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# A Note on Coloring Digraphs of Large Girth

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### Abstract

The digit of a digraph is the length of a shortest directed cycle. The dichromatic number  $\vec{\chi}(D)$  of a digraph D is the smallest size of a partition of the vertex-set into subsets inducing acyclic subgraphs. A conjecture by Harutyunyan and Mohar [7] states that  $\vec{\chi}(D) \leq \lceil \frac{\Delta}{4} \rceil + 1$  for every digraph D of digit at least 3 and maximum degree  $\Delta$ . The best known partial result by Golowich [5] shows that  $\vec{\chi}(D) \leq \frac{2}{5}\Delta + O(1)$ . In this short note we prove for every  $g \geq 2$  that if D is a digraph of digit at least 2g - 1 and maximum degree  $\Delta$ , then  $\vec{\chi}(D) \leq (\frac{1}{3} + \frac{1}{3g})\Delta + O_g(1)$ . This improves the bound of Golowich for digraphs without directed cycles of length at most 10.

### 1 Introduction

**Preliminaries.** All digraphs in this note are finite and do not contain loops or parallel arcs. Given a digraph D, we denote by V(D) its vertex-set and by A(D) the arc-set. A digraph is called *acyclic* if it does not contain directed cycles. By  $\Delta(D), \Delta^+(D), \Delta^-(D), \delta^+(D), \delta^-(D)$  we denote, respectively, the maximum degree in (the underlying graph of) D, and the extremal out- and indegrees in D. We furthermore denote by  $\tilde{\Delta}(D) = \max\{\sqrt{d^+(v)d^-(v)}|v \in V(D)\}$  the maximum geometric mean of the in- and out-degree of a vertex in D. Note that in case D has no cycles of length 2, the inequality of geometric and arithmetic mean shows that  $\tilde{\Delta}(D) \leq \frac{\Delta(D)}{2}$ . Given a vertex set  $X \subseteq V(D)$ , we denote by D[X] the induced subdigraph of D with vertex-set X and call X *acyclic* if D[X] is acyclic. By  $\vec{g}(D)$  we denote the *digirth* of D, that is, the shortest length of a directed cycle in D ( $\vec{g}(D) := \infty$  if D is acyclic). Given a a family  $A_1, \ldots, A_m$  of finite sets, a system of representatives of this family is a set  $X \subseteq \bigcup_{i=1}^m A_i$  such that  $X \cap A_i \neq \emptyset$  for all  $i \in [m]$ .

We deal with a notion of coloring for directed graphs introduced in 1982 by Neumann-Lara [13]. Given a digraph D, an *acyclic* coloring of D is a vertex-coloring in which all color classes are acyclic. The smallest number of colors sufficient for an acyclic coloring of D is denoted by  $\vec{\chi}(D)$ and called *dichromatic number* of D. This notion has received a fair amount of attention in the past two decades, see [1, 3, 4, 6, 8, 9, 12] for some recent results. As for undirected graphs, there is a Brooks-type upper bound on the dichromatic number of a digraph, see [11, 13], which implies  $\vec{\chi}(D) \leq \lfloor \frac{\Delta}{2} \rfloor$  for every digraph of girth at least 3 and maximum degree  $\Delta \geq 3$ . In this note, we are motivated by the following conjecture from [7], which claims that this Brook's type bound can be improved by a factor of 2 if we forbid directed cycles of length 2 in the digraph.

**Conjecture 1** (cf. [7], Conjecture 1.5). Let *D* be a digraph of digit at least 3 and maximum degree  $\Delta$ . Then  $\vec{\chi}(D) \leq \left\lceil \frac{\tilde{\Delta}(D)}{2} \right\rceil + 1 \leq \left\lceil \frac{\Delta}{4} \right\rceil + 1$ .

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Approaching their conjecture, in [7] Harutyunyan and Mohar proved that there is a small absolute constant  $\varepsilon > 0$  such that  $\vec{\chi}(D) \leq (1-\varepsilon)\tilde{\Delta}(D) \leq (\frac{1}{2}-\frac{\varepsilon}{2})\Delta(D)$  for every digraph D of digirth at least 3 and  $\tilde{\Delta}$  sufficiently large. Subsequently Golowich [5] improved the multiplicative constant in the upper bound, by showing that every digraph D of digirth at least 3 satisfies  $\vec{\chi}(D) \leq \frac{2}{5}\Delta(D) + O(1)$ . Our contribution is to further improve the multiplicative constant in this upper bound for digraphs without short directed cycles.

**Theorem 1.** Let  $g \ge 2$  a natural number, and let D be a digraph with  $\vec{g}(D) \ge 2g - 1$  and maximum degree  $\Delta$ . Then  $\vec{\chi}(D) \le (\frac{1}{3} + \frac{1}{3g})\Delta + (g + 1)$ .

## 2 Proof of Theorem 1

We need three auxiliary results by Neumann-Lara, by Aharoni, Berger and Kfir, and by Lovász.

**Lemma 2** (cf. [13], Theorem 5). Let  $k \in \mathbb{N}$  and let D be a (k+1)-critical digraph, that is,  $\vec{\chi}(D) = k+1$  but  $\vec{\chi}(D') \leq k$  for every proper subdigraph  $D' \subsetneq D$ . Then  $\delta^+(D), \delta^-(D) \geq k$ .

**Lemma 3** (cf. [2], Corollary II.13). Let D be a digraph of digit at least  $\gamma \geq 2$  and let  $V_1, V_2, \ldots, V_m$  be a partition of V(D). If  $|V_i| \geq \frac{\gamma}{\gamma-1}\Delta^+(D)$  for all  $i \in [m]$ , then there is a system X of representatives of  $V_1, \ldots, V_m$  which is acyclic in D.

**Lemma 4** ([10]). Let G be an undirected graph,  $k \in \mathbb{N}$ . Then V(G) admits a partition  $X_1, \ldots, X_k$  such that for every  $v \in X_i, i \in [k]$ , we have  $\deg_{G[X_i]}(v) \leq \frac{1}{k} \deg(v)$ .

The proof of Theorem 1 relies on the following bound on the dichromatic number for digraphs of large girth compared to their maximum out-degree.

**Lemma 5.** Let D be a digraph such that  $\vec{g}(D) > \Delta^+(D)$ . Then

$$\vec{\chi}(D) \le \left\lfloor \frac{\Delta(D)}{3} \right\rfloor + 2.$$

Proof. Abbreviate  $\Delta = \Delta(D)$  and  $\gamma = \vec{g}(D)$  and put  $k := \lfloor \frac{\Delta}{3} \rfloor + 1 > \frac{\Delta}{3}$ . By Lemma 4 there is a partition  $X_1, \ldots, X_k$  of V(D) such that for every  $i \in [m]$  we have  $\Delta(D[X_i]) \leq \frac{\Delta}{k} < 3$ . Hence,  $D[X_i]$  is a disjoint union of oriented paths and oriented cycles. For every i, let us denote by  $\vec{C_i}$ the set of all directed cycles in  $D[X_i]$  and put  $V' := \bigcup_{i \in [k], C \in \vec{C_i}} V(C)$ . We claim that there is an acyclic set X in D such that  $X \cap V(C) \neq \emptyset$  for all  $C \in \vec{C_i}$  and  $i \in [k]$ . To see this, note that  $|V(C)| \geq \gamma \geq \frac{\gamma}{\gamma-1}\Delta^+(D) \geq \frac{\gamma}{\gamma-1}\Delta^+(D[V'])$  for every  $C \in \vec{C_i}$  and  $i \in [k]$ . We can therefore apply Lemma 3 to the digraph D[V'] equipped with the partition  $(V(C)|C \in \vec{C_i}, i \in [k])$  to find a system of representatives X which is acyclic in D[V'] and thus in D. Next we claim that each of the sets  $X_i \setminus X, i \in [k]$  is acyclic in D. Indeed, the digraph  $D[X_i \setminus X] = D[X_i] - (X_i \cap X)$  is obtained from a disjoint union of oriented paths and cycles by removing at least one vertex from each directed cycle, and is therefore acyclic. Hence,  $X_1 \setminus X, X_2 \setminus X, \ldots, X_k \setminus X, X$  is a partition of V(D) into acyclic sets which certifies that  $\vec{\chi}(D) \leq k + 1 = \lfloor \frac{\Delta}{3} \rfloor + 2$ .

We can now complete the proof of Theorem 1 by applying Lemma 4 a second time.

Proof of Theorem 1. Let  $\ell := \left\lfloor \frac{\Delta}{3g} \right\rfloor + 1$ . By Lemma 4 there exists a partition  $Y_1, \ldots, Y_\ell$  of V(D) such that  $\Delta(D[Y_i]) \leq \frac{\Delta}{\ell} < 3g$  for every  $i \in [\ell]$ . We claim that for every  $i \in [\ell]$ , we have  $\vec{\chi}(D[Y_i]) \leq g+1$ . Suppose by way of a contradiction that  $\vec{\chi}(D[Y_i]) \geq g+2$  for some  $i \in [\ell]$ .

Consider a subgraph  $D_i$  of  $D[Y_i]$  with  $\vec{\chi}(D_i) \ge g + 2$  minimizing  $|V(D_i)| + |A(D_i)|$ . Clearly,  $D_i$  is (g+2)-critical, and thus  $\delta^-(D_i) \ge g + 1$  by Lemma 2. Hence we have

$$\Delta^{+}(D_{i}) \leq \Delta(D_{i}) - \delta^{-}(D_{i}) \leq \Delta(D[Y_{i}]) - \delta^{-}(D_{i}) \leq (3g - 1) - (g + 1) = 2g - 2 < \vec{g}(D) \leq \vec{g}(D_{i}) \leq d(D_{i}) \leq d(D_{i$$

We can therefore apply Lemma 5 to obtain  $\vec{\chi}(D_i) \leq \lfloor \frac{3g-1}{3} \rfloor + 2 = g + 1$ , which is the desired contradiction. This shows that indeed we have  $\vec{\chi}(D[Y_i]) \leq g + 1$  for all  $i \in [\ell]$ . The claim now follows from

$$\vec{\chi}(D) \le \sum_{i=1}^{\ell} \vec{\chi}(D[Y_i]) \le (g+1) \left( \left\lfloor \frac{\Delta}{3g} \right\rfloor + 1 \right) \le \left( \frac{1}{3} + \frac{1}{3g} \right) \Delta + (g+1).$$

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