Partial domination of maximal outerplanar graphs

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Abstract

Several domination results have been obtained for maximal outerplanar graphs (mops). The classical domination problem is to minimize the size of a set S of vertices of an n-vertex graph G such that $G - N[S]$, the graph obtained by deleting the closed neighborhood of S, is null. A classical result of Chvátal is that the minimum size is at most $n/3$ if G is a mop. Here we consider a modification by allowing $G - N[S]$ to have isolated vertices and isolated edges only. Let $\iota_1(G)$ denote the size of a smallest set S for which this is achieved. We show that if G is a mop on $n \geq 5$ vertices, then $\iota_1(G) \leq n/5$. We also show that if n_2 is the number of vertices of degree 2, then $\iota_1(G) \leq \frac{n+n_2}{6}$ if $n_2 \leq \frac{n}{3}$, and $\iota_1(G) \leq \frac{n-n_2}{3}$ otherwise. We show that these bounds are best possible.

Keywords: Partial domination; Isolation number; Maximal outerplanar graph.
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1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote the *degree* of v in G by $d_G(v)$. The *open neighborhood* $N_G(v)$ of a vertex v of G is the set of neighbors of v, that is, $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *neighborhood* of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$. The *closed neighborhood* of a subset S of $V(G)$ is the set $N_G[S] = N_G(S) \cup S$. For $S \subseteq V(G)$, the

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subgraph of G induced by S is denoted by $G[S]$. The subgraph obtained from G by deleting all vertices in S (and all edges incident to vertices in S) is denoted by $G - S$.

A subset S of $V(G)$ is a *dominating set* of G if each vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number of* G is the size of a smallest dominating set of G and is denoted by $\gamma(G)$. Given a graph H, a subset S of $V(G)$ is an *H-isolating set of* G if $G - N_G[S]$ does not contain a copy of H. Obviously, S is a dominating set if $H = K_1$. The *H-isolation number of* G is the size of a smallest H-isolating set of G and is denoted by $\iota(G, H)$. If $H = K_{1,k+1}$ for some $k \geq 0$, then we may abbreviate $\iota(G, H)$ to $\iota_k(G)$. The study of isolating sets is an appealing and natural direction in domination theory that was introduced recently by Caro and Hansberg [1].

A *triangulated disc* is a simple planar graph whose interior faces are triangles. A *maximal outerplanar graph* is a triangulated disc whose exterior face (the unbounded face) contains all vertices. Hence, a maximal outerplanar graph can be embedded in the plane such that all vertices lie on the boundary of the exterior face and all interior faces are triangles. We abbreviate the term *maximal outerplanar graph* to *mop*. O'Rourke [14] pointed out that every mop has a unique Hamiltonian cycle. Thus, the Hamiltonian cycle of a mop is the boundary of the mop. This paper's notation and terminology on mops follows that of [10]; in particular, an edge belonging to the Hamiltonian cycle of a mop is called a *Hamiltonian edge*, while any other edge of the mop is called a *diagonal*. A *fan* of order $n \geq 3$, denoted F_n, is the mop obtained from a path P_{n-1} by adding a new vertex v and joining it to every vertex of the path. We say that v is the *center* of F_n.

Domination in mops has been extensively studied since 1975. In the classical paper [3], Chvátal essentially proved that the domination number of an n-vertex mop is at most $n/3$. Other proofs were obtained by Fish [7] and by Matheson and Tarjan [12]. Caro and Hansberg [1] proved that the $K_{1,1}$-isolation number of a mop of order $n \geq 4$ is at most $n/4$.

Theorem 1 *If G is a mop of order n, then the following hold:*
(a) [3, 7, 12] *If $n \geq 3$, then $\gamma(G) \leq \frac{n}{3}$.*
(b) [1] *If $n \geq 4$, then $\iota_0(G) \leq \frac{n}{4}$.*

For results on other types of domination in mops, we refer the reader to [1, 2, 4–6, 8, 9, 15].

Consider any n-vertex mop G. Theorem 1(a) tells us that there exists some $S \subseteq V(G)$ such that $|S| \leq n/3$ and $G - N_G[S]$ is null. Theorem 1(b) tells us that there exists some $S \subseteq V(G)$ such that $|S| \leq n/4$ and $G - N_G[S]$ has isolated vertices only. In this paper, we establish sharp upper bounds on the size of a smallest subset S of $V(G)$ such that $G - N_G[S]$ has isolated vertices and isolated edges only, that is, the edges of $G - N_G[S]$ form a matching (that is, they are pairwise disjoint). The bounds are detailed in the following two theorems and the proofs are given in Section 2.

Theorem 2 *If G is a mop of order $n \geq 5$, then*

$$\iota_1(G) \leq \frac{n}{5}.$$

Theorem 3 *If G is a mop of order $n \geq 5$ with n_2 vertices of degree 2, then*

$$\iota_1(G) \leq \begin{cases} \frac{n+n_2}{6} & \text{if } n_2 \leq \frac{n}{3}, \\ \frac{n-n_2}{3} & \text{otherwise.} \end{cases}$$

The bounds in Theorems 2 and 3 are sharp. For example, let $F_5^1, F_5^2, \ldots, F_5^t$ be $t \geq 2$ vertex disjoint fans of order 5. From each F_5^i, we choose a vertex of degree 2 and its neighbor of degree 3. Then, we join these $2t$ vertices by edges to form a mop. The resulting graph G_t is a mop of order $n = 5t$ with $n_2 = t$ vertices of degree 2. An example of the graph G_t with $t = 4$ is illustrated in Figure 1.

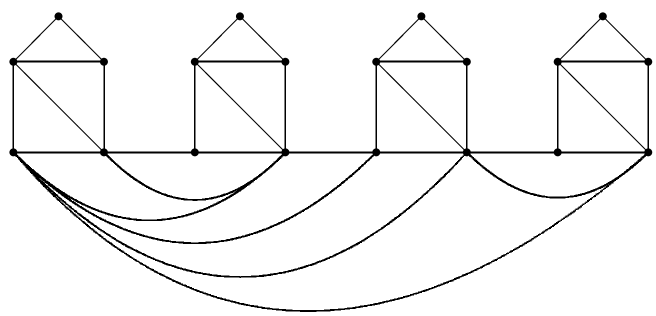

Figure 1 : The mop G_4, which satisfies $\iota_1(G_4) = n/5 = 20/5 = 4$ and $\iota_1(G_4) = (n + n_2)/6 = (20 + 4)/6 = 4$.

Clearly, for a $K_{1,2}$-isolating set S of G_t, we have $|S \cap V(F_5^i)| \geq 1$ because $G_t - N_{G_t}[S]$ does not contain $K_{1,2}$. Thus, $\iota_1(G_t) \geq t$. However, G_k has a $K_{1,2}$-isolating set containing the vertex of degree 4 of each F_5^i. Consequently, $\iota_1(G_t) = t = (5t)/5 = n/5$ and $\iota_1(G_t) = t = (5t + t)/6 = (n + n_2)/6$.

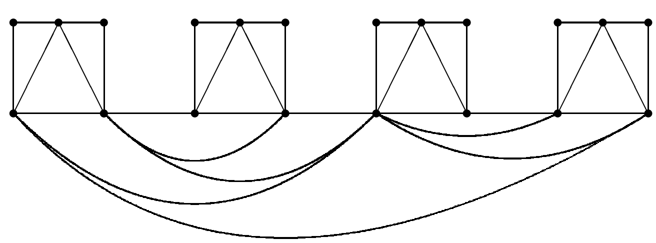

Figure 2 : The mop H_4, which satisfies $\iota_1(H_4) = (n - n_2)/3 = (20 - 8)/3 = 4$.

To see the sharpness of the bound $\iota_1(G) \leq \frac{n-n_2}{3}$, we first let $F_5^1, F_5^2, \ldots, F_5^t$ be $t \geq 2$ fans of order 5. We choose the two vertices of degree 3 from each F_5^i. Then we join these $2t$

vertices by edges to form a mop. The resulting graph H_t is a mop of order $n = 5t$ with $n_2 = 2t$ vertices of degree 2. An example of the graph H_t with $t = 4$ is illustrated in Figure 2. Clearly, for a $K_{1,2}$-isolating set S of H_t, we have $|S \cap V(F_5^i)| \geq 1$ because $H_t - N_{H_t}[S]$ does not contain $K_{1,2}$. Thus, $\iota_1(H_t) \geq t$. However, H_t has a $K_{1,2}$-isolating set containing the vertex of degree 4 of each F_5^i. Consequently, $\iota_1(H_t) = t = (5t - 2t)/3 = (n - n_2)/3$.

We conclude this section by comparing the bound $n/5$ with $(n+n_2)/6$ and $(n-n_2)/3$. We see that $n/5 < (n + n_2)/6$ when $n/5 < n_2$. Moreover, $n/5 < (n - n_2)/3$ when $n_2 < 2n/5$. Thus, the bound in Theorem 2 is better than that in Theorem 3 when $n/5 < n_2 < 2n/5$. We will show that the bound $n/5$ is sharp when $n/5 < n_2 < 2n/5$. Let F_1, F_2 and F_3 be vertex-disjoint fans of order 5 such that $V(F_1) = \{x_0, x_1, \ldots, x_4\}, V(F_2) = \{y_0, y_1, \ldots, y_4\}$ and $V(F_3) = \{z_0, z_1, \ldots, z_4\}$, where x_0, y_0, z_0 are the centers of the fans and $x_1, x_4, y_1, y_4, z_1, z_4$ are the vertices of degree 2 of the fans. Let A_{15} be the graph obtained by taking the union of F_1, F_2, F_3 and a mop that has $\{z_1, z_2, x_2, x_3, y_2, y_3\}$ as its vertex set and has z_1z_2, x_2x_3 and y_2y_3 among its Hamiltonian edges. Clearly, A_{15} is a mop of order 15 with 5 vertices of degree 2. This graph is illustrated in Figure 3. Then, we let $A_{15}^1, A_{15}^2, \ldots, A_{15}^t$ be t vertex-disjoint copies of A_{15}. The graph B_t is constructed by taking the union of $A_{15}^1, A_{15}^2, \ldots, A_{15}^t$ and a mop that has $\{x_3^1, y_3^1, x_3^2, y_3^2, \ldots, x_3^t, y_3^t\}$ as its vertex set and has $x_3^1y_3^1, x_3^2y_3^2, \ldots, x_3^ty_3^t$ among its Hamiltonian edges. Clearly, B_t is a mop of order $n = 15t$ with $n_2 = 5t$ vertices of degree 2. Thus, $n/5 < n_2 = 5n/15 < 2n/5$. Clearly, each $K_{1,2}$-isolating set of B_t needs at least one vertex from each fan F_j^i. Thus, $\iota_k(B_t) \geq 3t$. Moreover, $\{x_0^i \colon 1 \leq i \leq t\} \cup \{y_0^i \colon 1 \leq i \leq t\} \cup \{z_0^i \colon 1 \leq i \leq t\}$ is a $K_{1,2}$-isolating set of B. Thus, $\iota_1(B_t) = 3t = n/5$.

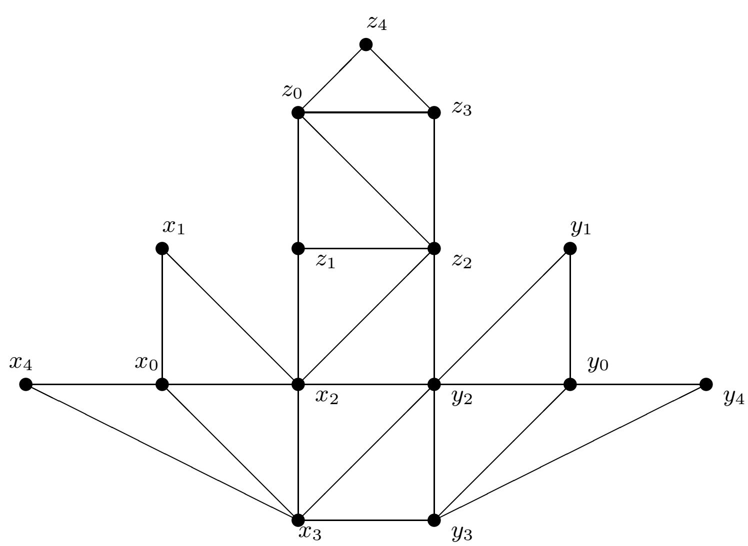

Figure 3 : The mop A_{15}.

2 Proofs of the upper bounds

In this section, we prove Theorems 2 and 3. We apply results of O'Rourke [13] in computational geometry that were used in a new proof by Lemańska, Zuazua and Zylinski [10] of an upper bound by Dorfling, Hattingh and Jonck [5] on the size of a total dominating set (a set S of vertices such that each vertex of the graph is adjacent to a vertex in S) of a mop. Before stating these results, we make a related straightforward observation that we will also use.

Given three mops G, G_1 and G_2, we say that a diagonal d of G *partitions G into G_1 and G_2* if G is the union of G_1 and G_2, $V(G_1) \cap V(G_2) = d$ and $E(G_1) \cap E(G_2) = \{d\}$.

Lemma 1 *If d is a diagonal of a mop G, then d partitions G into two mops G_1 and G_2.*

Proof. Let $x_1 x_2 \dots x_n x_1$ be the Hamiltonian cycle C of G. We may assume that $d = x_1 x_i$ for some $i \in \{2, 3, \dots, n\}$. Let C_1 be the cycle $x_1 x_2 \dots x_i x_1$ of G, and let C_2 be the cycle $x_1 x_i x_{i+1} \dots x_n x_1$. For each $i \in \{1, 2\}$, let G_i be the subgraph of G induced by $V(C_i)$. Then, $V(G_1) \cap V(G_2) = d$ and $E(G_1) \cap E(G_2) = \{d\}$. Each face of G_i is a face of G in the interior of C_i and hence a triangle; thus, G_i is a mop. Since G is a mop and $x_1 x_i$ is a diagonal of G, G has no edge with one vertex in $V(C_1) \backslash \{x_1, x_i\}$ and the other vertex in $V(C_2) \backslash \{x_1, x_i\}$. Thus, $E(G) = E(G_1) \cup E(G_2)$. $\square$

The next lemma was proved by Chvátal in [3] and is restated in [14, Lemma 1.1], and the lemma following it is an extension by O'Rourke [13] that has a central role in our proofs.

Lemma 2 ([3]) *If G is a mop of order $n \geq 8$, then G has a diagonal d that partitions it into two mops G_1 and G_2 such that G_1 has exactly* 4*,* 5 *or* 6 *Hamiltonian edges of G.*

Lemma 3 ([13]) *If G is a mop of order $n \geq 10$, then G has a diagonal d that partitions it into two mops G_1 and G_2 such that G_1 has exactly* 5*,* 6*,* 7 *or* 8 *Hamiltonian edges of G.*

For a graph G and an edge uv of G, the *edge contraction* of G along uv is the graph obtained from G by deleting u and v (and all incident edges), adding a new vertex x, and making x adjacent to the vertices in $N_G(\{u, v\}) \setminus \{u, v\}$ only. Recall that every mop can be embedded in a plane so that the exterior face contains all vertices. By looking at polygon corners as vertices, we have that a mop is a triangulation of a simple polygon, meaning that its boundary is the polygon and its interior faces are triangles.

Lemma 4 ([13]) *If G is a triangulation of a simple polygon P, e is a Hamiltonian edge of G, and G' is the edge contraction of G along e, then G' is a triangulation of some simple polygon P'.*

If G is a graph and $I \subseteq V(G)$ such that $uv \notin E(G)$ for every $u, v \in I$, then I is called an *independent set of G*.

For the proof of Theorem 3, we establish the following result about vertices of degree 2.

Lemma 5 *If G is a mop of order $n \geq 4$, then the set of vertices of G of degree 2 is an independent set of G of size at most $\frac{n}{2}$.*

Proof. Let V_2 be the set of vertices of G of degree 2. Let $x_0x_1 \dots x_{n-1}x_0$ be the unique Hamiltonian cycle of G and hence the boundary of the exterior face of G. For each $i \in \{0, 1, \dots, n-1\}$, the vertices $x_{i-1 \bmod n}$ and $x_{i+1 \bmod n}$ are neighbours of x_i.

Suppose $x_i, x_j \in V_2$ such that $x_ix_j \in E(G)$. For each $\ell \in \{-1, 0, 1, 2\}$, let $y_\ell = x_{i+\ell \bmod n}$. Since $n \geq 4$, the vertices y_{-1}, y_0, y_1, y_2 are distinct. Since $y_0 = x_i \in V_2$, we have $N_G(y_0) = \{y_{-1}, y_1\}$, so x_j is y_{-1} or y_1. We may assume that $x_j = y_1$. Since $x_j \in V_2$, we obtain $N_G(y_1) = \{y_0, y_2\}$. Together with $N_G(y_0) = \{y_{-1}, y_1\}$, this gives us that the path $y_{-1}y_0y_1y_2$ lies on an interior face of G, which contradicts the assumption that G is a mop.

Therefore, V_2 is an independent set of G. Thus, each edge of G contains at most one vertex in V_2. Let H be the set of Hamiltonian edges. For any $v \in V_2$ and $e \in H$, let $\chi(v, e) = 1$ if e contains v, and let $\chi(v, e) = 0$ otherwise. By the above, $\sum_{v \in V_2} \chi(v, e) \leq 1$ for each $e \in H$. If $x_i \in V_2$, then the edges containing x_i are $x_{i-1 \bmod n}x_i$ and $x_ix_{i+1 \bmod n}$, so $\sum_{e \in H} \chi(x_i, e) = 2$. We have

$$2|V_2| = \sum_{v \in V_2} 2 = \sum_{v \in V_2} \sum_{e \in H} \chi(v, e) = \sum_{e \in H} \sum_{v \in V_2} \chi(v, e) \leq \sum_{e \in H} 1 = n,$$

and hence $|V_2| \leq n/2$. $\square$

The next lemma lists other facts about vertices of degree 2, which we shall also use.

Lemma 6 *If G is a mop of order $n \geq 3$, then the following hold:*
(a) Each vertex of G is of degree at least 2.
(b) G has at least 2 vertices of degree 2.
(c) $G - v$ is a mop for any vertex v of G of degree 2.
(d) A graph H is a mop if $G = H - w$ for some $w \in V(H)$ such that $d_H(w) = 2$ and $N_H(w)$ is a Hamiltonian edge of G.

Proof. (a) This is immediate from the fact that G has a Hamiltonian cycle.

(b) We use induction on n. The result is trivial if $n = 3$. Suppose $n \geq 4$. Since G is a mop, G has a diagonal $d = xy$. By Lemma 1, G is the union of two mops G_1 and G_2 such that $V(G_1) \cap V(G_2) = d$ and $E(G_1) \cap E(G_2) = \{d\}$. By the induction hypothesis, for each $i \in \{1, 2\}$, G_i has two vertices $v_{i,1}$ and $v_{i,2}$ such that $d_{G_i}(v_{i,1}) = d_{G_i}(v_{i,2}) = 2$. By Lemma 5, for each $i \in \{1, 2\}$, we cannot have $d_{G_i}(x) = d_{G_i}(y) = 2$, so $v_{i,j_i} \notin \{x, y\}$ for some $j_i \in \{1, 2\}$. Therefore, we have $v_{1,j_1} \neq v_{2,j_2}$, $d_G(v_{1,j_1}) = d_{G_1}(v_{1,j_1}) = 2$ and $d_G(v_{2,j_2}) = d_{G_2}(v_{2,j_2}) = 2$.

(c) Let v be a vertex of G of degree 2. We may label the vertices $x_1, x_2, \dots, x_n$ so that $x_1x_2 \dots x_nx_1$ is the Hamiltonian cycle C of G and $x_n = v$. Since $d_G(x_n) = 2$, $N_G(x_n) = \{x_1, x_{n-1}\}$. The face F having $x_{n-1}x_n$ and x_nx_1 on its boundary must also have $x_{n-1}x_1$ on its boundary (as all interior faces are triangles), meaning that $x_{n-1}x_1 \in E(G)$. Thus, $x_1x_2 \dots, x_{n-1}x_1$ is a Hamiltonian cycle of $G - v$. Also, every interior face of G other than F is a face of $G - v$ (and a triangle). Therefore, $G - v$ is a mop.

(d) Let $x_1x_2\dots x_nx_1$ be the Hamiltonian cycle C of G. We may assume that $N_H(w) = \{x_1, x_2\}$. Let $C' = x_1wx_2\dots x_nx_1$. Then, C' is a Hamiltonian cycle of H. Let ρ be a plane drawing of G such that all vertices of G lie on the boundary C and all interior faces are triangles. Extend ρ to a plane drawing ρ' of H by putting w and the edges wx_1, wx_2 on the exterior of ρ. Then, C' is the boundary of ρ'. Also, the faces of ρ' are the faces of ρ together with the face bounded by wx_1, wx_2 and x_1x_2, so all faces of ρ' are triangles. Thus, H is a mop. $\square$

Lemmas 5 and 6(b) tell us that the number n_2 of vertices of degree 2 (of a mop) satisfies

$$2 \le n_2 \le \frac{n}{2}. \tag{1}$$

We show in passing that both bounds are sharp. For the upper bound, we start with any mop M with Hamiltonian cycle $x_1x_2\dots x_px_1$, add p new vertices $y_1, y_2, \dots, y_p$, and add the edges $y_1x_1, y_1x_2,\ y_2x_2, y_2x_3,\ \dots, y_px_p, y_px_1$, and hence the resulting graph G is a mop (by Lemma 6(d)), the vertices of G of degree 2 are $y_1, y_2, \dots, y_p$, and $p = \frac{|V(G)|}{2}$. For the lower bound, we start with a cycle $C = x_1x_2\dots x_nx_1$, set $p = \lceil n/2 \rceil$, and take G with $V(G) = V(C)$ and $E(G) = E(C) \cup \{x_2x_n, x_nx_3, x_3x_{n-1}, x_{n-1}x_4, \dots, x_{p-1}x_{n-p+3}, x_{n-p+3}x_p, x_px_{n-p+2}\}$, and hence the vertices of G of degree 2 are x_1 and x_{p+1}.

The next lemma settles Theorem 2 for $5 \le n \le 9$, and hence allows us to use Lemma 3 in the proof of Theorem 2.

Lemma 7 *If $5 \le n \le 9$ and G is a mop of order n, then $\iota_1(G) = 1 \le \frac{n}{5}$.*

Proof. By definition of a mop, $\iota_1(G) \ge 1$.

Suppose $n = 9$. Let $x_1x_2\dots x_9x_1$ be the unique Hamiltonian cycle of G and hence the boundary of the exterior face of G. By Lemma 2, G has a diagonal d such that G is the union of two mops G_1 and G_2 such that $V(G_1) \cap V(G_2) = d$, $E(G_1) \cap E(G_2) = \{d\}$, and G_1 has exactly ℓ Hamiltonian edges of G for some $\ell \in \{4, 5, 6\}$. We may assume that d is the edge $x_1x_{\ell+1}$ and that $V(G_1) = \{x_1, x_2, \dots, x_{\ell+1}\}$.

Suppose $\ell = 4$. Then, $V(G_2) = \{x_1, x_5, x_6, x_7, x_8, x_9\}$. Let (x_1, x_5, x_j) be the triangular face of G_2 containing the edge x_1x_5 (so $j \in \{6, 7, 8, 9\}$). Let $x' = x_5$ if $j = 9$, and let $x' = x_1$ otherwise. If $6 \le j \le 8$, then $x_1, x_2, x_5, x_j, x_9 \in N_G[x']$. If $j = 9$, then $x_1, x_4, x_5, x_6, x_9 \in N_G[x']$. Clearly, each component of $G - N_G[x']$ contains at most two vertices. Therefore, $\{x'\}$ is a $K_{1,2}$-isolating set of G. Hence, $\iota_1(G) = 1 < n/5$.

Suppose $\ell = 5$. Then, $x_6x_7, x_7x_8, x_8x_9, x_9x_1$ are the 4 Hamiltonian edges of G that belong to G_2. Thus, this case is identical to the case $\ell = 4$.

Suppose $\ell = 6$. Let (x_1, x_j, x_7) be the triangular face of G_1 containing the edge x_1x_7 (so $j \in \{2, 3, 4, 5, 6\}$). Suppose $j = 2$. Then, $x_2x_7 \in E(G)$. By Lemma 1, x_2x_7 partitions G into two mops H_1 and H_2 such that H_1 has exactly 4 Hamiltonian edges of G (namely, $x_7x_8, x_8x_9, x_9x_1, x_1x_2$), so the result follows as in the case $\ell = 4$. By symmetry, it suffices to consider the cases $j = 3$ and $j = 4$; that is, the cases $j = 6$ and $j = 5$ are identical to the cases $j = 2$ and $j = 3$, respectively. In both cases, each component of $G - N_G[x_j]$ contains at most two vertices, so $\{x_j\}$ is a $K_{1,2}$-isolating set of G.

Now suppose $n = 8$. Let uv be a Hamiltonian edge of G, and let H be the graph obtained by adding a new vertex w to G and adding the edges wu and wv. By Lemma 6(d), H is a mop of order 9, so $\iota_1(H) = 1 < n/5$. Let $\{x\}$ be a $K_{1,2}$-isolating set of H. If $x \neq w$, then $\{x\}$ is a $K_{1,2}$-isolating set of G. If $x = w$, then $\{u\}$ is a $K_{1,2}$-isolating set of G as $N_H[w] = \{u, v, w\} \subseteq N_H[u]$. Therefore, $\iota_1(G) = 1 < n/5$.

For $5 \leq i \leq 7$, we obtain the result for $n = i$ from the result for $n = i + 1$ in the same way we obtained the result for $n = 8$ from the result for $n = 9$. □

We now prove Theorems 2 and 3. Recall the statement of Theorem 2.

Theorem 2. *If G is a mop of order $n \geq 5$, then $\iota_1(G) \leq \frac{n}{5}$.*

Proof. We use induction on n. Lemma 7 establishes the base case $n = 5$ (in this case, G is a fan F_5, and its center is a $K_{1,2}$-isolating set of G) and also the case $6 \leq n \leq 9$. Suppose $n \geq 10$. We assume that if G' is a mop of order n' with $5 \leq n' < n$, then $\iota_1(G') \leq n'/5$.

Let $x_1 x_2 \dots x_n x_1$ be the unique Hamiltonian cycle C of G and hence the boundary of the exterior face of G. By Lemma 3, G has a diagonal d such that G is the union of two mops G_1 and G_2 such that $V(G_1) \cap V(G_2) = d$, $E(G_1) \cap E(G_2) = \{d\}$, and G_1 has exactly ℓ Hamiltonian edges of G for some $\ell \in \{5, 6, 7, 8\}$. We may assume that d is the edge $x_1 x_{\ell+1}$ and that $V(G_1) = \{x_1, x_2, \dots, x_{\ell+1}\}$. Note that the edges $x_1x_2, x_2x_3, \dots, x_\ell x_{\ell+1}$ are the ℓ Hamiltonian edges of G that belong to G_1. Let $(x_1, x_j, x_{\ell+1})$ be the triangular face of G_1 containing the edge $x_1 x_{\ell+1}$ (so $j \in \{2, 3, \dots, \ell\}$).

Claim 1 *If $\ell = 5$, then $\iota_1(G) \leq \frac{n}{5}$.*

Proof. Let G' be the graph obtained from G by deleting the vertices x_2, x_3, x_4, x_5 and contracting the edge $x_1 x_6$ to form a new vertex y (see Figure 4). Thus, G' is obtained from G_2 by contracting the edge $x_1 x_6$. By Lemma 4, G' is a mop.

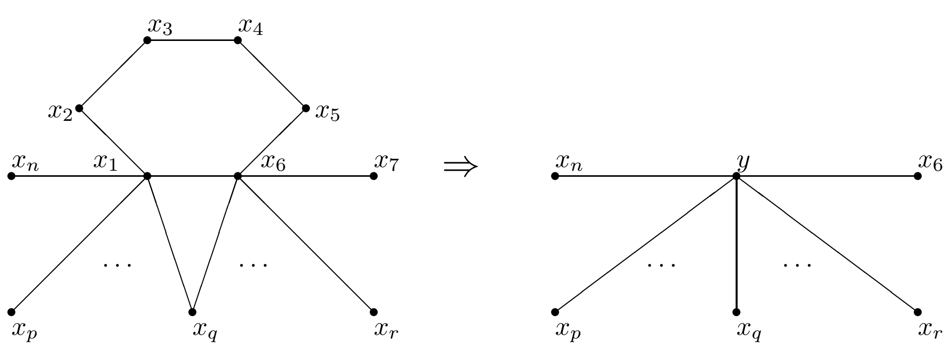

Figure 4 : The edge contraction of G_2.

Let n' be the order $n - 5 \geq 5$ of G'. By the induction hypothesis, $\iota_1(G') \leq n'/5 = n/5 - 1$.

Let S' be a smallest $K_{1,2}$-isolating set of G'. Then, $|S'| = \iota_1(G') \leq n/5 - 1$. We have that $\{x_1, x_6\}$ is a $\{K_{1,2}\}$-isolating set of G_1 and that $N_{G'}(y) \subseteq N_G(x_1) \cup N_G(x_6)$. Thus, if $y \in S'$, then $(S' \setminus \{y\}) \cup \{x_1, x_6\}$ is a $\{K_{1,2}\}$-isolating set of G, and hence $\iota_1(G) \leq (|S'|-1)+2 \leq n/5$. Suppose $y \notin S'$. Observe that $|V(G_1 - N_{G_1}[x_j])| \leq 2$. Thus, $G_1 - N_{G_1}[x_j]$ does not contain a copy of $K_{1,2}$. Moreover, $S' \cup \{x_j\}$ is a $K_{1,2}$-isolating set of G as x_j is adjacent to both x_1 and x_6. Therefore, $\iota_1(G) \leq |S'| + 1 \leq n/5$. (□)

Claim 2 *If $\ell = 6$, then $\iota_1(G) \leq \frac{n}{5}$.*

Proof. Let $G' = G_2$ and $n' = |V(G_2)|$. We have $V(G') = V(G) \setminus \{x_2, x_3, x_4, x_5, x_6\}$, so $n' = n - 5 \geq 5$. By the induction hypothesis, $\iota_1(G') \leq n'/5 = n/5 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Then, $|S'| = \iota_1(G') \leq n/5 - 1$. Suppose $j = 2$. Then, x_2x_7 is a diagonal of G that partitions G into two mops H_1 and H_2 with $H_1 = G_1 - x_1$. Since H_1 has exactly 5 Hamiltonian edges of G (namely, $x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7$), we have $\iota_1(G) \leq \frac{n}{5}$ by Claim 1. Now suppose $j \geq 3$. By symmetry, it suffices to consider the cases $j = 3$ and $j = 4$ (see Figure 5); that is, the cases $j = 6$ and $j = 5$ are identical to the cases $j = 2$ and $j = 3$, respectively. In both cases, the order of each component of $G_1 - N_{G_1}[x_j]$ is at most 2, so $G_1 - N_{G_1}[x_j]$ does not contain a copy of $K_{1,2}$. Since x_j is adjacent to x_1 and x_7, $S' \cup \{x_j\}$ is a $K_{1,2}$-isolating set of G, and hence $\iota_1(G) \leq |S'| + 1 \leq n/5$. (□)

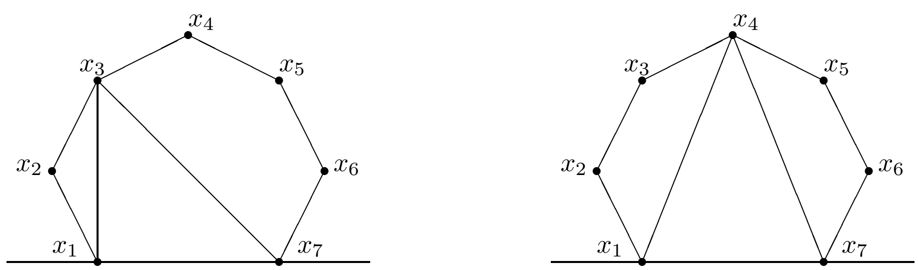

Figure 5

Claim 3 *If $\ell = 7$, then $\iota_1(G) \leq \frac{n}{5}$.*

Proof. Let $G' = G_2$ and $n' = |V(G_2)|$. We have $V(G') = V(G) \setminus \{x_2, x_3, x_4, x_5, x_6, x_7\}$, so $n' = n - 6 \geq 4$. Suppose $j = 2$. Then, x_2x_8 is a diagonal of G that partitions G into two mops H_1 and H_2 with $H_1 = G_1 - x_1$. Since H_1 has exactly 6 Hamiltonian edges of G (namely, $x_2x_3, x_3x_4, \ldots, x_7x_8$), we have $\iota_1(G) \leq \frac{n}{5}$ by Claim 2. Similarly, if $j = 3$, then x_3x_8 is a diagonal which partitions G into two mops H_1 and H_2 with $H_1 = G_1 - \{x_1, x_2\}$, and hence the result follows by Claim 1. Now suppose $j \geq 4$. By symmetry, we may assume that $j = 4$ (see Figure 6, left); that is, the cases $j = 7$, $j = 6$ and $j = 5$ are identical to the cases $j = 2$, $j = 3$ and $j = 4$, respectively.

The order of each component of $G_1 - N_{G_1}[x_4]$ is at most 2, so $G_1 - N_{G_1}[x_4]$ does not contain a copy of $K_{1,2}$. Suppose $n' = 4$. Then, $n = 10$ and $|V(G')| = 4$. Since x_4 is adjacent to both x_1 and x_8, $G' - N_G[x_4]$ does not contain a copy of $K_{1,2}$. Thus, $\{x_4\}$ is a $K_{1,2}$-isolating set of G, and hence $\iota_1(G) = 1 < n/5$. Now suppose $n' \geq 5$. By the induction hypothesis, $\iota_1(G') \leq n'/5 < n/5 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Then, since $S' \cup \{x_4\}$ is a $K_{1,2}$-isolating set of G, we have $\iota_1(G) \leq |S'| + 1 = \iota_1(G') + 1 < n/5$. (□)

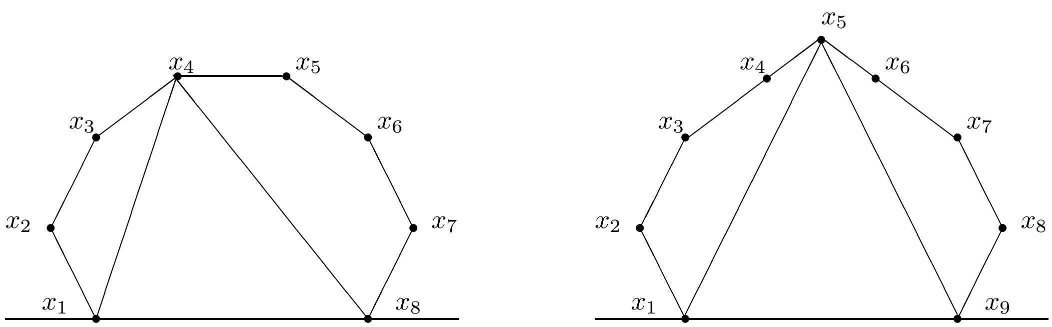

Figure 6

By Claims 1, 2 and 3, we may assume that $\ell = 8$. Similarly to the above, the cases $j = 2$, $j = 3$ and $j = 4$ follow by Claims 3, 2 and 1, respectively. Thus, we may assume that $j \geq 5$. Also similarly to the above, by symmetry, it suffices to consider $j = 5$ (see Figure 6, right).

The order of each component of $G_1 - N_{G_1}[x_5]$ is at most 2, so $G_1 - N_{G_1}[x_5]$ does not contain a copy of $K_{1,2}$. Let $G' = G_2$ and $n' = |V(G_2)|$. Then, $n' = n - 7$. Suppose $n' \leq 4$. Then, $n \leq 11$ and $|V(G')| \leq 4$. Since x_5 is adjacent to both x_1 and x_9, $G' - N_G[x_5]$ does not contain a copy of $K_{1,2}$. Thus, $\{x_5\}$ is a $K_{1,2}$-isolating set of G, and hence $\iota_1(G) = 1 < n/5$. Now suppose $n' \geq 5$. By the induction hypothesis, $\iota_1(G') \leq n'/5 < n/5 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Then, since $S' \cup \{x_5\}$ is a $K_{1,2}$-isolating set of G, we have $\iota_1(G) \leq |S'| + 1 = \iota_1(G') + 1 < n/5$. □

Recall the statement of Theorem 3.

Theorem 3. *If G is a mop of order $n \geq 5$ with n_2 vertices of degree 2, then*

$$\iota_1(G) \leq \begin{cases} \frac{n+n_2}{6} & \textit{when} \quad n_2 \leq \frac{n}{3}, \\ \\ \frac{n-n_2}{3} & \textit{otherwise}. \end{cases}$$

Proof. We first consider $n_2 > \frac{n}{3}$. Let V_2 be the set of vertices of G of degree 2. Let $G' = G - V_2$ and $n' = |V(G')|$. By Lemma 5, V_2 is an independent set of G, and $n_2 \leq n/2$. Since n' is an integer satisfying $n' = n - n_2 \geq n/2 \geq 5/2$, we have $n' \geq 3$. Since G' is obtained from G by deleting the vertices in V_2 one by one, G' is a mop by Lemma 6(c). Let S be a smallest dominating set of G'. By Theorem 1(a), $|S| \leq n'/3$. Since V_2 is an independent set of G, S is a $K_{1,2}$-isolating set of G. Thus, $\iota_1(G) \leq |S| \leq (n - n_2)/3$.

We now prove the bound for $n_2 \leq \frac{n}{3}$, using induction on n. We actually prove it for $n \geq 4$. If $n = 4$, then we trivially have $\iota_1(G) = 1$, so $\iota_1(G) \leq (n + n_2)/6$ by Lemma 6(b). By Theorem 2 and Lemma 6(b), we have $\iota_1(G) \leq 1 \leq (n + n_2)/6$ if $5 \leq n \leq 9$, and we have $\iota_1(G) \leq 2 \leq (n + n_2)/6$ if $n = 10$. We now consider $n \geq 11$ and assume that $\iota_1(G') \leq (n' + n_2')/6$ for any mop G' such that $|V(G')| = n' \geq 4$, G' has n_2' vertices of degree 2, and $n' + n_2' < n + n_2$.

Let $x_1x_2 \dots x_nx_1$ be the unique Hamiltonian cycle C of G and hence the boundary of the exterior face of G. By Lemma 3, G has a diagonal d such that G is the union of two mops G_1 and G_2 such that $V(G_1) \cap V(G_2) = d$, $E(G_1) \cap E(G_2) = \{d\}$, and G_1 has exactly ℓ Hamiltonian edges of G for some $\ell \in \{5, 6, 7, 8\}$. We may assume that d is the edge $x_1x_{\ell+1}$ and that $V(G_1) = \{x_1, x_2, \dots, x_{\ell+1}\}$. Note that the edges $x_1x_2, x_2x_3, \dots, x_\ell x_{\ell+1}$ are the ℓ Hamiltonian edges of G that belong to G_1. Let $(x_1, x_j, x_{\ell+1})$ be the triangular face of G_1 containing the edge $x_1x_{\ell+1}$ (so $j \in \{2, 3, \dots, \ell\}$).

Claim 4 *If $\ell = 5$, then $\iota_1(G) \leq \frac{n+n_2}{6}$.*

Proof. By Lemma 6(a), $d_{G_2}(v) \geq 2$ for each $v \in V(G_2)$. Since x_1 and x_6 are adjacent in G_2, Lemma 5 tells us that their degrees cannot be both 2. We may assume that $d_{G_2}(x_1) \geq 3$. This yields $d_{G_2}(x_1) + d_{G_2}(x_6) \geq 5$.

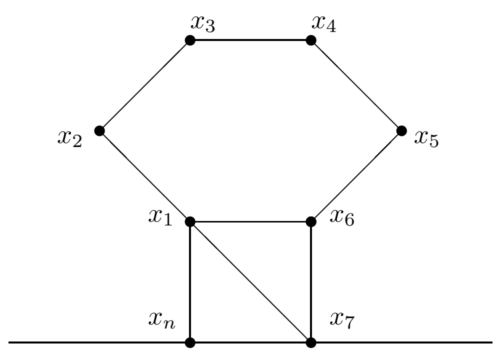

Figure 7

Claim 4.1 *If $d_{G_2}(x_1) + d_{G_2}(x_6) = 5$, then $\iota_1(G) \leq \frac{n+n_2}{6}$.*

Proof. Suppose $d_{G_2}(x_1) + d_{G_2}(x_6) = 5$. Then, $d_{G_2}(x_1) = 3$ and $d_{G_2}(x_6) = 2$. We have $x_1x_n, x_6x_7 \in E(C) \cap E(G_2)$. Since $x_1x_6, x_6x_7 \in E(G_2)$ and $d_{G_2}(x_6) = 2$, $N_{G_2}(x_6) = \{x_1, x_7\}$. Thus, since G_2 is a mop, the face having x_1x_6 and x_6x_7 on its boundary must also have x_1x_7 on its boundary (as all interior faces are triangles), that is, $x_1x_7 \in E(G_2)$ (see Figure 7). Together with $x_1x_6, x_1x_n \in E(G_2)$ and $d_{G_2}(x_1) = 3$, this gives us $N_{G_2}(x_1) = \{x_6, x_7, x_n\}$. Thus, since G_2 is a mop, the face having x_1x_7 and x_1x_n on its boundary must also have x_7x_n on its boundary, that is, $x_7x_n \in E(G_2)$ (see Figure 7). Let $G' = G - \{x_1, x_2, \dots, x_6\}$. Then,

$G' = (G_2 - x_6) - x_1$. Since $d_{G_2}(x_6) = 2$, $G_2 - x_6$ is a mop by Lemma 6(c). Since $d_{G_2 - x_6}(x_1) = 2$, G' is a mop by Lemma 6(c). Let $n' = |V(G')|$ and $n'_2 = |\{v \in V(G') \colon d_{G'}(v) = 2\}|$. We have $n' = n - 6 \geq 5$.

By Lemma 5, at most one of x_7 and x_n has degree 2 in G'. By Lemma 5, at most one of x_1 and x_6 has degree 2 in G_1, and hence, by Lemma 6(b), $d_{G_1}(x_h) = 2$ for some $h \in \{2, 3, 4, 5\}$. Since $x_h \in V(G_1) \backslash V(G_2)$, $d_G(x_h) = d_{G_1}(x_h)$. Therefore, $n'_2 \leq n_2$, and hence $n' + n'_2 \leq n + n_2 - 6$. By the induction hypothesis, $\iota_1(G') \leq (n' + n'_2)/6 \leq (n + n_2)/6 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Clearly, $|V(G_1 - N_{G_1}[x_j])| \leq 2$, so $G_1 - N_{G_1}[x_j]$ does not contain a copy of $K_{1,2}$. Since x_j is adjacent to both x_1 and x_6, it follows that $S' \cup \{x_j\}$ is a $K_{1,2}$-isolating set of G. Thus, we have $\iota_1(G) \leq |S'| + 1 = \iota_1(G') + 1 \leq (n + n_2)/6$. (□)

By Claim 4.1, we may assume that $d_{G_2}(x_1) + d_{G_2}(x_6) \geq 6$. Let G' be the graph obtained from G by deleting the vertices x_2, x_3, x_4, x_5 and contracting the edge x_1x_6 to form a new vertex y. Then, G' is obtained from G_2 by contracting x_1x_6. Thus, G' is a mop by Lemma 4. Let $n' = |V(G')|$ and $n'_2 = |\{v \in V(G') \colon d_{G'}(v) = 2\}|$. We have $n' = n - 5 \geq 6$.

Suppose that every vertex that has degree 2 in G' also has degree 2 in G. As in the proof of Claim 4.1, $d_{G_1}(x_h) = 2$ for some $h \in \{2, 3, 4, 5\}$, so $n'_2 \leq n_2 - 1$. Thus, $n' + n'_2 \leq n + n_2 - 6$. By the induction hypothesis, $\iota_1(G') \leq (n' + n'_2)/6 \leq (n + n_2)/6 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Then, $|S'| \leq (n + n_2)/6 - 1$. We can continue as in the proof of Claim 1 to obtain $\iota_1(G) \leq |S'| + 1 \leq (n + n_2)/6$.

Now suppose that G' has a vertex z such that $d_{G'}(z) = 2 \neq d_G(z)$. Then $z = y$ or $x_1, x_6 \in N_{G_2}(z)$.

Suppose $z = y$. As noted in the proof of Claim 4.1, $x_1x_n, x_6x_7 \in E(G_2)$. Thus, $x_7, x_n \in N_{G'}(y)$. Since $d_{G'}(y) = 2$, $N_{G'}(y) = \{x_7, x_n\}$. Since $d_{G_2}(x_1) + d_{G_2}(x_6) \geq 6$, we obtain $N_{G_2}(x_1) = \{x_6, x_7, x_n\}$ and $N_{G_2}(x_6) = \{x_1, x_7, x_n\}$. Since $N_{G_2}(x_1) = \{x_6, x_7, x_n\}$, x_1x_7 is a diagonal of G_2. By Lemma 1, we obtain $x_6x_n \notin E(G_2)$, which contradicts $N_{G_2}(x_6) = \{x_1, x_7, x_n\}$.

Therefore, $z \neq y$. Thus, $x_1, x_6 \in N_{G_2}(z)$. For each $i \in \{8, 9, \dots, n-1\}$ with $d_{G'}(x_i) = 2$, we have $N_{G'}(x_i) = N_G(x_i) = \{x_{i-1}, x_{i+1}\}$, so $z \neq x_i$. Thus, $z = x_7$ or $z = x_n$. By symmetry, we may assume that $z = x_7$. Since $x_7x_8 \in E(C) \cap E(G_2)$ and $x_1, x_6 \in N_{G_2}(x_7)$, $N_{G_2}(x_7) = \{x_1, x_6, x_8\}$. Thus, since G_2 is a mop, the face having x_1x_7 and x_7x_8 on its boundary must also have x_1x_8 on its boundary (as all interior faces are triangles), meaning that $x_1x_8 \in E(G_2)$. By Lemma 1, x_1x_8 partitions G into two mops H_1 and H_2 such that $V(H_2) = \{x_1, x_8, x_9, \dots, x_n\}$. Let $G' = H_2$, $n' = V(H_2)$ and $n'_2 = |\{v \in V(H_2) \colon d_{H_2}(v) = 2\}|$. We have $n' = n - 6 \geq 5$. By Lemma 5, for each $i \in \{1, 2\}$, at most one of x_1 and x_8 has degree 2 in H_i. By Lemma 6(b), $d_{H_1}(x_h) = 2$ for some $h \in V(H_1) \backslash \{x_1, x_8\}$, and hence $d_G(x_h) = 2$. Therefore, $n'_2 \leq n_2$, and hence $n' + n'_2 \leq n + n_2 - 6$. By the induction hypothesis, $\iota_1(G') \leq (n' + n'_2)/6 \leq (n + n_2)/6 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Let $x' = x_6$ if $j = 2$, and let $x' = x_1$ otherwise. If $3 \leq j \leq 5$, then $x_1, x_2, x_j, x_6, x_7, x_8 \in N_{H_1}[x']$. If $j = 2$, then $x_1, x_2, x_5, x_6, x_7 \in N_{H_1}[x']$. Since x_1x_6 is a diagonal of G, we have $x_3, x_4 \notin N_G(x_8)$ by Lemma 1. Therefore, $S' \cup \{x'\}$ is a $K_{1,2}$-isolating set of G. Thus, we have $\iota_1(G) \leq |S'| + 1 = \iota_1(G') + 1 \leq (n + n_2)/6$. (□)

Claim 5 *If $\ell = 6$, then $\iota_1(G) \le \frac{n+n_2}{6}$.*

Proof. As in the proof of Claim 2, we may assume that $j = 3$ or $j = 4$. In both cases, the order of each component of $G_1 - N_{G_1}[x_j]$ is at most 2, so $G_1 - N_{G_1}[x_j]$ does not contain a copy of $K_{1,2}$. Let $G' = G_2$, $n' = |V(G_2)|$ and $n_2' = |\{v \in V(G_2) \colon d_{G_2}(v) = 2\}|$. We have $V(G') = V(G) \setminus \{x_2, x_3, x_4, x_5, x_6\}$, so $n' = n - 5 \ge 6$.

We have $x_1x_j, x_jx_7 \in E(G_1)$. Since $3 \le j \le 4$, x_1x_j and x_jx_7 are diagonals of G. By Lemma 1, x_1x_j partitions G into two mops $G_{1,1}$ and $G_{1,2}$ with $V(G_{1,1}) = \{x_1, x_2, \ldots, x_j\}$, and x_jx_7 partitions G into two mops $G_{2,1}$ and $G_{2,2}$ with $V(G_{2,1}) = \{x_j, x_{j+1}, \ldots, x_7\}$. By Lemma 5, at most one of x_1 and x_j is of degree 2 in $G_{1,1}$, and at most one of x_j and x_7 is of degree 2 in $G_{2,1}$. Thus, by Lemma 6(b), $d_{G_{1,1}}(y) = 2$ for some $y \in V(G_{1,1})\backslash\{x_1, x_j\}$, and $d_{G_{2,1}}(z) = 2$ for some $z \in V(G_{2,1})\backslash\{x_j, x_7\}$. We have $d_G(y) = d_{G_{1,1}}(y) = 2$ and $d_G(z) = d_{G_{2,1}}(z) = 2$. Also, $y, z \notin V(G')$. Now $d_G(v) = d_{G'}(v)$ for each $v \in V(G')\backslash\{x_1, x_7\}$, and, by Lemma 6(b), at most one of x_1 and x_7 is of degree 2 in G'. Therefore, $n_2' \le n_2 - 1$.

We have $n' + n_2' \le n + n_2 - 6$. By the induction hypothesis, $\iota_1(G') \le (n' + n_2')/6 = (n + n_2)/6 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Then, $|S'| = \iota_1(G') \le (n+n_2)/6 - 1$. Since x_j is adjacent to x_1 and x_7, $S' \cup \{x_j\}$ is a $K_{1,2}$-isolating set of G, and hence $\iota_1(G) \le |S'| + 1 \le (n + n_2)/6$. (□)

Claim 6 *If $\ell = 7$, then $\iota_1(G) \le \frac{n+n_2}{6}$.*

Proof. As in the proof of Claim 3, we may assume that $j = 4$. Let $G' = G_2$ and $n' = |V(G_2)|$. We have $V(G') = V(G) \setminus \{x_2, x_3, x_4, x_5, x_6, x_7\}$, so $n' = n - 6 \ge 5$. By the same reasoning used for Claim 5, there are at least two vertices in $V(G)\backslash V(G')$ of degree 2 in G, at most one of x_1 and x_8 is of degree 2 in G', and $d_G(v) = d_{G'}(v)$ for each $v \in V(G')\backslash\{x_1, x_8\}$. Thus, $n_2' \le n_2 - 1$, and hence $n' + n_2' \le n + n_2 - 7$. By the induction hypothesis, $\iota_1(G') \le (n' + n_2')/6 < (n + n_2)/6 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Then, since $S' \cup \{x_4\}$ is a $K_{1,2}$-isolating set of G, we have $\iota_1(G) \le |S'| + 1 = \iota_1(G') + 1 < (n + n_2)/6$. (□)

By Claims 4, 5 and 6, we may assume that $\ell = 8$. As in the proof of Theorem 2 for the same case, we may assume that $j = 5$. Let $G' = G_2$ and $n' = |V(G_2)|$. We have $V(G') = V(G) \setminus \{x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, so $n' = n - 7 \ge 4$. By the same reasoning used for Claim 5, there are at least two vertices in $V(G)\backslash V(G')$ of degree 2 in G, at most one of x_1 and x_9 is of degree 2 in G', and $d_G(v) = d_{G'}(v)$ for each $v \in V(G')\backslash\{x_1, x_9\}$. Thus, $n_2' \le n_2 - 1$, and hence $n' + n_2' \le n + n_2 - 8$. By the induction hypothesis (recall that the first bound in the theorem is being proved for $n \ge 4$), $\iota_1(G') \le (n' + n_2')/6 < (n + n_2)/6 - 1$. Let S' be a smallest $K_{1,2}$-isolating set of G'. Then, since $S' \cup \{x_5\}$ is a $K_{1,2}$-isolating set of G, we have $\iota_1(G) \le |S'| + 1 = \iota_1(G') + 1 < (n + n_2)/6$. □

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