# Computational determination of the largest lattice polytope diameter 

Nathan Chadder ${ }^{1}$<br>Department of Computing and Software<br>McMaster University<br>Hamilton, Canada<br>Antoine Deza ${ }^{2}$<br>Department of Computing and Software<br>McMaster University<br>Hamilton, Canada


#### Abstract

A lattice $(d, k)$-polytope is the convex hull of a set of points in dimension $d$ whose coordinates are integers between 0 and $k$. Let $\delta(d, k)$ be the largest diameter over all lattice ( $d, k$ )-polytopes. We develop a computational framework to determine $\delta(d, k)$ for small instances. We show that $\delta(3,4)=7$ and $\delta(3,5)=9$; that is, we verify for $(d, k)=(3,4)$ and $(3,5)$ the conjecture whereby $\delta(d, k)$ is at most $\lfloor(k+1) d / 2\rfloor$ and is achieved, up to translation, by a Minkowski sum of lattice vectors.


Keywords: Lattice polytopes, edge-graph diameter, enumeration algorithm

[^0]
## 1 Introduction

Finding a good bound on the maximal edge-diameter of a polytope in terms of its dimension and the number of its facets is not only a natural question of discrete geometry, but also historically closely connected with the theory of the simplex method, as the diameter is a lower bound for the number of pivots required in the worst case. Considering bounded polytopes whose vertices are rational-valued, we investigate a similar question where the number of facets is replaced by the grid embedding size.

The convex hull of integer-valued points is called a lattice polytope and, if all the vertices are drawn from $\{0,1, \ldots, k\}^{d}$, it is referred to as a lattice $(d, k)$-polytope. Let $\delta(d, k)$ be the largest edge-diameter over all lattice $(d, k)$ polytopes. Naddef [7] showed in 1989 that $\delta(d, 1)=d$, Kleinschmidt and Onn [6] generalized this result in 1992 showing that $\delta(d, k) \leq k d$. In 2016, Del Pia and Michini [3] strengthened the upper bound to $\delta(d, k) \leq k d-\lceil d / 2\rceil$ for $k \geq 2$, and showed that $\delta(d, 2)=\lfloor 3 d / 2\rfloor$. Pursuing Del Pia and Michini's approach, Deza and Pournin [5] showed that $\delta(d, k) \leq k d-\lceil 2 d / 3\rceil-(k-3)$ for $k \geq 3$, and that $\delta(4,3)=8$. The determination of $\delta(2, k)$ was investigated independently in the early nineties by Thiele [8], Balog and Bárány [2], and Acketa and Žunić [1]. Deza, Manoussakis, and Onn [4] showed that $\delta(d, k) \geq$ $\lfloor(k+1) d / 2\rfloor$ for all $k \leq 2 d-1$ and proposed Conjecture 1.1.

Conjecture $1.1 \delta(d, k) \leq\lfloor(k+1) d / 2\rfloor$, and $\delta(d, k)$ is achieved, up to translation, by a Minkowski sum of lattice vectors.

In Section 2, we propose a computational framework which drastically reduces the search space for lattice $(d, k)$-polytopes achieving a large diameter. Applying this framework to $(d, k)=(3,4)$ and $(3,5)$, we determine in Section 3 that $\delta(3,4)=7$ and $\delta(3,5)=9$.
Theorem 1.2 Conjecture 1.1 holds for $(d, k)=(3,4)$ and $(3,5)$; that is, $\delta(3,4)=7$ and $\delta(3,5)=9$, and both diameters are achieved, up to translation, by a Minkowski sum of lattice vectors

Note that Conjecture 1.1 holds for all known values of $\delta(d, k)$ given in Table 1, and hypothesizes, in particular, that $\delta(d, 3)=2 d$. The new entries corresponding to $(d, k)=(3,4)$ and $(3,5)$ are entered in bold.

| $\lambda^{k}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 3 | 4 | 4 | 5 | 6 | 6 | 7 | 8 | 8 |
| 3 | 3 | 4 | 6 | 7 | $\mathbf{9}$ |  |  |  |  |  |
| 4 | 4 | 6 | 8 |  |  |  |  |  |  |  |
| $\vdots$ | $\vdots$ | $\vdots$ |  |  |  |  |  |  |  |  |
| $d$ | $d$ | $\left\lfloor\frac{3 d}{2}\right\rfloor$ |  |  |  |  |  |  |  |  |

Table 1
The largest possible diameter $\delta(d, k)$ of a lattice ( $d, k$ )-polytope

## 2 Theoretical and Computational Framework

Since $\delta(2, k)$ and $\delta(d, 2)$ are known, we consider in the remainder of the paper that $d \geq 3$ and $k \geq 3$. While the number of lattice $(d, k)$-lattice polytopes is finite, a brute force search is typically intractable, even for small instances. Theorem 2.1, which recalls conditions established in [5], allows to drastically reduce the search space.

Theorem 2.1 For $d \geq 3$, let $d(u, v)$ denote the distance between two vertices $u$ and $v$ in the edge-graph of a lattice $(d, k)$-polytope $P$ such that $d(u, v)=$ $\delta(d, k)$. For $i=1, \ldots, d$, let $F_{i}^{0}$, respectively $F_{i}^{k}$, denote the intersection of $P$ with the facet of the cube $[0, k]^{d}$ corresponding to $x_{i}=0$, respectively $x_{i}=k$. Then, $d(u, v) \leq \delta(d-1, k)+k$, and the following conditions are necessary for the inequality to hold with equality:
(1) $u+v=(k, k, \ldots, k)$,
(2) any edge of $P$ with $u$ or $v$ as vertex is $\{-1,0,1\}$-valued,
(3) for $i=1, \ldots, d, F_{i}^{0}$, respectively $F_{i}^{k}$, is a $(d-1)$-dimensional face of $P$ with diameter $\delta\left(F_{i}^{0}\right)=\delta(d-1, k)$, respectively $\delta\left(F_{i}^{k}\right)=\delta(d-1, k)$.

Thus, to show that $\delta(d, k)<\delta(d-1, k)+k$, it is enough to show that there is no lattice $(d, k)$-polytope admitting a pair of vertices $(u, v)$ such that $d(u, v)=$ $\delta(d, k)$ and the conditions (1), (2), and (3) are satisfied. The computational framework to determine, given $(d, k)$, whether $\delta(d, k)=\delta(d-1, k)+k$ is outlined below and illustrated for $(d, k)=(3,4)$ or $(3,5)$.

## Algorithm to determine whether $\delta(d, k)<\delta(d-1, k)+k$

Step 1: Initialization
Determine the set $\mathcal{F}$ of all the lattice $(d-1, k)$-polytopes $P$ such that $\delta(P)=$ $\delta(d-1, k)$. For example, for $(d, k)=(3,4)$, the determination of all the 335 lattice $(2,4)$-polygons $P$ such that $\delta(P)=4$ is straightforward.

## Step 2: Symmetries

Consider, up to the symmetries of the cube $[0, k]^{d}$, the possible entries for a pair of vertices $(u, v)$ such that $u+v=\{k, k, \ldots, k\}$. For example, for $(d, k)=$ $(3,4)$, the following 6 vertices cover all possibilities for $u$ up to symmetry: $(0,0,0),(0,0,1),(0,0,2),(0,1,1),(0,1,2)$, and $(0,2,2)$, where $v=(4,4,4)-u$.

Step 3: Shelling
For each of the possible pairs $(u, v)$ determined during Step 2 , consider all possible ways for $2 d$ elements of the set $\mathcal{F}$ determined during Step 1 to form the $2 d$ facets of $P$ lying on a facet of the cube $[0, k]^{d}$. For example, for $(d, k)=(3,4)$ and $u=(0,0,0)$, we must find 6 elements of $\mathcal{F}, 3$ with $(0,0)$ as a vertex, and 3 with $(4,4)$ as a vertex. In addition, if an edge of an element of $\mathcal{F}$ with $u$ or $v$ as vertex is not $\{-1,0,1\}$-valued, this element is disregarded.

Note that since the choice of an element of $\mathcal{F}$ defines the vertices of $P$ belonging to a facet of the cube $[0, k]^{d}$, the choice for the next element of $\mathcal{F}$ to form a shelling is significantly restricted. In addition, if the set of vertices and edges belonging to the current elements of $\mathcal{F}$ considered for a shelling includes a path from $u$ to $v$ of length at most $\delta(d-1, k)+k-1$, a shortcut between $u$ and $v$ exists and the last added elements of $\mathcal{F}$ can be disregarded.

Step 4. Inner points
For each choice of $2 d$ elements of $\mathcal{F}$ forming a shelling obtained during Step 3, consider the $\{1,2, \ldots, k-1\}$-valued points not in the convex hull of the vertices of the $2 d$ elements of $\mathcal{F}$ forming a shelling. Each such $\{1,2, \ldots, k-1\}$-valued point is considered as a potential vertex of $P$ in a binary tree. If the current set of edges includes a path from $u$ to $v$ of length at most $\delta(d-1, k)+k-1$, a shortcut between $u$ and $v$ exists and the corresponding node of the binary tree can be disregarded, and the the binary tree is pruned at this node.

A convex hull and diameter computation are performed for each node of the obtained binary tree. If there is a node yielding a diameter of $\delta(d-1, k)+k$
we can conclude that $\delta(d, k)=\delta(d-1, k)+k$. Otherwise, we can conclude that $\delta(d, k)<\delta(d-1, k)+k$. For example, for $(d, k)=(3,5)$, no choice of 6 elements of $\mathcal{F}$ forming a shelling such that $d(u, v) \geq 10$ exist, and thus Step 4 is not executed.

## 3 Computational Results

For $(d, k)=(3,4)$, a shelling exists for which path lengths are not decidable by the algorithm without convex hull computations. However, this shelling only achieves a diameter of 7 . For $(d, k)=(3,5)$ the algorithm stops at Step 3 , as there is no combination of 6 elements of $\mathcal{F}$ which form a shelling such that $d(u, v) \geq \delta(2,5)+5$. Thus, no convex hull computations are required for $(d, k)=(3,5)$. A shortcut from $u$ to $v$ is typically found early on in the shelling, which leads to the algorithm terminating quickly. Run on a 2009 Intel ${ }^{\circledR}$ Core ${ }^{\mathrm{TM}} 2$ Duo 2.20 GHz CPU, the algorithm is able to terminate for $(d, k)=$ $(3,4)$ and $(3,5)$ in under a minute. Consequently, $\delta(3,4)<8$ and $\delta(3,5)<10$. Since the Minkowski sum of $(1,0,0),(0,1,0),(0,0,1),(0,1,1),(1,0,1),(1,1,0)$, and $(1,1,1)$ forms a lattice $(3,4)$-polytope with diameter 7 , we conclude that $\delta(3,4)=7$. Similarly, since the Minkowski sum of $(1,0,0),(0,1,0),(0,0,1)$, $(0,1,1),(1,0,1),(1,1,0),(0,1,-1),(1,0,-1)$, and $(1,-1,0)$ forms, up to translation, a lattice $(3,5)$-polytope with diameter 9 , we conclude that $\delta(3,5)=9$. Computations for additional values of $\delta(d, k)$ are currently underway. In particular, the same algorithm may determine whether $\delta(d, k)=\delta(d-1, k)+k$ or $\delta(d-1, k)+k-1$ for $(d, k)=(5,3)$ and $(4,4)$ provided the set of all lattice $(d-1, k)$-polytopes achieving $\delta(d-1, k)$ is determined for $(d, k)=(5,3)$ and $(4,4)$. Similarly, the algorithm could be adapted to determine whether $\delta(d, k)<\delta(d-1, k)+k-1$ provided the set of all lattice $(d-1, k)$-polytopes achieving $\delta(d-1, k)$ or $\delta(d-1, k)-1$ is determined. For example, the adapted algorithm may determine whether $\delta(3,6)=10$.

## Acknowledgement

This work was partially supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant program (RGPIN-2015-06163).

## References

[1] Dragan Acketa and Jovis̆a Z̆unić, On the maximal number of edges of convex digital polygons included into an $m \times m$-grid, Journal of Combinatorial Theory A 69 (1995), 358-368.
[2] Antal Balog and Imre Bárány, On the convex hull of the integer points in a disc, Proceedings of the Seventh Annual Symposium on Computational Geometry (1991), 162-165.
[3] Alberto Del Pia and Carla Michini, On the diameter of lattice polytopes, Discrete and Computational Geometry 55 (2016), 681-687.
[4] Antoine Deza, George Manoussakis, and Shmuel Onn, Primitive zonotopes, Discrete and Computational Geometry (to appear).
[5] Antoine Deza and Lionel Pournin, Improved bounds on the diameter of lattice polytopes, arXiv:1610.00341 (2016).
[6] Peter Kleinschmidt and Shmuel Onn, On the diameter of convex polytopes, Discrete Mathematics 102 (1992), 75-77.
[7] Dennis Naddef, The Hirsch conjecture is true for ( 0,1 )-polytopes, Mathematical Programming 45 (1989), 109-110.
[8] Torsten Thiele, Extremalprobleme für Punktmengen, Master thesis, Freie Universität, Berlin, 1991.


[^0]:    1 Email: chaddens@mcmaster.ca
    2 Email: deza@mcmaster.ca

