# CHARACTERISING CIRCULAR-ARC CONTACT $B_{0}-$ VPG GRAPHS 

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#### Abstract

A contact $B_{0}-V P G$ graph is a graph for which there exists a collection of nontrivial pairwise interiorly disjoint horizontal and vertical segments in one-toone correspondence with its vertex set such that two vertices are adjacent if and only if the corresponding segments touch. It was shown in [15] that RECOGNition is NP-complete for contact $B_{0}-\mathrm{VPG}$ graphs. In this paper we present a minimal forbidden induced subgraph characterisation of contact $B_{0}-$ VPG graphs within the class of circular-arc graphs and provide a polynomial-time algorithm for recognising these graphs.


## 1. Introduction

Intersection graphs of various types of objects have been extensively studied in the last sixty years (see for example [30]). In [4], Asinowski et al. introduced the class of Vertex intersection graphs of Paths on a Grid (VPG graphs for short) which consists of those graphs whose vertices may be representated by paths on a grid in such a way that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point. It is not difficult to see that the class of VPG graphs coincides with that of string graphs [16], that is, intersection graphs of curves in the plane (see [4]).

A natural restriction which was forthwith considered consists in limiting the number of bends (i.e. 90 degrees turns at a grid-point) that the paths may have: a graph is a $B_{k}-V P G$ graph, for some integer $k \geq 0$, if one can assign a path on a grid having at most $k$ bends to each vertex such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point. Since their introduction, $B_{k^{-}}$ VPG graphs have received much attention (see for instance [3,4,8-11, 17, 19, 21, 22, 24]).

A notion closely related to intersection graphs is that of contact graphs. Such graphs can be seen as a special type of intersection graphs of geometrical objects in which these objects are pairwise interiorly disjoint. Similarly to intersections graphs, contact graphs of various types of objects have been extensively studied in the literature (see for instance $[1,2,12-14,17,25-27]$ ). In this paper, we are interested in the contact counterpart of VPG graphs, namely Contact graphs of Paths on a Grid (contact VPG graphs for short, also known as $C P G$ graphs) which are defined as follows. A graph $G$ is a contact VPG graph if the vertices of $G$ can be represented by a family of nontrivial and pairwise interiorly disjoint paths on a grid in such a way that two vertices are adjacent in $G$ if and only if the corresponding paths touch, that is, share a grid-point which is an endpoint of at least one of the two paths. Note that this class is hereditary, i.e., closed under vertex deletion. Similarly to VPG graphs, a contact $B_{k}-V P G$ graph is a contact VPG graph admitting a representation in which each path has at most $k$ bends. Clearly, any contact $B_{k}-\mathrm{VPG}$ graph is also a $B_{k}-\mathrm{VPG}$ graph.

In this paper, we focus solely on contact $B_{0}-\mathrm{VPG}$ graphs. It was shown in $[15,20]$ that recognising the class of contact $B_{0}-\mathrm{VPG}$ graphs is NP-complete, and the complete list of minimal forbidden induced subgraphs for the class is not yet known. Nevertheless,

[^0]characterisations of contact $B_{0}-\mathrm{VPG}$ graphs by minimal forbidden induced subgraphs are known when restricted to some graph classes such as chordal, $P_{5}$-free, $P_{4}$-tidy, treecographs [6, 7]; furthermore, most of those characterisations lead to polynomial-time recognition algorithms within the class. It is also known that every bipartite planar graph is contact $B_{0}-\mathrm{VPG}[14]$. We here provide a characterisation of contact $B_{0}-\mathrm{VPG}$ graphs by minimal forbidden induced subgraphs within the class of circular-arc graphs, i.e., intersection graphs of arcs of a circle [23,28] (see Section 4), and a polynomial-time recognition algorithm for this class (see Section 5). We first give some terminology in Section 2 and some preliminary results in Section 3.

## 2. Basic definitions

Let $G$ be a finite, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any $W \subseteq V(G)$, we denote by $G[W]$ the subgraph of $G$ induced by $W$.

Let $N(v)$ be the set of neighbours of $v \in V(G)$ and $N[v]=N(v) \cup\{v\}$. A vertex is simplicial if its neighbours are pairwise adjacent. If $H$ is an induced subgraph of $G$ and $v$ a vertex of $G$, we denote by $N_{H}(v)$ the set $N(v) \cap V(H)$ and by $G-H$ the graph $G[V(G)-V(H)]$.

Let $v$ and $w$ be two vertices of $G$. The graph $G^{\prime}$ obtained by the contraction of $v$ and $w$ has vertex set $V(G)-\{w\}$ and edge set $(E(G)-\{w z: z \in N(w)\}) \cup\{v z: z \in$ $N(w), z \neq v\}$.

Let $A, B \subseteq V(G)$. We say that $A$ is complete to $B$ if every vertex of $A$ is adjacent to every vertex of $B$; and $A$ is anticomplete to $B$ if no vertex of $A$ is adjacent to a vertex of $B$. A stable set is a set of pairwise nonadjacent vertices. A graph $G$ is bipartite if $V(G)$ can be partitioned into two stable sets $V_{1}, V_{2}$; and $G$ is complete bipartite if $V_{1}$ is complete to $V_{2}$. We denote by $K_{r, s}$ the complete bipartite graph with $\left|V_{1}\right|=r$ and $\left|V_{2}\right|=s$. The claw is the complete bipartite graph $K_{1,3}$. The bipartite claw is the graph arising by subdividing the three edges of the claw.

We denote by $K_{r}(r \geq 0)$ the complete graph on $r$ vertices; $K_{3}$ will be also called a triangle. A clique in $G$ is a subset of vertices which induces a complete subgraph. A diamond, also known as $K_{4}-e$, is the graph obtained from $K_{4}$ by removing exactly one edge.

Let $P$ be a path in $G$. We denote by $P=v_{1} \ldots v_{k}$ the fact that $V(P)=\left\{v_{1}, \ldots, v_{k}\right\}$ and $v_{i}$ is adjacent to $v_{i+1}$ for $1 \leq i \leq k-1$. Vertices $v_{1}$ and $v_{k}$ are the extreme vertices of $P$, while vertices in $V(P)-\left\{v_{1}, v_{k}\right\}$ are the internal vertices of $P$. Similarly, let $C$ be a cycle in $G$. We denote by $C=v_{1} \ldots v_{k}$ the fact that $V(C)=\left\{v_{1}, \ldots, v_{k}\right\}$ and $v_{i}$ is adjacent to $v_{i+1}$ for $1 \leq i \leq k$, where indexes should be understood modulo $k$ (throughout the paper). An edge joining two nonconsecutive vertices of a path or a cycle in a graph is called a chord. An induced path is a chordless path in a graph. Likewise, an induced cycle is a chordless cycle in a graph. A hole is an induced cycle of length at least 4. A graph is chordal if it does not contain any hole. A hole is odd if it has an odd number of vertices, and even, otherwise.

Let $G$ and $H$ be two graphs. We say that $G$ is $H$-free if $G$ does not contain an induced subgraph isomorphic to $H$. If $\mathcal{H}$ is a family of graphs, we say that $G$ is $\mathcal{H}$-free if $G$ is $H$-free for every $H \in \mathcal{H}$.

A graph $G$ is a circular-arc graph if it is the intersection graph of a set $\mathcal{S}$ of arcs on a circle, i.e., if there exists a one-to-one correspondence between the vertices of $G$ and the $\operatorname{arcs}$ of $\mathcal{S}$ such that two vertices of $G$ are adjacent if and only if the corresponding arcs in $\mathcal{S}$ intersect. Circular-arc graphs can be recognised in linear time [29], and have been
characterised recently by a family of obstacles [18]. Previously, partial characterisations by minimal forbidden induced subgraphs were presented in [32] and [5].

## 3. Preliminary results

We first introduce some known families of minimal forbidden induced subgraphs for the class of contact $B_{0}-$ VPG graphs.

(A) $H_{0}$

(в) $K_{5}$

(c) $K_{4}-e$

Figure 1. Some forbidden induced subgraphs for contact $B_{0}-$ VPG graphs.

Lemma 3.1. [6, 15] $H_{0}, K_{5}$ and $K_{4}-e$ are not contact $B_{0}-V P G$.
Let $\mathcal{T}$ [6] be the family of graphs containing $H_{0}$ (see Figure 1) as well as all graphs that can be partitioned into a nontrivial tree $T$ of maximum degree at most three and the disjoint union of triangles, in such a way that each triangle is complete to a vertex $v$ of $T$ and anticomplete to $T-\{v\}$, every leaf $v$ of $T$ is complete to exactly two triangles, every vertex $v$ of degree two in $T$ is complete to exactly one triangle, and vertices of degree three in $T$ have no neighbours outside $T$ (see Figure 2).


Figure 2. An example of a graph in $\mathcal{T}$.
Theorem 3.2. [6] Let $G$ be a chordal graph. Then, $G$ is a contact $B_{0}-V P G$ graph if and only if $G$ is $\left\{\mathcal{T}, K_{5}, K_{4}-e\right\}$-free.

Let $\mathcal{F}_{1}$ be the family of graphs in $\mathcal{T}$ such that the tree $T$ is a path.


Figure 3. $\mathcal{F}_{1}$ : The family of graphs in $\mathcal{T}$ such that the tree $T$ is a path.

Lemma 3.3. The graphs in $\mathcal{F}_{1}$ are not contact $B_{0}-V P G$.
Proof. $\mathcal{F}_{1}$ is a subfamily of $\mathcal{T}$ and it was shown in [6] that no graph in $\mathcal{T}$ is contact $B_{0}-\mathrm{VPG}$.

It is easy to see that $\left\{H_{0}\right\} \cup \mathcal{F}_{1}$ is the family of graphs of $\mathcal{T}$ that do not contain a bipartite claw as induced subgraph (if a graph in $\mathcal{T}$ contains an induced bipartite claw then the tree $T$ must contain a vertex of degree three, and conversely, if $T$ contains a vertex of degree three then the graph contains an induced bipartite claw). Since the bipartite claw is not a circular-arc graph [32], we have the following corollary.
Corollary 3.4. Let $G$ be a (bipartite claw)-free chordal graph. Then, $G$ is a contact $B_{0}-V P G$ graph if and only if $G$ is $\left\{\mathcal{F}_{1}, H_{0}, K_{5}, K_{4}-e\right\}-$ free.

The next result easily follows.
Corollary 3.5. Let $G$ be a chordal circular-arc graph. Then, $G$ is a contact $B_{0}-V P G$ graph if and only if $G$ is $\left\{\mathcal{F}_{1}, H_{0}, K_{5}, K_{4}-e\right\}$-free.

In [5], circular-arc graphs are characterised within some graph classes including, among others, the class of diamond-free graphs. The following is a straightforward corollary of Theorem 16 in [5].
Corollary 3.6. Let $G$ be a diamond-free circular-arc graph that contains a hole. If $C=v_{1} \ldots v_{k}$ is a hole of $G$, then the vertices of $G-C$ can be partitioned into $2 k$ (possibly empty) pairwise anticomplete sets $U_{1}, \ldots, U_{k}, S_{1}, \ldots, S_{k}$ such that the following conditions hold.

- For each $i=1, \ldots, k, G\left[U_{i}\right]$ is the disjoint union of cliques and for each $u \in U_{i}$, $N_{C}(u)=\left\{v_{i}\right\}$.
- For each $i=1, \ldots, k, G\left[S_{i}\right]$ is a clique and for each $s \in S_{i}, N_{C}(s)=\left\{v_{i}, v_{i+1}\right\}$.

Remark 3.7. In this framework, $G$ is further $\left\{K_{5}, H_{0}\right\}$-free if and only if $\left|S_{i}\right| \leq 2$ for each $i=1, \ldots, k$, the cliques in each $U_{i}, i=1, \ldots, k$, have size at most three, the number of triangles in each $U_{i}, i=1, \ldots, k$, is at most two, and it is at most one if either $S_{i-1}$ or $S_{i}$ are of size two, and zero if both $S_{i-1}$ and $S_{i}$ are of size two.

We use the following to further simplify the structure of the graphs under consideration.
Lemma 3.8. [6] Let $G$ be a $\left\{K_{5}, K_{4}-e\right\}$-free graph. ${ }^{1}$ If $G$ is a minimal non contact $B_{0}-V P G$ graph, then every simplicial vertex of $G$ has degree exactly three.


Figure 4. Types of a vertex $v_{i}$ in a hole $v_{1}, \ldots, v_{k}$.
In accordance with Corollary 3.6, Remark 3.7, and Lemma 3.8, Figure 4 illustrates the different cases that may arise for a vertex $v_{i}$ in a hole $v_{1}, \ldots, v_{k}$ of a $\left\{K_{5}, H_{0}, K_{4}-e\right\}-$ free circular-arc graph $G$ which is minimally not contact $B_{0}-\mathrm{VPG}$.

- Type 0: $U_{i}=S_{i}=S_{i-1}=\emptyset$.
- Type 1: $U_{i}$ induces a triangle and $S_{i}=S_{i-1}=\emptyset$.
- Type 2: $U_{i}=\emptyset$ and $\max \left\{\left|S_{i}\right|,\left|S_{i-1}\right|\right\}=2$.
- Type 3: $U_{i}$ induces a triangle and $\max \left\{\left|S_{i}\right|,\left|S_{i-1}\right|\right\}=2$.
- Type 4: $G\left[U_{i}\right]$ is the disjoint union of two triangles.

[^1]
## 4. Characterisation

We will call line (vertical or horizontal) a 0 -bend path on the grid in a contact $B_{0}-$ VPG representation of a graph $G$ so as to avoid confusion with paths in $G$. In a contact $B_{0}-\mathrm{VPG}$ representation of a graph, a corner is a point of the grid that belongs to a vertical and a horizontal line.

Lemma 4.1. The number of corners in a contact $B_{0}-V P G$ representation of a hole is even.

Proof. Let us colour the vertices of the hole according to the representation: a vertex is coloured red (resp. blue) if it is represented by a vertical (resp. horizontal) line. A corner is then determined by two consecutive vertices of the hole that receive different colours. Since the hole starts and ends at the same vertex, and thus, with the same colour, the number of corners is even.
Lemma 4.2. Let $C$ be an odd hole. In every contact $B_{0}-V P G$ representation of $C$ there are two lines that correspond to consecutive vertices and have the same direction (both vertical or both horizontal).

Proof. Assume the contrary. Then every pair of consecutive vertices in $C$ determines a corner in its contact $B_{0}-\mathrm{VPG}$ representation. But the number of pairs of consecutive vertices in an odd hole is odd, which contradicts Lemma 4.1.

Lemma 4.3. Let $G$ be a contact $B_{0}-V P G$ graph admitting a representation in which the lines $\ell_{v}$ and $\ell_{w}$ corresponding to two adjacent vertices $v$ and $w$ have the same direction. Then the graph $G^{\prime}$ obtained by contracting $v$ and $w$ is also contact $B_{0}-V P G$.

Proof. A representation of $G^{\prime}$ can be obtained by combining $\ell_{v}$ and $\ell_{w}$ into a single line.

Corollary 4.4. Let $C$ be an odd hole of a contact $B_{0}-V P G$ graph $G$. Then there are two consecutive vertices of $C$ such that their contraction yields a contact $B_{0}-V P G$ graph.

As noticed in previous work $[15,20]$, any contact $B_{0}-\mathrm{VPG}$ representation of a $K_{4}$ necessarily contains a point where coincide one endpoint of each of the lines representing the four vertices. We say that this endpoint of the line is taken by the $K_{4}$, which implies in particular that it cannot be the contact point with a line corresponding to a neighbour outside this $K_{4}$. It follows that if $\ell$ is a line representing a vertex of Type 4 and $\ell^{\prime}$ is a line representing one of its neighbour outside the $K_{4} \mathrm{~s}$, then the contact point of $\ell$ and $\ell^{\prime}$ is an interior point of $\ell$ and an endpoint of $\ell^{\prime}$; in particular, it is a corner.

Let $\mathcal{F}_{2}$ be the family of graphs that are an even hole where one of its vertices is of Type 4 and every other vertex of the hole is of Type 1 (see Figure 5).
Lemma 4.5. The graphs in $\mathcal{F}_{2}$ are not contact $B_{0}-V P G$.
Proof. Let $G$ be a graph in $\mathcal{F}_{2}$. Let $C=v_{1} \ldots v_{k}$ be an even hole of $G$ such that $v_{1}$ is the vertex of Type 4 and $v_{2}, \ldots, v_{k}$ are of Type 1 .

Suppose that there is a contact $B_{0}-\mathrm{VPG}$ representation of $G$ and let $\ell_{1}, \ldots, \ell_{k}$ be the lines corresponding to the vertices $v_{1}, \ldots, v_{k}$, respectively. Then, every $\ell_{i}$ with $2 \leq i \leq k$, has one endpoint taken by its corresponding $K_{4}$, and $\ell_{1}$ has both endpoints taken. It follows that $\ell_{1}$ and $\ell_{2}$ meet at an interior point of $\ell_{1}$ which is an endpoint of $\ell_{2}$; and we conclude by induction that for any $i \geq 2, \ell_{i}$ and $\ell_{i+1}$ meet at an interior


Figure 5. $\mathcal{F}_{2}$ : The family of graphs that are an even hole where one of its vertices is of Type 4 and every other vertex of the hole is of Type 1.
point of $\ell_{i}$ which is an endpoint of $\ell_{i+1}$. We then reach a contradiction as $\ell_{k}$ and $\ell_{1}$ should meet at an interior point of $\ell_{k}$ which is an endpoint of $\ell_{1}$.

Let $\mathcal{F}_{3}$ be the family of graphs that are an odd hole where every vertex of the hole is of Type 1 (see Figure 6).


Figure 6. $\mathcal{F}_{3}$ : The family of graphs that are an odd hole where every vertex of the hole is of Type 1.

Lemma 4.6. The graphs in $\mathcal{F}_{3}$ are not contact $B_{0}-V P G$.
Proof. Let $G$ be a graph in $\mathcal{F}_{3}$, with odd hole $C$. If $G$ is contact $B_{0}-\mathrm{VPG}$, by Corollary 4.4, there are two consecutive vertices of $C$ such that their contraction yields a contact $B_{0}-$ VPG graph. But by contracting any two consecutive vertices of $C$ we get a graph in $\mathcal{F}_{2}$, which is not contact $B_{0}-$ VPG by Lemma 4.5 .

Let $\mathcal{F}_{4}$ be the family of graphs that are an odd hole containing at least one vertex of Type 4, where "between" every pair of "consecutive" vertices of Type 4, there is only one vertex of Type 0 and no vertices of Type 2 nor 3 . We say that a pair of vertices $v_{i}, v_{j}$ (possibly the same) of Type 4 are "consecutive" if no vertex in the path $v_{i+1}, \ldots, v_{j-1}$ is of Type 4; and a vertex "between" $v_{i}$ and $v_{j}$ is any vertex in the path $v_{i+1}, \ldots, v_{j-1}$ (see Figure 7).

Lemma 4.7. The graphs in $\mathcal{F}_{4}$ are not contact $B_{0}-V P G$.
Proof. It follows from Corollary 4.4 and the fact that by contracting two consecutive vertices of $C$, we obtain as an induced subgraph either $H_{0}$, a graph of $\mathcal{F}_{1}$, or a graph of $\mathcal{F}_{2}$, which are not contact $B_{0}-$ VPG by Lemmas 3.1, 3.3, and 4.5.

Let $\mathcal{F}_{5}$ be the family of graphs that are an even hole where two of its vertices are of Type 3 and all the other vertices of the hole are of Type 1 (see Figure 8).
Lemma 4.8. The graphs in $\mathcal{F}_{5}$ are not contact $B_{0}-V P G$.

(A)

(B)

Figure 7. $\mathcal{F}_{4}$ : The family of graphs that are an odd hole containing at least one vertex of Type 4, where "between" every pair of "consecutive" vertices of Type 4, there is only one vertex of Type 0 and no vertices of Type 2 nor 3.


Figure 8. $\mathcal{F}_{5}$ : The family of graphs that are an even hole where two of its vertices are of Type 3 and all the other vertices of the hole are of Type 1 .

Proof. Let $G$ be a graph in $\mathcal{F}_{5}$. Let $C=v_{1} \ldots v_{k}$ be an even hole of $G$ such that $v_{1}$ and $v_{2}$ are of Type 3, and $v_{3}, \ldots, v_{k}$ are of Type 1 .
Suppose that there is a contact $B_{0}-\mathrm{VPG}$ representation of $G$ and let $\ell_{1}, \ldots, \ell_{k}$ be the lines corresponding to the vertices $v_{1}, \ldots, v_{k}$, respectively. Then, every $\ell_{i}$ with $3 \leq i \leq k$, has one endpoint taken by its corresponding $K_{4}$, and $\ell_{1}$ and $\ell_{2}$ have a common endpoint while their other endpoint taken. It follows that $\ell_{2}$ and $\ell_{3}$ meet at an interior point of $\ell_{2}$ which is an endpoint of $\ell_{3}$; and we conclude by induction that for $i \geq 3, \ell_{i}$ and $\ell_{i+1}$ meet at an interior point of $\ell_{i}$ which is an endpoint of $\ell_{i+1}$. We then reach a contradiction as $\ell_{k}$ and $\ell_{1}$ should meet at an interior point of $\ell_{k}$ which is an endpoint of $\ell_{1}$.

Let $G$ be an $H_{0}$-free graph containing a hole $C=v_{1} \ldots v_{k}$, such that the vertices of $G-C$ can be partitioned into $2 k$ (possibly empty) pairwise anticomplete sets $U_{1}, \ldots, U_{k}, S_{1}, \ldots, S_{k}$, where for each $i=1, \ldots, k$ and for each $u \in U_{i}, N_{C}(u)=\left\{v_{i}\right\}$, and for each $s \in S_{i}, N_{C}(s)=\left\{v_{i}, v_{i+1}\right\}$; moreover, $G\left[U_{i}\right]$ is either empty, or consists of one or two disjoint triangles; and $G\left[S_{i}\right]$ is either empty or a clique of size two. Notice that the vertices of $C$ can be classified into Type 0, Type 1, Type 2, Type 3, and Type 4. We say that an orientation of some of the edges of $C$ is feasible if
(1) no edge is oriented both ways;
(2) if $S_{i} \neq \emptyset$ then $v_{i} v_{i+1}$ is not oriented;
(3) if $v_{i}$ is of Type 4, then $v_{i-1} v_{i}$ and $v_{i+1} v_{i}$ are oriented this way.
(4) if $v_{i}$ is of Type 3 and $S_{i} \neq \emptyset$ (resp. $S_{i-1} \neq \emptyset$ ), then $v_{i-1} v_{i}$ (resp. $v_{i+1} v_{i}$ ) is oriented this way;
(5) if $v_{i}$ is of Type 1 , then at least one of $v_{i-1} v_{i}$ and $v_{i+1} v_{i}$ is oriented this way.
(6) if $C$ is odd, at least one edge of $C$ is not oriented.

Lemma 4.9. Let $G$ and $C$ be defined as above. If $C$ admits a feasible orientation then $G$ is a contact $B_{0}-V P G$ graph.
Proof. The representation of $G$ is based on the "staircase" scheme, illustrated in Figure 9 , with $\left\lfloor\frac{k-2}{2}\right\rfloor$ steps representing vertices $v_{1} \ldots v_{k}$ where the lines in the figure are in clockwise order.


Figure 9. Sketch of a staircase contact $B_{0}$-VPG representation of a hole admitting a feasible orientation (the endpoints of a line are marked by an arrow).


Figure 10. Staircase contact $B_{0}-\mathrm{VPG}$ representation of an even hole where all vertices are of Type 1 and the edges are oriented clockwise.

More specifically, we build a staircase contact $B_{0}-$ VPG representation of $G$ given a feasible orientation of $C$, as follows. If $C$ is even, the base of the staircase consists of one line only; and if $C$ is odd, the base of the staircase is formed by two lines corresponding to vertices $v_{i}, v_{i+1}$ such that the edge $v_{i} v_{i+1}$ is not oriented (in the dotted circle of Figure 9 is shown the contact point when $S_{i}$ is nonempty). For every other $i=1, \ldots, k$, the corner formed by the lines corresponding to $v_{i}$ and $v_{i+1}$ is drawn as shown in the dashed circles of Figure 9 (rotated or reflected according to the position of the corner in the staircase), where (a) represents the orientation $v_{i} v_{i+1}$, (b) represents the orientation $v_{i+1} v_{i}$, and (c) represents $v_{i} v_{i+1}$ not oriented. The short lines within the dashed/dotted circles in Figure 9 represent the vertices in $U_{i}$ and $S_{i}$ that may exist. An example of a staircase contact $B_{0}-$ VPG representation is shown in Figure 10.

Theorem 4.10. Let $G$ be a circular-arc graph that is not chordal. Let $\mathcal{F}=\mathcal{F}_{1} \cup \mathcal{F}_{2} \cup$ $\mathcal{F}_{3} \cup \mathcal{F}_{4} \cup \mathcal{F}_{5} \cup\left\{H_{0}, K_{4}-e, K_{5}\right\}$. Then, $G$ is a contact $B_{0}-V P G$ graph if and only if $G$ is $\mathcal{F}$-free.
Proof. If $G$ is not $\mathcal{F}$-free, then, by Lemmas 3.1, 3.3, 4.5, 4.6, 4.7 and 4.8, $G$ is not contact $B_{0}-\mathrm{VPG}$.

Now assume $G$ is $\mathcal{F}$-free and let $C=v_{1} \ldots v_{k}$ be a hole of $G$ (which exists as $G$ is not chordal). It then follows from Corollary 3.6 that the vertices of $G-C$ can be partitioned into $2 k$ (possibly empty) pairwise anticomplete sets $U_{1}, \ldots, U_{k}, S_{1}, \ldots, S_{k}$ such that for each $i=1, \ldots, k, G\left[U_{i}\right]$ is the disjoint union of cliques and for each $u \in U_{i}$, $N_{C}(u)=\left\{v_{i}\right\} ; G\left[S_{i}\right]$ is a clique and for each $s \in S_{i}, N_{C}(s)=\left\{v_{i}, v_{i+1}\right\}$. Furthermore, by Remark 3.7, we have that for each $i=1, \ldots, k,\left|S_{i}\right| \leq 2$, and the cliques in $U_{i}$ have size at most three; moreover, the number of triangles in $U_{i}$ is at most two, and it is at most one if either $S_{i-1}$ or $S_{i}$ are of size two, and zero if both $S_{i-1}$ and $S_{i}$ are of size two. By Lemma 3.8, we may assume henceforth that for each $i=1, \ldots, k,\left|S_{i}\right|$ is either zero or two, and that $U_{i}$ is either empty or the disjoint union of triangles, which allows us to classify the vertices according to their neighbourhood outside $C$ as Type 0 , Type 1, Type 2, Type 3, or Type 4.

By Lemma 4.9, it suffices to show that $C$ admits a feasible orientation. To this end, consider the connected components of $C$ restricted to the vertices of Type 1.

Case 1: Every vertex of $C$ is of Type 1 (the only connected component is a hole).
If $C$ is odd, then $G$ is a graph in $\mathcal{F}_{3}$, a contradiction. Thus, $C$ is even and orienting every edge as $v_{i} v_{i+1}$ produces a feasible orientation of the edges of $C$.

Case 2: There is only one connected component $P$, which is a path, and only one vertex in $C-P$.

Suppose without loss of generality this vertex is $v_{1}$. Notice that $v_{1}$ cannot be of Type 2 or 3 as every vertex of Type 2 or 3 has a neighbour of Type 2 or 3 . If $v_{1}$ is Type 4, then $G$ is either a graph in $\mathcal{F}_{2}$ or contains a graph in $\mathcal{F}_{3}$ as induced subgraph, a contradiction. Thus, $v_{1}$ is of Type 0 and orienting every edge as $v_{j} v_{j+1}$, for $j=1, \ldots, k-1$, while keeping $v_{k} v_{1}$ not oriented, produces a feasible orientation of the edges of $C$.

Case 3: There is only one connected component $P$, which is a path, and only two vertices in $C-P$.

Notice that these two vertices are necessarily adjacent; thus, we may assume without loss of generality that $v_{k}$ and $v_{1}$ are the only two vertices in $C-P$.

If both are of Type 3, then $G$ contains as induced subgraph either a graph in $\mathcal{F}_{3}$ or a graph in $\mathcal{F}_{5}$ (according to the parity of $C$ ), a contradiction. If $v_{1}$ is Type 3 and $v_{k}$ is Type 2, a feasible orientation of $C$ is obtained by orienting every edge as $v_{j+1} v_{j}$, for $j=1, \ldots, k-1$, and keeping $v_{k} v_{1}$ not oriented (the case where $v_{1}$ is of Type 2 and $v_{k}$ is of Type 3 is symmetric). The same orientation remains feasible if $v_{1}$ and $v_{k}$ are both of Type 2, although in this case, $v_{2} v_{1}$ need not be oriented.

Note that $v_{k}$ and $v_{1}$ cannot both be of Type 4 for otherwise they would induce a graph in $\mathcal{F}_{1}$, a contradiction. Suppose first that one of them is Type 4 and the other Type 0 . Then $C$ must be even as $G$ would otherwise be a graph in $\mathcal{F}_{4}$, a contradiction. Assuming that $v_{1}$ is of Type 4 and $v_{k}$ is of Type 0 (the other case is symmetric), a feasible orientation of $C$ is obtained by orienting the edges as $v_{j+1} v_{j}$, for $j=1, \ldots, k-1$, and $v_{k} v_{1}$. The same orientation remains feasible if both $v_{1}$ and $v_{k}$ are of Type 0 , although in this case, edges $v_{2} v_{1}$ and $v_{k} v_{1}$ need not be oriented (note that at least one of them should not be oriented when $C$ is odd).

Case 4: None of the above.
Let $P$ be a (possibly trivial) connected component of $C$ restricted to the vertices of Type 1 (if any). Since we are in neither of the above cases, $P$ is a path and there exist exactly two vertices $u$ and $w$ in $P-C$ having neighbours in $P$. Moreover, $u$ and $w$ are not adjacent. Since $G$ is $\mathcal{F}_{1}-$ free, at least one of them is neither of Type 4 nor of Type 3, say $u$ without loss of generality. Orienting the edge joining $u$ and $P$ towards $P$
and the edges of $P$ in the same direction (clockwise or counter-clockwise), we obtain a partial orientation in which every vertex of $P$ has one incoming edge. By repeating the process for each connected component, we obtain at the end an orientation satisfying the following properties.

- No edge of $C$ is oriented both ways.
- No edge of $C$ incident to a vertex of Type 3 or Type 4 is oriented.
- No edge $v_{i} v_{i+1}$ of $C$ such that $S_{i} \neq \emptyset$, is oriented.
- No edge of $C$ with two endpoints of Type 0 is oriented.
- Every vertex of Type 1 in $C$ has one incoming edge of $C$.

Next, we orient every edge incident to a vertex $v$ of Type 4 towards $v$, and for every vertex $w$ of Type 3 we define the orientation $u w$, where $u$ is the neighbour of $w$ having no common neighbour with $w$. Since $G$ is $\mathcal{F}_{1}$-free, this orientation is well defined (no edge is incident to two vertices of Types 3 or 4). After this second round of orientation, four of the five properties mentioned above are maintained and the property "no edge of $C$ incident to a Type 3 or Type 4 is oriented" is replaced by "every vertex of Type 3 (resp. Type 4) in $C$ has one (resp. two) incoming edge(s) of $C$.". A sketch of the orientation process can be found in Figure 11.


Figure 11. Building a feasible orientation of the edges of the hole. Vertices in grey may or may not be present.

Thus, in order to ensure that the obtained orientation is a feasible orientation, there remains to show that if $C$ is odd, then there is at least one nonoriented edge. Since this property holds if there are vertices of Type 2 or Type 3, we are left with the case where $C$ odd and only has vertices of Type 4,1 , and 0 . Since $G$ is not in $\mathcal{F}_{4}$, either there exist two adjacent vertices of Type 0 (in which case, the edge joining them is not oriented), or there is a path $P$ of vertices of Type 1 such that the two vertices $u, v$ of $C-P$ having neighbours in $P$ are of Type 0 . By the rules defined above, none of the edges joining $u$ and $v$ to $P$ was oriented during the second phase, and one of them was left not oriented during the first phase, which concludes the proof.

Combining Corollary 3.5 and Theorem 4.10, we have the following result.
Theorem 4.11. Let $G$ be a circular-arc graph. Let $\mathcal{F}=\mathcal{F}_{1} \cup \mathcal{F}_{2} \cup \mathcal{F}_{3} \cup \mathcal{F}_{4} \cup \mathcal{F}_{5} \cup$ $\left\{H_{0}, K_{4}-e, K_{5}\right\}$. Then, $G$ is a contact $B_{0}-V P G$ graph if and only if $G$ is $\mathcal{F}$-free.

## 5. Algorithm

In order to recognise the class of contact $B_{0}-$ VPG graphs within circular-arc graphs, we first check whether the graph is chordal, which can be done in polynomial time [31]. If it is the case, we can apply the recognition algorithm of [6], whose output is either a contact $B_{0}-\mathrm{VPG}$ representation or a forbidden induced subgraph. Otherwise, we obtain a hole in the graph, and we either find an induced $K_{4}-e$ in the graph or we can compute the structure of the graph with respect to this hole, as described in Corollary 3.6. Each of these steps can be performed in polynomial time.

Once we have computed those different sets, it is easy to check whether it contains either $K_{5}$ or $H_{0}$ or none of them. In the latter case, we can use Lemma 3.8 to disregard the simplicial vertices of degree one or two, since its proof also suggests how to include them in case we obtain a contact $B_{0}-\mathrm{VPG}$ representation of the remaining part of the graph.

The remainder of the recognition algorithm is largely based on the proofs of Lemma 4.9 and Theorem 4.10. We first follow the steps in the proof of Theorem 4.10 to either build a feasible orientation of the hole or find a forbidden induced subgraph. In case we obtained a feasible orientation of the hole, we follow the proof of Lemma 4.9 in order to obtain a contact $B_{0}-\mathrm{VPG}$ representation of the graph.

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## References

[1] N. Aerts and S. Felsner. Vertex contact graphs of paths on a grid. In D. Kratsch and I. Todinca, editors, Proceedings of the International Workshop on Graph-Theoretic Concepts in Computer Science 2014, volume 8747 of Lecture Notes in Computer Science, pages 56-68, 2014.
[2] N. Aerts and S. Felsner. Vertex contact representations of paths on a grid. Journal of Graph Algorithms and Applications, 19(3):817-849, 2015.
[3] L. Alcón, F. Bonomo, and M.P. Mazzoleni. Vertex intersection graphs of paths on a grid: characterization within block graphs. Graphs and Combinatorics, 33(4):653-664, 2017.
[4] A. Asinowski, E. Cohen, M.C. Golumbic, V. Limouzy, M. Lipshteyn, and M. Stern. Vertex intersection graphs of paths on a grid. Journal of Graph Algorithms and Applications, 16(2):129150, 2012.
[5] F. Bonomo, G. Durán, L. N. Grippo, and M. D. Safe. Partial characterizations of circular-arc graphs. Journal of Graph Theory, 61(4):289-306, 2009.
[6] F. Bonomo, M.P. Mazzoleni, M.L. Rean, and B. Ries. Characterising chordal contact $B_{0}$-VPG graphs. In J. Lee, G. Rinaldi, and A. Ridha Mahjoub, editors, Proceedings of the International Symposium on Combinatorial Optimization 2018, volume 10856 of Lecture Notes in Computer Science, pages 89-100, 2018.
[7] F. Bonomo, M.P. Mazzoleni, M.L. Rean, and B. Ries. On some special classes of contact $B_{0}-\mathrm{VPG}$ graphs. arXiv e-prints, page arXiv:1807.07372, Jul 2018.
[8] S. Chaplick, E. Cohen, and J. Stacho. Recognizing some subclasses of vertex intersection graphs of 0 -bend paths in a grid. In P. Kolman and J. Kratochvíl, editors, Proceedings of the International Workshop on Graph-Theoretic Concepts in Computer Science 2011, volume 6986 of Lecture Notes in Computer Science, pages 319-330, 2011.
[9] S. Chaplick, V. Jelínek, J. Kratochvíl, and T. Vyskocil. Bend-bounded path intersection graphs: Sausages, noodles, and waffles on a grill. In M.C. Golumbic, M. Stern, A. Levy, and G. Morgenstern, editors, Proceedings of the International Workshop on Graph-Theoretic Concepts in Computer Science 2012, volume 7551 of Lecture Notes in Computer Science, pages 274-285, 2012.
[10] E. Cohen, M.C. Golumbic, and B.Ries. Characterizations of cographs as intersection graphs of paths on a grid. Discrete Applied Mathematics, 178:46-57, 2014.
[11] E. Cohen, M.C. Golumbic, W.T. Trotter, and R.Wang. Posets and VPG graphs. Order, 33(1):3949, 2016.
[12] N. de Castro, F.J. Cobos, J.C. Dana, A. Márquez, and M. Noy. Triangle-free planar graphs as segments intersection graphs. In J. Kratochvíl, editor, Proceedings of the International Symposium on Graph Drawing and Network Visualization 1999, volume 1731 of Lecture Notes in Computer Science, pages 341-350, 1999.
[13] H. de Fraysseix and P. Ossona de Mendez. Representations by contact and intersection of segments. Algorithmica, 47(4):453-463, 2007.
[14] H. de Fraysseix, P. Ossona de Mendez, and J. Pach. Representation of planar graphs by segments. Intuitive Geometry, 63:109-117, 1991.
[15] Z. Deniz, E. Galby, A. Munaro, and B. Ries. On contact graphs of paths on a grid. In T. Biedl and A. Kerren, editors, Proceedings of the International Symposium on Graph Drawing and Network Visualization 2018, volume 11282 of Lecture Notes in Computer Science, pages 317-330, 2018.
[16] G. Ehrlich, S. Even, and R. Tarjan. Intersection graphs of curves in the plane. Journal of Combinatorial Theory. Series B, 21:8-20, 1976.
[17] S. Felsner, K. Knauer, G.B. Mertzios, and T. Ueckerdt. Intersection graphs of L-shapes and segments in the plane. Discrete Applied Mathematics, 206:48-55, 2016.
[18] M. Francis, P. Hell, and J. Stacho. Forbidden structure characterization of circular-arc graphs and a certifying recognition algorithm. In P. Indyk, editor, Proceedings of the 26th Annual ACMSIAM Symposium on Discrete Algorithms, pages 1708-1727, San Diego, CA, 2015.
[19] M. Francis and A. Lahiri. VPG and EPG bend-numbers of Halin graphs. Discrete Applied Mathematics, 215:95-105, 2016.
[20] E. Galby, A. Munaro, and B. Ries. CPG graphs: Some structural and hardness results. CoRR, abs/1903.01805, 2019.
[21] M.C. Golumbic and B. Ries. On the intersection graphs of orthogonal line segments in the plane: characterizations of some subclasses of chordal graphs. Graphs and Combinatorics, 29:499-517, 2013.
[22] D. Gonçalves, L. Isenmann, and C. Pennarun. Planar graphs as L-intersection or L-contact graphs. In Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms, pages 172184. SIAM, 2018.
[23] H. Hadwiger, H. Debrunner, and V. Klee. Combinatorial Geometry in the Plane. New York: Holt Rinehardt and Winston, 1964.
[24] D. Heldt, K. Knauer, and T. Ueckerdt. On the bend-number of planar and outerplanar graphs. Discrete Applied Mathematics, 179:109-119, 2014.
[25] P. Hliněný. Classes and recognition of curve contact graphs. Journal of Combinatorial Theory. Series B, 74(1):87-103, 1998.
[26] P. Hliněný. The maximal clique and colourability of curve contact graphs. Discrete Applied Mathematics, 81(1):59-68, 1998.
[27] P. Hliněný. Contact graphs of line segments are NP-complete. Discrete Mathematics, 235(1):95106, 2011.
[28] V. Klee. What are the intersection graphs of arcs in a circle? The American Mathematical Monthly, 76(7):810-813, 1969.
[29] R. McConnell. Linear-time recognition of circular-arc graphs. Algorithmica, 37(2):93-147, 2003.
[30] T.A. McKee and F.R. McMorris. Topics in Intersection Graph Theory. SIAM, Philadelphia, 1999.
[31] D. Rose, R. Tarjan, and G. Lueker. Algorithmic aspects of vertex elimination on graphs. SIAM Journal on Computing, 5:266-283, 1976.
[32] W.T. Trotter and J.I. Moore. Characterization problems for graphs, partially ordered sets, lattices, and families of sets. Discrete Mathematics, 16:361-381, 1976.

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[^1]:    ${ }^{1}$ In [6], the lemma is stated for chordal graphs but the proof does not use this hypothesis.

