Mixing time of the switch Markov chain and stable degree sequences

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Abstract

The switch chain is a well-studied Markov chain which can be used to sample approximately uniformly from the set $\Omega(d)$ of all graphs with a given degree sequence d. Polynomial mixing time (rapid mixing) has been established for the switch chain under various conditions on the degree sequences. Amanatidis and Kleer introduced the notion of strongly stable families of degree sequences, and proved that the switch chain is rapidly mixing for any degree sequence from a strongly stable family. Using a different approach, Erdős et al. recently extended this result to the (possibly larger) class of P-stable degree sequences, introduced by Jerrum and Sinclair in 1990. We define a new notion of stability for a given degree sequence, namely k-stability, and prove that if a degree sequence d is 8-stable then the switch chain on $\Omega(d)$ is rapidly mixing. We also provide sufficient conditions for P-stability, strong stability and 8-stability. Using these sufficient conditions, we give the first proof of P-stability for various families of heavy-tailed degree sequences, including power-law degree sequences, and show that the switch chain is rapidly mixing for these families.

We further extend these notions and results to directed degree sequences.

1 Introduction

Given a finite set of discrete objects Ω and a distribution π over Ω , how can we efficiently sample an object from Ω according to distribution π ? This is a classical problem in theoretical computer science, with many applications in other research fields such as statistics, engineering, and different branches of sciences. In most situations, Ω is a very large set, and

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it is not possible, given limited time and computation power, to enumerate all objects in Ω and compute the probability of each object under π .

There are various general methods developed to solve this problem. The most commonly applied method is Markov Chain Monte Carlo (MCMC). Using MCMC, one needs to define a Markov chain with state space Ω and stationary distribution π . Then, output an object in Ω after running the Markov chain sufficiently long time. The MCMC-based algorithms are approximate samplers. The challenge is to obtain an upper bound on the so-called mixing time, the minimum number of steps required to run the Markov chain so that the distribution of the output differs from π by a prescribed tolerance, say ϵ , in total variation distance. The techniques used to bound the mixing time are problem-specific.

In this paper, we consider the problem of uniformly sampling a graph on vertex set $[n] = \{1, 2, ..., n\}$ with a specified degree sequence, where $n \geq 1$ is a positive integer. A degree sequence is a sequence $\mathbf{d} = (d_1, ..., d_n)$ of nonnegative integers with even sum. We say \mathbf{d} is graphical if there exists a simple graph on the vertex set [n] such that vertex i has degree d_i for every $i \in [n]$. Let $\Omega(\mathbf{d})$ be the set of all graphs with vertex set [n] and degree sequence \mathbf{d} . Hence, we will study the problem with $\Omega = \Omega(\mathbf{d})$ and with π the uniform distribution over $\Omega(\mathbf{d})$. This problem has many practical applications for researchers who use graphs to model complex discrete systems.

The first MCMC approach to uniformly sampling graphs with a given degree was given by Jerrum and Sinclair [19] in 1990. They defined a Markov chain to perform approximately uniform sampling from a set of graphs $\Omega'(\mathbf{d})$ which contains $\Omega(\mathbf{d})$. Every graph in $\Omega'(\mathbf{d})$ has degree sequence which is either \mathbf{d} , or a very small perturbation of \mathbf{d} . Rejection sampling is performed until the output of the Markov chain belongs to $\Omega(\mathbf{d})$. Hence the expected runtime of their algorithm is polynomial precisely when $\Omega(\mathbf{d})$ is at least a polynomial fraction of $\Omega'(\mathbf{d})$. Jerrum and Sinclair introduced the notion of P-stability to capture this condition. See Section 6 for a precise definition of P-stability.

The switch Markov chain $\mathcal{M}(\boldsymbol{d})$ (or switch chain) has state space $\Omega(\boldsymbol{d})$ and makes a transition by replacing a pair of edges by another pair of edges (possibly the same pair), ensuring that the resulting graph is simple and has degree sequence \boldsymbol{d} . A more formal definition is given in Section 3.1. This chain is appealing because, unlike the Jerrum–Sinclair chain, every sample belongs to $\Omega(\boldsymbol{d})$. Many authors have studied the mixing time of the switch chain for particular families of degree sequences, bipartite degree sequences and directed degree sequences [7, 17, 10, 11, 12, 18, 21, 27]. Amanatidis and Kleer [1] gave an ingenious comparison argument to prove that the switch chain has polynomial mixing time for strongly stable classes of degree sequences (See Section 6 for a precise definition of strongly stability.) The classes of degree sequences for which the switch chain for graphs was known to be rapidly mixing (before [1]) are all strongly stable, so the theorem of [1] can be seen as a common generalisation of these results. Amanatidis and Kleer also adapt the definition of strongly stable to classes of bipartite degree sequences, and prove that the switch chain for bipartite graphs has polynomial mixing time for strongly stable classes of bipartite degree sequences. This gives a common generalisation of the results in [11, 21, 27] for the bipartite switch chain.

All strongly stable degree classes are P-stable, but it is not known whether the converse holds. Recently, Erdős et al. [8] proved that the switch chain has polynomial mixing time (rapid mixing) for all P-stable degree sequences (also in the bipartite and directed setting).

Hence this result extends that of [1] from strongly stable to P-stable classes, and to directed degree sequences. However, P-stability is not a necessary condition for rapid mixing of the switch chain, as shown recently by Erdős et al. [9].

In this paper, we define a new notion of stability for degree sequences. For any vector $\boldsymbol{x} \in \mathbb{R}^n$ let $\|\boldsymbol{x}\|_1 = \sum_{j=1}^n |x_j|$ denote the ℓ_1 -norm of \boldsymbol{x} . Necessarily, for any graphical degree sequence \boldsymbol{d} , we have $\boldsymbol{d} \in \mathbb{N}^n$ and $\|\boldsymbol{d}\|_1$ even. Let $M(\boldsymbol{d})$ denote $\|\boldsymbol{d}\|_1$.

Definition 1.1. Given a positive integer k and nonnegative real number α , we say a graphical degree sequence d is (k, α) -stable if

$$|\Omega(\mathbf{d}')| \leq M(\mathbf{d})^{\alpha} |\Omega(\mathbf{d})|$$

for every graphical sequence \mathbf{d}' with $\|\mathbf{d}' - \mathbf{d}\|_1 \leq k$. Let $\mathcal{D}_{k,\alpha}$ be the set of all degree sequences that are (k,α) -stable. We say that a family \mathcal{D} of degree sequences is k-stable if there exists a constant $\alpha > 0$ such that $\mathcal{D} \subseteq \mathcal{D}_{k,\alpha}$.

Obviously,

for all
$$k$$
, $\mathcal{D}_{k,\alpha_1} \subseteq \mathcal{D}_{k,\alpha_2}$ if $\alpha_1 \le \alpha_2$, (1.1)

for all
$$\alpha$$
, $\mathcal{D}_{k_1,\alpha} \subseteq \mathcal{D}_{k_2,\alpha}$ if $k_1 \ge k_2$. (1.2)

Our definition of 2-stability is equivalent to the notion of P-stability, introduced by Jerrum and Sinclair [19]. We prove this in Proposition 6.2. It is not clear whether k-stability is equivalent to P-stability for fixed k > 2. Relationships between P-stability, strong stability and 8-stability will be further discussed in Section 6.

Suppose that \mathcal{D} is an 8-stable family of degree sequences. One of our main results is the following:

The switch chain on
$$\Omega(\mathbf{d})$$
 mixes in polynomial time for all $\mathbf{d} \in \mathcal{D}$. (1.3)

A more accurate statement is given in Theorem 2.1.

We are usually interested in sequences of degree sequences d(n), indexed by the positive integers $n \in \mathbb{Z}_{\geq 1}$, or $n \in \mathcal{I}$ where \mathcal{I} is an infinite subset of $\mathbb{Z}_{\geq 1}$. Due to the technical nature of the definition of P-stability, strong stability and k-stability, it is in general not easy to determine if a sequence of degree sequences $(d(n))_{n \in \mathcal{I}}$ is stable. Jerrum, Sinclair and McKay [20] gave two sufficient conditions for a family of degree sequences to be P-stable. Their first condition, [20, Theorem 8.1], is an inequality involving the maximum and minimum degrees and n, while their second condition [20, Theorem 8.3] is a "more refined" inequality which also involves the number of edges. Using [20, Theorem 8.1], they verified that the family of regular degree sequences is P-stable. They also verified P-stability of several other families of "moderate" degree sequences, in the sense that the degrees are not too far from being regular.

In this paper, we will give a sufficient condition, Condition 1, for a degree sequence to be 8-stable. We will also give a sufficient condition, Condition 2, for *P*-stability and strong stability. Condition 2 is slightly weaker than Condition 1. While our conditions apply to

moderate degree sequences, they work particularly well for heavy-tailed degree sequences. We will prove that the following families of degree sequences all satisfy Condition 1 (and thus also Condition 2), and thus they are "stable" in the sense of P-stability, strong stability and 8-stability:

- power-law degree sequences with exponent greater than 2;
- bi-regular degree sequences permitting both constant degrees and degrees linear in n;
- other heavy-tailed degree sequences examined in [15].

Our theorems apply to a possibly smaller class of degree sequences than those of [8]. (See Section 6 for a comparison between 8-stability, P-stability and strong stability.) However, our approach is much simpler and requires less additional machinery than the arguments of [1, 8]. We make minimal changes to the multicommodity flow argument given in [7, 18], presenting a new counting argument to prove the so-called "critical lemma". Furthermore, our sufficient conditions Condition 1, Condition 2 enable us to expand the class of degree sequences known to be P-stable, subsuming the sufficient condition from [18, Theorem 1.1].

Remark. Power-law degree sequences, and other heavy-tailed degree sequences, are of particular importance in network science. The more refined sufficient condition for P-stability given in [20, Theorem 8.3] can be used to show that the family of "power-law distribution-bounded" degree sequences (defined in Section 5.3) is P-stable when the average degree is at most twice the minimum degree. The requirement that the average degree is at most twice the minimum degree effectively excludes most of the interesting power-law degree sequences, i.e. power-law degree sequences with minimum degree 1 and average degree greater than 2. The first author and Wormald mentioned in [16] that power-law degree sequences with exponent greater than 2 can be shown to be P-stable. However, that assertion has not been proved, in [16] or elsewhere. We establish the 8-stability, and thus the P-stability, of power-law degree sequences in Section 5.

To conclude this section, we discuss approaches to sampling from $\Omega(d)$ which are not based on Markov chains. McKay and Wormald [25] gave an algorithm, based on an operation called switchings, which performs exactly uniform sampling from $\Omega(d)$ in expected polynomial time when the maximum degree is not too large. This result has been improved upon by Wormald and the first author [14], who achieved expected runtime $O(d^3n)$ for d-regular degree sequences when $d = o(\sqrt{n})$. Wormald and the first author also adapted their approach to power law degree sequences[16]. Very recently these results were improved further by Arman, Wormald and the first author [3], who presented an algorithm with expected runtime which is $O(nd+d^4)$ for d-regular graphs with $d = o(\sqrt{n})$, and is O(n) for the same class of power-law degree sequences considered in [16] (roughly speaking, those with power-law exponent greater than 2.88).

Switchings-based approximate sampling algorithms with very fast expected runtime (linear or sub-quadratic) have been given by Bayati, Kim and Saberi [4], Kim and Vu [22], Steger and Wormald [32] and Zhao [34]. The error in the output distribution of these algorithms

are functions of n and tend to zero as n grows. Unlike Markov chain algorithms, this error cannot be made smaller by increasing the runtime of the algorithm.

2 Main results

Given d, let $\mu \in S_n$ be a permutation such that $d_{\mu(1)} \geq d_{\mu(2)} \geq \cdots d_{\mu(n)}$. Define

$$\Delta(\mathbf{d}) = d_{\mu(1)}, \quad J(\mathbf{d}) = \sum_{i=1}^{d_{\mu(1)}} d_{\mu(i)}.$$

That is, $\Delta(\mathbf{d})$ is the largest entry of \mathbf{d} , while $J(\mathbf{d})$ is the sum of the $\Delta(\mathbf{d})$ largest entries of \mathbf{d} . First we state (1.3) in a more formal and accurate form.

Theorem 2.1. Suppose that the graphical degree sequence \mathbf{d} is $(8, \alpha)$ -stable. Write $M = M(\mathbf{d})$ and $\Delta = \Delta(\mathbf{d})$. Then the switch chain on $\Omega(\mathbf{d})$ mixes in polynomial time, with mixing time $\tau(\varepsilon)$ which satisfies

$$\tau(\varepsilon) \le 12 \,\Delta^{14} \, n^6 \, M^{3+\alpha} \, \Big(\frac{1}{2} M \log M + \log(\varepsilon^{-1}) \Big).$$

Our next theorem gives a sufficient condition for a degree sequence to be (8,8)-stable, and a weaker condition which implies both P-stability and strong stability.

Theorem 2.2. (a) Let d be a graphical degree sequence which satisfies

Condition 1:
$$M(d) > 2J(d) + 18\Delta(d) + 56$$
.

Then \mathbf{d} is (8,8)-stable.

(b) Suppose that \mathcal{D} is a family of degree sequences such that every $\mathbf{d} \in \mathcal{D}$ satisfies

Condition 2:
$$M(\mathbf{d}) > 2J(\mathbf{d}) + 6\Delta(\mathbf{d}) + 2$$
.

Then \mathcal{D} is both P-stable, and strongly stable.

Remark. We did not try to optimise the coefficient of $\Delta(d)$ and the constant term in the assumptions of Theorem 2.2. With more careful treatment in the proof, these numbers can be reduced.

As a further corollary we have the following mixing time bound on $\Omega(d)$ when d satisfies the condition in Theorem 2.2(a).

Theorem 2.3. Assume that $\mathbf{d} = (d_1, \dots, d_n)$ is a graphical degree sequence which satisfies Condition 1. Write $M = M(\mathbf{d})$ and $\Delta = \Delta(\mathbf{d})$. Then the switch chain on $\Omega(\mathbf{d})$ mixes in polynomial time, with mixing time $\tau(\varepsilon)$ which satisfies

$$\tau(\varepsilon) \le 12 \,\Delta^{14} \,n^6 \,M^{11} \left(\frac{1}{2} M \log M + \log(\varepsilon^{-1})\right). \tag{2.1}$$

Proof. The result follows immediately from Theorems 2.1 and 2.2.

Remark. The mixing time bounds in Theorem 2.1 and Theorem 2.3 are probably far from tight, and are not the main focus of our work. It is likely that the mixing time bound that can be inferred from the results of Erdős et al [8] is lower than our bound. (It is harder to extract an explicit upper bound on the mixing time of the switch chain from [1], so we do not make any comparison with their bound.)

The structure of the paper is as follows. After reviewing the necessary Markov chain definitions and outlining the multicommodity flow argument from [18], we prove Theorem 2.1 in Section 3 by employing a new counting argument. The proof of Theorem 2.2(a) is given in Section 4, while the proof of Theorem 2.2(b) is deferred to Section 6. In Section 5 we apply Theorem 2.2 to give the first proof of P-stability for several families of heavy-tailed degree sequences, and provide an explicit upper bound for the mixing time of the switch chain for degree sequences from these families. In Section 6, P-stability and strong stability will be formally defined and compared with our new notion of k-stability. Then the proof of Theorem 2.2(b) will be presented. Finally, some analogous results for the switch chain on directed graphs are established in Section 7.

3 Polynomial mixing for 8-stable degree sequences

The aim of this section is the proof of Theorem 2.1. First we must review some background.

Let \mathcal{M} be a reversible Markov chain with finite state space Ω , transition matrix P and stationary distribution π . The *total variation distance* between two probability distributions σ , σ' on Ω is

$$d_{\text{TV}}(\sigma, \sigma') = \frac{1}{2} \sum_{x \in \Omega} |\sigma(x) - \sigma'(x)|.$$

The mixing time $\tau(\varepsilon)$ is defined by

$$\tau(\varepsilon) = \max_{x \in \Omega} \min\{T \ge 0 \mid d_{\text{TV}}(P_x^t, \pi) \le \varepsilon \text{ for all } t \ge T\}$$

where P_x^t is the distribution of the state X_t of \mathcal{M} after t steps from the initial state $X_0 = x$. We say that the Markov chain \mathcal{M} is rapidly mixing, or has polynomial mixing time, if $\tau(\varepsilon)$ is bounded from above by some polynomial in $\log(|\Omega|)$ and $\log(\varepsilon^{-1})$.

Sinclair [31] introduced multicommodity flow as a generalisation of the canonical path method. Let \mathcal{G} be the directed graph underlying a Markov chain \mathcal{M} , so that xy is an edge of \mathcal{G} if and only if P(x,y) > 0. A flow in \mathcal{G} is a function $f: \mathcal{P} \to [0,\infty)$ such that

$$\sum_{p \in \mathcal{P}_{xy}} f(p) = \pi(x)\pi(y) \quad \text{ for all } x, y \in \Omega, \ x \neq y.$$

Here we define \mathcal{P}_{xy} to be the set of all simple directed paths from x to y in \mathcal{G} . In practice, when defining a multicommodity flow, we will set f(p) = 0 for most paths, and focus on a much smaller set of directed paths which will each carry a positive amount of flow. Let

 $\mathcal{P} = \bigcup_{x \neq y} \mathcal{P}_{xy}$. Extend f to a function on oriented edges by setting $f(e) = \sum_{p \ni e} f(p)$, so that f(e) is the total flow routed through e. Write $Q(e) = \pi(x)P(x,y) = \pi(y)P(y,x)$ for the edge e = xy. The quantity Q(e) is well-defined, by the reversibility of \mathcal{M} . Let $\ell(f)$ be the length of the longest path with f(p) > 0, and let $\rho(e) = f(e)/Q(e)$ be the load of the edge e. The maximum load of the flow is $\rho(f) = \max_{e} \rho(e)$. Using Sinclair [31, Proposition 1 and Corollary 6'], the mixing time of \mathcal{M} can be bounded above by

$$\tau(\varepsilon) \le \rho(f)\ell(f) \left(\log(1/\pi^*) + \log(\varepsilon^{-1})\right) \tag{3.1}$$

where $\pi^* = \min\{\pi(x) \mid x \in \Omega\}.$

3.1 The switch chain and multicommodity flow

The switch Markov chain $\mathcal{M}(\boldsymbol{d})$ (or switch chain) has state space $\Omega(\boldsymbol{d})$ and transitions given by the following procedure: from the current state $G \in \Omega(\boldsymbol{d})$, choose an unordered pair of distinct non-adjacent edges uniformly at random, say $F = \{\{x,y\},\{z,w\}\}\}$, and choose a perfect matching F' from the set of three perfect matchings of (the complete graph on) $\{x,y,z,w\}$, chosen uniformly at random. If $F' \cap (E(G) \setminus F) = \emptyset$ then the next state is the graph G' with edge set $(E(G) \setminus F) \cup F'$, otherwise the next state is G' = G.

Define $M_2(\mathbf{d}) = \sum_{j=1}^n d_j(d_j - 1)$. If $P(G, G') \neq 0$ and $G \neq G'$ then $P(G, G') = 1/(3a(\mathbf{d}))$, where

$$a(\mathbf{d}) = \binom{M(\mathbf{d})/2}{2} - \frac{1}{2} M_2(\mathbf{d})$$
(3.2)

is the number of unordered pairs of distinct nonadjacent edges in G. This shows that the switch chain $\mathcal{M}(\mathbf{d})$ is symmetric, and it is aperiodic since $P(G, G) \geq 1/3$ for all $G \in \Omega(\mathbf{d})$.

Cooper, Dyer and the second author [7] defined and analysed a multicommodity flow for the switch chain for regular degree sequences. They proved that the switch chain has polynomial mixing time for $\Omega(\mathbf{d})$ whenever $\mathbf{d} = (d, d, \ldots, d)$ is a d-regular sequence, for any d = d(n). The second author and Sfragara [18] showed how the analysis could be extended to irregular degree sequences which were not too dense. Surprisingly, as the proof given in [7] is quite long and technical, there was only one lemma which relied on regularity in its proof. In [18] a new argument was provided for this "critical lemma", leading to the extended rapid mixing result for irregular degree sequences.

To understand the purpose of the critical lemma, we provide a brief outline of the multicommodity flow argument from [7, 18]. The analysis of the multicommodity flow is discussed in Section 3.2, and the purpose of the critical lemma is given in (3.4). Then in Section 4 we will give a new counting proof (Lemma 1.3) which establishes the critical lemma when d is 8-stable.

Let $G, G' \in \Omega(\mathbf{d})$ be two graphs and let $G \triangle G'$ be the symmetric difference of G and G', treated as a 2-coloured graph (with edges from $G \setminus G'$ coloured blue and edges from $G' \setminus G$ coloured red, say). We define a set of directed paths from G to G', and assign a value f(p) to each of these paths, so that f is a flow.

- Define a bijection from the set of blue edges incident at v to the set of red edges incident at v, for each vertex $v \in \{1, ..., n\}$. The vector of these bijections is called a pairing ψ , and the set of all possible pairings is denoted $\Psi(G, G')$.
- The pairing gives a canonical way to decompose the symmetric difference $G\triangle G'$ into a sequence of simpler closed alternating walks, called 1-circuits and 2-circuits.
- Each 1-circuit or 2-circuit is processed in a canonical way, in order, to give a segment of the canonical path $\gamma_{\psi}(G, G')$.

Thus, for each $(G, G') \in \Omega(\mathbf{d})^2$ and each $\psi \in \Psi(G, G')$, we define a (canonical) path $\gamma_{\psi}(G, G')$ from G to G'. For full details see [7, Section 2.1].

Next, the value of the flow along this path is defined as follows:

$$f(\gamma_{\psi(G,G')}) = \frac{1}{|\Omega(\mathbf{d})|^2 |\Psi(G,G')|}$$
(3.3)

and f(p) = 0 for any other directed path from G to G'. Recall that $\mathcal{P}_{G,G'}$ is defined to be the set of all directed paths from G to G', in the underlying digraph of the switch chain. Summing f(p) over all $p \in \mathcal{P}_{G,G'}$ gives $1/|\Omega(\mathbf{d})|^2 = \pi(G)\pi(G')$, as required for a valid flow. This flow from G to G' has been equally shared among all paths in $\{\gamma_{\psi}(G, G') \mid \psi \in \Psi(G, G')\}$.

3.2 Encodings and the critical lemma

Definition 3.1. An encoding L of a graph $Z \in \Omega(\mathbf{d})$ is an edge-labelled graph on n vertices, with edge labels in $\{-1, 1, 2\}$, such that

- (i) the sum of edge-labels around vertex j equals d_j for all $j \in [n]$,
- (ii) the edges with labels -1 or 2 form a subgraph of one of the 10 graphs shown in Figure 1. (In the figure, "?" represents a label which may be either -1 or 2.)

An edge labelled -1 or 2 is a defect edge.

In the analysis of the multicommodity flow in [7, 18], encodings play an important role. Given $G, G', Z \in \Omega(\mathbf{d})$, identify each of Z, G, G' with their symmetric 0-1 adjacency matrix and define the matrix L by

$$L + Z = G + G'$$
.

Then L corresponds to an edge-labelled graph, which we also denote by L. It follows from the next result, proved in [18], that the edge-labelled graph L satisfies the definition of encoding given above. We call L the encoding of Z with respect to (G, G').

Lemma 3.2 ([18, Lemma 2.1(ii)]). Let $(G, G') \in \Omega(d)^2$ and $\psi \in \Psi(G, G')$, and suppose that (Z, Z') is a transition of the switch chain which forms part of the canonical path $\gamma_{\psi}(G, G')$. Then the encoding L of Z with respect to (G, G') has at most four defect edges, which must form a subgraph of one of the 10 possible edge-labelled graphs shown in Figure 1.

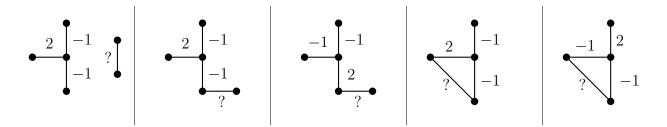


Figure 1: The defect edges in L form a subgraph of one of these graphs, as proved in [18, Lemma 2.1(ii)]. The "?" stands for a defect edge which may be labelled -1 or 2.

Note that this result holds for any degree sequence d, as the restrictions on d which are needed for the rapid mixing result in [18] only arise in the proof of the "critical lemma".

Given $Z \in \Omega(\mathbf{d})$, say that encoding L is consistent with Z if L + Z only takes entries in $\{0, 1, 2\}$ (again identifying L and Z with their adjacency matrices). Equivalently, L is consistent with Z if any edge which has label -1 in L must also be an edge of Z. Let $\mathcal{L}^{\dagger}(Z)$ be the set of encodings which are consistent with Z. The task of the critical lemma is to prove that

$$|\mathcal{L}^{\dagger}(Z)|$$
 is at most polynomially larger than $|\Omega(\mathbf{d})|$ (3.4)

for all $Z \in \Omega(\mathbf{d})$ and for \mathbf{d} which satisfy some condition required to make the proof work. (The set of encodings considered in the critical lemma is slightly different in each of [7, 18] and the present work, but this does not matter as long as the set contains all encodings L which arise along a canonical path (that is, encodings corresponding to some state Z which belongs to a path $\gamma_{\psi}(G, G')$).

Arguing as in [18, Theorem 1.1], for example, it can be shown that the switch chain is rapidly mixing on $\Omega(\mathbf{d})$ for any \mathbf{d} which satisfies the condition of the critical lemma. We encapsulate this argument into the following result, which can then be used as a black box.

Theorem 3.3. Let \mathbf{d} be a graphical degree sequence. Write $M = M(\mathbf{d})$ and $\Delta = \Delta(\mathbf{d})$. Suppose that there exists a function $g(\mathbf{d})$, which depends only on \mathbf{d} , such that

$$|\mathcal{L}^{\dagger}(Z)| \leq g(\boldsymbol{d}) |\Omega(\boldsymbol{d})|$$

for all $Z \in \Omega(\mathbf{d})$. Then the mixing time $\tau(\varepsilon)$ of the switch chain on $\Omega(\mathbf{d})$ satisfies

$$\tau(\varepsilon) \le \frac{1}{2}g(\mathbf{d}) \Delta^{14} M^3 \left(\frac{1}{2}M \log M + \log(\varepsilon^{-1})\right).$$

Proof. We follow the structure of the argument used to prove [18, Theorem 1.1], working towards an application of (3.1). As noted in [18, Equation (1)],

$$|\Omega(\boldsymbol{d})| \leq \exp\left(\frac{1}{2}M\log M\right)$$
.

Hence, as the stationary distribution π is uniform, the smallest stationary probability π^* satisfies

$$\log(1/\pi^*) = \log(|\Omega(\mathbf{d})|) \le \frac{1}{2}M\log M. \tag{3.5}$$

Next, $\ell(f) \leq M/2$ since each transition along a canonical path from G to G' replaces an edge of G by an edge of G'.

Let e = (Z, Z') be a transition of the switch chain. Then

$$1/Q(e) = 6 a(\mathbf{d}) |\Omega(\mathbf{d})| \le M^2 |\Omega(\mathbf{d})|,$$

using (3.2). (Note, the factor $|\Omega(\mathbf{d})|$ was missing in the corresponding bound in the proof of [18, Theorem 1.1], so we correct that typographical error here.)

The argument in [18] used a set of encodings $\mathcal{L}^*(Z)$ which is a proper subset of $\mathcal{L}^{\dagger}(Z)$. Specifically, $\mathcal{L}^*(Z)$ is the set of all encodings in $\mathcal{L}^{\dagger}(Z)$ which also satisfy the conclusions of [18, Lemma 2.2]. Then [18, Lemma 2.3] proves that for the flow f and transition e,

$$f(e) \le \Delta^{14} \frac{|\mathcal{L}^*(Z)|}{|\Omega(\boldsymbol{d})|^2}.$$

(Again, we emphasise that this is still true for degree sequences that do not satisfy the condition of the rapid mixing theorem from [18], since that condition was only used in the proof of the "critical lemma".) We may replace $\mathcal{L}^*(Z)$ with $\mathcal{L}^{\dagger}(Z)$ to obtain a weaker upper bound on f(e), and hence

$$\rho(f) = \max_{e} \frac{f(e)}{Q(e)} \le \Delta^{14} M^2 \max_{Z \in \Omega(\boldsymbol{d})} \frac{|\mathcal{L}^{\dagger}(Z)|}{|\Omega(\boldsymbol{d})|} \le g(\boldsymbol{d}) \, \Delta^{14} \, M^2.$$

Substituting this bound, the bound $\ell(f) \leq M/2$ and (3.5) into (3.1) completes the proof. \square

3.3 Proof of Theorem 2.1

Next we give a new counting proof, which establishes the "critical lemma" when d is 8-stable.

Lemma 3.4. Assume that the graphical degree sequence \mathbf{d} is $(8, \alpha)$ -stable for some nonnegative real number α . Then for any $Z \in \Omega(\mathbf{d})$,

$$|\mathcal{L}^{\dagger}(Z)| \leq 24 \, n^6 M(\boldsymbol{d})^{\alpha} \, |\Omega(\boldsymbol{d})|.$$

Proof. We know that $|\mathcal{L}^{\dagger}(Z)|$ is bounded from above by the number of edge-labelled graphs satisfying conditions (i) and (ii) from Definition 3.1. First, we bound the number of ways to choose the defect edges in Z. These defect edges must form a subgraph of one of the 10 edge-labelled graphs shown in Figure 1, recalling that "?" can be either -1 or 2. Each of these 10 graphs has 4 edges and at most 6 vertices. If H has ℓ vertices then the number of injections $\varphi: V(H) \to [n]$ is at most n^{ℓ} . It follows that the number of ways to choose H and φ is at most

$$10\left(n^6 + 4n^5 + 6n^4 + 4n^2 + 1\right) \le 24n^6$$

since $n \geq 4$ (or no switch is possible).

Now let \mathcal{E} be the set of chosen (labelled) defect edges. We bound the number of ways to complete \mathcal{E} to an encoding $L \in \mathcal{L}^{\dagger}(Z)$. All edges in $L \setminus \mathcal{E}$ are labelled 1. For all $j \in [n]$, the

number of edges in $L \setminus \mathcal{E}$ incident with vertex j is equal to $d'_j = d_j - x_j$, where x_j is the sum of edge-labels from \mathcal{E} around vertex j. Let $\mathbf{d}' = (d'_1, \ldots, d'_n)$ and observe that $\|\mathbf{d}'\|_1$ is even. The number of encodings $L \in \mathcal{L}^{\dagger}(Z)$ such that the set of defect edges given by \mathcal{E} is at most $|\Omega(\mathbf{d}')|$. Checking through all possible graphs H, we confirm that $\|\mathbf{d} - \mathbf{d}'\|_1 \leq 8$ always. (As an example, let H be the first option in Figure 1, with "?" replaced by 2. There are three vertices with $d'_j = d_j + 2$, two with $d'_j = d_j - 1$, and $d'_j = d_j$ for all other vertices. Hence $\|\mathbf{d} - \mathbf{d}'\|_1 = 8$ in this case. The subgraph formed from H by deleting one of the (-1)-defect edges also satisfies $\|\mathbf{d} - \mathbf{d}'\|_1 = 8$.) Since \mathbf{d} is $(8, \alpha)$ -stable, we have $|\Omega(\mathbf{d}')| \leq M(\mathbf{d})^{\alpha} |\Omega(\mathbf{d})|$. The result follows as there are at most $24n^6$ ways to fix \mathcal{E} .

Proof of Theorem 2.1. This follows by combining Theorem 3.3 and Lemma 3.4. \Box

4 Stable degree sequences and proof of Theorem 2.2

Assume that d is graphical. For positive integers k, let

$$\mathcal{N}_k(d) = \{ d' \in \mathbb{N}^n : \|d'\|_1 \equiv 0 \pmod{2}, \|d' - d\|_1 \le k \}.$$
 (4.1)

Lemma 4.1. Suppose that every graphical degree sequence $\mathbf{d}' \in \mathcal{N}_6(\mathbf{d})$ is $(2, \alpha)$ -stable. Then \mathbf{d} is $(8, 4\alpha)$ -stable.

Proof. Assume that \boldsymbol{d}' is a degree sequence such that $\|\boldsymbol{d}-\boldsymbol{d}'\|_1 \leq 8$. Then we can find $\boldsymbol{d}' = \boldsymbol{d}^{(8)}, \boldsymbol{d}^{(6)}, \boldsymbol{d}^{(4)}, \boldsymbol{d}^{(2)}, \boldsymbol{d}^{(0)} = \boldsymbol{d}$, such that $\boldsymbol{d}^{(i)} \in \mathcal{N}_i(\boldsymbol{d})$ and $\|\boldsymbol{d}^{(i+1)} - \boldsymbol{d}^{(i)}\|_1 \leq 2$ for every $i \in \{0, 2, 4, 6\}$. By assumption, $|\Omega(\boldsymbol{d}^{(i+1)})| \leq M^{\alpha} |\Omega(\boldsymbol{d}^{(i)})|$ for every $i \in \{0, 2, 4, 6\}$. Consequently $|\Omega(\boldsymbol{d}')| \leq M^{4\alpha} |\Omega(\boldsymbol{d})|$, and the assertion follows.

A directed 2-path in a graph G is an ordered triple of distinct vertices (a, b, c) such that ab and bc are edges in G. For any graph G with degree sequence \mathbf{d} , and any $v \in G$, the number of directed 2-paths which start at v (that is, with v = a) is at most

$$\sum_{i=1}^{d_v} (d_{\mu(i)} - 1) \le \sum_{i=1}^{d_{\mu(1)}} (d_{\mu(i)} - 1) \le J(\boldsymbol{d}) - \Delta(\boldsymbol{d}). \tag{4.2}$$

The key result of this section is the following, which will be proved in Section 4.1.

Theorem 4.2. If $M(d) > 2J(d) + 6\Delta(d) + 2$ then **d** is (2, 2)-stable.

Using Theorem 4.2, we can now prove Theorem 2.2(a). (The proof of Theorem 2.2(b) is deferred to the end of Section 6.)

Proof of Theorem 2.2(a). First suppose that \mathbf{d} satisfies Condition 1 and let $\mathbf{d}' \in \mathcal{N}_6(\mathbf{d})$. Then $M(\mathbf{d}') \geq M(\mathbf{d}) - 6$, and $\Delta(\mathbf{d}') \leq \Delta(\mathbf{d}) + 6$. Also $J(\mathbf{d}') \leq J(\mathbf{d}) + 6\Delta(\mathbf{d}) + 6$, with equality when $\Delta(\mathbf{d}') = \Delta(\mathbf{d}) + 6$ and the largest $\Delta(\mathbf{d}')$ entries of \mathbf{d} all equal $\Delta(\mathbf{d})$. Applying Theorem 4.2 to \mathbf{d}' shows that \mathbf{d}' is (2,2)-stable, and therefore \mathbf{d} is (8,8)-stable, by Lemma 4.1. This establishes part (a) of Theorem 2.2.

We conclude this section by proving Theorem 4.2.

4.1 Proof of Theorem 4.2

The switching method is a way to bound or approximate the ratio of the sizes of large, finite sets, introduced by McKay [24]. It has been used to obtain asymptotic enumeration formulae for sparse graphs with given degrees, see [26], and also forms the basis for fast exactly-uniform sampling algorithms, see [14, 25]. Our proof of Theorem 4.2 uses three simple switching arguments.

Proof of Theorem 4.2. For all $t \in [n]$ we write e_t to denote the vector with 1 in the t'th position and zeroes elsewhere. Let $\mathbf{d}' \in \mathcal{N}_2(\mathbf{d})$ such that $\|\mathbf{d} - \mathbf{d}'\|_1 = 2$. We will prove that $|\Omega(\mathbf{d}')| \leq M(\mathbf{d})^2 |\Omega(\mathbf{d})|$, which implies that \mathbf{d} is (2, 2)-stable, as claimed.

We will consider three cases for d':

$$d' = d - e_i - e_j$$
, $d' = d + e_i + e_j$, $d' = d + e_i - e_j$.

Write $u \sim v$ to indicate that vertices u and v are adjacent.

Case 1: $\mathbf{d}' = \mathbf{d} + \mathbf{e}_i + \mathbf{e}_j$. There are two subcases: $i \neq j$ and i = j.

Case 1a: $i \neq j$. We will use the (i^-, j^-) -switching shown on the left of Figure 2. This switching converts a graph $G' \in \Omega(\mathbf{d}')$ to a graph $G \in \Omega(\mathbf{d})$. To perform the switching, choose an ordered set of vertices (u_1, u_2, u_3, u_4) in G' such that

- (a) the four vertices are distinct from i, j and are pairwise distinct except that $u_1 = u_4$ is permitted;
- (b) $i \sim u_1, u_2 \sim u_3, j \sim u_4$;
- (c) u_1 is not adjacent with u_2 and u_3 is not adjacent with u_4 .

Then the switching deletes edges iu_1 , ju_4 and u_2u_3 , and add edges u_1u_2 and u_3u_4 . The resulting graph G has degree sequence d.

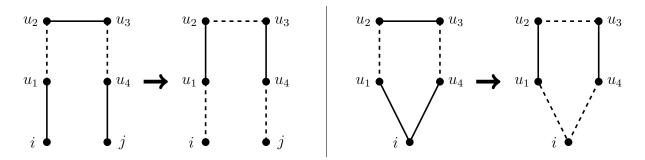


Figure 2: The (i^-, j^-) -switching. Left: Case 1a with $i \neq j$. Right: Case 1b with i = j.

Let $f_{(i^-,j^-)}(G')$ denote the number of ways to perform an (i^-,j^-) -switching to G'. As $G' \in \Omega(\mathbf{d}')$, there are $d_i + 1$ ways to choose u_1 , $d_j + 1$ ways to choose u_4 , and at most $M(\mathbf{d}') = M(\mathbf{d}) + 2$ ways to choose (u_2, u_3) . Hence $f_{(i^-,j^-)}(G) \leq (d_i + 1)(d_j + 1)(M(\mathbf{d}) + 2)$.

Among all such choices, if condition (a) above fails, we say there is a vertex collision. To obtain a lower bound, we use inclusion-exclusion and need to exclude the choices where $\{u_2, u_3\} \cap \{i, j, u_1, u_4\} \neq \emptyset$, and the choices where $u_1 \sim u_2$ or $u_3 \sim u_4$. Given u_1, u_4 , the number of choices for (u_2, u_3) such that $\{u_2, u_3\} \cap \{i, j, u_1, u_4\} \neq \emptyset$ is at most $8\Delta(\boldsymbol{d}) + 4$, noting that $d'_i, d'_j \leq \Delta(\boldsymbol{d}) + 1$. Next, we bound the number of choices for (u_2, u_3) where there is no vertex collision but $u_1 \sim u_2$ or $u_3 \sim u_4$. Since $\boldsymbol{d'}$ agrees with \boldsymbol{d} in all components except for i and j, adapting (4.2) shows that the number of 2-paths in G' which start at u_1 and avoid both i and j is at most

$$\sum_{\ell=1}^{d_{u_1}-1} (d_{\pi(\ell)}-1) \le \sum_{\ell=1}^{\Delta(\boldsymbol{d})-1} (d_{\pi(\ell)}-1) \le J(\boldsymbol{d}) - \Delta(\boldsymbol{d}).$$

Hence, given u_1 and u_4 , the number of choices for (u_2, u_3) where there is no vertex collision but $u_1 \sim u_2$ (or similarly $u_3 \sim u_4$) is at most $J(\mathbf{d}) - \Delta(\mathbf{d})$. It follows that, for every $G' \in \Omega(\mathbf{d}')$,

$$f_{(i^-,j^-)}(G') \ge (d_i+1)(d_j+1)(M(\mathbf{d})+2-8\Delta(\mathbf{d})-4-2(J(\mathbf{d})-\Delta(\mathbf{d}))$$

= $(d_i+2)(d_i+1)(M(\mathbf{d})-2J(\mathbf{d})-6\Delta(\mathbf{d})-2),$

which is positive by the assumption of the theorem.

Next, given $G \in \Omega(\mathbf{d})$, we estimate $b_{(i^-,j^-)}(G)$, the number of ways to create G by performing an (i^-,j^-) -switching to a graph in $\Omega(\mathbf{d}')$. To estimate $b_{(i^-,j^-)}(G)$ we count the number of inverse (i^-,j^-) -switchings that can be performed to G. An inverse (i^-,j^-) -switching chooses vertices (u_1,u_2,u_3,u_4) such that

- (a) the four vertices are distinct from i, j and are pairwise distinct except that $u_1 = u_4$ is permitted;
- (b) iu_1, u_2u_3 and ju_4 are not edges;
- (c) $u_1 \sim u_2$ and $u_3 \sim u_4$.

Trivially we have $b_{(i^-,j^-)}(G') \leq M(\mathbf{d})^2$. Hence

$$\frac{|\Omega(\mathbf{d}')|}{|\Omega(\mathbf{d})|} \le \frac{\max_{G \in \Omega(\mathbf{d})} b_{(i^-,j^-)}(G)}{\min_{G' \in \Omega(\mathbf{d}')} f_{(i^-,j^-)}(G')} \le \frac{M(\mathbf{d})^2}{(d_i+1)(d_j+1)(M(\mathbf{d})-2J(\mathbf{d})-6\Delta(\mathbf{d})-2)} \le M(\mathbf{d})^2.$$

Case 1b: i=j. We use the $(2i^-)$ -switching, which is defined the same as the (i^-, j^-) -switching except that i=j, and now we require that $u_1 \neq u_4$. This switching is shown on the right of Figure 2. Trivially, we have $f_{2i^-}(G') \leq (d_i+2)(d_i+1)(M(\boldsymbol{d})+2)$. Given u_1 and u_4 , the number of choices for (u_2, u_3) such that $\{u_2, u_3\} \cap \{i, u_1, u_4\} \neq \emptyset$ is at most $2 \cdot (3\Delta(\boldsymbol{d}) + 2) = 6\Delta(\boldsymbol{d}) + 4$, noting that $d'_i \leq \Delta(\boldsymbol{d}) + 2$. The number of choices where there is no vertex collision but $u_1 \sim u_2$ or $u_3 \sim u_4$ is at most $2(J(\boldsymbol{d}) - \Delta(\boldsymbol{d}))$, as shown in Case 1a. (Again, this uses the fact that \boldsymbol{d}' agrees with \boldsymbol{d} everywhere except i and j.) Therefore

$$f_{2i^{-}}(G') \ge (d_i + 2)(d_i + 1)(M(\mathbf{d}) + 2 - (6\Delta(\mathbf{d}) + 4) - 2(J(\mathbf{d}) - \Delta(\mathbf{d})))$$

= $(d_i + 2)(d_i + 1)(M(\mathbf{d}) - 2J(\mathbf{d}) - 4\Delta(\mathbf{d}) - 2),$

which is positive by the theorem assumption. We also have the trivial upper bound $b_{2i^-}(G) \leq M(\mathbf{d})^2$. It follows that

$$\frac{|\Omega(\mathbf{d}')|}{|\Omega(\mathbf{d})|} \le \frac{\max_{G \in \Omega(\mathbf{d})} b_{(i^-,j^-)}(G)}{\min_{G' \in \Omega(\mathbf{d}')} f_{(i^-,j^-)}(G')} \le \frac{M(\mathbf{d})^2}{(d_i+2)(d_i+1)(M(\mathbf{d})-2J(\mathbf{d})-4\Delta(\mathbf{d})-2)} \le M(\mathbf{d})^2.$$

Case 2: $\mathbf{d}' = \mathbf{d} - \mathbf{e}_i - \mathbf{e}_j$. Again there are two subcases: $i \neq j$ and i = j.

Case 2a: $i \neq j$. We will use the (i^+, j^+) -switching. This switching chooses an ordered set of vertices (u_1, u_2) in $G' \in \Omega(\mathbf{d}')$ such that

- (a) the four vertices u_1, u_2, i, j are pairwise distinct;
- (b) $u_1 \sim u_2$;
- (c) u_1 is not adjacent with i and u_2 is not adjacent with j.

The switching deletes the edge u_1u_2 and adds edges iu_1 and ju_2 . The result is a graph $G \in \Omega(\mathbf{d})$. See the left hand side of Figure 3.

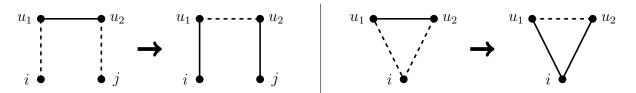


Figure 3: The (i^+, j^+) -switching. Left: Case 2a with $i \neq j$. Right, Case 2b with i = j.

Obviously, $f_{(i^+,j^+)}(G) \leq M(\mathbf{d}') = M(\mathbf{d}) - 2$. To apply inclusion-exclusion, we need to subtract the number of choices where (a) or (c) is violated. The number of choices where a vertex collision occurs (that is, where (a) is violated) is at most $4\Delta(\mathbf{d}') \leq 4\Delta(\mathbf{d})$. The number of choices without vertex collision but where $i \sim u_1$ (or $j \sim u_2$) is at most $J(\mathbf{d}) - \Delta(\mathbf{d})$, as shown in Case 1a. Hence,

$$f_{(i^+,j^+)}(G') \ge M(d) - 2 - 4\Delta(d) - 2(J(d) - \Delta(d)) = M(d) - 2J(d) - 2\Delta(d) - 2$$

which is positive by the assumption of the theorem. We also have the trivial upper bound

$$b_{(i^+,j^+)}(G) \le d_i d_j.$$

Hence,

$$\frac{|\Omega(\boldsymbol{d}')|}{|\Omega(\boldsymbol{d})|} \le \frac{\max_{G \in \Omega(\boldsymbol{d})} b_{(i^+,j^+)}(G)}{\min_{G' \in \Omega(\boldsymbol{d}')} f_{(i^+,j^+)}(G')} \le \frac{d_i d_j}{M(\boldsymbol{d}) - 2J(\boldsymbol{d}) - 2\Delta(\boldsymbol{d}) - 2} \le M(\boldsymbol{d})^2.$$

Case 2b: i = j. We will use the $2i^+$ -switching, which is the same as the (i^+, j^+) -switching but with i = j. See the right hand side of Figure 3. Arguing as in Case 2a gives

$$f_{2i^+}(G') \ge M(\mathbf{d}) - 2 - 2\Delta(\mathbf{d}) - 2(J(\mathbf{d}) - \Delta(\mathbf{d})) = M(\mathbf{d}) - 2J(\mathbf{d}) - 2.$$

Together with the trivial upper bound $b_{2i^+}(G) \leq d_i(d_i - 1)$ we have

$$\frac{|\Omega(\boldsymbol{d}')|}{|\Omega(\boldsymbol{d})|} \le \frac{\max_{G \in \Omega(\boldsymbol{d})} b_{2i^+}(G)}{\min_{G' \in \Omega(\boldsymbol{d}')} f_{2i^+}(G')} \le \frac{d_i(d_i - 1)}{M(\boldsymbol{d}) - 2J(\boldsymbol{d}) + 2} \le M(\boldsymbol{d})^2.$$

Case 3: $\mathbf{d}' = \mathbf{d} + \mathbf{e}_i - \mathbf{e}_j$ and $i \neq j$. In this case we will use the (i^-, j^+) -switching shown in Figure 4. The switching chooses vertices (u_1, u_2, u_3) in $G' \in \Omega(\mathbf{d}')$ such that

- (a) the five vertices i, j, u_1, u_2, u_3 are all distinct;
- (b) $i \sim u_1$ and $u_2 \sim u_3$;
- (c) u_1 is not adjacent to u_2 and u_3 is not adjacent to j.

The switching then replaces edges iu_1 and u_2u_3 by edges u_1u_2 and u_3j . The resulting graph G belongs to $\Omega(\mathbf{d})$.

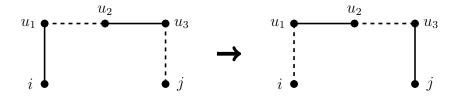


Figure 4: The (i^-, j^+) -switching used in Case 3.

Trivially, we have $f_{(i^-,j^+)}(G') \leq (d_i+1)M(\boldsymbol{d})$, noticing that $d'_i = d_i+1$ and $M(\boldsymbol{d}') = M(\boldsymbol{d})$. Given u_1 , the number of choices of (u_2,u_3) where a vertex collision occurs is at most $6\Delta(\boldsymbol{d})$, noticing that $d'_j \leq \Delta(\boldsymbol{d}) - 1$. The number of choices where no vertex collision occurs but $u_1 \sim u_2$ (or $u_3 \sim j$) is at most $J(\boldsymbol{d}) - \Delta(\boldsymbol{d})$. Hence,

$$f_{(i^-,j^+)}(G') \ge (d_i+1)(M(\mathbf{d})-6\Delta(\mathbf{d})-2(J(\mathbf{d})-\Delta(\mathbf{d}))) = (d_i+1)(M(\mathbf{d})-2J(\mathbf{d})-4\Delta(\mathbf{d})),$$

which is positive by the theorem assumption. Consequently, as $b_{(i^-,j^+)}(G) \leq d_j M(\mathbf{d})$, we have

$$\frac{|\Omega(\boldsymbol{d}')|}{|\Omega(\boldsymbol{d})|} \leq \frac{\max_{G \in \Omega(\boldsymbol{d})} b_{(i^-,j^+)}(G)}{\min_{G' \in \Omega(\boldsymbol{d}')} f_{(i^-,j^+)}(G')} \leq \frac{d_j M(\boldsymbol{d})}{(d_i+1)(M(\boldsymbol{d})-2J(\boldsymbol{d})-4\Delta(\boldsymbol{d}))} < M(\boldsymbol{d})^2.$$

Combining all cases together completes the proof of the theorem.

Remark. Even though Case 2 is the reverse of Case 1, we did not use the reverse switching of Case 1 for Case 2. This is because the switching in Case 2 switches fewer edges than the reverse of the switching in Case 1, and as a result, it yields a better lower bound on $f_{(i^+,j^+)}(G')$. The reader may wonder why we did not use the reverse switching in Case 2 for Case 1. This is because in the reverse switching in Case 2, we have to restrict to $u_1 \neq u_2$, and this restriction would result in a larger error and a useless lower bound on $f_{(i^-,j^-)}(G')$.

5 Applications

Theorem 4.2 is tailored for treating heavy-tailed degree sequences, and we discuss several such families below. First we remark that as a straightforward corollary of Theorem 2.3 we deduce that the switch chain mixes in polynomial time for any d-regular degree sequence with $d^* < n/2 - 9 - 28/d^*$, where $d^* = \min\{d, n - 1 - d\}$. (The parameter d^* arises by complementation.) The bound obtained on the mixing time is given by (2.1). However, rapid mixing was established for *all* regular degree sequences by Cooper, Dyer and Greenhill [7], and the upper bound on the mixing time given in [7] is smaller than that obtained from (2.1).

In [15], the first author and Wormald studied asymptotic enumeration of $\Omega(\mathbf{d})$ for heavy-tailed degree sequences. A few particular families of heavy-tailed degree sequences were discussed in [15], including power-law sequences. In this section, we examine the mixing time of the switch chain on $\Omega(\mathbf{d})$ for these families.

A degree sequence d is said to be *nontrivial* if every component of d is positive. For sampling over $\Omega(d)$, it is sufficient to consider only nontrivial d. Define

$$M_2(\mathbf{d}) = \sum_{i=1}^n d_i(d_i - 1).$$

We drop d from the notations M(d), $M_2(d)$, J(d) and $\Delta(d)$ when there is no confusion. In all theorems of this section, we say a family of degree sequences is *stable* if it is both 8-stable and strongly stable, and thus is also P-stable.

5.1 Degree sequences with an upper bound on $M_2(d)$

The first example in [15] are degree sequences where M_2 does not grow too fast with M. An asymptotic enumeration result was given in [15] when $M_2 = o(M^{9/8})$. Under a much weaker condition we show a polynomial-time mixing bound for the switch chain on $\Omega(\mathbf{d})$.

Theorem 5.1. Suppose that the graphical degree sequence d satisfies

$$M(\mathbf{d}) > 2\sqrt{\Delta(\mathbf{d})(M(\mathbf{d}) + M_2(\mathbf{d}))} + 18\Delta(\mathbf{d}) + 56.$$

$$(5.1)$$

Then the switch chain on $\Omega(\mathbf{d})$ mixes in polynomial time, with mixing time bounded by (2.1). Furthermore, the family of all degree sequences \mathbf{d} which satisfy (5.1) is stable.

Proof. It is sufficient to verify Condition 1 of Theorem 2.2. Observe that

$$M + M_2 = \sum_{i=1}^{n} d_i^2 \ge \sum_{i=1}^{\Delta} d_i^2.$$

By the Cauchy-Schwartz inequality,

$$\Delta\left(\sum_{i=1}^{\Delta}d_i^2\right) \ge \left(\sum_{i=1}^{\Delta}d_i\right)^2 = J^2.$$

Hence $J \leq \sqrt{\Delta(M+M_2)}$. It follows that Condition 1 holds. The assertions of the theorem follow from Theorems 2.2 and 2.3.

Since $\Delta \leq \sqrt{2M_2}$, it follows from Theorem 5.1 that the switch chain mixes in polynomial time whenever $M_2 \leq c M^{4/3}$ for some sufficiently small constant c > 0.

5.2 Power-law density-bounded sequences

Two types of power-law degree sequences were considered in [15]. We say \mathbf{d} is a power-law density-bounded sequence with parameter γ if there exists C > 0 such that the number of components in \mathbf{d} having value i is at most $Ci^{-\gamma}n$ for all $i \geq 1$ and for all n. This version of power-law degree sequences have been considered by many people in the literature, see for example [6]. An asymptotic enumeration result was given in [15] for such \mathbf{d} when $\gamma > 5/2$. In the following theorem, we show that the switch chain mixes in polynomial time on $\Omega(\mathbf{d})$ for such \mathbf{d} if $\gamma > 2$.

Theorem 5.2. Let \mathbf{d} be a nontrivial power-law density-bounded degree sequence with parameter $\gamma > 2$. Then the switch chain on $\Omega(\mathbf{d})$ mixes in polynomial time, with mixing time bounded by (2.1). Furthermore, the family of all nontrivial power-law density-bounded degree sequences \mathbf{d} with parameter $\gamma > 2$ is stable.

Proof. By definition, the maximum degree satisfies $\Delta \leq (Cn)^{1/\gamma}$. Let $j = cn^{1/\gamma}$ where c > 0 is a sufficiently small constant. Then $\sum_{i \geq j} ni^{-\gamma} \geq \Delta$. That is, there are at least Δ components of \boldsymbol{d} which are bounded below by j. Hence

$$J \le \sum_{i > j} Ci^{1-\gamma} n = O(n^{2/\gamma}).$$

Now $M = \Theta(n)$ as **d** is nontrivial. Then $J + \Delta = o(M)$ as $\gamma > 2$. This implies that Condition 1 holds, and the statements of the theorem follow by Theorems 2.2 and 2.3. \square

When $\gamma > 5/2$, a polynomial bound on the mixing time of the switch chain was given in [18]. To the best of our knowledge, the rest of Theorem 5.2 is new.

5.3 Power-law distribution-bounded sequences

Let $F_{\gamma}(i) = \sum_{j \geq i} j^{-\gamma} = O(i^{1-\gamma})$. We say that \mathbf{d} is a power-law distribution-bounded sequence with parameter γ if there exists C > 0 such that the number of components of \mathbf{d} taking value at least i is at most $CF_{\gamma}(i)$ n for all i and n. Power-law distribution-bounded sequences behave similarly to power-law density-bounded degree sequences, but allow longer tails (higher maximum degree). The maximum component of a power-law density-bounded sequence with parameter γ is of order $n^{1/\gamma}$, whereas the maximum component of a power-law distribution-bounded sequence with parameter γ is of order $n^{1/(\gamma-1)}$. Power-law distribution-bounded sequences, introduced in [15], are more realistic models for degree sequences of many real world networks. In particular, this includes degree sequences composed of n i.i.d. copies of power-law variables, which were considered in [29, 33].

The following theorem bounds the mixing time of the switch chain on $\Omega(\mathbf{d})$ where \mathbf{d} is power-law distribution-bounded with parameter $\gamma > 2$.

Theorem 5.3. Let \mathbf{d} be a nontrivial power-law distribution-bounded degree sequence with parameter $\gamma > 2$. Then the switch chain on $\Omega(\mathbf{d})$ mixes in polynomial time, with mixing time bounded by (2.1). Furthermore, the family of all nontrivial power-law distribution-bounded degree sequences \mathbf{d} with parameter $\gamma > 2$ stable.

Proof. By definition, the maximum degree Δ is of order at most $n^{1/(\gamma-1)}$. It has been shown in [16, eq. (54)] that $J \leq (Cn)^{(2\gamma-3)/(\gamma-1)^2}$. Again we have $J + \Delta = o(M)$, which implies that Condition 1 holds when $\gamma > 2$. The statements of the theorem follow by Theorems 2.2 and 2.3.

5.4 Bi-regular degree sequences

Another example examined in [15] is d whose components only take two values δ and Δ . This family is related to the "multi-star graphs" considered by Zhao [34], if all star centres have the same degree.

Theorem 5.4. Assume that all components of d take value in $\{\delta, \Delta\}$ where $\delta \leq \Delta$, and let ℓ be the number of vertices with degree Δ . Assume that one of the following conditions holds.

(a)
$$\ell \ge \Delta$$
 and $\Delta \ell + (n - \ell)\delta > 2\Delta^2 + 18\Delta + 56$.

(b)
$$\ell < \Delta \text{ and } n\delta > (\Delta - \delta)\ell + 2\delta\Delta + 18\Delta + 56.$$

Then the switch chain on $\Omega(\mathbf{d})$ mixes in polynomial time, with mixing time bounded by (2.1). Furthermore, the family of all graphical degree sequences \mathbf{d} which satisfy (a) or (b) is stable.

Proof. It is easy to check that if (a) or (b) holds then Condition 1 holds. The assertions follow by Theorems 2.2 and 2.3.

5.5 Long-tailed power-law degree sequences

The first author and Wormald [15] introduced long-tailed power-law degree sequences, which allow even longer tails than the power-law degree sequences. We say d follows a long-tailed power law with parameters (α, β, γ) if there is a constant C > 0 such that for every n,

- each component of **d** is non-zero and either at most C, or at least n^{α} ;
- for every integer $i \ge 1$, the number of components of d whose value is at least in^{α} but less than $(i+1)n^{\alpha}$ is at most $Cn^{\beta}i^{-\gamma}$.

Note that a long-tailed power-law degree sequence with $\alpha = 0$ and $\beta = 1$ is the same as a power-law density-bounded degree sequence.

Theorem 5.5. Assume that a nontrivial graphical degree sequence d follows a long-tailed power law with parameters (α, β, γ) . Further suppose that one of the following conditions holds:

(a)
$$1 < \gamma \le 2$$
 and $\alpha + 2\beta/\gamma < 1$;

(b)
$$\gamma > 2$$
 and $\alpha + \beta < 1$.

Then the switch chain on $\Omega(\mathbf{d})$ mixes in polynomial time, with mixing time bounded by (2.1). Furthermore, the family of all graphical degree sequences \mathbf{d} which follow a long-tailed power law with parameters (α, β, γ) such that (a) or (b) holds is stable.

Proof. As d is nontrivial, we have $M = \Omega(n)$. We will verify that if the parameters (α, β, γ) satisfy (a) or (b) then $\Delta = o(n)$ and J = o(n). This will imply that Condition 1 holds, and then the assertions of the theorem follow by Theorems 2.2 and 2.3.

Let $\hat{i} = Cn^{\beta/\gamma}$ for some sufficiently large constant C > 0. By definition, $\Delta = O(\hat{i}n^{\alpha}) = O(n^{\alpha+\beta/\gamma})$. Let \mathcal{H} denote the set of vertices with degree at least n^{α} and let H denote the sum of the degrees of the vertices in \mathcal{H} . Suppose that $d_{\mu(1)} \geq d_{\mu(2)} \geq \cdots \geq d_{\mu(n)}$ for some permutation $\mu \in S_n$, and let $\mathcal{U} = \{\mu(1), \dots, \mu(\Delta)\}$ be the set of the first Δ vertices with respect to μ . Then no vertex outside of \mathcal{U} has higher degree than a vertex in \mathcal{U} . Furthermore, \mathcal{U} contains at most all vertices in \mathcal{H} and Δ vertices in $[n] \setminus \mathcal{H}$, whose degrees are at most C. It follows immediately that $J \leq H + C\Delta$.

First consider the case $1 < \gamma \le 2$. The additional condition $\alpha + 2\beta/\gamma < 1$ ensures that $\Delta = o(n)$. We also have

$$H = O(n^{\alpha+\beta}) \sum_{i=1}^{\hat{i}} i^{1-\gamma} = O(\log n) \cdot n^{\alpha+\beta} n^{(\beta/\gamma)(2-\gamma)} = O(\log n) \cdot n^{\alpha+2\beta/\gamma} = o(n),$$

where the factor $\log n$ above accounts for the case $\gamma = 2$. It follows then that $J \leq H + C\Delta = o(n)$ when (a) holds.

Next consider the case $\gamma > 2$. The additional condition $\alpha + \beta < 1$ ensures that $\Delta = o(n)$. Moreover,

$$H = O(n^{\alpha+\beta}) \sum_{i=1}^{\hat{i}} i^{1-\gamma} = O(n^{\alpha+\beta}) = o(n).$$

This implies that J = o(n) when (b) holds, completing the proof.

6 Comparison of different notions of stability

Jerrum and Sinclair [19] initiated the study of stable degree sequences, introducing the notion of P-stability. Given a graphical degree sequence $\mathbf{d} = (d_1, \dots, d_n)$, let

$$\Omega'(\boldsymbol{d}) = \bigcup_{\boldsymbol{d}'} \Omega(\boldsymbol{d}')$$

where \mathbf{d}' ranges over the set of graphical degree sequences $\mathbf{d}' = (d'_1, \dots, d'_n)$ such that $\|\mathbf{d}' - \mathbf{d}\|_1 \leq 2$ and $d'_j \leq d_j$ for all $j \in [n]$.

Definition 6.1 ([19]). A family \mathcal{D} of degree sequences is P-stable if there exists a polynomial p such that for all positive integers n and for every degree sequence $\mathbf{d} = (d_1, \dots, d_n) \in \mathcal{D}$

$$|\Omega'(\boldsymbol{d})| \le p(n) |\Omega(\boldsymbol{d})|.$$

Trivially, any finite set of degree sequences is P-stable. Recall that a degree sequence is nontrivial if every component is positive. We now prove that P-stability is equivalent to 2-stability for families of nontrivial degree sequences.

Proposition 6.2. A family of nontrivial degree sequence is P-stable if and only if it is 2-stable.

Proof. Observe that $\Omega'(\mathbf{d}) \subseteq \mathcal{N}_2(\mathbf{d})$, and that $\Omega'(\mathbf{d})$ is a union over at most n^2 degree sequences \mathbf{d}' . Therefore, if \mathcal{D} is 2-stable then there exists some constant $\alpha > 0$ such that

$$|\Omega'(\boldsymbol{d})| \le n^2 M(\boldsymbol{d})^{\alpha} |\Omega(\boldsymbol{d})| \le n^{2+2\lceil \alpha \rceil} |\Omega(\boldsymbol{d})|.$$

This shows that \mathcal{D} is P-stable with polynomial $p(n) = n^{2+2\lceil \alpha \rceil}$, and hence 2-stability implies P-stability.

For the converse, let \mathcal{D} be a P-stable family of nontrivial degree sequences with polynomial p(n). For any $\mathbf{d} \in \mathcal{D}$ we have $M(\mathbf{d}) \geq n$, since \mathbf{d} is nontrivial. We will show that

$$|\Omega(\mathbf{d}')| \le n^2 |\Omega'(\mathbf{d})|$$
 for all $\mathbf{d} \in \mathcal{D}$ and $\mathbf{d}' \in \mathcal{N}_2(\mathbf{d})$. (6.1)

This yields our proposition as then, for all $d \in \mathcal{D}$ and all $d' \in N_2(d)$,

$$|\Omega(\mathbf{d}')| \le n^2 |\Omega'(\mathbf{d})| \le n^2 p(n) |\Omega(\mathbf{d})| \le M(\mathbf{d})^{\alpha} |\Omega(\mathbf{d})|$$

for some constant $\alpha > 0$, using P-stability for the second inequality. It is sufficient to consider \mathbf{d}' such that $d'_i \geq d_i + 1$ for some i, as otherwise $\Omega(\mathbf{d}') \subseteq \Omega'(\mathbf{d})$.

Case 1: $d'_i = d_i + 1$ and $d'_j = d_j + 1$ for some $i \neq j$. Define a switching from $\Omega(\mathbf{d}')$ to $\Omega'(\mathbf{d})$ as follows. For any $G \in \Omega(\mathbf{d}')$, the switching deletes an edge incident to i and an edge incident to j. The resulting graph G' is in $\Omega'(\mathbf{d})$. Each G can be switched to at least one graph in $\Omega'(\mathbf{d})$, and each graph in $\Omega'(\mathbf{d})$ can be produced by at most n^2 graphs in $\Omega(\mathbf{d}')$ using this switching. Hence $|\Omega(\mathbf{d}')| \leq n^2 |\Omega'(\mathbf{d})|$. Therefore (6.1) holds in this case.

The other cases where $d'_i = d_i + 2$ for some i, or $d'_i = d_i + 1$ and $d'_j = d_j - 1$ for some $i \neq j$ can be analysed in a similar way. If $d'_i = d_i + 2$ for some i then the switching chooses and deletes two distinct edges incident with i, while if $d'_i = d_i + 1$ and $d'_j = d_j - 1$ then the switching deletes an edge incident with i. In both cases, each graph in $\Omega'(\mathbf{d})$ can be produced by at most n graphs in $\Omega(\mathbf{d}')$ in this way. Therefore (6.1) also holds in these cases, completing the proof.

The following proposition follows immediately by (1.2) and the equivalence of 2-stability and P-stability.

Proposition 6.3. If a family of degree sequences is 8-stable then it is P-stable.

A slightly adjusted definition of P-stability was studied by Jerrum, Sinclair and McKay [20]. We remark that the definition of P-stability from [20] is equivalent to 2-stability, and thus is equivalent to the original Jerrum–Sinclair definition of P-stability. The equivalence can be proved by a similar argument as in Proposition 6.2.

Jerrum and Sinclair [19] introduced the idea of P-stability in order to prove a rapid mixing result for a Markov chain for sampling graphs with given degree sequence. This Markov chain, which we will call the JS chain, has state space $\Omega'(\boldsymbol{d})$, where \boldsymbol{d} is the target degree sequence. The JS chain produces an almost uniform sample from $\Omega'(\boldsymbol{d})$, which is rejected if the output is not an element of $\Omega(\boldsymbol{d})$. The expected number of restarts required is polynomial if and only if the ratio $|\Omega'(\boldsymbol{d})|/|\Omega(\boldsymbol{d})|$ is bounded above by some polynomial in n, leading to the definition of P-stable families.

For our purposes, we just need to know that the possible transitions $G \mapsto G'$ of the JS chain are as follows:

- (insertion): G' is obtained from G by inserting an edge. Here G' has degree d.
- (deletion): G' is obtained from G by deleting an edge of G. Here G has degree d.
- (hinge-flip): G' is obtained from G by deleting an edge uv and inserting an edge wv. Here neither G nor G' have degree d, as vertex w has degree $d_w - 1$ in G and vertex u has degree $d_u - 1$ in G'.

All such pairs (G, G') correspond to a transition of the JS chain, and there are no other transitions. (The terminology "hinge-flip" comes from [1].)

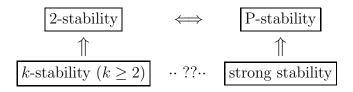
Amanatidis and Kleer [1] defined a stronger notion of stability, which they called *strong* stability, as follows. Say that two graphs G, H are at distance r in the JS chain if H can be obtained from G using at most r transitions. Let $dist(G, \mathbf{d})$ be the minimum distance of G from an element of $\Omega(\mathbf{d})$, and define

$$k_{\mathrm{JS}}(\boldsymbol{d}) = \max_{G' \in \Omega'(\boldsymbol{d})} \mathrm{dist}(G, \boldsymbol{d}).$$

That is, from any element of $\Omega'(d)$ it is possible to reach an element of $\Omega(d)$ using at most r transitions of the JS chain.

Definition 6.4. A family \mathcal{D} of graphical degree sequences is strongly stable if there is a constant ℓ such that $k_{JS}(\mathbf{d}) \leq \ell$ for all $\mathbf{d} \in \mathcal{D}$.

Amanatidis and Kleer proved that every strongly stable family is P-stable [1, Proposition 3], and that the switch chain has polynomial mixing time for all degree sequences from a strongly stable family [1, Theorem 4]. Hence, the strong stability is a (possibly) stronger notion than the P-stability. However, we do not know the relationship between P-stability and k-stability for k > 2. To conclude, we have the following relations between the various notions of stability under discussions:



Next, we prove that Condition 2, which is a sufficient condition for (2, 2)-stability, also implies strongly stability.

Proof of Theorem 2.2(b). It suffices to observe that each switching operation used in the proof of Theorem 4.2 can be implemented using a sequence of at most three transitions of the Jerrum-Sinclair chain. This implies that \mathcal{D} is strongly stable, as every $\mathbf{d} \in \mathcal{D}$ satisfies $k_{JS}(\mathbf{d}) \leq 3$. Specifically, an (i^-, j^-) -switching (Case 1) can be implemented by a deletion followed by two hinge flips, an (i^+, j^+) -switching (Case 2) can be implemented by a hinge flip followed by an insertion, and an (i^-, j^+) -switching (Case 3) can be implemented using two hinge flips.

7 Directed graphs

A directed graph (or digraph) G = (V, A) consists of a set V of vertices and a set A = A(G) of arcs (directed edges). A directed degree sequence is a pair $(\mathbf{d}^-, \mathbf{d}^+)$ of sequences of nonnegative integers,

$$d^- = (d_1^-, \dots, d_n^-), \qquad d^+ = (d_1^+, \dots, d_n^+)$$

such that $\sum_{j=1}^n d_j^- = \sum_{j=1}^n d_j^+$. The sequence is digraphical if there exists a directed graph with vertex set [n] such that d_j^- is the in-degree and d_j^+ is the out-degree of vertex j, for all $j \in [n]$. Write $\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)$ for the set of all directed graphs with directed degree sequence $(\boldsymbol{d}^-, \boldsymbol{d}^+)$. Note that loops are not permitted, so a directed graph (digraph) in $\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)$ corresponds to a bipartite graph on vertex set $\{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_n\}$ such that u_j has degree d_j^- and v_j has degree d_j^+ and (u_j, v_j) is not an edge, for $j \in [n]$. This connection between bipartite graphs and directed graphs is very well-known and has a long history [13, 30]. Again, we are interested in sequences of directed degree sequences $(\boldsymbol{d}^-(n), \boldsymbol{d}^+(n))$ indexed by n, but usually drop the dependence on n from our notation. We note that Erdős et al. gave a definition of P-stability for directed degree sequences, see [8, Definition 7.4]. We will use a variant of this definition.

The directed switch Markov chain, denoted by $\mathcal{M}(\boldsymbol{d}^-, \boldsymbol{d}^+)$, has state space $\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)$ and transitions described by the following procedure: from the current digraph $G \in \Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)$, choose an unordered pair $\{(i,j),(k,\ell)\}$ of distinct arcs of G uniformly at random. If i,j,k,ℓ are distinct and $\{(i,j),(k,\ell)\}\cap A(G)=\emptyset$ then delete the arcs $(i,j),(k,\ell)$ from G and add the arcs $(i,\ell),(k,j)$ to obtain the new state; otherwise, remain at G.

The directed switch chain is symmetric but it is not always irreducible, unlike the switch chain for directed graphs. We say that the directed degree sequence $(\boldsymbol{d}^-, \boldsymbol{d}^+)$ is switch-irreducible if the directed switch chain on $\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)$ is irreducible. The directed switch chain is ergodic on $\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)$ for any switch-irreducible digraphical degree sequence $(\boldsymbol{d}^-, \boldsymbol{d}^+)$, with uniform stationary distribution. Berger and Müller-Hanneman [5] and LaMar [23] provide characterisations which can be applied to test whether a given directed degree sequence $(\boldsymbol{d}^-, \boldsymbol{d}^+)$ is switch-irreducible. Rather than restricting to switch-irreducible sequences, some authors allow the directed switch chain to occasionally perform an additional operation, known as a triple swap [11], which reverses the arcs of a directed 3-cycle. With the addition of this operation, the chain is irreducible for all directed degree sequences. However, we use the directed switch chain as defined above, with no triple swaps.

Let

$$M(\boldsymbol{d}^-, \boldsymbol{d}^+) = \sum_{j=1}^n d_j^- = \sum_{j=1}^n d_j^+$$

be the number of arcs in a directed graph with directed degree sequence (d^-, d^+) , and let

$$\Delta^{-}(\mathbf{d}^{-}) = \max\{d_{1}^{-}, \dots d_{n}^{-}\}, \quad \Delta^{+}(\mathbf{d}^{+}) = \max\{d_{1}^{+}, \dots d_{n}^{+}\}.$$

Write

$$\Delta(\boldsymbol{d}^-, \boldsymbol{d}^+) = \max\{\Delta^-(\boldsymbol{d}^-), \Delta^+(\boldsymbol{d}^+)\}.$$

For any positive integer k and nonnegative real number α , we say that the digraphical degree sequence $(\boldsymbol{d}^-, \boldsymbol{d}^+)$ is (k, α) -stable if

$$|\Omega(\boldsymbol{d}'^{-}, \boldsymbol{d}'^{+})| \leq M(\boldsymbol{d}^{-}, \boldsymbol{d}^{+})^{\alpha} |\Omega(\boldsymbol{d}^{-}, \boldsymbol{d}^{+})|$$

for every digraphical degree sequence $(\boldsymbol{d}^{'-},\boldsymbol{d}^{'+})$ such that

$$\|\boldsymbol{d}^{'-} - \boldsymbol{d}^{-}\|_{1} + \|\boldsymbol{d}^{'+} - \boldsymbol{d}^{+}\|_{1} \le k.$$

We say a family of digraphical degree sequences \mathcal{D} is k-stable if there exists a fixed $\alpha > 0$ such that all degree sequences in \mathcal{D} are (k, α) -stable.

We adapt the definition of P-stability for directed degree sequences from [8, Definition 7.4]. Given a digraphical degree sequence $(\mathbf{d}^-, \mathbf{d}^+)$, let

$$\Omega'(\boldsymbol{d}^-,\boldsymbol{d}^+) = \bigcup_{(\boldsymbol{d}'^-,\boldsymbol{d}'^+)} \Omega(\boldsymbol{d}'^-,\boldsymbol{d}'^+)$$

where $(\boldsymbol{d}'^-, \boldsymbol{d}'^+)$ ranges over the set of digraphical degree sequences with $d_j^{'-} \leq d_j^-$ and $d_j^{'+} \leq d_j^+$ for all $j \in [n]$, such that

$$\|\boldsymbol{d}^{'-} - \boldsymbol{d}^{-}\|_{1} + \|\boldsymbol{d}^{'+} - \boldsymbol{d}^{+}\|_{1} \leq 2.$$

Definition 7.1. A family \mathcal{D} of directed degree sequences is P-stable if there exists a polynomial p such that for all positive integers n and for every directed degree sequence $(\mathbf{d}^-, \mathbf{d}^+) \in \mathcal{D}$,

$$|\Omega'(\boldsymbol{d}^-, \boldsymbol{d}^+)| \le p(n) |\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)|.$$

Note that a family of digraphical degree sequences is P-stable if and only if it is 2-stable, as can be shown by adapting the proof of Proposition 6.2. We will establish the directed analogue of Theorem 2.1, stated below.

Theorem 7.2. Let $(\mathbf{d}^-, \mathbf{d}^+)$ be a digraphical switch-irreducible directed degree sequence. Write $M = M(\mathbf{d}^-, \mathbf{d}^+)$ and $\Delta = \Delta(\mathbf{d}^-, \mathbf{d}^+)$. If $(\mathbf{d}^-, \mathbf{d}^+)$ is $(12, \alpha)$ -stable then the directed switch chain on $\Omega(\mathbf{d}^-, \mathbf{d}^+)$ mixes in polynomial time, with mixing time $\tau(\varepsilon)$ which satisfies

$$\tau(\varepsilon) \le 1200 \, \Delta^{16} \, n^6 \, M^{3+\alpha} \, \Big(M \log M + \log(\varepsilon^{-1}) \Big).$$

Given a directed degree sequence $(\boldsymbol{d}^-, \boldsymbol{d}^+)$, suppose that $\rho, \xi \in S_n$ are permutations such that $d_{\mu(1)}^- \geq d_{\mu(2)}^- \geq \cdots \geq d_{\mu(n)}^-$ and $d_{\xi(1)}^+ \geq d_{\xi(2)}^+ \geq \cdots \geq d_{\xi(n)}^+$. Let

$$J^{-}(\boldsymbol{d}^{-},\boldsymbol{d}^{+}) = \sum_{\ell=1}^{\Delta^{-}(\boldsymbol{d}^{-})} d_{\xi(\ell)}^{+}, \qquad J^{+}(\boldsymbol{d}^{-},\boldsymbol{d}^{+}) = \sum_{\ell=1}^{\Delta^{+}(\boldsymbol{d}^{+})} d_{\mu(\ell)}^{-}.$$

These will turn out to be the appropriate directed analogues of the parameter $J(\mathbf{d})$ used for undirected graphs. When it causes no confusion, we drop $(\mathbf{d}^-, \mathbf{d}^+)$ from notation such at $M(\mathbf{d}^-, \mathbf{d}^+)$, etc.

Theorem 7.3. Let d be a digraphical directed degree sequence which satisfies

$$M > J^{-} + J^{+} + 8(\Delta^{-} + \Delta^{+}) + 48.$$
 (7.1)

Then \mathbf{d} is (12, 12)-stable.

Again, we have not attempted to optimise the coefficients in (7.1). Combining Theorems 7.2 and 7.3 immediately establishes the following.

Theorem 7.4. Assume that $(\mathbf{d}^-, \mathbf{d}^+)$ is a digraphical switch-irreducible directed degree sequence which satisfies (7.1). Then the switch chain on $\Omega(\mathbf{d}^-, \mathbf{d}^+)$ mixes in polynomial time, with mixing time $\tau(\varepsilon)$ which satisfies

$$\tau(\varepsilon) \le 600 \,\Delta^{16} \, n^6 \, M^{15} \, \Big(M \log M + \log(\varepsilon^{-1}) \Big). \tag{7.2}$$

We do not believe that the mixing time bounds in Theorem 7.2 and Theorem 7.4 are tight.

7.1 Multicommodity flow and mixing time

In [17, 18], the mixing time of the directed switch chain is analysed using a multicommodity flow argument. The flow is an extension of the one used in [7], following the steps described in Section 3.1 in the undirected case. Again, a set $\Psi(G, G')$ of pairings is defined for each $(G, G') \in \Omega(\mathbf{d}^-, \mathbf{d}^+)^2$, and a canonical path $\gamma_{\psi}(G, G')$ is defined from G to G', where each step of the path is a transition of the directed switch chain. For our purposes, it is sufficient to understand the structure of the encodings which arise along these paths.

Definition 7.5. An encoding L of a digraph $Z \in \Omega(\mathbf{d}^-, \mathbf{d}^+)$ is an arc-labelled digraph on n vertices, with arc labels in $\{-1, 1, 2\}$, such that

- (i) the sum of arc-labels on arcs into vertex j equals d_j^- and the sum of arc-labels on arcs out of vertex j equals d_i^+ , for all $j \in [n]$;
- (ii) the arcs with labels -1 or 2 form a subdigraph of one of the digraphs shown in Figure 5.

An arc labelled -1 or 2 is a defect arc.

Given $G, G', Z \in \Omega(\mathbf{d}^-, \mathbf{d}^+)$, identify each of Z, G, G' with their 0-1 adjacency matrix and define the matrix L by

$$L + Z = G + G'$$

just as we did for undirected graphs. Then L corresponds to an arc-labelled digraph, which we also denote by L. It follows from the next result that the arc-labelled digraph L is an encoding, which we call the encoding of Z with respect to (G, G').

Lemma 7.6 ([18, Lemma 3.3(ii)]). Given $G, G' \in \Omega(\mathbf{d}^-, \mathbf{d}^+)$ with symmetric difference $G \triangle G'$, let (Z, Z') be a transition on the canonical path from G to G' with respect to the pairing $\psi \in \Psi(G, G')$. Let L be the encoding of Z with respect to (G, G'). Then there are at most five defect arcs in L. The digraph consisting of the defect arcs in L must form a subdigraph of one of the possible labelled digraphs shown in Figure 5, up to the symmetries described below.

Define the arc-reversal operator ζ , which acts on a digraph G by reversing every arc in G; that is, replacing (u,v) by (v,u) for every arc $(u,v) \in A(G)$. In Figure 5, $\{\mu,\nu\} = \{-1,2\}$ and $\{\xi,\omega\} = \{-1,2\}$ independently, giving four symmetries obtained by exchanging these pairs. We can also apply the operation ζ to reverse the orientation of all arcs. Hence each digraph shown in Figure 5 represents up to eight possible digraphs.

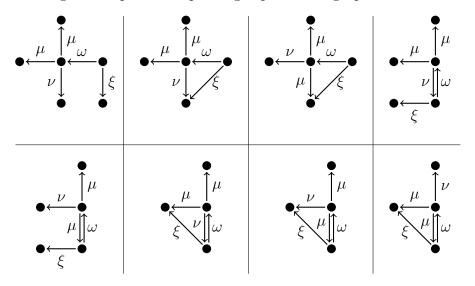


Figure 5: Possible configurations of defect arcs, up to symmetries. See [18, Lemma 3.3(ii)]

Given $Z \in \Omega(\mathbf{d}^-, \mathbf{d}^+)$, we say that encoding L is consistent with Z if L + Z only takes entries in $\{0, 1, 2\}$. Let $\mathcal{L}(Z)$ be the set of encodings which are consistent with Z and satisfy (i), (ii) from Definition 7.5 above. This is the same set of encodings used in [18].

Theorem 7.7. Let $(\mathbf{d}^-, \mathbf{d}^+)$ be a digraphical switch-irreducible directed degree sequence. Suppose that there exists a function $g(\mathbf{d}^-, \mathbf{d}^+)$, which depends only on $(\mathbf{d}^-, \mathbf{d}^+)$, such that

$$|\mathcal{L}(Z)| \leq g(\boldsymbol{d}^-, \boldsymbol{d}^+) \, |\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)|$$

for all $Z \in \Omega(\mathbf{d}^-, \mathbf{d}^+)$. Then the mixing time $\tau(\varepsilon)$ of the switch chain on $\Omega(\mathbf{d}^-, \mathbf{d}^+)$ satisfies

$$\tau(\varepsilon) \le 2g(\mathbf{d}^-, \mathbf{d}^+) \,\Delta^{16} \, M^3 \, \Big(M \log M + \log(\varepsilon^{-1}) \Big).$$

Proof. We follow the structure of the proof of [18, Theorem 1.2]. It follows from the bipartite model of directed graphs that

$$|\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)| \le M! \le \sqrt{2\pi M} \left(\frac{M}{e}\right)^M.$$

Therefore the smallest stationary probability π^* satisfies $\log(1/\pi^*) = \log |\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)| \leq M \log M$. Next, observe that $\ell(f) \leq M$ since each transition along a canonical path from G to G' replaces an edge of G by an edge of G'. Finally, if e = (Z, Z') is a transition of the directed switch chain then $1/Q(e) = \binom{M}{2} |\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)| \leq \frac{1}{2} M^2 |\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)|$. Given a transition e = (Z, Z') of the directed switch chain, the load f(e) on the transition satisfies

$$f(e) \le 4 \Delta^{16} \frac{|\mathcal{L}(Z)|}{|\Omega(\mathbf{d}^-, \mathbf{d}^+)|^2},$$

as proved in [18, Lemma 3.4]. (Again, the degree condition in the statement of [18, Theorem 1.2] is only needed in the proof of the critical lemma, so this bound on f(e) holds even when $(\mathbf{d}^-, \mathbf{d}^+)$ does not satisfy that condition. This gives

$$\rho(f) = \max_{e} \frac{f(e)}{Q(e)} \leq 2\Delta^{16} M^2 \max_{Z \in \Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)} \frac{|\mathcal{L}(Z)|}{|\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)|} \leq 2g(\boldsymbol{d}^-, \boldsymbol{d}^+) \Delta^{16} M^2.$$

Substituting these expressions into (3.1) completes the proof.

Lemma 7.8. Assume that the graphical degree sequence \mathbf{d} is $(12, \alpha)$ -stable for some nonnegative real number α . Then for any $Z \in \Omega(\mathbf{d}^-, \mathbf{d}^+)$,

$$|\mathcal{L}(Z)| \le 600 \, n^6 M^{\alpha} \, |\Omega(\mathbf{d}^-, \mathbf{d}^+)|.$$

Proof. By definition, $|\mathcal{L}(Z)|$ is the number of arc-labelled directed graphs which satisfy conditions (i) and (ii), as well as some other constraints. First we bound the number of ways to choose the defect arcs of Z, by first choosing a subdigraph H of one of the digraphs in Figure 5, and then mapping each vertex of H to a distinct vertex of H. There are 64 different digraphs shown in Figure 5, after considering all symmetries. Each of these digraphs has 5 arcs and at most 6 vertices. If a subdigraph of H has ℓ vertices then the number of injective functions $\varphi: V(H) \to [n]$ is at most n^{ℓ} . By considering the maximum number of vertices in a subdigraph consisting of a given number of arcs, we find that the number of ways to choose H and φ is at most

$$64\left(n^6 + 5n^6 + 10n^5 + 10n^4 + 5n^2 + 1\right) \le 64n^6\left(1 + 5 + \frac{5}{2} + \frac{5}{8} + \frac{5}{256} + \frac{1}{4096}\right)$$
$$\le 600 \, n^6.$$

The first inequality follows since $n \geq 4$, as otherwise there are no switch operations available.

Let \mathcal{E} denote the chosen defect arcs with their labels. We now bound the number of ways to complete \mathcal{E} to an encoding L. All arcs in $L \setminus \mathcal{E}$ are labelled 1, and the number of arcs into vertex j in $L \setminus \mathcal{E}$ equals $d'_j = d^-_j - x^-_j$, where x^-_j is the sum of arc-labels from arcs into j which belong to \mathcal{E} . Similarly, the number of arcs out of vertex j in $L \setminus \mathcal{E}$ equals $d'_j = d^+_j - x^+_j$, for all $j \in [n]$. Note that $\|d'^-\|_1 = \|d'^+\|_1$. Hence the number of valid encodings L given \mathcal{E} is at most $|\Omega(d'^-, d'^+)|$. By considering cases we see that for all possible sets \mathcal{E} ,

$$\|\boldsymbol{d}^{'-} - \boldsymbol{d}^{-}\|_{1} + \|\boldsymbol{d}^{'+} - \boldsymbol{d}^{+}\|_{1} \le 12.$$

Since $(\mathbf{d}^-, \mathbf{d}^+)$ is $(12, \alpha)$ -stable, the result follows.

7.2 Sufficient condition for stability

Assume that $(\mathbf{d}^-, \mathbf{d}^+)$ is a digraphical degree sequence. For positive integers k, let

$$\mathcal{N}_k(\boldsymbol{d}^-, \boldsymbol{d}^+) = \{(\boldsymbol{d}^{'-}, \boldsymbol{d}^{'+}) \in \mathbb{N}^n \times \mathbb{N}^n : \|\boldsymbol{d}^{'-}\|_1 = \|\boldsymbol{d}^{'+}\|_1, \|\boldsymbol{d}^{'-} - \boldsymbol{d}^-\|_1 + \|\boldsymbol{d}^{'+} - \boldsymbol{d}^+\| \le k\}.$$

The following analogue of Lemma 4.1 is proved using the same ideas (proof omitted).

Lemma 7.9. Suppose that every digraphical degree sequence $(\boldsymbol{d}^{'-}, \boldsymbol{d}^{'+}) \in \mathcal{N}_{10}(\boldsymbol{d}^{-}, \boldsymbol{d}^{+})$ is $(2, \alpha)$ -stable. Then $(\boldsymbol{d}^{-}, \boldsymbol{d}^{+})$ is $(12, 6\alpha)$ -stable.

Now we state the sufficient condition for (2,2)-stability, which is the directed analogue of Theorem 4.2. Luckily, the switching argument from Theorem 4.2 can be fairly easily adapted to the directed setting.

Theorem 7.10. If $(\mathbf{d}^-, \mathbf{d}^+)$ satisfies $M > J^- + J^+ + 3(\Delta^- + \Delta^+) + 3$ then $(\mathbf{d}^-, \mathbf{d}^+)$ is (2, 2)-stable.

Proof. Let $(\mathbf{d}'^-, \mathbf{d}'^+) \in \mathcal{N}_2(\mathbf{d}^-, \mathbf{d}^+) \setminus (\mathbf{d}^-, \mathbf{d}^+)$. There are only four cases: either $\mathbf{d}'^- = \mathbf{d}^- + \mathbf{e}_i$ and $\mathbf{d}'^+ = \mathbf{d}^+ + \mathbf{e}_j$, or $\mathbf{d}'^- = \mathbf{d}^- - \mathbf{e}_i$ and $\mathbf{d}'^+ = \mathbf{d}^+ - \mathbf{e}_j$, or $\mathbf{d}'^- = \mathbf{d} + \mathbf{e}_i - \mathbf{e}_j$, or $\mathbf{d}'^+ = \mathbf{d} + \mathbf{e}_i - \mathbf{e}_j$, for some $i, j \in [n]$. These cases are the directed analogues of the cases from the proof of Theorem 4.2, presented in Section 4.1. We do not give full details but summarise the calculations.

In Case 1, the directed (i^-, j^-) -switching is shown in Figure 6, with the subcase $i \neq j$ on the left and i = j on the right. This switching converts a graph $G' \in \Omega(\mathbf{d}'^- + \mathbf{e}_i, \mathbf{d}'^+ + \mathbf{e}_j)$ to a graph $G \in \Omega(\mathbf{d}^-, \mathbf{d}^+)$. Note that the arcs alternate in orientation, as well as in their presence/absence, at each vertex other than i, j. Again we allow $u_1 = u_4$.

Let $f_{(i^-,j^-)}(G')$ be the number of ways to perform an (i^-,j^-) -switching to $G' \in \Omega(\boldsymbol{d}^- + \boldsymbol{e}_i, \boldsymbol{d}^+ + \boldsymbol{e}_j)$. There are $(d_i^- + 1)(d_j^+ + 1)$ ways to choose u_1 and u_4 . Then there are $M(\boldsymbol{d}^-, \boldsymbol{d}^+) + 1$ choices for the arc (u_3,u_2) , but we must subtract $4(\Delta^-(\boldsymbol{d}^-) + \Delta^+(\boldsymbol{d}^+) + 2)$ for the possible vertex coincidences. Furthermore we must subtract the number of choices where there are no vertex coincidences but the arc (u_1,u_2) is present, or the arc (u_3,u_4) is present. The number of choices such that arc (u_1,u_2) is present is at most

$$\sum_{\ell=1}^{d_{u_1}^+-1} (d_{\mu(\ell)}^- - 1) \le \sum_{\ell=1}^{\Delta^+(\boldsymbol{d}^+)-1} (d_{\mu(\ell)}^- - 1) \le J^+(\boldsymbol{d}^-, \boldsymbol{d}^+) - \Delta^+(\boldsymbol{d}^+)$$

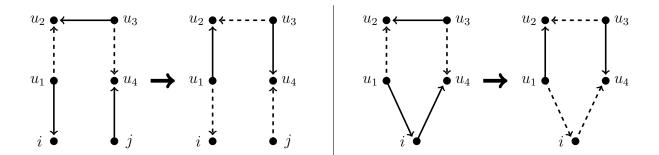


Figure 6: The directed (i^-, j^-) -switching, when $i \neq j$ (left) and when i = j (right).

and similarly, the number of choices such that (u_3, u_4) is present is at most $J^-(\mathbf{d}^-, \mathbf{d}^+) - \Delta^-(\mathbf{d}^-)$. Therefore

$$f_{(i^-,j^-)}(G') \ge (d_i^- + 1)(d_j^+ + 1)\Big(M - J^- - J^+ - 3\Delta^- - 3\Delta^+ - 3\Big),$$

which is positive by the assumption of the theorem. The number of ways to create $G \in \Omega(\mathbf{d}^-, \mathbf{d}^+)$ using a (i^-, j^-) -switching is at most $M(\mathbf{d}^-, \mathbf{d}^+)^2$, and we conclude that

$$\frac{|\Omega(\boldsymbol{d}^- + \boldsymbol{e}_i, \boldsymbol{d}^+ + \boldsymbol{e}_j)|}{|\Omega(\boldsymbol{d}^-, \boldsymbol{d}^+)|} \le M(\boldsymbol{d}^-, \boldsymbol{d}^+)^2.$$

The argument when i = j is similar and also leads to an upper bound of $M(\mathbf{d}^-, \mathbf{d}^+)^2$. A similar analysis shows that $M(\mathbf{d}^-, \mathbf{d}^+)^2$ is also an upper bound in the other cases, by adapting the switchings and arguments from Theorem 4.2.

Now we can use Theorem 7.10 to prove Theorem 7.3.

Proof of Theorem 7.3. First suppose that d satisfies (7.1) and let $d' \in \mathcal{N}_{10}(d)$. Then

$$\max_{i \in [n]} \{ |d_i^{'-} - d_i^{-}|, |d_i^{'+} - d_i^{+}| \} \le 5,$$

and thus $M(d'^-, d'^+) \ge M(d^-, d^+) - 5$. Also

$$\Delta(\boldsymbol{d}^{'-}) + \Delta(\boldsymbol{d}^{'+}) \leq \Delta(\boldsymbol{d}^{-}) + \Delta(\boldsymbol{d}^{+}) + 10$$

and

$$J^{-}(\mathbf{d}^{'-}, \mathbf{d}^{'+}) + J^{+}(\mathbf{d}^{'-}, \mathbf{d}^{'+}) \le J^{-}(\mathbf{d}^{-}, \mathbf{d}^{+}) + J^{+}(\mathbf{d}^{-}, \mathbf{d}^{+}) + 5(\Delta(\mathbf{d}^{-}) + \Delta(\mathbf{d}^{+})) + 10.$$

Applying Theorem 7.10 to d' shows that d' is (2,2)-stable, and therefore d is (12,12)-stable, by Lemma 7.9. This completes the proof.

7.3 Digraphical power-law sequences

Recall the definition of power-law distribution-bounded and power-law density-bounded degree sequences from Sections 5.2 and 5.3. Let $D: \mathbb{N} \to \mathbb{N}$. We say that a sequence $\boldsymbol{a} \in \mathbb{N}^n$ is D(n)-bounded if every component of \boldsymbol{a} is at most D(n). Now consider digraphical sequences $(\boldsymbol{d}^-, \boldsymbol{d}^+)$ where \boldsymbol{d}^+ is power-law distribution-bounded, or power-law density bounded, with exponent $\gamma > 1$, whereas \boldsymbol{d}^- is Δ^- -bounded (in the sense defined above), where Δ^- is some function of n. Such degree sequences include models where \boldsymbol{d}^+ is composed of i.i.d. copies of power-law random variables, and \boldsymbol{d}^- is composed of i.i.d. copies of truncated power-law random variables [28]. Directed degree sequences of this type are important because some social networks such as Twitter are directed graphs [2], where the indegrees (i.e. the number of followers) appear to follow a power law with parameter below 2, whereas the majority of the outdegrees follows a power law, but with a much smaller maximum outdegree compared with the maximum indegree. In the next two theorems we show that random directed graphs with such degree sequences can be sampled in polynomial time using the directed switch chain. We say a digraphical degree sequence is nontrivial if there is no i such that $d_i^+ = d_i^- = 0$. If $(\boldsymbol{d}^-, \boldsymbol{d}^+)$ is nontrivial then $M(\boldsymbol{d}^-, \boldsymbol{d}^+) = \frac{1}{2}(\|\boldsymbol{d}^-\|_1 + \|\boldsymbol{d}^+\|_1) \geq n$.

Theorem 7.11. Let $(\mathbf{d}^-, \mathbf{d}^+)$ be a digraphical, switch-irreducible nontrivial degree sequence such that \mathbf{d}^+ is power-law density-bounded with parameter $\gamma > 1$, and $\Delta^- = o(n^{(\gamma-1)/\gamma})$. Then the directed switch chain on $\Omega(\mathbf{d}^-, \mathbf{d}^+)$ mixes in polynomial time, with mixing time bounded by (7.2).

Proof. By definition, $\max\{J^-, J^+\} \leq \Delta^-\Delta^+$. Since $\Delta^+ = O(n^{1/\gamma})$ and $M = \Theta(n)$ by the definition of power-law density-bounded sequence, it follows that $(\boldsymbol{d}^-, \boldsymbol{d}^+)$ is (12, 12)-stable by Theorem 7.3. The assertion then follows by Theorem 7.4.

Theorem 7.12. Let $(\mathbf{d}^-, \mathbf{d}^+)$ be a digraphical, switch-irreducible directed degree sequence such that \mathbf{d}^+ is power-law distribution-bounded with parameter $\gamma > 2$, and $\Delta^- = o(n^{(\gamma-2)/(\gamma-1)})$. Then the directed switch chain on $\Omega(\mathbf{d}^-, \mathbf{d}^+)$ mixes in polynomial time, with mixing time bounded by (7.2).

Proof. The proof is the same as that of Theorem 7.11 except that with power-law distribution-bounded sequences we have $\Delta^+ = O(n^{1/(\gamma-1)})$ and $M = \Theta(n)$.

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