edge coloring $c: E \rightarrow$ $\{1,2, \ldots, p\}$, a pair $s, t$ of vertices of $G$, and an integer $k>0$.
Question: Is there a proper subset $S \subset V$ such that $s \in S, t \notin S$ and $|c(\partial S)| \leq k ?$

Analogously, the problem of finding a subset $S \subset V$ such that $|c(\partial S)| \geq$ $|c(\partial T)|$ for every $T \subset V$ is called Maximum Colored Cut, and its decision version is stated as follows.

Maximum Colored Cut
Instance: A graph $G=(V, E)$ with an edge coloring $c: E \rightarrow$ $\{1,2, \ldots, p\}$ and an integer $k>0$.
Question: Is there a proper subset $S \subset V$ such that $|c(\partial S)| \geq k$ ?
Note that the classical (simple) Maximum Cut problem [18] is the particular case of Maximum Colored Cut when $c: E \rightarrow \mathbb{N}$ is an injective function. Therefore, for the Maximum Colored Cut problem we are interested in analyzing its complexity on graph classes $\mathcal{C}$ for which Maximum Cut is solvable in polynomial time.

In addition, we are also interested in the complexity of determining if the input graph has a subset $S \subset V$ such that $|c(\partial S)|=p$, i.e., if there is an edge cut $\partial S$ using all the colors; we call this problem Colorful Cut.

Colorful Cut
Instance: A graph $G=(V, E)$ with an edge coloring $c: E \rightarrow$ $\{1,2, \ldots, p\}$.
Question: Is there a proper subset $S \subset V$ such that $|c(\partial S)|=p$ ?
Complexity issues related to Minimum Colored ( $s, t$ )-Cut and Minimum Colored Cut have been widely investigated in recent years (cf. [1,

4, 10, 11, 17, 27, 29, 30, 31, 32).
Given an edge-colored graph $G$, the span of a color of $G$ is the number of connected components of the subgraph induced by the edges of this color.

When the span of each color equals one, Minimum Colored Cut is equivalent to Minimum Cut in Hypergraphs, since each color class can be seen as a hyperedge. Therefore, it can be solved in polynomial time because, as observed in [24], the hypergraph cut function is symmetric and submodular (cf. [23, 24]). Coudert et al. [10] showed that Minimum Colored Cut can be solved in polynomial time when the number of edges of each color is bounded by a given constant, but Minimum Colored $(s, t)$ Cut is NP-hard even when each color contains at most two edges. Blin et al. [1] presented a randomized algorithm that returns an optimal colored cut of $G$ with probability at least $\left(|V|^{2 k}\right)^{-1}$, where $k$ is the maximum span of $G$. Sevaral approximation and hardness results for Minimum Colored $(s, t)$-Cut are presented in [27, 29, 30, 32]. Zhang and Fu [31] showed that Minimum Colored $(s, t)$-Cut is NP-hard even if the maximum length of any path is equal to two; and when restricted to disjoint-path graphs, MinIMUM Colored $(s, t)$-CuT can be solved in polynomial time if the number of edges of each color is at most two. Regarding the parameterized complexity of Minimum Colored $(s, t)$-Cut, Fellows et al. 17] showed that the problem is $\mathrm{W}[2]$-hard when parameterized by the number of colors of the solution, and W[1]-hard when parameterized by the number of edges of the solution. Coudert et al. [11] showed that Minimum Colored Cut can be solved in time $2^{k} \cdot n^{O(1)}$, where $k$ is the number of colors with span larger than one.

The main goal of this work is to present a complexity analysis of Maximum Colored Cut, which, to the best of our knowledge, was missing in the literature. As Colorful Cut is a particular case of Maximum Colored Cut, our hardness results deal with Colorful Cut, while the tractable cases will be presented for Maximum Colored Cut.

The remainder of the article is organized as follows. In Section 2 we provide several NP-completeness results for restricted versions of ColorfuL Cut, and in Section 3 we present cubic kernels for Maximum Colored Cut parameterized either by $p$ or by $k$. We use standard graph-theoretic notation; see [13] for any undefined notation. For the basic definitions of parameterized complexity, such as fixed-parameter tractability, W[2]-hardness, para-NP-hardness, or (polynomial) kernelization, we refer the reader to [12, 14].

## 2. NP-completeness results for Colorful Cut

Hadlock [19] proved that (simple) Maximum Cut is polynomial-time solvable on planar graphs. In this section we prove, among other results, the NP-completeness of Colorful Cut on a particular subclass of planar graphs. We start with general planar graphs, and then we discuss how the construction can be modified to get stronger hardness results.

Theorem 1. Colorful Cut is NP-complete on planar graphs.
Proof. Let $I=(U, C)$ be an instance of 3-sat. We construct in polynomial time a planar instance $G=(V, E)$ with an edge coloring $c$ such that $I=$ $(U, C)$ is satisfiable if and only if $(G, c)$ has an edge cut using all colors of $c$.

With each clause $c_{j}=(x \vee y \vee z) \in C$ we associate a $K_{3}$ where each edge is labeled by a literal of $c_{j}$ together with its occurrence in $C$, obtaining a graph $G^{\prime}$ with $m$ connected components (see Figure 1(a)), each of them associated with a clause of $C$. Starting with $G^{\prime}$, we construct a multigraph $G$ equipped with a coloring $c$, for short $G^{c}$, where each literal in a clause of $C$ is associated with a colored (multi)edge as follows:

- For each pair $\left\{x_{i}^{j}, \overline{x i}^{k}\right\}$, where the integers $j$ and $k$ represent occurrences of the literals $x_{i}$ and $\overline{x_{i}}$ in the clauses of $C$, respectively, create a color denoted by $S_{i}^{j, k}$.
- The edge labeled with $x_{i}^{j}$ in $G^{\prime}$ is replaced with parallel edges colored with $S_{i}^{j, k}$ in $G^{c}$ for all $k$. Analogously, the edge labeled by ${\overline{x_{i}}}^{k}$ in $G^{\prime}$ is replaced with parallel edges in $G^{c}$ colored with $S_{i}^{j, k}$ for all $j$.

Figure 1 illustrates the graphs $G^{\prime}$ and $G^{c}$ associated with an instance $I$.
Without loss of generality, we may assume that all variables have both positive and negative literals in $I$ (if not, the clauses containing such variables are trivially satisfiable and can be removed). From a truth assignment $A_{I}$ of $I$, we can construct a colorful cut of $G^{c}$ as follows. For each clause $c_{j}$ of $C$, pick arbitrarily one edge $\{v, w\}$ corresponding to a true literal of $C$. Then, put $v$ and $w$ in the same part of the partition, leaving the remaining vertex of the clause in the other part. This procedure gives a cut using all colors. Indeed, for each true literal $x_{i}^{j}$, there is at least one false literal ${\overline{x_{i}}}^{k}$ that places the color $S_{i}^{j, k}$ in the cut.


Figure 1: Graphs $G^{\prime}$ and $G^{c}$ associated with $\left(x_{1} \vee \overline{x_{2}} \vee \overline{x_{3}}\right) \wedge\left(\overline{x_{1}} \vee x_{2} \vee x_{3}\right) \wedge\left(\overline{x_{1}} \vee \overline{x_{2}} \vee x_{3}\right)$.


Figure 2: Edge cut associated with the instance $\left(x_{1} \vee \overline{x_{2}} \vee \overline{x_{3}}\right) \wedge\left(\overline{x_{1}} \vee x_{2} \vee x_{3}\right) \wedge\left(\overline{x_{1}} \vee \overline{x_{2}} \vee x_{3}\right)$ with truth assignment $x_{1}=x_{2}=x_{3}=1$.

Figure 2 illustrates a colorful cut for the multigraph $G^{c}$ associated with the instance of Figure 1, considering the truth assignment $A_{I}=x_{1}=x_{2}=$ $x_{3}=1$ as previously described.

Conversely, suppose that $G^{c}$ has a colorful cut. Without loss of generality, we may assume that each $K_{3}$ has a cut edge Indeed, as the cut has all the colors of the edge coloring, if there is some $K_{3}$ in a part of the partition, we can choose any vertex of this clique and place this vertex in the other part, without prejudice, because all the colors are still in the cut. Beginning with this cut, we construct a truth assignment $A_{I}$ that satisfies $I$, putting $x_{i}=1$ if at least one of the edges associated with some $x_{i}^{j}$ is inside a part of the partition, and $x_{i}=0$ otherwise. Note that this assignment is well-defined: there is no pair of literals $\left\{x_{i}^{j}, \overline{x_{i}}{ }^{k}\right\}$ such that the edges corresponding to both literals are inside a part of the partition, otherwise the color $S_{i}^{j, k}$ is missing
and the cut does not contain all colors. Besides that, each $K_{3}$ has the edges corresponding to some literal inside a part of the partition, which defines a truth assignment for $I$.

Finally, we can transform $G^{c}$ into a simple graph replacing each edge $\{v, w\}$ colored with $S_{i}^{j, k}$ by a path $\{v, x, y, w\}$ such that $c(\{x, y\})=S_{i}^{j, k}$ and the remaining edges of this path receive new different colors.

Figure 3 illustrates the simple graph $H$ obtained from $G^{c}$, where the new colors are labeled with numbers. It is not difficult to see that the multigraph $G^{c}$ has a colorful cut if and only if the associated graph $H$ has a colorful cut.


Figure 3: Graph $H$ associated with $G^{c}$.
Figure 4 illustrates a colorful cut of $H$.


Figure 4: Colorful cut of $H$.
Several NP-hard problems, such as (simple) Maximum Cut, are polynomialtime solvable on bounded treewidth graphs [3]. An important class of graphs that belong to the intersection of planar graphs and graphs of bounded treewidth is the class of $K_{4}$-minor-free graphs [2, 15].

A graph $G$ is called series-parallel if it can be obtained from a $K_{2}$ by applying a sequence of operations, where each operation is either to duplicate an edge (i.e., replace an edge with two parallel edges) or to subdivide an edge (i.e., replace an edge with a path of length two). A graph $G$ is $K_{4}$-minor free if and only if each 2 -connected component of $G$ is a series-parallel graph [2, 15].
$K_{4}$-minor-free graphs are planar because $K_{5}$ and $K_{3,3}$ have $K_{4}$ as a minor. In addition, $K_{4}$-minor-free graphs are the graphs with treewidth two [2].

We can modify the graph obtained in the construction presented in the proof of Theorem 11 in order to obtain the following corollary.

Corollary 2. Colorful Cut remains NP-complete even when the input graph $G$ satisfies simultaneously the following properties:

1. $G$ is $K_{4}$-minor-free.
2. $G$ is connected.
3. $G$ has maximum degree three.
4. Each color class of $G$ contains at most two edges.

Proof. Let $H$ be an instance of Colorful Cut constructed as described in the proof of Theorem 1. First observe that each connected component of $H$ (clause gadget) can be obtained from a $K_{2}$ by either duplicating an edge or subdividing an edge. Therefore, each connected component of $H$ is series-parallel.

In order to make the graph $H$ be connected and with bounded degree just create a binary tree $T$ with $m$ leaves and add edges by connecting a vertex with maximum degree of each gadget clause of $H$ to a distinct leaf of $T$. Assigning a new distinct color for each edge previously created, it holds that $H$ is $K_{4}$-minor-free and each color class of $H$ contains at most two edges (as in the proof of Theorem 11). Finally, each vertex $v$ of degree 4 can be replaced by a $P_{3}$ where each pedant vertex is neighbor of two vertices that were adjacent to $v$ and came from the same edge in $G^{\prime}$ (the edges of these $P_{3}$ 's also get new colors).

As the set of edges that we add in the graph induces a tree having a new color for each edge, it is easy to see that the modified graph has a colorful cut if and only if the original graph has a colorful cut.

Clearly, every bipartite graph has a colorful cut. Thus, it is natural to ask about the complexity of the problem on graphs with small odd cycle transversal.

Corollary 3. Colorful Cut remains NP-complete even when the input graph $G$ has odd cycle transversal number one.

Proof. It is enough to pick a vertex of each gadget of $(G, c)$, constructed as described in the proof of Theorem 1, and identify them into a single vertex.

Note that Maximum Cut is trivial on complete graphs, and that it is polynomial time solvable on cographs [3]. By adding a new vertex and edges colored with a new color, we can construct a hard instance in order to show the NP-completeness of Colorful Cut on complete graphs.

Theorem 4. Colorful Cut is NP-complete on complete graphs.
Proof. Given an instance ( $G, c$ ) of Colorful Cut, we create another instance $\left(G^{\prime}, c^{\prime}\right)$ such that $G^{\prime}$ is a clique as follows. Start from $(G, c)$, add all the missing edges to $G$, add a new vertex $v$ adjacent to all the vertices of $G$, add give to the edges in $E\left(G^{\prime}\right) \backslash E(G)$ the same color, different from the colors appearing in $E(G)$. Clearly, this new color appears in all the maximum colored cuts of $G^{\prime}$, and therefore $\left(G^{\prime}, c^{\prime}\right)$ has a colorful cut if and only if $(G, c)$ has one.

Note that if each color class of a graph $G$ induces a $K_{2}$, then $G$ has a colorful cut if and only if $G$ is bipartite, which can be decided in polynomial time. The next result shows that this is best possible, in the sense that Colorful Cut is NP-complete when each color class induces either a $K_{2}$ or a $K_{3}$.

Theorem 5. Colorful Cut is NP-complete when each color class induces a clique of size at most three.

Proof. For this proof we use a reduction from Not All Equal 3-sat (NAE 3-SAT), which is NP-complete [25]. Let $I=(U, C)$ be an instance of NAE 3 -SAT such that $U=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $C=\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}$.

The construction of an instance $(G, c)$ is given by the following procedure:

- For each clause $c_{j}=(x, y, z) \in C$, construct a clique $\left\{(x)_{j},(y)_{j},(z)_{j}\right\}$ with all the edges colored with color $j$.
- For each variable $u_{i} \in U$, add two new vertices $a_{i}$ and $b_{i}$ to $V$, such that $a_{i}$ is only adjacent to all positive occurrences of $u_{i}$, and $b_{i}$ is only adjacent to all negative occurrences of the same variable.
- For each variable $u_{i} \in U$, add an edge joining the vertices (in the clause cliques) corresponding to the first positive occurrence and the first negative occurrence of $u_{i}$.
- Excluding the edges of the clause cliques, all other edges are colored with new different integers strictly greater than $m$.

Figure 5 illustrates the instance $(G, c)$ of Colorful Cut associated with an instance $I=(U, C)$ of Not All Equal 3-sat.


Figure 5: Graph associated with the formula $\left(u_{1} \vee u_{2} \vee \overline{u_{3}}\right) \wedge\left(\overline{u_{1}} \vee \overline{u_{2}} \vee \overline{u_{3}}\right) \wedge\left(\overline{u_{1}} \vee u_{2} \vee \overline{u_{3}}\right)$.
At this point, it is not difficult to see that $I=(U, C)$ is a satisfiable instance of NAE 3 -SAT if and only if $(G, c)$ has a colorful cut. Indeed, suppose first that $I=(U, C)$ is a satisfiable instance of NAE 3 -sAT, and let $\eta$ be a truth assignment of $U$ that satisfies $I$. A colorful cut $\partial S$ in $G^{c}$ is obtained as follows: if $u_{i}=1$, then put all its positive occurrences together with $b_{i}$ in $S$, and put all its negative occurrences together with $a_{i}$ in $V \backslash S$. By construction, all the colors greater than $m$ are in the cut $\partial S$. Furthermore, each of the colors $j \leq m$ is in the cut because in each clause there is always a true and a false occurrence.

Conversely, suppose that $(G, c)$ contains a colorful cut $\partial S$. All the clause cliques have vertices in different parts of the partition, because the colors $j$ with $1 \leq j \leq m$ only appear in those clique edges. Thus we can produce a truth assignment $\eta$ by setting to true to those literals corresponding to the clique vertices $\left\{x_{j}, y_{j}, z_{j}\right\}$ that belong to $S$, and by setting to false otherwise. This is a consistent truth assignment because the edge joining the first positive and negative occurrences of the variable $u_{i}$ (if any) must be a
cut edge, that is, its exclusive color must be in the cut, which means that those occurrences must be in different parts of the partition, thus having opposite truth assignments. As all positive occurrences of $u_{i}$ are adjacent to the vertex $a_{i}$ and those edges have pairwise different colors presented in the cut, it forces all positive occurrences of $u_{i}$ to be in the same part of the partition, receiving the same truth assignment. Analogously, we can prove that all negative occurrences of $u_{i}$ must be in the same part of the partition, getting the same truth assignment.

Figure 6 illustrates the colorful cut of $(G, c)$ from the naE 3-SAT instance $I=(U, C)$ with $U=\left\{u_{1}, u_{2}, u_{3}\right\}$ and $C=\left\{\left(u_{1} \vee u_{2} \vee \overline{u_{3}}\right),\left(\overline{u_{1}} \vee \overline{u_{2}} \vee \overline{u_{3}}\right),\left(\overline{u_{1}} \vee\right.\right.$ $\left.\left.u_{2} \vee \overline{u_{3}}\right)\right\}$, satisfying the truth assignment $\overline{u_{1}}=u_{2}=u_{3}=1$.


Figure 6: Colorful cut of $G^{c}=(V, E, f)$ from Figure 5 corresponding to the truth assignment $\overline{u_{1}}=u_{2}=u_{3}=1$.

## 3. Polynomial kernelization of Maximum Colored Cut

From the results presented in Section 2 it follows that Maximum Colored Cut is para-NP-hard (see [12, 14]) parameterized by any of these parameters: treewidth, neighborhood diversity, genus, degeneracy, odd cycle transversal number, $p-k$, and several combinations of such parameters. In contrast to these results, next we show the fixed-parameter tractability of Maximum Colored Cut when parameterized by either $k$ or $p$, by means of the existence of a polynomial kernel.

Theorem 6. Maximum Colored Cut admits a cubic kernel parameterized by the number of colors.

Proof. First recall that a cut of a graph $G$ is a bipartite subgraph of $G$. The following claim is an easy fact.
Claim 1. Let $H=\left(V_{1}, V_{2}, E\right)$ be a bipartite graph having $\beta$ edges and no isolated vertices. The maximum number of edges having endpoints in the same part that can be added to $H$ is $2\binom{\beta}{2}$, corresponding to the case where $E$ induces a matching.

Now, suppose that $\lambda$ is the maximum number of colors in a cut of $G=$ ( $V, E$ ) and let $S \subset V$ be a set such that $|c(\partial S)|=\lambda$. Forming a bipartite graph $H$ by selecting exactly one edge of each color class in $[S, V \backslash S$, by Claim 1 it follows that any color class that is not in $H$ has at most $2\binom{\lambda}{2}$ edges, otherwise $\lambda$ would not be maximum. Let $E_{i} \subseteq E$ be the set of edges colored with color $i$. As $\lambda \leq p$, if $\left|E_{i}\right|>2\binom{p}{2}$, then color $i$ appears in any maximum colored cut. Such a property gives us the following reduction rule:
$\star$ If for some color $i,\left|E_{i}\right|>2\binom{p}{2}$, decrease by one the number of colors and replace $G$ by $G\left[E \backslash E_{i}\right]$.

The exhaustive application of Rule $\star$ yields a kernel of size $O\left(p^{3}\right)$.
Before our last result, we need the following lemma.
Lemma 7. Any simple graph $G=(V, E)$ with an edge coloring $c: E \rightarrow$ $\{1,2, \ldots, p\}$ has an edge cut $\partial S$ such that $|c(\partial S)| \geq \frac{p}{2}$.

Proof. Let $G^{\prime}$ be an uncolored graph obtained from $G$ by keeping one arbitrary edge from each color. Then the lemma follows by applying to $G^{\prime}$ the well-known property that any graph with at least $m$ edges contains a bipartite subgraph with at least $\frac{m}{2}$ edges [16].

Corollary 8. Maximum Colored Cut admits a cubic kernel parameterized by the cost of the solution.

Proof. If $k \geq p / 2$, by Lemma 7 we conclude that we are dealing with a Yes-instance. Otherwise, $k<p / 2$, and applying Rule $\star$ exhaustively yields a kernel of size $O\left(k^{3}\right)$.

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