On Star 5-Colorings of Sparse Graphs

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Abstract

A *star k-coloring* of a graph G is a proper (vertex) k-coloring of G such that the vertices on a path of length three receive at least three colors. Given a graph G, its *star chromatic number*, denoted $\chi_s(G)$, is the minimum integer k for which G admits a star k-coloring. Studying star coloring of sparse graphs is an active area of research, especially in terms of the maximum average degree of a graph; the *maximum average degree*, denoted $\mathrm{mad}(G)$, of a graph G is $\max\left\{\frac{2|E(H)|}{|V(H)|} : H \subset G\right\}$. It is known that for a graph G, if $\mathrm{mad}(G) < \frac{8}{3}$, then $\chi_s(G) \le 6$ [18], and if $\mathrm{mad}(G) < \frac{18}{7}$ and its girth is at least 6, then $\chi_s(G) \le 5$ [7]. We improve both results by showing that for a graph G, if $\mathrm{mad}(G) \le \frac{8}{3}$, then $\chi_s(G) \le 5$. As an immediate corollary, we obtain that a planar graph with girth at least 8 has a star 5-coloring, improving the best known girth condition for a planar graph to have a star 5-coloring [18, 21].

1 Introduction

All graphs in this paper are simple. Given a graph G, a *proper k-coloring* of G is a partition of its vertex set $V(G)$ into k parts such that there is no edge with both endpoints in the same part. In other words, each color class induces an empty graph.

As a generalization of proper coloring, Grünbaum [14] introduced the notion of *acyclic coloring*, which is a proper coloring satisfying the additional constraint that the vertices on a cycle (of any length) receive at least three colors. In other words, the union of two color classes induces an acyclic graph. One of the most interesting results regarding acyclic coloring is the result by Borodin [4], which states that every planar graph admits an acyclic coloring with five colors. This resolved a conjecture in the initial paper [14] of Grünbaum where he showed that five colors is necessary to acyclically

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color certain planar graphs; Kostochka and Mel′nikov [17] even constructed a bipartite planar graph requiring five colors when acyclically colored. In contrast, the famous Four Color Theorem [2,3] states that every planar graph has a proper 4-coloring.

In [14], Grünbaum also raised the question of proper coloring with the additional constraint that the vertices on a path of length three receive at least three colors. In other words, the union of two color classes induces a star forest. Although Grünbaum gave no specific name for this type of coloring, this coloring is now known as *star coloring*, ever since the term was first coined by Albertson et al. [1]. To be precise, a *star k-coloring* of a graph G is a proper k-coloring of G where the vertices on a path of length three receive at least three colors. The *star chromatic number* of a graph G, denoted $\chi_s(G)$, is the minimum k for which G admits a star k-coloring. Since a star forest is also an acyclic graph, star coloring is a strengthening of acyclic coloring. Acyclic coloring and star coloring have been an active area of research, and we direct the readers to a thorough survey by Borodin [5] for the rich literature. There is also an edge-coloring analogue for star coloring; for recent progress on star edge-coloring subcubic graphs, see [12, 15, 19, 20].

In this paper, we are interested in star colorings of sparse graphs, where sparsity is measured in terms of the maximum average degree. The *maximum average degree* of a graph G, denoted $\text{mad}(G)$, is the maximum of the average degrees of all its subgraphs, that is, $\text{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subset G\right\}$. Since a planar graph G with girth at least g satisfies $\text{mad}(G) < \frac{2g}{g-2}$, a result regarding graphs with bounded maximum average degree implies that planar graphs with certain girth conditions can reach the same conclusion, see [11].

Grünbaum proved that planar graphs are star 2304-colorable in [14] back in 1973, and after 45 years the best result so far is by Albertson et al. [1] where they showed that all planar graphs are star 20-colorable. They also constructed a planar graph that requires at least ten colors to be star colored, and for a given girth g, they constructed a planar graph with girth g that requires at least four colors to be star colored. Moreover, they investigated the star chromatic number for planar graphs with certain girth constraints, where they proved that a planar graph G with girth at least 5 and 7 satisfies $\chi_s(G) \leq 16$ and $\chi_s(G) \leq 9$, respectively, improving upon some bounds in [13]. Timmons [21] and Kündgen and Timmons [18] continued the study as they obtained results that imply a planar graph is star 4-, 5-, 6-, 7-, 8-colorable if its girth is at least 14, 9, 8, 7, 6, respectively. Sufficient conditions on girth to guarantee that a planar graph is star 4-colorable have received much attention due to its relation to the Four Color Theorem [2,3]. In particular, Bu et al. [7] improved the girth constraint to 13, and Brandt et al. [6] has the current best bound showing that a planar graph with girth at least 10 has a star 4-coloring.

Lower bounds on the girth constraints have also been investigated. In particular, a planar graph with girth 7 and 5 that requires 5 and 6 colors to be star colored has been constructed in [21] and [18], respectively. See Table 1 for a summary of lower and upper bounds on the star chromatic number of a planar graph with a given girth constraint.

girth	3	4	5	6	7	8	9	≥ 10
upper bound	20 [1]	20 [1]	16 [1]	8 [18]	7 [18]	6 [18]	5 [21]	4 [6]
lower bound	10 [1]	10 [1]	6 [18]	5 [21]	5 [21]	4 [1]	4 [1]	4 [1]

Table 1: Table of best known results.

Various results above are also true for the maximum average degree setting [6, 18, 21] and the list version setting [7, 8, 10, 18]. Researchers have also investigated star coloring for bipartite planar graphs [16] and subcubic graphs [9, 10]. In particular, we explicitly state the following two theorems:

Theorem 1.1 ([7])**.** *For a graph G, if* $\mathrm{mad}(G) < \frac{18}{7}$ *and its girth is at least 6, then G is star 5-colorable.*

Theorem 1.2 ([18])**.** *For a graph G, if* $\mathrm{mad}(G) < \frac{8}{3}$*, then G is star 6-colorable.*

Our main theorem improves both aforementioned theorems. Note that Theorems 1.1 and 1.2 imply that a planar graph with girth at least 8 and 9 has a star coloring with 6 and 5 colors, respectively. Likewise, as a direct consequence of our main result, we improve the best known girth condition for a planar graph to be star 5-colorable. We now present our main result and its direct consequence:

Theorem 1.3. *For a graph G, if* $\mathrm{mad}(G) \leq \frac{8}{3}$*, then G is star 5-colorable.*

Corollary 1.4. *A planar graph with girth at least 8 is star 5-colorable.*

We actually prove a stronger statement, guaranteeing a certain partition of the vertices that implies the existence of a star 5-coloring. For a positive integer k, a *k-independent set* of a graph G is a subset S of $V(G)$ such that a pair of vertices in S has distance at least $k+1$ in G. For two disjoint sets A and B, let $A \sqcup B$ denote the disjoint union of A and B. We say a graph G has an *$\mathcal{FII}$-partition* $F \sqcup I_\alpha \sqcup I_\beta$, if F, I_α, I_β is a partition of $V(G)$, each of I_α and I_β induces a 2-independent set, and F induces a forest. Since a forest is star 3-colorable (by picking a root and coloring the vertices according to the distance to the root modulo three), if a graph G has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$, then G is star 5-colorable; use three colors on F, one color on I_α, and one color on I_β. The above idea of using a 2-independent set first appeared in Albertson et al. [1]. Hence, in order to prove Theorem 1.3, it is sufficient to show the following Theorem 1.5.

Theorem 1.5. *For a graph G, if* $\mathrm{mad}(G) \leq \frac{8}{3}$*, then G has an $\mathcal{FII}$-partition.*

The paper is organized as follows. The proof of Theorem 1.5 is split into Sections 2, 3, and 4. Section 2 lays out the discharging rules and reducible configurations. Sections 3 and 4 provide the proofs of the reducible configurations. We conclude with questions and tightness bounds in Section 5.

We list some important definitions used in this paper. We use $[n]$ to denote the set $\{1, \dots, n\}$. Let G be a graph. For $S \subset V(G)$, let $G - S$ denote the graph obtained from G by deleting the vertices in S. If $S = \{x\}$, then we denote $G - S$ by $G - x$. Likewise, in order to improve the readability of the paper, we often drop the braces and commas to denote a set and use '+' for the set operation '$\cup$'. For instance, given $A \subset V(G)$ and $x, y, z \in V(G)$, we use $A + x - y$ and $A - z + xy$ to denote $(A \cup \{x\}) \setminus \{y\}$ and $(A \setminus \{z\}) \cup \{x, y\}$, respectively.

A *d^+-vertex*, *d-vertex*, *d^--vertex* is a vertex of degree at least d, exactly d, at most d, respectively. Given a vertex x, a neighbor of x with degree at least d, exactly d, at most d is called a *d^+-neighbor*, *d-neighbor*, *d^--neighbor*, respectively. For $S \subset V(G)$, a vertex in a set S is called an *S-vertex*. Similarly, we say u is an *S-neighbor* of a vertex v if $u \in N_G(v) \cap S$. A *pendent k-cycle* is a cycle of length k where all its vertices except one vertex x are 2-vertices; we also say this cycle is *at the vertex x*. A 3-cycle is also called a *triangle*.

We finish this section with observations, which is frequently used in the proof.

Lemma 1.6. *Let H be a graph with an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$.*

(i) *If H has an induced subgraph isomorphic to J_1 in Figure 1 and $v^* \in I_\alpha$, then $w_1 \in I_\beta$.*

(ii) *If H has an induced subgraph isomorphic to J_2 in Figure 1, then $v^* \in F$.*

Figure 1: The graphs J_1 and J_2.

Proof. To show (i), suppose to the contrary that $w_1 \notin I_\beta$. Then $w_1 \in F$, and so one vertex from each pendent triangle at w_1 is in I_β, which is a contradiction since I_β is a 2-independent set. To show (ii), suppose to the contrary that $v^* \notin F$, say I_α. By (i), $w_1 \in I_\beta$, and again by (i), $w_2 \in I_\alpha$, which is a contradiction since I_α is a 2-independent set. $\square$

In order to check the maximum average degree of a graph, we often use the potential function. For a graph H, let $\rho_H : 2^{V(H)} \to \mathbb{Z}$ be the function such that $\rho_H(A) = 4|A| - 3|E(H[A])|$ for $A \subset V(H)$, called the *potential function* for H. We use the following, which is straightforward from the definition:

$$\rho_H(A) \geq 0 \text{ for all } A \subset V(H) \text{ if and only if } \mathrm{mad}(H) \leq \frac{8}{3}.$$

For $I \subset V(H)$, let

$$\rho^*_H(I) = \min\{\rho_H(K) \mid I \subset K \subset V(H)\}.$$

For brevity, we often drop the braces and commas to denote $\rho^*_H(A)$, such as $\rho^*_H(ab)$ instead of $\rho^*_H(\{a, b\})$. An easy counting argument shows that for subsets A and B of $V(H)$, $\rho_H(A) + \rho_H(B) \geq \rho_H(A \cup B) + \rho_H(A \cap B)$. This further implies the following:

$$\rho^*_H(A) + \rho^*_H(B) \geq \rho^*_H(A \cup B) + \rho^*_H(A \cap B). \tag{1.1}$$

For a graph H with $\mathrm{mad}(H) \leq \frac{8}{3}$ and disjoint subsets S, T of $V(H)$, if T contains all vertices not in S that are adjacent to an S-vertex, then since $\rho_H(S \cup T) \geq 0$, we have

$$\rho^*_{H-S}(T) \geq -4 \cdot |S| + 3 \cdot (\text{the number of edges in } H \text{ incident with an } S\text{-vertex}). \tag{1.2}$$

Finally, for graphs H and J with $\mathrm{mad}(H), \mathrm{mad}(J) \leq \frac{8}{3}$, let H' be the graph obtained from H by attaching J to a vertex, that is, identifying a vertex v of H and a vertex w of J. If $\rho^*_H(v) \geq k \geq 0$ and $\rho^*_J(w) \geq 4 - k$, then attaching J decreases the potential by at most k, and so $\mathrm{mad}(H') \leq \frac{8}{3}$. Throughout the paper, we often attach a pendent triangle, J_1, and J_2, which decreases the potential by at most 1, 1, and 2, respectively.

2 Discharging Procedure

Throughout the figures in the paper, the degree of a solid (black) vertex is the number of incident edges drawn in the figure, whereas a hollow (white) vertex means a 2^+-vertex. For a graph H, let $V^*(H)$ be the set of vertices of H except the 2-vertices on a pendent cycle. Suppose to the contrary that a counterexample G exists to Theorem 1.5; namely, $\mathrm{mad}(G) \leq \frac{8}{3}$ but G has no $\mathcal{FII}$-partition. Choose G to be a minimum counterexample with respect to

(1) $|V^*(G)|$ is minimum, (2) $|V(G)|$ is minimum.

We provide a list of subgraphs where each subgraph does not appear in G; each subgraph is also referred to as a *reducible configuration*. We first define the following sets, see Figure 2.

$$
\begin{aligned}
W_2 &= \{x \in V(G) : x \text{ is a 2-vertex not on a pendent triangle}\} \\
W_3 &= \{x \in V(G) : x \text{ is a 3-vertex with two 2-neighbors}\} \\
W_4 &= \{x \in V(G) : x \text{ is a 4-vertex on one pendent triangle}\} \\
W_5 &= \{x \in V(G) : x \text{ is a 5-vertex on two pendent triangles}\} \\
V_k &= \{x \in V(G) : x \text{ is a } k\text{-vertex not in } W_k\} \qquad (k \in \{3,4,5,6\}).
\end{aligned}
$$

For brevity, we use W_{ij}, W_{ijk}, and W_{2345} to denote $W_i \cup W_j$, $W_i \cup W_j \cup W_k$, and $W_2 \cup W_3 \cup W_4 \cup W_5$, respectively.

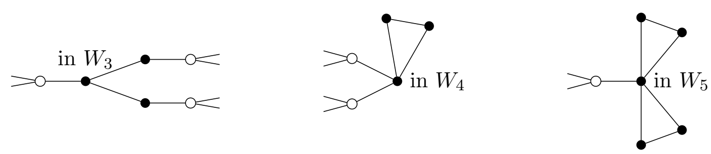


Figure 2: An illustration of a vertex in W_3, W_4, and W_5.

The following is a list of reducible configurations. We will use **[C1]**-**[C10]** to show that every vertex has final charge exactly $\frac{8}{3}$ after applying our discharging rules. The configurations **[C′1]**-**[C′5]** are utilized in the final step to reach a contradiction. We postpone the proofs to Sections 3 and 4.

[C1] (Lemma 3.1) A 1^--vertex.

[C2] (Lemma 3.5) Two adjacent 2-vertices not on a pendent triangle.

[C3] (Lemma 3.2) A 3-vertex with only 2-neighbors.

[C4] (Lemma 3.2) A 3-vertex on a pendent triangle.

[C5] (Lemma 3.4) Two adjacent W_3-vertices.

[C6] (Lemma 3.8) A 3-vertex with a 2-neighbor and a W_3-neighbor.

[C7] (Lemma 3.9) A 3-vertex with two W_3-neighbors.

[C8] (Lemma 3.7) A W_4-vertex with a W_{2345}-neighbor.

[C9] (Lemma 3.6) A W_5-vertex with either a 3-neighbor or a W_{25}-neighbor.

[C10] (Lemma 3.10) A 7-vertex on three pendent triangles with a W_{235}-neighbor.

[C′1] (Lemma 4.1) A cycle consisting of W_{23}-vertices.

[C′2] (Lemma 4.6) A cycle consisting of $(V_3 \cup W_4)$-vertices where every V_3-vertex has a W_{23}-neighbor.

[C′3] (Lemma 4.2) A V_4-vertex with all W_{235}-neighbors that has either two W_2-neighbors or two W_5-neighbors.

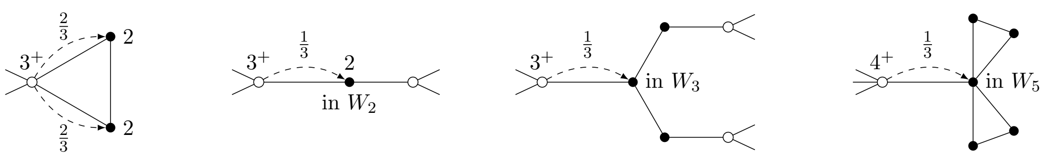
[C′4] (Lemma 4.4) A 5-vertex on one pendent triangle with three W_{235}-neighbors where two are W_2-neighbors.

[C′5] (Lemma 4.3) A 6-vertex on two pendent triangles with a W_{235}-neighbor and a different W_{25}-neighbor.

We will use the discharging method. For each vertex v of G, let the *initial charge* $\mu(v)$ of v be its degree, namely, $\mu(v) = \deg_G(v)$. Note that the average initial charge (over all vertices) is at most $\frac{8}{3}$ since $\mathrm{mad}(G) \le \frac{8}{3}$. Next, we distribute the charge according to the following *discharging rules*, which are designed so that the total charge is preserved, to obtain the *final charge* $\mu^*(v)$ at each vertex v. See Figure 3.

Discharging Rules

R1 A 3^+-vertex sends charge $\frac{2}{3}$ to each of its 2-neighbors on a pendent cycle.
R2 A 3^+-vertex sends charge $\frac{1}{3}$ to each of its W_2-neighbors.
R3 A 3^+-vertex sends charge $\frac{1}{3}$ to each of its W_3-neighbors.
R4 A 4^+-vertex sends charge $\frac{1}{3}$ to each of its W_5-neighbors.



R1 **R2** **R3** **R4**

Figure 3: An illustration of the discharging rules.

First, we will show that the final charge at each vertex is at least $\frac{8}{3}$. Let u be a k-vertex of G, and label the neighbors $u_1, \dots, u_k$ of u in such a way that $\deg_G(u_1) \le \cdots \le \deg_G(u_k)$. By **[C1]**, $k \ge 2$. Note that W_3 is the set of all 3-vertices with exactly two 2-neighbors by **[C3]**, and moreover W_3 is an independent set by **[C5]**. Also, by **[C2]**, all pendent cycles of G are triangles, and a 2-vertex with a 2-neighbor is on a pendent triangle.

(1) Assume $\deg_G(u) = 2$. Note that a 2-vertex does not send any charge by the discharging rules. If u is on a pendent triangle, then it has a 3^+-neighbor, which sends charge $\frac{2}{3}$ to u by **R1**. Thus, the final charge $\mu^*(u)$ of u is at least $2 + \frac{2}{3} = \frac{8}{3}$. If u is not on a pendent triangle, then both neighbors of u are 3^+-vertices by **[C2]**, so $\mu^*(u) = 2 + 2 \cdot \frac{1}{3} = \frac{8}{3}$ by **R2**.

(2) Assume $\deg_G(u) = 3$. By **[C3]**, u has at most two 2-neighbors, so the neighbor u_3 is a 3^+-vertex. Also, u is not on a pendent triangle by **[C4]**, so u sends charge $\frac{1}{3}$ to each W_{23}-neighbor.

(2-1) Suppose u has exactly two 2-neighbors u_1 and u_2, that is, $u \in W_3$. So u sends charge $\frac{1}{3}$ to each of u_1 and u_2 by **R2**. By **[C5]**, $u_3 \notin W_3$ and so u does not send any charge to u_3. However, by **R3** u_3 sends charge $\frac{1}{3}$ to u. Hence $\mu^*(u) = 3 - 2 \cdot \frac{1}{3} + \frac{1}{3} = \frac{8}{3}$.

(2-2) Suppose u has exactly one 2-neighbor u_1. So u sends charge $\frac{1}{3}$ to u_1 by **R2**. By **[C6]**, $u_2, u_3 \notin W_3$, and so u does not send any charge to u_2, u_3. Hence, $\mu^*(u) = 3 - \frac{1}{3} = \frac{8}{3}$.

(2-3) Suppose u has no 2-neighbors. So u sends charge only to W_3-neighbors. By **[C7]**, u has at most one W_3-neighbor. Thus, it sends charge at most $\frac{1}{3}$ by **R3**, and so $\mu^*(u) \ge 3 - \frac{1}{3} = \frac{8}{3}$.

(3) Assume $\deg_G(u) = 4$. If u is on two pendent triangles, then the entire graph is formed by identifying two triangles at one vertex, which has an $\mathcal{FII}$-partition. If $u \in W_4$, then by **[C8]**, u sends charge only to neighbors on the pendent triangle. Thus, u sends charge $\frac{2}{3}$ to each of its 2-neighbors on the pendent triangle by **R1**, so, $\mu^*(u) = 4 - 2 \cdot \frac{2}{3} = \frac{8}{3}$. If u is not on a pendent triangle, then it sends charge at most $\frac{1}{3}$ to each of its neighbors by **R2**-**R4**, so $\mu^*(u) \geq 4 - 4 \cdot \frac{1}{3} = \frac{8}{3}$.

(4) Assume $\deg_G(u) = 5$. If $u \in W_5$, then by **[C9]**, the neighbor u_5 is a 4^+-vertex, which is not in W_5. By **R4**, u_5 sends charge $\frac{1}{3}$ to u. Since $u_5 \notin W_5$, u sends no charge to u_5. By **R1**, u sends charge $\frac{2}{3}$ to each of its 2-neighbors on a pendent triangle, hence, $\mu^*(u) = 5 - 4 \cdot \frac{2}{3} + \frac{1}{3} = \frac{8}{3}$. If u is on exactly one pendent triangle, then u sends charge $\frac{2}{3}$ to each of its 2-neighbors on a pendent triangle by **R1** and sends charge at most $\frac{1}{3}$ to each of the other neighbors by **R2**-**R4**, so $\mu^*(u) \geq 5 - 2 \cdot \frac{2}{3} - 3 \cdot \frac{1}{3} = \frac{8}{3}$. If u not on a pendent triangle, then $\mu^*(u) \geq 5 - 5 \cdot \frac{1}{3} = \frac{10}{3} > \frac{8}{3}$.

(5) Assume $\deg_G(u) \geq 6$. Suppose u is on exactly k pendent triangles. If $\deg_G(u) = 2k$, then the entire graph is formed by identifying k triangles at one vertex, which has an $\mathcal{FII}$-partition. Thus, $\deg_G(u) \geq 2k+1$, and so $\mu^*(u) \geq \deg_G(u) - (2k) \cdot \frac{2}{3} - (\deg_G(u) - 2k) \cdot \frac{1}{3} = \frac{2\deg_G(u) - 2k}{3} \geq \frac{\deg_G(u)+1}{3}$. So $\mu^*(u) \geq 3 > \frac{8}{3}$ when $\deg_G(u) \geq 8$.

(5-1) Suppose $\deg_G(u) = 7$. By **[C10]**, u is on at most two pendent triangles or u has a neighbor who does not receive charge from u, so $\mu^*(u) > \frac{8}{3}$.

(5-2) Suppose $\deg_G(u) = 6$. If u is on at most one pendent triangles, so $\mu^*(u) \geq 6 - 2 \cdot \frac{2}{3} - 4 \cdot \frac{1}{3} = \frac{10}{3} > \frac{8}{3}$. If u is on two pendent triangles, then $\mu^*(u) \geq 6 - 4 \cdot \frac{2}{3} - 2 \cdot \frac{1}{3} = \frac{8}{3}$.

From (1)-(5), we conclude that the final charge of every vertex is at least $\frac{8}{3}$. If there is a vertex whose final charge is more than $\frac{8}{3}$, then the average charge is more than $\frac{8}{3}$, which is a contradiction. Hence, every vertex has final charge exactly $\frac{8}{3}$, which further implies the following **[P1]**-**[P5]**.

[P1] By (5), there is no 7^+-vertex. Moreover, every 6-vertex v is on exactly two pendent triangles and has two W_{235}-neighbors. Together with **[C′5]**, v has two W_3-neighbors.

[P2] By (4), every V_5-vertex v is on exactly one pendent triangle and has three W_{235}-neighbors. Moreover, by **[C′4]**, v has at most one W_2-neighbor.

[P3] By (3), every V_4-vertex v has only W_{235}-neighbors. Moreover, by **[C′3]**, v has at most one W_5-neighbor and at most one W_2-neighbor.

[P4] By (4), **[C8]**, **[P1]**, **[P2]**, and **[P3]**, every W_4-vertex v has two V_3-neighbors.

[P5] By (2), every V_3-vertex v has exactly one W_{23}-neighbor. Moreover, by **[C9]**, **[P1]**, **[P2]**, and **[P3]**,, the other two neighbors of v are in $V_3 \cup W_4$.

If $V_3 \cup W_4 \neq \emptyset$, then **[P4]** and **[P5]** imply that $G[V_3 \cup W_4]$ is 2-regular. Yet, this contradicts **[C′2]**, so $V_3 \cup W_4 = \emptyset$. Hence, $V(G) = T \sqcup W_{235} \sqcup \mathcal{V}_{4^+}$, where $\mathcal{V}_{4^+} = V_4 \cup V_5 \cup V_6$ and T denotes the set of all 2-vertices on pendent triangles of G. Note that by **[P1]**, **[P2]**, and **[P3]**, $\mathcal{V}_{4^+}$ is an independent set.

By **[C3]**, **[C5]**, and **[C′1]**, $G[W_{23}]$ is the union of vertex-disjoint paths. Let Z be the set of isolated vertices of $G[W_{23}]$ and let F_0 be the set of non-isolated vertices of $G[W_{23}]$. Note that by the definition of W_3, $Z \subset W_2$.

In the following, we will reach a contradiction by finding an $\mathcal{FII}$-partition of G. We partition each of $\mathcal{V}_{4^+}$, W_5, and T as in the following (1)-(3). See Figure 4 for an illustration.

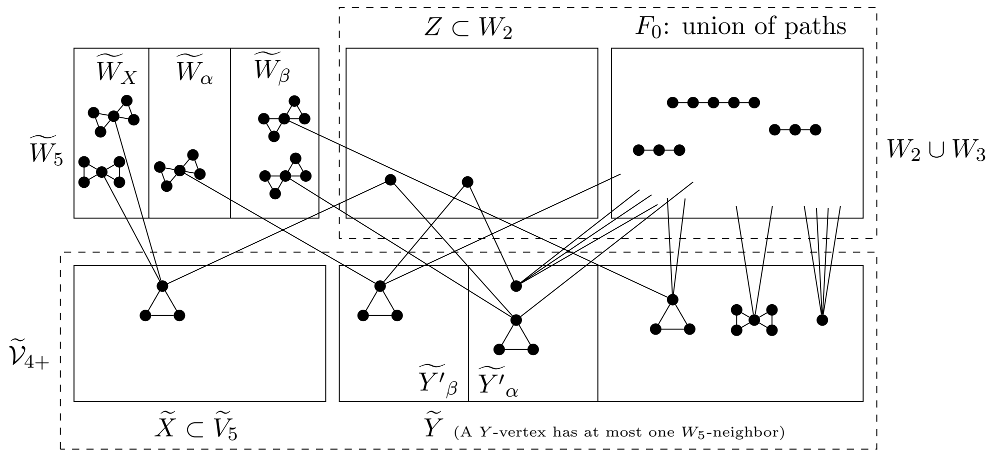


Figure 4: The structure of G. For $A \subset V(G)$, $\widetilde{A}$ means $(N_G(A) \cap T) \cup A$.

(1) Partition $\mathcal{V}_{4+}$ into $X \sqcup Y_\alpha \sqcup Y_\beta$. Let

$$\begin{aligned} X &= \{x \in \mathcal{V}_{4+} \mid x \text{ has two } W_5\text{-neighbors}\}, \\ Y &= \{y \in \mathcal{V}_{4+} \mid y \text{ has at most one } W_5\text{-neighbor}\}. \end{aligned}$$

Note that $X \subset V_5$ by **[P1]**, **[P2]**, and **[P3]**. Let $Y' = \{y \in Y \mid y \text{ has a } Z\text{-neighbor}\}$. **[P1]**, **[P2]**, and **[P3]** also imply that every Y'-vertex has exactly one Z-neighbor, and every $(Y \setminus Y')$-vertex has only $(F_0 \cup W_5)$-neighbors. Since $\mathcal{V}_{4+}$ is an independent set, a Y'-vertex has degree one in $G[Y' \cup Z]$. Since each Z-vertex is a 2-vertex, each component of $G[Y' \cup Z]$ is a path of length at most two. Moreover, each Y'-vertex is an endpoint of some nontrivial component of $G[Y' \cup Z]$, which implies that each component of $G[Y' \cup Z]$ contains at most two Y'-vertices. Partition Y' into two sets Y'_α and Y'_β so that for every component C of $G[Y' \cup Z]$, $|C \cap Y'_\gamma| \le 1$ for every $\gamma \in \{\alpha, \beta\}$. Now let $Y_\alpha = Y'_\alpha \cup (Y \setminus Y')$ and $Y_\beta = Y'_\beta$ so that $X \sqcup Y_\alpha \sqcup Y_\beta$ is a partition of $\mathcal{V}_{4+}$.
(2) Partition W_5 into $W_X \sqcup W_\alpha \sqcup W_\beta$. The following three sets partition W_5, since each W_5-vertex has exactly one neighbor in $X \cup Y$:

$$\begin{aligned} W_X &= \{w \in W_5 \mid w \text{ has an } X\text{-neighbor}\}, \\ W_\alpha &= \{w \in W_5 \mid w \text{ has a } Y_\beta\text{-neighbor}\}, \\ W_\beta &= \{w \in W_5 \mid w \text{ has a } Y_\alpha\text{-neighbor}\}. \end{aligned}$$

(3) Partition T into $T_X \sqcup T_\alpha \sqcup T_\beta$. Choose a 2-vertex from each pendent triangle at a $(W_X \cup X)$-vertex, and partition the chosen 2-vertices into two sets T_α and T_β so that each of T_α and T_β is a 2-independent set. This is possible since a 2-vertex on a pendent triangle at an X-vertex and a 2-vertex on a pendent triangle at a W_X-vertex have distance at least three. Let $T_X = T \setminus (T_\alpha \cup T_\beta)$.

We will now show that $F \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G, where

$$F = W_{23} \cup X \cup W_X \cup T_X, \qquad I_\alpha = Y_\alpha \cup W_\alpha \cup T_\alpha, \qquad I_\beta = Y_\beta \cup W_\beta \cup T_\beta.$$

Consider F. Since W_{23} is the disjoint union of paths and X has at most one W_{23}-neighbor, $W_{23} \cup X$ induces a forest. Moreover, each pendent triangle containing a T_X-vertex also contains a vertex not in F. Hence, F induces a forest.

Suppose that there are two vertices $u, v \in I_\alpha \cup I_\beta$ with distance at most two. By the definition of T_α and T_β, if $u \in T_\alpha$ (resp. T_β), then $v \notin I_\alpha$ (resp. $v \notin I_\beta$). Suppose that $u, v \in Y \cup W_\alpha \cup W_\beta$. Since u and v have distance at most two, at least one of u and v are in Y. Suppose that $u, v \in Y$. Since every vertex in $F_0 \cup W_5$ has at most one $\mathcal{V}_{4^+}$-neighbor, it follows that $u, v \in Y'$ and they have a common Z-neighbor, which implies u, v are in the same component of $G[Y \cup Z]$. By way of construction, either $u \in Y'_\alpha$ and $v \in Y'_\beta$, or $v \in Y'_\alpha$ and $u \in Y'_\beta$. Lastly, suppose $u \in W_\alpha \cup W_\beta$ and $v \in Y$. Since u and v have distance at most two, u and v are adjacent. By the definition of W_α and W_β, u and v are not in the same I_γ for some $\gamma \in \{\alpha, \beta\}$.

Hence, we have shown that $F \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G, which is the final contradiction.

3 Reducible Configurations [C1]-[C10]

In this section, we prove that **[C1]**-**[C10]** cannot exist in G. Recall that the induction is on (1) $|V^*(G)|$, the number of vertices of G except the 2-vertices on pendent cycles, and (2) $|V(G)|$, the number of vertices of G.

In all lemmas and claims, we often end up with an $\mathcal{FII}$-partition of G, which is a contradiction.

Lemma 3.1. *In G, there is no 1^--vertex.* **[C1]**

Proof. If G has a 1^--vertex x, then by the minimality of G, $G - x$ has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$, which implies that $(F + x) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. □

Lemma 3.2. *In G, there is no 3-vertex that either has only 2-neighbors or is on a pendent triangle.* **[C3]**, **[C4]**

Proof. Let v_1, v_2, v_3 be the neighbors of a 3-vertex v. Suppose to the contrary that v has only 2-neighbors. Let for each $i \in [3]$, let z_i be the neighbor of v_i other than v. Note that $z_i = z_j$ for some $i \neq j$ is possible, but it does not affect the following argument. Let $S = \{v, v_1, v_2, v_3\}$ and $H = G - S$. By the minimality of G, H has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. If neither $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$ nor $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G, then $z_i \in I_\alpha$ and $z_j \in I_\beta$ for some $i, j \in [3]$. Now, $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G.

Suppose to the contrary that v is on a pendent triangle vv_1v_2. Let $S = \{v_1, v_2\}$ and $H = G - S$. By the minimality of G, H has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. If $v \in I_\alpha \cup I_\beta$, then $(F + v_1v_2) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. So assume $v \in F$, and now either $(F + v_1) \sqcup (I_\alpha + v_2) \sqcup I_\beta$ or $(F + v_1) \sqcup I_\alpha \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G. □

Lemma 3.3. *In G, the following statements hold:*

(i) *There is no triangle $x_1x_2x_3$ such that $x_3 \in W_2$, x_1, x_2 are 3-vertices, and for each $i \in [2]$, the neighbor of x_i other than x_1, x_2 is either a 3^--vertex or a W_4-vertex.*

(ii) *There is no 4-cycle $x_1x_2x_3x_4$ such that $x_2 \in W_2$, $x_1, x_3 \in W_3$, and x_4 is a 3^--vertex.*

Proof. Suppose to the contrary that such a cycle exists. We use the labels as in Figure 5.

(i) Let $S = \{x_1, x_2, x_3\}$ and $H = G - S$. Since $\rho^*_H(z_1) + \rho^*_H(z_2) \geq \rho^*_H(z_1z_2) \geq= -4 \cdot 3 + 3 \cdot 5 = 3$ by (1.1) and (1.2), without loss of generality assume $\rho^*_H(z_1) \geq 2$. Then $\text{mad}(H') \leq \frac{8}{3}$, where H' is the graph obtained from H by attaching J_2 to z_1. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$. Let $F = F' \cap V(H)$ $I_\alpha = I'_\alpha \cap V(H)$, and $I_\beta = I'_\beta \cap V(H)$. By Lemma 1.6

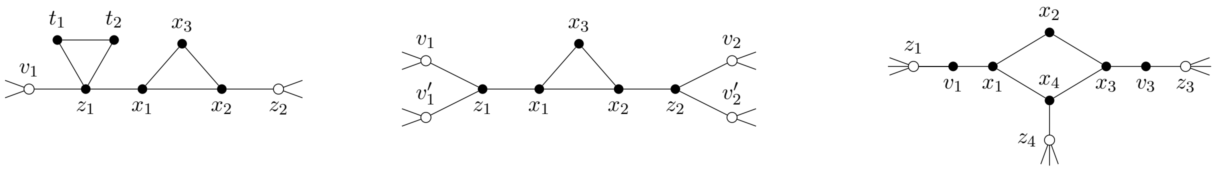


Figure 5: An illustration for Lemma 3.3

(ii), $z_1 \in F$. If $z_2 \notin F$, then either $(F + x_1x_2) \sqcup I_\alpha \sqcup (I_\beta + x_3)$ or $(F + x_1x_2) \sqcup (I_\alpha + x_3) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. Thus, $\{z_1, z_2\} \subset F$. If $z_1 \in W_4$, then we may assume that $t_2 \in I_\alpha$, and then either $(F + x_2x_3) \sqcup (I_\alpha + x_1) \sqcup I_\beta$ or $(F + x_1x_2) \sqcup (I_\alpha + x_3) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If z_1 is a 3-vertex, then either $(F + x_1x_2) \sqcup (I_\alpha + x_3) \sqcup I_\beta$, $(F + x_2x_3) \sqcup I_\alpha \sqcup (I_\beta + x_1)$, or $(F + x_2x_3) \sqcup (I_\alpha + x_1) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G.

(ii) Let $S = \{x_1, x_2, x_3, x_4, v_1, v_3\}$. By the minimality of G, there is an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of $G - S$. If $z_1 \in F$, then $(F + S - x_1x_3) \sqcup (I_\alpha + x_1) \sqcup (I_\beta + x_3)$, $(F + S - x_1) \sqcup (I_\alpha + x_1) \sqcup I_\beta$, or $(F + S - x_1) \sqcup I_\alpha \sqcup (I_\beta + x_1)$ is an $\mathcal{FII}$-partition of G. Thus, $z_1, z_3 \notin F$, and now $(F + S - x_2) \sqcup (I_\alpha + x_2) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. □

Lemma 3.4. *In G, there are no two adjacent W_3-vertices.* **[C5]**

Proof. Suppose to the contrary that $x, y \in W_3$ are adjacent. We use the labels as in the left figure of Figure 6. Note that x_1, x_2, y_1, y_2 are distinct by Lemma 3.3 (i). It might happen that $z_i = z_j$ for some $i \neq j$, nonetheless the following arguments are still valid. Let $H = G - S$ where $S = \{x, y, x_1, x_2, y_3, y_4\}$. By the minimality of G, H has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. If $z_1, z_2, z_3, z_4 \notin F$, then $(F' + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. Suppose that at least one z_i is in F. Without loss of generality, we may assume $z_1, z_2 \notin I_\alpha$. Since $(F + S - x) \sqcup (I_\alpha + x) \sqcup I_\beta$ is not an $\mathcal{FII}$-partition of G, $z_3, z_4 \in F$. Then $(F + S - xy) \sqcup (I_\alpha + x) \sqcup (I_\beta + y)$ is an $\mathcal{FII}$-partition of G. □

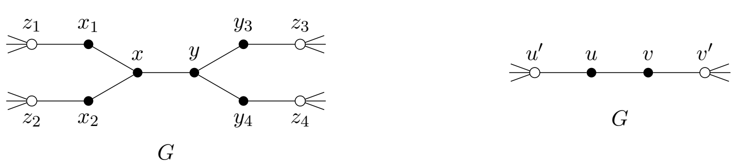


Figure 6: An illustration for Lemmas 3.4 and 3.5.

Lemma 3.5. *In G, there are no two adjacent 2-vertices not on a pendent triangle.* **[C2]**

Proof. Let u and v be adjacent 2-vertices not on a pendent triangle, and use the labels as in the right figure of Figure 6. Let $H = G - \{u, v\}$. Since $\rho^*_H(u') + \rho^*_H(v') \geq 1$ by (1.1) and (1.2), without loss of generality, we may assume $\rho^*_H(u') \geq 1$. Now consider the graph H' obtained by attaching a pendent triangle $u'xy$ to u'. Then $\mathrm{mad}(H') \leq \frac{8}{3}$. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$. Let $F = F' \cap V(H)$, $I_\alpha = I'_\alpha \cap V(H)$, and $I_\beta = I'_\beta \cap V(H)$. If either $u' \notin F$ or $v' \notin F$, then $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If $u', v' \in F$, then without loss of generality we may assume $x \in I_\alpha$. Now, $(F + v) \sqcup (I_\alpha + u) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. □

Lemma 3.6. *In G, there is no W_5-vertex with either a 3-neighbor or a W_{25}-neighbor.* **[C9]**

Proof. For $v \in W_5$, let vt_1t_2 and vt_3t_4 be the two pendent triangles at v, and let v_1 be the neighbor of v that is not on a pendent triangle. Suppose to the contrary that v_1 is either a 3-vertex or a W_{25}-vertex. If $v_1 \in W_5$, then the entire graph G is a subgraph of the graph J_2 in Figure 1. Yet, J_2 has an $\mathcal{FII}$-partition, and therefore G has an $\mathcal{FII}$-partition.

Assume v_1 is a 3^--vertex. Let $H = G - \{t_1, t_2, t_3, t_4, v\}$. By the minimality of G, H has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. If $v_1 \in F$, then $(F + vt_1t_3) \sqcup (I_\alpha + t_2) \sqcup (I_\beta + t_4)$ is an $\mathcal{FII}$-partition of G. If $v_1 \notin F$ and we cannot move v_1 to F, then the neighbors of v_1 in H are in F. Without of generality assume $v_1 \in I_\alpha$. Now $(F + t_1t_2t_3t_4) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G. $\square$

Lemma 3.7. *In G, there is no vertex $v \in W_4$ with a W_{2345}-neighbor.* **[C8]**

Proof. Suppose to the contrary there is a vertex $v \in W_4$ with a W_{2345}-neighbor. We use the labels as in Figure 7.

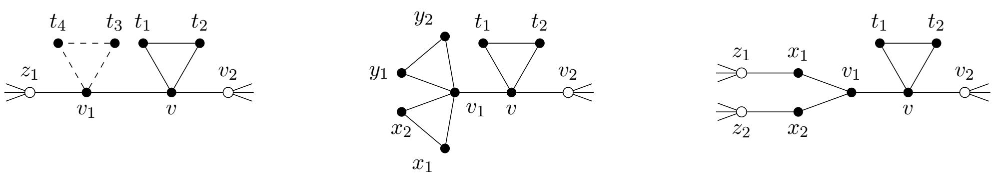


Figure 7: An illustration for Lemma 3.7.

Assume $v_1 \in W_{25}$. Let $H = G - \{t_1, t_2\}$. By the minimality of G, H has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. Since $(F + t_1t_2) \sqcup I_\alpha \sqcup I_\beta$ is not an $\mathcal{FII}$-partition of G, we have $v \in F$. Also, since neither $(F + t_2) \sqcup (I_\alpha + t_1) \sqcup I_\beta$ nor $(F + t_2) \sqcup I_\alpha \sqcup (I_\beta + t_1)$ is an $\mathcal{FII}$-partition of G, without loss of generality, we may assume $v_1 \in I_\alpha$ and $v_2 \in I_\beta$. If $v_1 \in W_2$, then $(F + v_1t_2) \sqcup (I_\alpha - v_1 + t_1) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If $v_1 \in W_4$, then we may assume $t_3, t_4 \in F$, and so $(F + v_1t_2 - t_4) \sqcup (I_\alpha - v_1 + t_1t_4) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If $v_1 \in W_5$, then we may assume $\{x_1, x_2, y_1, y_2\} \subset F$ and so G has an $\mathcal{FII}$-partition $(F + v_1t_2 - x_2y_2) \sqcup (I'_\alpha + t_1x_2 - v_1) \sqcup (I'_\beta + y_2)$.

Now suppose $v_1 \in W_3$. Let $S = \{v, v_1, t_1, t_2, x_1, x_2\}$ and let $H = G - S$.

Claim 3.7.1. *For every $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H, $z_1, z_2 \in F$ and $v_2 \notin F$.*

Proof. If $v_2 \in F$, then $(F + S - t_1) \sqcup (I_\alpha + t_1) \sqcup I_\beta$, $(F + S - t_1v_1) \sqcup (I_\alpha + v_1) \sqcup (I_\beta + t_1)$, or $(F + S - t_1v_1) \sqcup (I_\alpha + t_1) \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G. Thus $v_2 \notin F$, and we may assume $v_2 \in I_\beta$. If $z_i \notin F$ for some i, then $(F + S - t_1) \sqcup (I_\alpha + t_1) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. $\square$

Suppose that $\rho^*_H(v_2) \geq 2$. Let H' be the graph obtained from H by identifying v_2 and v^* in the graph J_2 in Figure 1. Then $\text{mad}(H') \leq \frac{8}{3}$. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. Now, $v_2 \notin F$ by Claim 3.7.1, but this contradicts Lemma 1.6 (ii). Hence, $\rho^*_H(v_2) \leq 1$. By (1.1) and (1.2),

$$\rho^*_H(z_1) + \rho^*_H(z_2) + 1 \geq \rho^*_H(z_1) + \rho^*_H(z_2) + \rho^*_H(v_2) \geq \rho^*_H(z_1z_2v_2) \geq -4 \cdot 6 + 3 \cdot 9 = 3.$$

Thus, we may assume that $\rho^*_H(z_1) \geq 1$. Let H' be the graph obtained from H by attaching a pendent triangle to z_1. Then $\text{mad}(H') \leq \frac{8}{3}$. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. Thus, by Claim 3.7.1, we may assume $v_2 \in I_\beta$ and $z_1, z_2 \in F$. By considering the pendent triangle of H' at z_1, we know either $(F + S - t_1x_1) \sqcup (I_\alpha + t_1x_1) \sqcup I_\beta$ or $(F + S - t_1x_1) \sqcup (I_\alpha + t_1) \sqcup (I_\beta + x_1)$ is an $\mathcal{FII}$-partition of G. $\square$

Lemma 3.8. *In G, there is no 3-vertex with a 2-neighbor and a W_3-neighbor.* **[C6]**

Proof. Suppose to the contrary that v is a 3-vertex with a 2-neighbor and a W_3-neighbor. We use the labels as in Figure 8. Note that it is easy to check x_1, x_2 are distinct from v_2. Let $H = G - S$, where $S = \{v, v_1, v_2, x_1, x_2\}$.

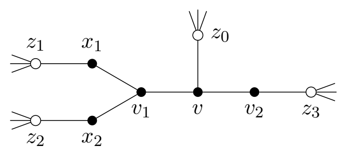


Figure 8: An illustration for Lemma 3.8.

Claim 3.8.1. *For every $\mathcal{FII}$-partition of $F \sqcup I_\alpha \sqcup I_\beta$ of H, we know $z_3 \in F$. Moreover, if $z_0 \in I_\alpha$ (resp. I_β), then one of z_1 and z_2 is in F and the other is in I_β (resp. I_α).*

Proof. If $z_3 \notin F$, then $(F + S) \sqcup I_\alpha \sqcup I_\beta$, $(F + S - v_1) \sqcup (I_\alpha + v_1) \sqcup I_\beta$, or $(F + S - v_1) \sqcup I_\alpha \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G. Hence, $z_3 \in F$. Now, assume $z_0 \in I_\alpha$. Since neither $(F + S - v_1) \sqcup I_\alpha \sqcup (I_\beta + v_1)$ nor $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G, one of z_1 and z_2 is in F and the other is in I_β. $\square$

Claim 3.8.2. *The following statements hold:*

(i) *Let H' be the graph obtained from H by attaching a pendent triangle T to z_0. If H' has an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$, then $z_0 \notin F'$.*

(ii) *Let H' be the graph obtained from H by adding an edge $z_i z_j$ for some $i, j \in \{1, 2, 3\}$. If H' has an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$, then $z_0 \in F'$.*

Proof. For an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$ of H', let $F = F' \cap V(H)$, $I_\alpha = I'_\alpha \cap V(H)$, and $I_\beta = I'_\beta \cap V(H)$. Since $F \sqcup I_\alpha \sqcup I_\beta$ is also an $\mathcal{FII}$-partition of H, Claim 3.8.1 implies $z_3 \in F$.

(i) Suppose to the contrary that $z_0 \in F$. Without loss of generality, we may assume a 2-vertex on T belongs to I'_α. Thus, either $(F' + S - v) \sqcup (I'_\alpha + v) \sqcup I'_\beta$ or $(F + S - vv_1) \sqcup (I_\alpha + v) \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G.

(ii) Suppose to the contrary that $z_0 \notin F$, say $z_0 \in I_\alpha$. By Claim 3.8.1, without loss of generality, assume $z_1 \in F$ and $z_2 \in I_\beta$. If $z_i z_j = z_1 z_2$, then $(F + S - x_1) \sqcup I_\alpha \sqcup (I_\beta + x_1)$ is an $\mathcal{FII}$-partition of G. If $z_i z_j = z_1 z_3$, then $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If $z_i z_j = z_2 z_3$, then $(F + S - v_2) \sqcup I_\alpha \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G. $\square$

Let $Z = \{z_0, z_1, z_2, z_3\}$. By (1.2), $\rho^*_H(Z) \geq -4 \cdot 5 + 3 \cdot 8 = 4$.

Claim 3.8.3. *The following statements hold:*

(a) $\rho^*_H(z_i) = 1$ *for* $i \in \{0, 3\}$.

(b) $\rho^*_H(z_i z_j) \geq 2$ *for* $i, j \in \{0, 1, 2, 3\}$ *with* $i \neq j$.

Proof. Instead of proving (a) and (b) separately, we show the following (1)-(4):

(1) $\rho^*_H(z_3) \leq 1$. (2) $\rho^*_H(z_0) \leq 1$, (3) $\rho^*_H(z_i z_1 z_2) \leq 3$ for $i \in \{0, 3\}$, (4) $\rho^*_H(z_i z_0 z_3) \leq 3$ for $i \in \{1, 2\}$.

We argue it is sufficient to show (1)-(4). From (3) and (1.1), $\rho^*_H(z_3) + 3 \geq \rho^*_H(z_3) + \rho^*_H(z_0z_1z_2) \geq \rho^*_H(Z) \geq 4$. Thus $\rho^*_H(z_3) \geq 1$, which implies $\rho^*_H(z_3) = 1$ by (1). Similarly, (2) and (3) imply $\rho^*_H(z_0) = 1$. Hence, (1), (2), and (3) imply (a). We now show how (3) and (4) imply (b). Note that (3) and (4) are equivalent to $\rho^*_H(z_iz_jz_k) \leq 3$ for three distinct $i, j, k \in \{0, 1, 2, 3\}$. For $i, j \in \{0, 1, 2, 3\}$, by (3) and (4), since (1.1) implies

$$\rho^*_H(z_iz_j)+6 \geq \rho^*_H(z_iz_j)+\rho^*_H(Z\setminus\{z_i\})+\rho^*_H(Z\setminus\{z_j\}) \geq \rho^*_H(z_iz_j)+\rho^*_H(Z)+\rho^*_H(Z\setminus\{z_i, z_j\}) \geq 2\rho^*_H(Z) = 8,$$

we know $\rho^*_H(z_iz_j) \geq 2$. Hence, it is sufficient to show (1)-(4).

In each case, we will define a graph H' from H so that $\mathrm{mad}(H') \leq \frac{8}{3}$ and $|V^*(H')| < |V^*(G)|$. By the minimality of G, H' has an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$. Let $F = F' \cap V(H)$, $I_\alpha = I'_\alpha \cap V(H)$, and $I_\beta = I'_\beta \cap V(H)$. Claim 3.8.1 implies $z_3 \in F$.

(1) Suppose to the contrary that $\rho^*_H(z_3) \geq 2$. Let H' be the graph obtained from H by attaching two pendent triangles to z_3. Since $\rho^*_H(z_3) \geq 2$, $\mathrm{mad}(G') \leq \frac{8}{3}$. Note that, since $z_3 \in F$, z_3 has only F-neighbors in H. Assume $z_0 \in F$. If $z_1, z_2 \notin I_\alpha$, then $(F + S - v_1v_2) \sqcup (I_\alpha + v_1) \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G. Otherwise, either $(F + S - v_2) \sqcup I_\alpha \sqcup (I_\beta + v_2)$ or $(F + S - v_1v_2) \sqcup (I_\alpha + v_2) \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G. Without loss of generality, assume $z_0 \in I_\alpha$. By Claim 3.8.1, $(F + S - v_2) \sqcup I_\alpha \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G.

(2) Suppose to the contrary that $\rho^*_H(z_0) \geq 2$. Let H' be the graph obtained from H by attaching J_1 and a pendent triangle to z_0. Note that $\mathrm{mad}(H') \leq \frac{8}{3}$ since $\rho^*_H(z_0) \geq 2$. By Claim 3.8.2 (i), $z_0 \notin F$. Without loss of generality, assume $z_0 \in I_\alpha$. By Claim 3.8.1, $z_1 \in F$ and $z_2 \in I_\beta$. Also, z_0 has a I'_β-neighbor by Lemma 1.6 (i). Thus, $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G.

(3) Suppose to the contrary that $\rho^*_H(z_iz_1z_2) \geq 4$ for some $i \in \{0, 3\}$. By (1), (2), and (1.1), $1 + \rho^*_H(z_1z_2) \geq \rho^*_H(z_i) + \rho^*_H(z_1z_2) \geq 4$, so $\rho^*_H(z_1z_2) \geq 3$. Therefore,

$$\rho^*_H(z_i) = 1, \rho^*_H(z_1z_2) \geq 3, \rho^*_H(z_iz_1z_2) \geq 4. \tag{3.1}$$

Let H' be the graph obtained from H by attaching one pendent triangle T to z_i and adding an edge z_1z_2. Note that by (3.1), $\mathrm{mad}(H') \leq \frac{8}{3}$. By Claim 3.8.2 (i) and (ii), it must be that $i = 3$.

If $z_0 \in F$, then we may assume a 2-vertex of T belongs to I'_α. Furthermore, if $(F + S - v_1v_2) \sqcup (I_\alpha + v_2) \sqcup (I_\beta + v_1)$ is not an $\mathcal{FII}$-partition of G, then either $(F + S - v_2x_1) \sqcup (I_\alpha + v_2) \sqcup (I_\beta + x_1)$ or $(F + S - v_2x_2) \sqcup (I_\alpha + v_2) \sqcup (I_\beta + x_2)$ is an $\mathcal{FII}$-partition of G. Now, without loss of generality, assume $z_0 \in I_\alpha$. By Claim 3.8.2 (ii), we may assume $z_1 \in F$ and $z_2 \in I_\beta$. Now, $(F + S - x_1) \sqcup I_\alpha \sqcup (I_\beta + x_1)$ is an $\mathcal{FII}$-partition of G.

(4) Without loss of generality, suppose to the contrary that $\rho^*_H(z_0z_1z_3) \geq 4$. Since (1.1) implies $\rho^*_H(z_0) + \rho^*_H(z_1z_3) \geq \rho^*_H(z_0z_1z_3) \geq 4$, together with (a) (which is true since (1), (2), and (3) are proved), we know $\rho^*_H(z_1z_3) \geq 3$. Therefore,

$$\rho^*_H(z_0) = 1, \rho^*_H(z_1z_3) \geq 3, \rho^*_H(z_0z_1z_3) \geq 4. \tag{3.2}$$

Note that by (3.2), $\mathrm{mad}(H') \leq \frac{8}{3}$, where H' is the graph obtained from H by attaching one pendent triangle to z_0 and adding an edge z_1z_3. By the minimality of G, H' has an $\mathcal{FII}$-partition, which is a contradiction by Claim 3.8.2 (i) and (ii). □

By Claim 3.8.3 (a) and (1.1), we have $\rho^*_H(z_0z_1z_2) \geq \rho^*_H(Z) - \rho^*_H(z_3) \geq 3$. In addition, for $i \in \{1, 2\}$, since $\rho^*_H(z_i) + 1 = \rho^*_H(z_i) + \rho^*_H(z_0) \geq \rho^*_H(z_0z_i) \geq 2$, we have $\rho^*_H(z_i) \geq 1$. Therefore, by Claim 3.8.3

$$\rho^*_H(z_0) = 1, \rho^*_H(z_1) \geq 1, \rho^*_H(z_2) \geq 1, \rho^*_H(z_0z_1) \geq 2, \rho^*_H(z_0z_2) \geq 2, \rho^*_H(z_1z_2) \geq 2, \rho^*_H(z_0z_1z_2) \geq 3.$$

Then $\mathrm{mad}(H') \le \frac{8}{3}$, where H' is the graph obtained from H by attaching a pendent triangle to each of z_0, z_1, z_2. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. By Claim 3.8.2 (i), $z_0 \notin F$. Together with Claim 3.8.1, we may assume $z_0 \in I_\alpha$, $z_1 \in I_\beta$, and $z_2 \in F$. By considering the pendent triangle at z_2, either $(F + S - x_2) \sqcup (I_\alpha + x_2) \sqcup I_\beta$ or $(F + S - x_2) \sqcup I_\alpha \sqcup (I_\beta + x_2)$ is an $\mathcal{FII}$-partition of G. □

Lemma 3.9. *In G, there is no 3-vertex with two W_3-neighbors.* **[C7]**

Proof. Suppose to the contrary that there is a 3-vertex v with two W_3-neighbors. We use the labels as in Figure 9. By Lemma 3.3 (ii), all x_i's are distinct. Let $H = G - S$ where $S = \{v, v_1, v_2, x_1, x_2, x_3, x_4\}$.

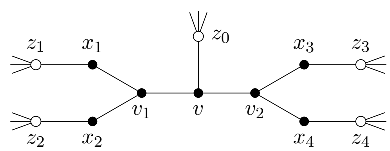


Figure 9: An illustration for Lemma 3.9.

Claim 3.9.1. *For every $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H, the following statements hold:*

(i) *if $z_0 \in F$, then exactly one of z_1, z_2 is in F, exactly one of z_3, z_4 is in F, and $\{z_1, z_2, z_3, z_4\} \cap I_\gamma = \emptyset$ for some $\gamma \in \{\alpha, \beta\}$.*

(ii) *if $z_0 \in I_\gamma$ for some $\gamma \in \{\alpha, \beta\}$ and $\{z_1, z_2, z_3, z_4\} \not\subset F$, then exactly one of z_1, z_2 is in F, exactly one of z_3, z_4 is in F, and $\{z_1, z_2, z_3, z_4\} \cap I_\gamma = \emptyset$.*

Proof. (i) Assume $z_0 \in F$. If $z_1, z_2 \notin F$, then $(F + S - v_2) \sqcup (I_\alpha + v_2) \sqcup I_\beta$, $(F + S - v_2) \sqcup I_\alpha \sqcup (I_\beta + v_2)$, or $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If $z_1, z_2 \in F$, then $(F + S - v_1) \sqcup (I_\alpha + v_1) \sqcup I_\beta$, $(F + S - v_1 v_2) \sqcup (I_\alpha + v_1) \sqcup (I_\beta + v_2)$, or $(F + S - v_1 v_2) \sqcup (I_\alpha + v_2) \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G. Thus exactly one of z_1 and z_2 is in F. By symmetry, exactly one of z_3 and z_4 is in F. Now, if $\{z_1, z_2, z_3, z_4\} \cap I_\gamma \neq \emptyset$ for each $\gamma \in \{\alpha, \beta\}$, then either $(F + S - v_1 v_2) \sqcup (I_\alpha + v_1) \sqcup (I_\beta + v_2)$ or $(F + S - v_1 v_2) \sqcup (I_\alpha + v_2) \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G.

(ii) Without loss of generality, assume $z_0 \in I_\alpha$. If $z_1, z_2 \in F$, then $|\{z_3, z_4\} \cap F| \le 1$, so $(F + S - v_1) \sqcup I_\alpha \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G. If $z_1, z_2 \notin F$, then either $(F + S) \sqcup I_\alpha \sqcup I_\beta$ or $(F + S - v_2) \sqcup I_\alpha \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G. Thus, exactly one of z_1, z_2 is in F. By symmetry, exactly one of z_3 and z_4 is in F. If $\{z_1, z_2, z_3, z_4\} \cap I_\alpha \neq \emptyset$, then either $(F + S - v_1) \sqcup I_\alpha \sqcup (I_\beta + v_1)$ or $(F + S - v_2) \sqcup I_\alpha \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G. □

In the proof of each of the following cases, we will define a graph H' by modifying H so that $\mathrm{mad}(H') \le \frac{8}{3}$ and $|V^*(H')| < |V^*(G)|$. By the minimality of G, H' has an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$. Let $F = F' \cap V(H)$, $I_\alpha = I'_\alpha \cap V(H)$, and $I_\beta = I'_\beta \cap V(H)$. Then we can apply Claim 3.9.1. Note that by (1.2), we know $\rho^*_H(Z) \ge -4 \cdot 7 + 3 \cdot 11 = 5$, where $Z = \{z_0, z_1, z_2, z_3, z_4\}$.

Claim 3.9.2. $\rho^*_H(z_1 z_2) = \rho^*_H(z_3 z_4) = 2$ *and* $\rho^*_H(z_0) = 1$.

Proof. By (1.1), since $\rho^*_H(z_1 z_2) + \rho^*_H(z_3 z_4) + \rho^*_H(z_0) \ge \rho^*_H(Z) \ge 5$, it is sufficient to show $\rho^*_H(z_1 z_2) \le 2$, $\rho^*_H(z_3 z_4) \le 2$, and $\rho^*_H(z_0) \le 1$.

Suppose to contrary that $\rho^*_H(z_1 z_2) \ge 3$. Let H' be the graph obtained from H by adding an edge $z_1 z_2$. Note that since $\rho^*_H(z_1 z_2) \ge 3$, $\mathrm{mad}(H') \le \frac{8}{3}$. If $z_0 \in F$, then by Claim 3.9.1 (i), we may assume that $z_1, z_3 \in F$ and $z_2, z_4 \in I_\alpha$, and therefore $(F + S - x_1v_2) \sqcup (I_\alpha + x_1) \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G. Suppose that $z_0 \notin F$, say $z_0 \in I_\alpha$. If $\{z_1, z_2, z_3, z_3\} \not\subset F$, then by Claim 3.9.1 (ii), we may assume that $z_1, z_3 \in F$, $z_2, z_4 \in I_\beta$, and therefore $(F + S - x_1) \sqcup I_\alpha \sqcup (I_\beta + x_1)$ is an $\mathcal{FII}$-partition of G. If $\{z_1, z_2, z_3, z_3\} \subset F$, then $(F + S - v_2) \sqcup I_\alpha \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G. Therefore, $\rho^*_H(z_1z_2) \leq 2$, and by symmetry, $\rho^*_H(z_3z_4) \leq 2$.

In the following, we will show $\rho^*_H(z_0) \leq 1$ in two steps. First we show $\rho^*_H(z_0) \leq 2$, and then show $\rho^*_H(z_0) \neq 2$. Suppose to the contrary that $\rho^*_H(z_0) \geq 3$. Let H' be the graph obtained from H by attaching J_2 and one pendent triangle to z_0. See the left figure of Figure 10. Note that $\mathrm{mad}(H') \leq \frac{8}{3}$ since $\rho^*_H(z_0) \geq 3$. By Lemma 1.6 (i), we know $z_0 \in F$. So Claim 3.9.1 (i) applies, and moreover, by considering the pendent triangle at z_0, we know that either $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ or $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. Hence, $\rho^*_H(z_0) \leq 2$.

Now suppose that $\rho^*_H(z_0) = 2$. By (1.1),

$$\begin{aligned}
\rho^*_H(z_0z_1) + \rho^*_H(z_0z_2) + \rho^*_H(z_0z_3) + \rho^*_H(z_0z_4) &\geq \rho^*_H(z_0z_1z_2) + \rho^*_H(z_0) + \rho^*_H(z_0z_3z_4) + \rho^*_H(z_0) \\
&\geq \rho^*_H(Z) + 3\rho^*_H(z_0) \geq 5 + 3 \cdot 2 = 11,
\end{aligned}$$

so without loss of generality, we may assume $\rho^*_H(z_0z_1) \geq 3$. Since $\rho^*_H(z_0) + \rho^*_H(z_1) \geq \rho^*_H(z_0z_1)$ by (1.1) and we already have $\rho^*_H(z_0) \leq 2$, it follows that $\rho^*_H(z_1) \geq 1$. Hence,

$$\rho^*_H(z_0) = 2, \rho^*_H(z_1) \geq 1, \rho^*_H(z_0z_1) \geq 3. \tag{3.3}$$

Let H' be the graph obtained from H by attaching J_1 and one pendent triangle T_0 to z_0, and attaching one pendent triangle T_1 to z_1. See the middle figure of Figure 10. Note that by (3.3), $\mathrm{mad}(H') \leq \frac{8}{3}$. If $z_0 \in F$, then by considering T_0, Claim 3.9.1 (i) implies that either $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$ or $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G. Suppose that $z_0 \notin F$, say $z_0 \in I_\alpha$. By Lemma 1.6 (i), the neighbor of z_0 in J_1 is in I'_β. If $\{z_1, z_2, z_3, z_4\} \not\subset F$, then by Claim 3.9.1 (ii), $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G. If $\{z_1, z_2, z_3, z_4\} \subset F$, then by considering T_1, either $(F + S - x_1v_2) \sqcup (I_\alpha + x_1) \sqcup (I_\beta + v_2)$ or $(F + S - x_1v_2) \sqcup I_\alpha \sqcup (I_\beta + x_1v_2)$ is an $\mathcal{FII}$-partition of G. Thus, $\rho^*_H(z_0) \neq 2$ and so $\rho^*_H(z_0) \leq 1$. □

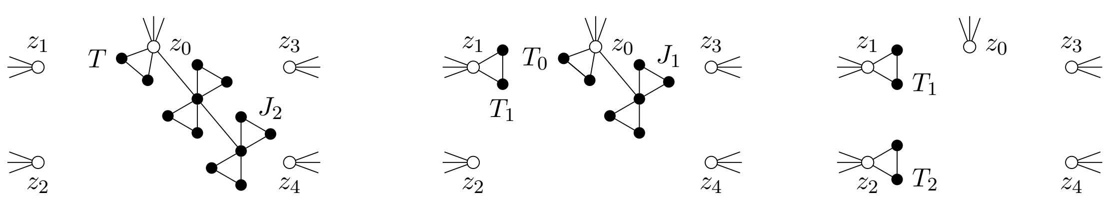


Figure 10: An illustration for Lemma 3.9.

Claim 3.9.3. *Either $\rho^*_H(z_1) = 0$ or $\rho^*_H(z_2) = 0$, and also either $\rho^*_H(z_3) = 0$ or $\rho^*_H(z_4) = 0$.*

Proof. Suppose to the contrary that $\rho^*_H(z_1) \geq 1$ and $\rho^*_H(z_2) \geq 1$. Note that $\rho^*_H(z_1z_2) = 2$ by Claim 3.9.2. Let H' be the graph obtained from H by attaching one pendent triangle to each of z_1 and z_2. See the right figure of Figure 10. Note that since $\rho^*_H(z_1), \rho^*_H(z_2) \geq 1$, and $\rho^*_H(z_1z_2) = 2$, $\mathrm{mad}(H') \leq \frac{8}{3}$. If $z_0 \in F$, then by Claim 3.9.1 (i) we may assume that $z_1, z_3 \in F$ and $z_2, z_4 \in I_\alpha$, and therefore either $(F + S - x_1v_2) \sqcup (I_\alpha + x_1) \sqcup (I_\beta + v_2)$ or $(F + S - x_1v_2) \sqcup I_\alpha \sqcup (I_\beta + x_1v_2)$ is an $\mathcal{FII}$-partition of G. Suppose that $z_0 \notin F$, say $z_0 \in I_\alpha$. If $\{z_1, z_2, z_3, z_3\} \not\subset F$, then by Claim 3.9.1 (ii), we may assume $z_1 \in F$, and therefore either $(F+S-x_1) \sqcup (I_\alpha + x_1) \sqcup I_\beta$ or $(F+S-x_1) \sqcup I_\alpha \sqcup (I_\beta + x_1)$ is an $\mathcal{FII}$-partition of G. If $\{z_1, z_2, z_3, z_3\} \subset F$, then either $(F+S-x_1v_2) \sqcup (I_\alpha + x_1) \sqcup (I_\beta + v_2)$ or $(F+S-x_1v_2) \sqcup I_\alpha \sqcup (I_\beta + x_1v_2)$ is an $\mathcal{FII}$-partition of G. Hence, either $\rho^*_H(z_1) = 0$ or $\rho^*_H(z_2) = 0$, and by symmetry, either $\rho^*_H(z_3) = 0$ or $\rho^*_H(z_4) = 0$. $\square$

By Claim 3.9.3, we may assume that $\rho^*_H(z_1) = \rho^*_H(z_3) = 0$. Let H' be the graph obtained from H by adding a path of length two between z_2 and z_4, and for each of z_2 and z_4, attach one pendent triangle T_2 and T_4, respectively. Note that $\text{mad}(H') \le \frac{8}{3}$, since adding a path of length two decreases the potential by 2, and the following inequalities, which follow from Claim 3.9.2 and (1.1):

$$\begin{aligned}
&\rho^*_H(z_2) = \rho^*_H(z_1) + \rho^*_H(z_2) \ge \rho^*_H(z_1z_2) = 2\\
&\rho^*_H(z_4) = \rho^*_H(z_3) + \rho^*_H(z_4) \ge \rho^*_H(z_3z_4) = 2\\
&\rho^*_H(z_2z_4) + 1 = \rho^*_H(z_2z_4) + \rho^*_H(z_1) + \rho^*_H(z_3) + \rho^*_H(z_0) \ge \rho^*_H(Z) \ge 5.
\end{aligned}$$

If $z_2, z_4 \notin F$, then we may assume $z_2 \in I_\alpha$ and $z_4 \in I_\beta$ since z_2 and z_4 have distance two in H'. Therefore, $(F+S-v_1v_2) \sqcup (I_\alpha + v_2) \sqcup (I_\beta + v_1)$, $(F+S-v_1) \sqcup I_\alpha \sqcup (I_\beta + v_1)$, or $(F+S-v_2) \sqcup (I_\alpha + v_2) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. Suppose that $z_2 \in F$. Without loss of generality, we may assume a 2-vertex on T_2 belongs to I'_α, where $F' \sqcup I'_\alpha \sqcup I'_\beta$ is an $\mathcal{FII}$-partition of H'. If $z_0 \in F$, then by Claim 3.9.1 (i), either $(F+S-x_2v_2) \sqcup (I_\alpha + x_2) \sqcup (I_\beta + v_2)$ or $(F+S-x_2v_2) \sqcup (I_\alpha + x_2v_2) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. Suppose that $z_0 \notin F$. If $\{z_1, z_2, z_3, z_4\} \not\subset F$, then by Claim 3.9.1 (ii), $(F+S-x_2) \sqcup (I_\alpha + x_2) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If $\{z_1, z_2, z_3, z_4\} \subset F$, then $(F+S-x_2v_2) \sqcup (I_\alpha + x_2) \sqcup (I_\beta + v_2)$ is an $\mathcal{FII}$-partition of G. $\square$

Lemma 3.10. *In G, there is no 7-vertex on three pendent triangles with a W_{235}-neighbor.* **[C10]**

Proof. Let v be a 7-vertex on three pendent triangles where v_1 is the neighbor of v not on a pendent triangle. If $v_1 \in W_5$, then G is a graph with twelve vertices, and it is easy to find an $\mathcal{FII}$-partition of G. Suppose $v_1 \in W_{23}$, which implies that v_1 is a 3^--vertex. We use the labels as in Figure 11.

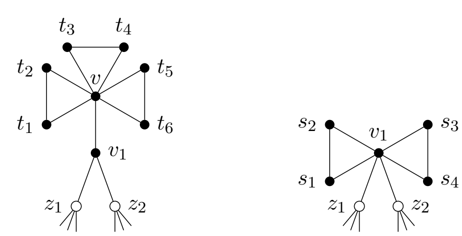


Figure 11: An illustration for Lemma 3.10.

Let $S = \{t_1, \dots, t_6\}$ and let $H := G - (S \cup \{v\})$. By (1.1) and (1.2), $\rho^*_H(v_1) \ge -4 \cdot 7 + 3 \cdot 10 = 2$. Then $\text{mad}(H') \le \frac{8}{3}$, where H' be the graph obtained from H by attaching two pendent triangles to v_1. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. If $v_1 \in I_\alpha$, then since $(F+S+v_1) \sqcup (I_\alpha - v_1 + v) \sqcup I_\beta$ is not an $\mathcal{FII}$-partition of G, we know $z_1, z_2 \in F$. Now, $(F+S) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G. If $v_1 \in F$, then by considering two pendent triangles of H', we have $z_1, z_2 \in F$, and so $(F+S) \sqcup (I_\alpha + v) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. $\square$

4 Reducible Configurations [C′1]-[C′5]

Lemma 4.1. *The subgraph of G induced by W_{23} is a forest.* **[C′1]**

Proof. Suppose to the contrary that there is a cycle C consisting of W_{23}-vertices. By **[C2]** and **[C5]**, C is an even cycle such that a W_3-vertex and a W_2-vertex appear alternatively. Let $C : u_1v_1u_2v_2 \dots u_kv_k$ $(k \geq 2)$ where $u_i \in W_2$ and $v_i \in W_3$. Let z_i be the neighbor of v_i not on C, and let $Z = \{z_1, \dots, z_k\}$. Let $H = G - V(C)$. Since $|V^*(H)| < |V^*(G)|$, by the minimality of G, H has an $\mathcal{FII}$-partition.

Claim 4.1.1. *For every $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H, either $Z \subset F$ or $Z \cap F = \emptyset$.*

Proof. Let $Z \cap F = \{z_{i_1}, z_{i_2}, \dots, z_{i_t}\}$ where $i_1 < \cdots < i_t$. Suppose to the contrary that $1 \leq t \leq k-1$. Without loss of generality, assume $z_k \in I_\alpha$ and $i_1 = 1$ so that $z_{i_1} = z_1 \in F$. If $t = 1$, then $(F + V(C) - u_1) \sqcup I_\alpha \sqcup (I_\beta + u_1)$ is an $\mathcal{FII}$-partition of G. Now assume $t \geq 2$. Add u_1 to I_β. For each $s \in \{2, 3, \dots, t\}$, add u_{i_s} to either I_α or I_β one by one according to the following rule: add u_{i_s} to I_α and I_β if either $u_{i_s - 1}$ or z_{i_s-1} is in I_β and I_α, respectively. Note that both $u_{i_s-1} \in I_\alpha$, $z_{i_s-1} \in I_\beta$ and $u_{i_s-1} \in I_\beta$, $z_{i_s-1} \in I_\alpha$ cannot happen since u_{i_s-1} is added to either I_α or I_β if and only if $z_{i_s-1} \in F$. Also, since $t \leq k-1$, u_{i_t} and u_1 has distance at least three. Now add all vertices in $V(C) \setminus \{u_{i_1}, \dots, u_{i_t}\}$ to F, which results in an $\mathcal{FII}$-partition of G. □

Claim 4.1.2. *Let H' be a graph with an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$.*

(i) *If H' is the graph obtained from H by attaching a pendent triangle T to some z_i, then $Z \cap F' = \emptyset$.*

(ii) *If H' is the graph obtained from H by attaching J_1 in Figure 1 to some z_i, then $Z \subset F'$.*

Proof. Let $F \sqcup I_\alpha \sqcup I_\beta$ be a restriction of $F' \sqcup I'_\alpha \sqcup I'_\beta$ to $V(H)$, which is an $\mathcal{FII}$-partition of H. By Claim 4.1.1, either $Z \subset F$ or $Z \cap F = \emptyset$. Hence, either $Z \subset F'$ or $Z \cap F' = \emptyset$.
(i) Suppose to the contrary that $Z \cap F' \neq \emptyset$, so $Z \subset F'$. Without loss of generality, let $z_i = z_1$ and assume a 2-vertex on T belongs to I'_α. Let $U_1 = \{u_i \mid i \text{ is odd }, 3 \leq i \leq k\}$ and $U_2 = \{u_i \mid i \text{ is even }, 4 \leq i \leq k\}$. Then $(F + v_2v_3 \dots v_k + u_1u_2) \sqcup (I_\alpha + U_1) \sqcup (I_\beta + U_2 + v_1)$ is an $\mathcal{FII}$-partition of G.
(ii) Suppose to the contrary that $Z \not\subset F'$, so $Z \cap F' = \emptyset$. Without loss of generality assume $z_1 \in I'_\alpha$. By Lemma 1.6 (i), the center of J_1 is in I'_β, so G has an $\mathcal{FII}$-partition $(F + V(C) - v_1) \sqcup I_\alpha \sqcup (I_\beta + v_1)$. □

(Case 1) Suppose $\rho^*_H(z_i) \geq 2$ for some i. Let H'' be the graph obtained from H by attaching a pendent triangle T and the graph J_1 in Figure 1 to z_i. Then $\text{mad}(H'') \leq \frac{8}{3}$. Since $|V^*(H'')| < |V^*(G)|$, by the minimality of G, H'' has an $\mathcal{FII}$-partition. Note that we may apply both Claim 4.1.2 (i) and (ii) to H'' and conclude both $Z \cap F' = \emptyset$ and $Z \subset F$, which is a contradiction.
(Case 2) Suppose $\rho^*_H(z_i) \leq 1$ for all $i \in [k]$. By (1.1) and (1.2), $\sum_{i=1}^k \rho^*_H(z_i) \geq \rho^*_H(Z) \geq -4 \cdot 2k + 3 \cdot 3k \geq k$, and so $\rho^*_H(z_i) = 1$ for all i. Then we have $\rho^*_H(z_1z_2) \geq 2$, since

$$\rho^*_H(z_1z_2) + k - 2 = \rho^*_H(z_1z_2) + \sum_{i \geq 3} \rho^*_H(z_i) \geq \rho^*_H(Z) \geq k.$$

Let H'' be the graph obtained from G by attaching a pendent triangle T to z_1 and attaching the graph J_1 in Figure 1 to z_2. As in the previous case, $\text{mad}(H'') \leq \frac{8}{3}$. Since $|V^*(H'')| < |V^*(G)|$, by the minimality of G, H'' has an $\mathcal{FII}$-partition. Note that we may apply both Claim 4.1.2 (i) and (ii) to H'' and conclude both $Z \cap F' = \emptyset$ and $Z \subset F$, which is a contradiction. □

The following lemma implies [$\mathbf{C'3}$].

Lemma 4.2. *In G, there is no V_4-vertex v satisfying one of the following:*

(i) *v has two W_5-neighbors.*

(ii) *v has three W_2-neighbors and the last neighbor is in W_{235}.*

(iii) *v has two W_2-neighbors and the other two neighbors are in W_{235}.*

Proof. Let u_1, u_2, u_3, u_4 be neighbors of a vertex $v \in V_4$.
(i) Suppose to the contrary that $u_1, u_2 \in W_5$. Let $S = N_G[u_1] \cup N_G[u_2]$ and $H = G - S$. Let H' be the graph obtained from H by adding an edge u_3u_4. By (1.1) and (1.2), we have $\rho^*_H(u_3u_4) \geq -4 \cdot 11 + 3 \cdot 16 = 4$, and so $\mathrm{mad}(H') \leq \frac{8}{3}$. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$, which is also an $\mathcal{FII}$-partition of H. From each u_i where $i \in [2]$, let t_i, t'_i be 2-vertices from different pendent triangles on u_i. Now, $(F + S - t_1t'_1t_2t'_2) \sqcup (I_\alpha + t_1t_2) \sqcup (I_\beta + t'_1t'_2)$ is an $\mathcal{FII}$-partition of G.
(ii) Suppose to the contrary that $u_1, u_2, u_3 \in W_2$ and $u_4 \in W_{235}$. We use the labels as in Figure 12. Let $H = G - S$, where

$$S = \begin{cases} \{v, u_1, u_2, u_3\} \cup N_G[u_4] & \text{if } u_4 \in W_5 \\ \{v, u_1, u_2, u_3, x_4, x'_4\} & \text{if } u_4 \in W_3 \\ \{v, u_1, u_2, u_3, u_4\} & \text{if } u_4 \in W_2. \end{cases}$$

By the minimality of G, H has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. If $u_4 \in W_5$, then since $(F + S - tt') \sqcup$

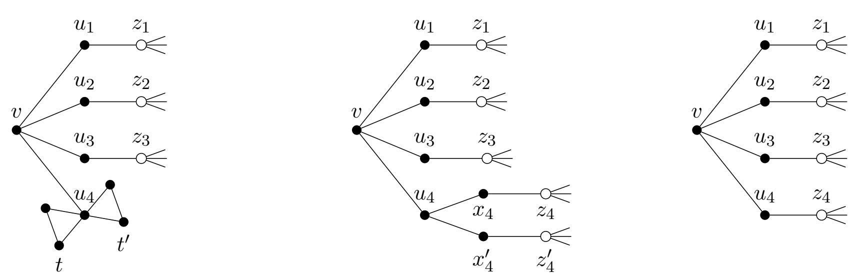


Figure 12: An illustration of Lemma 4.2 (ii).

$(I_\alpha + t) \sqcup (I_\beta + t')$ is not an $\mathcal{FII}$-partition of G, at least two z_i's are in F. Then either $(F + S - vu_4) \sqcup (I_\alpha + v) \sqcup (I_\beta + u_4)$ or $(F + S - vu_4) \sqcup (I_\alpha + u_4) \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G.

Suppose that $u_4 \in W_3$. If at most one of z_1, z_2, z_3 is in F, then $(F + S - u_4) \sqcup I_\alpha \sqcup (I_\beta + u_4)$, $(F + S - u_4) \sqcup (I_\alpha + u_4) \sqcup I_\beta$, or $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If two of z_1, z_2, z_3 are in F, then $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$, $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$, $(F + S - vu_4) \sqcup (I_\alpha + v) \sqcup (I_\beta + u_4)$, or $(F + S - vu_4) \sqcup (I_\alpha + u_4) \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G.

Now suppose that $u_4 \in W_2$.

Claim 4.2.1. *For every $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H, exactly two of z_i's are in F.*

Proof. If at most one z_i is in F, then G has an $\mathcal{FII}$-partition $(F + S) \sqcup I_\alpha \sqcup I_\beta$. If at least three $z'_i s$ are in F, then either $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$ or $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G. $\square$

By (1.2), we have $\rho^*_H(z_1z_2z_3z_4) \geq 4$.

Claim 4.2.2. (a) *$\rho^*_H(z_i) \leq 2$ for every $i \in [4]$ where $i \neq j$.*

(b) *$\rho^*_H(z_iz_j) \leq 3$ for every $i, j \in [4]$ where $i \neq j$.*

(c) *If $\rho^*_H(z_iz_j) = 3$ for $i, j \in [4]$ where $i \neq j$, then $\rho^*_H(z_iz_jz_k) = 3$ for every $k \in [4] \setminus \{i, j\}$.*

(d) *There are no distinct $i, j, k \in [4]$ such that attaching a pendent triangle at each of z_i, z_j, z_k results in a graph H' satisfying* $\mathrm{mad}(H') \leq \frac{8}{3}$.

Proof. In the proof of each case, we define a graph H' by modifying H so that $\mathrm{mad}(H') \leq \frac{8}{3}$ and $|V^*(H')| < |V^*(G)|$. By the minimality of G, H' has an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$. Let $F = F' \cap V(H)$, $I_\alpha = I'_\alpha \cap V(H)$, and $I_\beta = I'_\beta \cap V(H)$.

(a) Suppose to the contrary that $\rho^*_H(z_1) \geq 3$. Let H' be the graph obtained from H by attaching J_2 and a pendent triangle to z_1. Note that since $\rho^*_H(z_1) \geq 3$ we know $\mathrm{mad}(H') \leq \frac{8}{3}$. By Lemma 1.6 (ii), $z_1 \in F$. By considering a pendent triangle T_1, together with Claim 4.2.1, either $(F + S - u_1) \sqcup (I_\alpha + u_1) \sqcup I_\beta$ or $(F + S - u_1) \sqcup I_\alpha \sqcup (I_\beta + u_1)$ is an $\mathcal{FII}$-partition of G.

(b) Suppose to the contrary that $\rho^*_H(z_3z_4) \geq 4$. Since $\rho^*_H(z_3) + \rho^*_H(z_4) \geq \rho^*_H(z_3z_4)$ by (1.1), we may assume that $\rho^*_H(z_3) \geq 2$. Let H' be the graph obtained from H by adding an edge z_3z_4 and attaching J_1 to z_3. Since $\rho^*_H(z_3z_4) \geq 4$ and $\rho^*_H(z_3) \geq 2$, we know $\mathrm{mad}(H') \leq \frac{8}{3}$. Suppose $z_3 \in F$. By Claim 4.2.1, exactly one of z_1, z_2, z_4 is in F. If $z_4 \in F$, then $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If $z_4 \notin F$, then either $(F + S - u_3) \sqcup (I_\alpha + u_3) \sqcup I_\beta$ or $(F + S - u_3) \sqcup I_\alpha \sqcup (I_\beta + u_3)$ is an $\mathcal{FII}$-partition of G. Now suppose $z_3 \notin F$, say $z_3 \in I_\alpha$. By Lemma 1.6 (i), by considering the center of J_1, we know $z_4 \in F$. Now, G has an $\mathcal{FII}$-partition $(F + S - u_4) \sqcup (I_\alpha + u_4) \sqcup I_\beta$.

(c) Suppose to the contrary that $\rho^*_H(z_3z_4) = 3$ and $\rho^*_H(z_1z_3z_4) \geq 4$. By (1.1), $\rho^*_H(z_1) + \rho^*_H(z_3z_4) \geq \rho^*_H(z_1z_3z_4) \geq 4$ and so $\rho^*_H(z_1) \geq 1$. Let H' be the graph obtained from H by adding a pendent triangle T_1 at z_1 and an edge z_3z_4. If $z_4 \in F$, then since $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is not an $\mathcal{FII}$-partition of G, we may assume $z_3 \in I_\alpha$. Now, $(F + S - u_4) \sqcup (I_\alpha + u_4) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. If $z_3, z_4 \notin F$, then $z_1 \in F$. By considering a pendent triangle T_1, either $(F + S - u_1) \sqcup (I_\alpha + u_1) \sqcup I_\beta$ or $(F + S - u_1) \sqcup I_\alpha \sqcup (I_\beta + u_1)$ is an $\mathcal{FII}$-partition of G.

(d) Suppose to the contrary that attaching a pendent triangle T_i to each of z_1, z_2, z_3 results in a graph H' satisfying $\mathrm{mad}(H') \leq \frac{8}{3}$. By Claim 4.2.1, we may assume $z_1 \in F$ and a 2-vertex on T_1 is in I_α. Now, $(F + S - u_1) \sqcup (I_\alpha + u_1) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. $\square$

If $\rho^*_H(z_i) = 0$ for two integers $i \in [4]$, say $\rho^*_H(z_1) = \rho^*_H(z_2) = 0$, then by (1.1) and (1.2), $\rho^*_H(z_1) + \rho^*_H(z_2) + \rho^*_H(z_3z_4) \geq \rho^*_H(z_1z_2z_3z_4) \geq 4$. This implies $\rho^*_H(z_3z_4) \geq 4$, which is a contradiction to Claim 4.2.2 (b). Hence, we may assume that $\rho^*_H(z_i) \geq 1$ for $i \in [3]$. If $\rho^*_H(z_1z_2), \rho^*_H(z_2z_3), \rho^*_H(z_1z_3) \leq 1$, then $3 \geq \rho^*_H(z_1z_2) + \rho^*_H(z_2z_3) + \rho^*_H(z_1z_3) \geq 2\rho^*_H(z_1z_2z_3)$, which implies $\rho^*_H(z_1z_2z_3) = 1$. Therefore $\rho^*_H(z_4) \geq 3$, which is a contradiction to Claim 4.2.2 (a). Hence, we may assume $\rho^*_H(z_1z_2) \geq 2$. If $\rho^*_H(z_1z_2z_3) \leq 2$, then by (1.1)

$$\begin{aligned}
\rho^*_H(z_1z_2z_4) &\geq \rho^*_H(z_1z_2z_3z_4) + \rho^*_H(z_1z_2) - \rho^*_H(z_1z_2z_3) \geq 6 - 2 = 4, \\
\rho^*_H(z_1z_4), \rho^*_H(z_2z_4) &\geq \rho^*_H(z_4) \geq \rho^*_H(z_1z_2z_3z_4) - \rho^*_H(z_1z_2z_3) \geq 2.
\end{aligned}$$

Thus, attaching one pendent triangle at each z_i for $i \in \{1, 2, 4\}$ results in a graph H' with $\mathrm{mad}(H') \leq \frac{8}{3}$, which is a contradiction to Claim 4.2.2 (d). Hence, $\rho^*_H(z_1z_2z_3) \geq 3$. Note that by (1.1)

$$\rho^*_H(z_2z_3) + \rho^*_H(z_1z_3) \geq \rho^*_H(z_1z_2z_3) + \rho^*_H(z_3) \geq 3 + 1 = 4. \tag{4.1}$$

If $\rho^*_H(z_2z_3) = 2$, then $\rho^*_H(z_1z_3) \geq 2$ by (4.1). Thus, attaching one pendent triangle at each z_i for $i \in \{1,2,3\}$ results in a graph H' with $\text{mad}(H') \leq \frac{8}{3}$, which is a contradiction to Claim 4.2.2 (d). Hence $\rho^*_H(z_2z_3) \neq 2$, and by symmetry, $\rho^*_H(z_1z_3) \neq 2$. We may assume $\rho^*_H(z_2z_3) = 1$. Note that $\rho^*_H(z_1z_3z_4) \geq \rho^*_H(z_1z_3) = 3$ where the second equality is from (4.1) and Claim 4.2.2 (b). Together with Claim 4.2.2 (d), we have $\rho^*_H(z_1z_2z_3) = 3$, and therefore by (1.1)

$$\begin{aligned}
\rho^*_H(z_4) &\geq \rho^*_H(z_1z_2z_3z_4) - \rho^*_H(z_1z_2z_3) \geq 1,\\
\rho^*_H(z_1z_4) &\geq \rho^*_H(z_1z_2z_3z_4) - \rho^*_H(z_2z_3) \geq 3,\\
\rho^*_H(z_3z_4) &\geq \rho^*_H(z_1z_2z_3z_4) + \rho^*_H(z_3) - \rho^*_H(z_1z_2z_3) \geq 4+1-3 = 2.
\end{aligned}$$

Thus, attaching one pendent triangle at each z_i for $i \in \{1,3,4\}$ results in a graph H' satisfying $\text{mad}(H') \leq \frac{8}{3}$, which is a contradiction to Claim 4.2.2 (d).

(iii) Suppose to the contrary that $u_1, u_2 \in W_2$ and $u_3, u_4 \in W_{235}$. By (i) and (ii), we may assume that $u_3 \in W_3$ and $u_4 \in W_{35}$. We use the labels as in Figure 13. Let $S = N_G[v] \cup N_G[u_3] \cup N_G[u_4]$. By the minimality of G, $H = G - S$ has an $\mathcal{FII}$-partition. Let Z be the set of all z_i's and z_i''s.

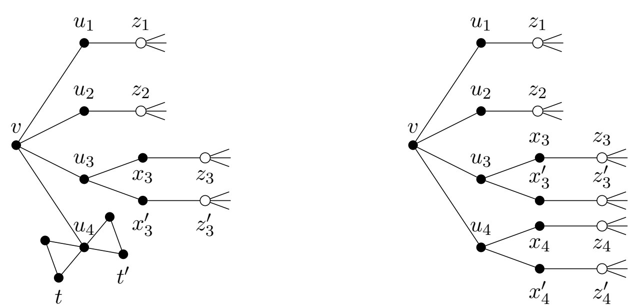


Figure 13: An illustration for Lemma 4.2 (iii).

Claim 4.2.3. *For every $\mathcal{FII}$-partition $H \sqcup I_\alpha \sqcup I_\beta$ of H, $Z \subset F$.*

Proof. Suppose to the contrary that $Z \not\subset F$. Suppose $u_4 \in W_5$. If $z_1 \notin F$, say $z_1 \in I_\alpha$, then since neither $(F + S - u_3u_4) \sqcup (I_\alpha + u_3) \sqcup (I_\beta + u_4)$ nor $(F + S - u_3u_4) \sqcup (I_\alpha + u_4) \sqcup (I_\beta + u_3)$ is an $\mathcal{FII}$-partition of G, we may assume $z_3 \in I_\alpha$ and $z_3' \in I_\beta$. Yet, $(F + S - tt') \sqcup (I_\alpha + t) \sqcup (I_\beta + t')$ is an $\mathcal{FII}$-partition of G. Therefore, $z_1, z_2 \in F$, and since $Z \not\subset F$, we know $\{z_3, z_3'\} \not\subset F$. Now, $(F + S - vu_4) \sqcup (I_\alpha + v) \sqcup (I_\beta + u_4)$ is an $\mathcal{FII}$-partition of G.

Suppose that $u_4 \in W_3$. If $z_3 \notin F$, then we may assume $z_3 \in I_\alpha$. Since neither $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$ nor $(F + S - vu_4) \sqcup (I_\alpha + v) \sqcup (I_\beta + u_4)$ is an $\mathcal{FII}$-partition of G, we may assume $z_1 \in I_\alpha$. Similarly, we conclude $z_2 \in I_\beta$. Now, $(F + S) \sqcup I_\alpha \sqcup I_\beta$, $(F + S - u_4) \sqcup (I_\alpha + u_4) \sqcup I_\beta$, or $(F + S - u_4) \sqcup I_\alpha \sqcup (I_\beta + u_4)$ is an $\mathcal{FII}$-partition of G. Therefore, $z_3, z_3', z_4, z_4' \in F$. Since $Z \not\subset F$, we know $\{z_1, z_2\} \not\subset F$. Now, $(F + S - u_3u_4) \sqcup (I_\alpha + u_3) \sqcup (I_\beta + u_4)$ is an $\mathcal{FII}$-partition of G. ☐

Note that by (1.2), if $u_4 \in W_5$, then $\rho^*_H(Z) \geq -4 \cdot 11 + 3 \cdot 16 = 4 = |Z|$, and if $u_4 \in W_3$, then $\rho^*_H(Z) \geq -4 \cdot 9 + 3 \cdot 14 = 6 = |Z|$.

Suppose $\rho^*_H(z_1z_2) \geq 3$. Let H' be the graph obtained from H by adding an edge z_1z_2. Then $\text{mad}(H') \leq \frac{8}{3}$. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. By Claim 4.2.3, $Z \subset F$, so G has an $\mathcal{FII}$-partition $(F + S - u_3u_4) \sqcup (I_\alpha + u_3) \sqcup (I_\beta + u_4)$.

Now suppose $\rho^*_H(z_1z_2) \leq 2$. Then $\sum_{z\in Z\setminus\{z_1,z_2\}} \rho^*_H(z) \geq |Z| - 2$, since $\rho^*_H(z_1z_2) + \sum_{z\in Z\setminus\{z_1,z_2\}} \rho^*_H(z) \geq |Z|$ by (1.1). Without loss of generality assume $\rho^*_H(z_3) \geq 1$. Let H' be the graph obtained from H by attaching a pendent triangle T to z_3. Then $\text{mad}(H') \leq \frac{8}{3}$. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. By Claim 4.2.3, we have $Z \subset F$. By considering the pendent triangle T, either $(F + S - vu_4x_3) \sqcup (I_\alpha + x_3u_4) \sqcup (I_\beta + v)$ or $(F + S - vu_4x_3) \sqcup (I_\alpha + v) \sqcup (I_\beta + x_3u_4)$ is an $\mathcal{FII}$-partition of G. $\square$

Lemma 4.3. *In G, there is no 6-vertex on two pendent triangles with a W_{235}-neighbor and a different W_{25}-neighbor.* **[C′5]**

Proof. Suppose to the contrary that there is a 6-vertex v on exactly two pendent triangles with a W_{25}-neighbor u_1 and a different W_{235}-neighbor u_2. If $u_2 \in W_5$, then by considering an $\mathcal{FII}$-partition of $G - (N_G[v] \cup N_G[u_2])$, it is easy to find an $\mathcal{FII}$-partition of G. Assume $u_2 \in W_{23}$. We use the labels as in Figure 14.

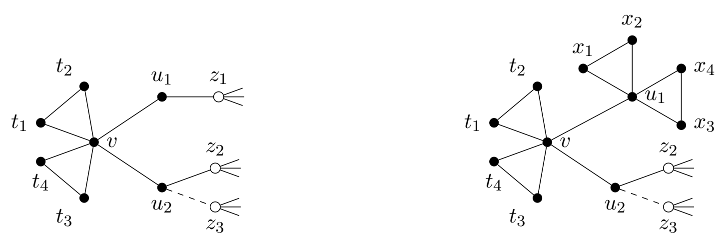


Figure 14: An illustration of Lemma 4.3.

If $u_1 \in W_2$, then let $S = N_G[v]$, and if $u_1 \in W_5$, then let $S = N_G[v] \cup N_G[u_1]$. By the minimality of G, $H = G - S$ has an $\mathcal{FII}$-partition. When u_2 is a 2-vertex, we ignore z_3 in Claim 4.3.1.

Claim 4.3.1. *For every $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H, $\{z_2, z_3\} \subset F$.*

Proof. Suppose to the contrary that $\{z_2, z_3\} \not\subset F$. Without loss of generality, assume $z_2 \in I_\alpha$. If $u_1 \in W_5$, then $(F + S - t_1t_3x_1x_3) \sqcup (I_\alpha + t_1x_1) \sqcup (I_\beta + t_3x_3)$ is an $\mathcal{FII}$-partition of G. If $u_1 \in W_2$, then since $(F + S - t_1t_3) \sqcup (I_\alpha + t_1) \sqcup (I_\beta + t_3)$ is not an $\mathcal{FII}$-partition of G, we conclude $z_1, z_3 \in F$. Now, $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G. $\square$

Suppose $u_2 \in W_2$. Let $F \sqcup I_\alpha \sqcup I_\beta$ be an $\mathcal{FII}$-partition of H. By Claim 4.3.1, $z_2 \in F$. If $u_1 \in W_5$, then $(F + S - vu_1) \sqcup (I_\alpha + u_1) \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G. If $u_1 \in W_2$, then either $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$ or $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G.

Suppose $u_2 \in W_3$. By (1.2), if $u_1 \in W_2$ then $\rho^*_H(z_2z_3) \geq 3$, and if $u_1 \in W_5$ then $\rho^*_H(z_2z_3) \geq 4$. Let H' be the graph obtained from H by adding an edge z_2z_3. Then $\text{mad}(H') \leq \frac{8}{3}$. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$, which is also an $\mathcal{FII}$-partition of H. By Claim 4.3.1, $z_2, z_3 \in F$. If $u_1 \in W_2$, then either $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$ or $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G. If $u_1 \in W_5$, then $(F + S - vu_1) \sqcup (I_\alpha + v) \sqcup (I_\beta + u_1)$ is an $\mathcal{FII}$-partition of G. $\square$

Lemma 4.4. *In G, there is no 5-vertex v on one pendent triangle with three W_{235}-neighbors where two are W_2-neighbors.* **[C′4]**

Proof. Let v be a 5-vertex on one pendent triangle with three W_{235}-neighbors where two are W_2-neighbors. We use the labels as in Figure 15. Let $H = G - S$, where

$$S = \begin{cases} \{v, t_1, t_2, u_1, u_2, u_3\} \cup N_G(u_3) & \text{if } u_3 \in W_5 \\ \{v, t_1, t_2, u_1, u_2, u_3\} & \text{if } u_3 \in W_2 \\ \{v, t_1, t_2, u_1, u_2, x_3, x_4\} & \text{if } u_3 \in W_3. \end{cases}$$

By the minimality of G, H has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. If $u_3 \in W_5$, then since neither

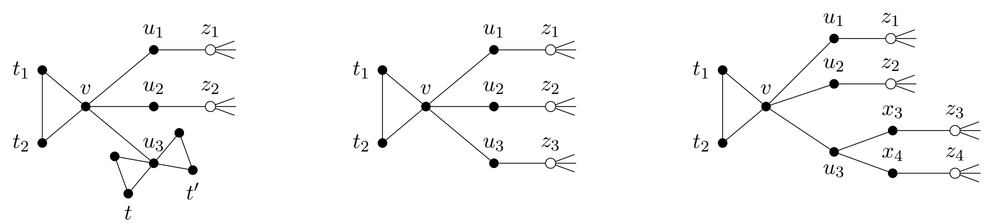


Figure 15: An illustration for Lemma 4.4.

$(F + S - vu_3) \sqcup (I_\alpha + v) \sqcup (I_\beta + u_3)$ nor $(F + S - vu_3) \sqcup (I_\alpha + u_3) \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G, we have $z_1, z_2 \notin F$. Thus, G has an $\mathcal{FII}$-partition $(F + S - t_1tt') \sqcup (I_\alpha + t) \sqcup (I_\beta + t't_1)$. If $u_3 \in W_2$, then since neither $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$, nor $(F + S - v) \sqcup I_\alpha \sqcup (I_\beta + v)$ is an $\mathcal{FII}$-partition of G, we have $z_1, z_2 \notin F$. Thus, G has an $\mathcal{FII}$-partition $(F + S - t_1) \sqcup (I_\alpha + t_1) \sqcup I_\beta$.

Now suppose that $u_3 \in W_3$. If $z_i \in F$ for some $i \in [2]$, then we may assume that $z_1, z_2 \notin I_\alpha$ and so either $(F + S - v) \sqcup (I_\alpha + v) \sqcup I_\beta$ or $(F + S - vu_3) \sqcup (I_\alpha + v) \sqcup (I_\beta + u_3)$ is an $\mathcal{FII}$-partition of G. If $z_1, z_2 \notin F$, then either $(F + S - t_1) \sqcup (I_\alpha + t_1) \sqcup I_\beta$ or $(F + S - t_1u_3) \sqcup (I_\alpha + t_1) \sqcup (I_\beta + u_3)$ is an $\mathcal{FII}$-partition of G. □

Lemma 4.5. *In G, there is no W_3-vertex u with a 3-neighbor such that a 2-neighbor of u has only 3^--neighbors.*

Proof. Suppose to the contrary that there is a vertex $u \in W_3$ with a 3-neighbor z_1 and a 2-neighbor x_2 with only 3^--vertices. We use the label as in Figure 16. By Lemmas 3.8 and 4.1, all z_i's are distinct. Let $S = \{u, x_2, x_3\}$, and $H = G - S$.

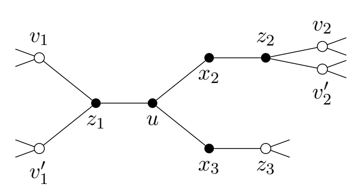


Figure 16: An illustration for Lemma 4.5

Claim 4.5.1. *For every $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H, $z_1, z_3 \in F$ and $z_2 \notin F$.*

Proof. If at most one z_i is in F, then $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. Suppose that $Z \subset F$. Since neither $(F + x_2x_3) \sqcup (I_\alpha + u) \sqcup I_\beta$ nor $(F + x_2x_3) \sqcup I_\alpha \sqcup (I_\beta + u)$ is an $\mathcal{FII}$-partition of G, we may assume $v_1 \in I_\alpha$ and $v_1' \in I_\beta$. Since $(F + S) \sqcup I_\alpha \sqcup I_\beta$ is not an $\mathcal{FII}$-partition of G, either v_2 or v_2'

is in F. Now, either $(F + ux_3) \sqcup (I_\alpha + x_2) \sqcup I_\beta$ or $(F + ux_3) \sqcup I_\alpha \sqcup (I_\beta + x_2)$ is an $\mathcal{FII}$-partition of G. Hence, exactly two of z_i's are in F. Suppose to contrary that $z_2 \in F$. Since none of $(F + S) \sqcup I_\alpha \sqcup I_\beta$, $(F + ux_3) \sqcup (I_\alpha + x_2) \sqcup I_\beta$, or $(F + ux_3) \sqcup I_\alpha \sqcup (I_\beta + x_2)$ is an $\mathcal{FII}$-partition of G, we may assume that $z_1 \in I_\alpha$, $v_2 \in I_\beta$, and $v_2' \in F$. Since $(F + x_2x_3) \sqcup I_\alpha \sqcup (I_\beta + u)$ is not an $\mathcal{FII}$-partition of G, either v_1 or v_1' is in I_β. Now, $(F + z_1x_2x_3) \sqcup (I_\alpha + u - z_1) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G. □

Suppose $\rho^*_H(z_2) \geq 2$. Let H' be the graph obtained from H by attaching J_2 to z_2. Since $|V^*(H')| < |V^*(G)|$, by the minimality of G, H' has an $\mathcal{FII}$-partition $F' \sqcup I'_\alpha \sqcup I'_\beta$. By Lemma 1.6 (ii), $z_2 \in F'$, which is a contradiction to Claim 4.5.1. Hence, $\rho^*_H(z_2) \leq 1$.

Since $\rho^*_H(z_1) + \rho^*_H(z_2) + \rho^*_H(z_3) \geq 3$ by (1.1) and (1.2), we have $\rho^*_H(z_1) + \rho^*_H(z_3) \geq 2$. Suppose $\rho^*_H(z_3) \geq 1$. Let H' be the graph obtained from H by attaching a pendent triangle T to z_3. Then H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. By Claim 4.5.1, $z_3 \in F$. By considering a 2-vertex of T not in F, we know either $(F + ux_2) \sqcup (I_\alpha + x_3) \sqcup I_\beta$ or $(F + ux_2) \sqcup I_\alpha \sqcup (I_\beta + x_3)$ is an $\mathcal{FII}$-partition of G. Hence, $\rho^*_H(z_3) = 0$, and therefore $\rho^*_H(z_1) \geq 2$.

Now, let H' be the graph obtained from H by attaching two pendent triangles T_1 and T_2 to z_1. Then H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. By Claim 4.5.1, $z_1 \in F$. By considering 2-vertices on T_1, T_2, we know that $v_1, v_1' \in F$. Hence, either $(F + x_2x_3) \sqcup (I_\alpha + u) \sqcup I_\beta$ or $(F + x_2x_3) \sqcup I_\alpha \sqcup (I_\beta + u)$ is an $\mathcal{FII}$-partition of G. □

Lemma 4.6. *In G, there is no cycle C consisting of $(V_3 \cup W_4)$-vertices such that every V_3-vertex on C has a W_{23}-neighbor.* **[C′2]**

Proof. Suppose to the contrary that there is such a cycle $C : u_1u_2 \dots u_k$. For $j \in \{2, 3\}$, let

$$X_j = \{u \in V(C) \cap V_3 \mid \text{the neighbor of } u \text{ not on } C \text{ is a } W_j\text{-vertex}\}.$$

We use the labels as in the left figure of Figure 17; in particular, we label the neighbors of $(X_2 \cup X_3)$-vertices and their neighbors. We first consider the case where all of the v_i's, t_i's, and t_i''s are distinct. The other case when some vertices are identical is presented afterwards.

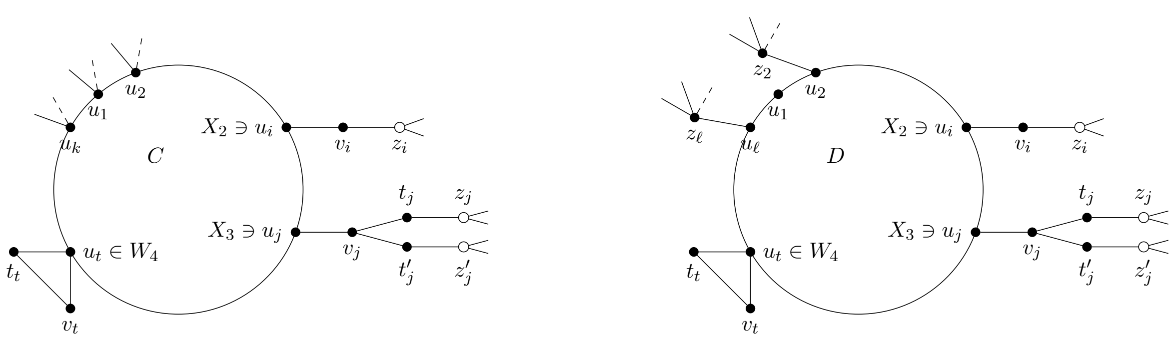


Figure 17: An illustration for Lemma 4.6

Suppose that all of the v_i's, t_i's, and t_i''s are distinct. Let V be the set of all v_i's, T be the set of all t_i's and t_i''s, and Z be the set of all z_i's and z_i''s. Let $S = V(C) \cup V \cup T$ and $H = G - S$.

Claim 4.6.1. *If $k = 5$, then $V(C) \not\subset X_2$.*

Proof. Suppose to the contrary that $k = 5$ and $V(C) = X_2$. Since $\sum_i \rho^*_H(z_i) \geq -10 \cdot 4 + 15 \cdot 3 = 5$ by (1.1) and (1.2), we may assume $\rho^*_H(z_1) \geq 1$. Let H' be the graph obtained from H by attaching a pendent triangle T_1 to z_1. By the minimality of G, H' has an $\mathcal{FII}$-partition, which also gives an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$ of H. If $Z \subset F$, then by considering a 2-vertex on T_1, either $(F + S - v_1u_2u_4) \sqcup (I_\alpha + u_2) \sqcup (I_\beta + v_1u_4)$ or $(F + S - v_1u_2u_4) \sqcup (I_\alpha + v_1u_4) \sqcup (I_\beta + u_2)$ is an $\mathcal{FII}$-partition of G. If $z_i \notin F$ for some i, then it is easy to find a partition Y_0, Y_1, Y_2 of $V(C) \setminus \{u_i\}$ such that $(F + V + u_i + Y_0) \sqcup (I_\alpha + Y_1) \sqcup (I_\beta + Y_2)$ is an $\mathcal{FII}$-partition of G. □

By the minimality of G, H has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. For $j \in \{2, 3\}$, let $X^F_j = \{u_i \in X_j \mid z'_i, z_i \in F\}$, and let $X^{\alpha\beta}_j = X_j \setminus X^F_j$.

Claim 4.6.2. *There exists an $i \in [k]$ such that $u_i \in X^F_3 \cup W_4$.*

Proof. Suppose to the contrary that for every i, $u_i \in X_2 \cup X^{\alpha\beta}_3$. If $k \leq 4$, then it is not hard to find a partition Y_1, Y_2, Y_3 of $V(C)$ such that $(F + T + V + Y_1) \sqcup (I_\alpha + Y_2) \sqcup (I_\beta + Y_3)$ is an $\mathcal{FII}$-partition of G. Assume $k \geq 5$.

If $u_i \in X^{\alpha\beta}_2$ for every i, then we may assume $z_1 \in I_\alpha$, and so $(F + S - u_1) \sqcup I_\alpha \sqcup (I_\beta + u_1)$ is an $\mathcal{FII}$-partition of G. Hence, $u_i \notin X^{\alpha\beta}_2$ for some i, so $u_i \in X^F_2 \cup X^{\alpha\beta}_3$. We have two cases: (1) $u_i \in X^{\alpha\beta}_2$ and $u_j \in X^F_2 \cup X^{\alpha\beta}_3$ for some i, j, and (2) $u_i \in X^F_2 \cup X^{\alpha\beta}_3$ for every i.

(Case 1) Without loss of generality, assume $u_k \in X^{\alpha\beta}_2$ and $u_1 \in X^F_2 \cup X^{\alpha\beta}_3$. We first find a partition of $V(G)$ by performing the following algorithm. First, add all vertices of $X^{\alpha\beta}_2 \cup T \cup V$ to F, and add u_1 to I_α. For $i \in [k-2]$, if $u_1, \ldots, u_i$ are determined, but u_{i+1} is not yet, then do the following:

If $u_i \notin F$, then add u_{i+1} to F. Otherwise, for $\gamma \in \{\alpha, \beta\}$ satisfying $u_{i-1} \notin I_\gamma$, add u_{i+1} to I_γ.

Note that by the algorithm, $u_k, u_2 \in F$. Since the resulting partition $F \sqcup I_\alpha \sqcup I_\beta$ is not an $\mathcal{FII}$-partition of G, $u_{k-1} \in I_\alpha$, and therefore $u_{k-2} \in F$. Since $(F + u_{k-1}) \sqcup (I_\alpha - u_{k-1}) \sqcup I_\beta$ must not be an $\mathcal{FII}$-partition of G, $u_{k-2} \in X^F_2 \cup X^{\alpha\beta}_3$. Also, since $F \sqcup (I_\alpha - u_{k-1}) \sqcup (I_\beta + u_{k-1})$ is not an $\mathcal{FII}$-partition of G, this implies $u_{k-3} \in I_\beta$, and therefore $u_{k-4} \in F$. Note that this implies $k \geq 6$. Now, $(F - u_{k-2} + u_{k-1}) \sqcup (I_\alpha - u_{k-1} + u_{k-2}) \sqcup I_\beta$ is an $\mathcal{FII}$-partition of G.

(Case 2) Suppose $u_j \in X^F_2 \cup X^{\alpha\beta}_3$ for every j. If $k = 5$, then by Claim 4.6.1, we may assume that $u_1 \in X^{\alpha\beta}_3$, and therefore $(F + S - u_2u_4) \sqcup (I_\alpha + u_2) \sqcup (I_\beta + u_4)$, $(F + S - v_1u_2u_4) \sqcup (I_\alpha + u_2) \sqcup (I_\beta + v_1u_4)$, or $(F + S - v_1u_2u_4) \sqcup (I_\alpha + v_1u_4) \sqcup (I_\beta + u_2)$ is an $\mathcal{FII}$-partition of G.

Assume $k \neq 5$. For $a \in \{0, 1, 2\}$, let $Y_a = \{u_i \mid i \equiv a \pmod 3\}$. Then $(F + T + V + Y_0) \sqcup (I_\alpha + Y_1) \sqcup (I_\beta + Y_2)$ is an $\mathcal{FII}$-partition of G where Y_0, Y_1, Y_2 is a partition of $V(C)$ defined as the following: (i) if $k \equiv 0 \pmod 3$, then no modifications to the Y_a's; (ii) if $k \equiv 1 \pmod 3$, then modify the Y_a's so that the last three vertices satisfy $u_{k-2}, u_k \in Y_0$ and $u_{k-1} \in Y_2$; (iii) if $k \equiv 2 \pmod 3$, then modify the Y_a's so that the last seven vertices satisfy $u_{k-6}, u_{k-4}, u_{k-2}, u_k \in Y_0$, $u_{k-3} \in Y_1$, and $u_{k-1}, u_{k-5} \in Y_2$. □

If $u_i \in X^F_3 \cup W_4$ for every i, then $(F + T + V(C) - u_1) \sqcup (I_\alpha + V) \sqcup (I_\beta + u_1)$ is an $\mathcal{FII}$-partition of G. Hence, we assume that $u_i \notin X^F_3 \cup W_4$ for some i. Together with Claim 4.6.2, we may assume $u_k \in X_2 \cup X^{\alpha\beta}_3$ and $u_1 \in X^F_3 \cup W_4$. For simplicity, let $Q = \{v_i \in V \mid u_i \in X^F_3 \cup W_4\}$. We find an $\mathcal{FII}$-partition of G by performing the following algorithm.

Step 1. Add v_1 to I_α, add u_1, u_2 to F, and add all undetermined vertices in $S - (Q \cup X^F_2 \cup X^{\alpha\beta}_3)$ to F.

Step 2. If vertices in $\{u_j \mid j \leq i\} \cup \{v_j \mid j \leq i\}$ are determined, but either $u_{i+1} \in X_2^F \cup X_3^{\alpha\beta}$ or $v_{i+1} \in Q$ is not determined, then do the following: For $v_{i+1} \in Q$, add v_{i+1} to exactly one of I_α or I_β that does not contain u_i. For $u_{i+1} \in X_2^F \cup X_3^{\alpha\beta}$, as long as $u_i \in F$ and there is $\gamma \in \{\alpha, \beta\}$ such that $\{u_{i-1}, u_i, v_i\} \cap I_\gamma = \emptyset$, add u_{i+1} to I_γ. Otherwise, add u_{i+1} to F.

Note that the resulting partition $F \sqcup I_\alpha \sqcup I_\beta$ obtained by the algorithm is not an $\mathcal{FII}$-partition of G; the problem arises because of u_k. If neither $F \sqcup I_\alpha \sqcup I_\beta$ nor $F \sqcup (I_\alpha - v_1) \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G, then $u_k \in F$ and $|\{v_k, u_{k-1}\} \cap F| \geq 1$. If $F \sqcup I_\alpha \sqcup I_\beta$ is not an $\mathcal{FII}$-partition of G, then $|\{v_2, u_3\} \cap F| \geq 1$. Then since neither $(F - u_1) \sqcup I_\alpha \sqcup (I_\beta + u_1)$ nor $(F - u_1) \sqcup (I_\alpha - v_1 + u_1) \sqcup (I_\beta + v_1)$ is an $\mathcal{FII}$-partition of G, we obtain $|\{v_2, u_3\} \cap F| = |\{v_k, u_{k-1}\} \cap F| = 1$. If $v_k \notin F$, then either $(F - u_1) \sqcup (I_\alpha + v_k) \sqcup (I_\beta - v_k + u_1)$ or $(F - u_1) \sqcup (I_\alpha - v_1 v_k) \sqcup (I_\beta + v_k u_1)$ is an $\mathcal{FII}$-partition of G. Thus, $v_k \in F$, $u_{k-1} \notin F$, and $u_k \in X_2^F \cup X_3^{\alpha\beta}$. This also implies that $u_{k-1} \in X_2^F \cup X_3^{\alpha\beta}$ and $v_{k-1} \in F$. If neither $(F - u_k) \sqcup (I_\alpha - v_1 + u_k) \sqcup (I_\beta + v_1)$ nor $(F - u_k) \sqcup I_\alpha \sqcup (I_\beta + u_k)$ is an $\mathcal{FII}$-partition of G, then $u_{k-2} \notin F$. Now, either $(F + u_{k-1} - u_k) \sqcup (I_\alpha - v_1 u_{k-1} + u_k) \sqcup (I_\beta + v_1)$ or $(F + u_{k-1} - u_k) \sqcup I_\alpha \sqcup (I_\beta + u_k)$ is an $\mathcal{FII}$-partition of G.

Now we consider the case where some vertices v_i's, t_i's, and t'_i's are not distinct. We mimic the previous case when they are all distinct, but we use a different cycle to proceed with the argument. By Lemma 4.5, for $u_i \in X_3$, we know $t_i \notin \{v_j, t_j\}$ for some $j \neq i$. By Lemma 3.3 (i), $v_i \neq v_{i+1}$. Hence, there are two different indices i and j where $u_i, u_j \in X_2$, $v_i = v_j$, and the distance between u_i and u_j along C is at least 2. Take such i and j so that the distance between u_i and u_j along C is minimum, and consider the cycle $u_i u_{i+1} \dots u_j v_j$; we abuse notation and relabel this cycle as $D : u_1 u_2 \dots u_\ell$ $(4 \leq \ell \leq k)$ where u_1 is a 2-vertex. Namely, all vertices in $V(D) \setminus \{u_1\}$ are in $V_3 \cup W_4$, a vertex in $(V(D) \cap V_3) \setminus \{u_2, u_\ell\}$ has a W_{23}-neighbor, and the neighbors of u_2 and u_ℓ not on D are in $V_3 \cup W_4$. Let z_2 and z_ℓ be the neighbor of u_2 and u_ℓ, respectively, not on D. See the right figure of Figure 17 for an illustration. Redefine the following sets: $V = \{v_i \mid u_i \in V(D)\}$, $T = \{t_i, t'_i \mid u_i \in V(D)\}$, $Z = \{z_i, z'_i \mid u_i \in V(D)\}$. Also, restrict X_j to be $X_j \cap V(D)$. Note that by the choice of D, all of the v_i's, t_i's, and t'_i's are distinct.

Let $S = V(D) \cup V \cup T$. By the minimality of G, the graph $H = G - S$ has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. For $j \in \{2, 3\}$, let $X_j^F = \{u_i \in X_j \mid z'_i, z_i \in F\}$, and let $X_j^{\alpha\beta} = X_j \setminus X_j^F$. For simplicity, let $Q = \{v_i \in V \mid u_i \in X_3^F \cup W_4\}$. We attempt to find an $\mathcal{FII}$-partition of G by performing the following algorithm.

Step 1. Add u_1, u_2, u_ℓ to F. If $z_2 \notin F$, then $u_3 \in F$. (If $z_2 \in F$, then leave u_3 undetermined.) Add all undetermined vertices in $S - (Q \cup X_2^F \cup X_3^{\alpha\beta})$ to F.

Step 2. Same as Step 2 of the previous case.

Let $F \sqcup I_\alpha \sqcup I_\beta$ be a resulting partition by the algorithm. Note that we had a choice to choose either I_α or I_β when we determined u_i or v_i for the very first instance of Step 2. Hence, the algorithm can also produce the partition $F \sqcup I'_\alpha \sqcup I'_\beta$ where $I'_\alpha = (I_\alpha - S) \cup (I_\beta \cap S)$ and $I'_\beta = (I_\beta - S) \cup (I_\alpha \cap S)$. If $z_\ell \in F$ and z_ℓ has both an I_α-neighbor and an I_β-neighbor, then, since $z_\ell \in V_3 \cup W_4$, either $F \sqcup I_\alpha \sqcup I_\beta$, $(F - u_1) \sqcup (I_\alpha + u_1) \sqcup I_\beta$, or $(F - u_1) \sqcup I_\alpha \sqcup (I_\beta + u_1)$ is an $\mathcal{FII}$-partition of G. Suppose that $z_\ell \in F$ and z_ℓ has no I_β-neighbor. Since neither $(F - u_\ell) \sqcup I_\alpha \sqcup (I_\beta + u_\ell)$ nor $(F - u_\ell) \sqcup I'_\alpha \sqcup (I'_\beta + u_\ell)$ is an $\mathcal{FII}$-partition of G, we have $\{u_{\ell-1}, v_{\ell-1}, u_{\ell-2}\} \cap I_\alpha \neq \emptyset$ and $\{u_{\ell-1}, v_{\ell-1}, u_{\ell-2}\} \cap I_\beta \neq \emptyset$. If $u_{\ell-1} \notin F$, then $v_{\ell-1} \in F$, and so $u_{\ell-2} \notin F$. This implies that either $(F + u_{\ell-1} - u_\ell) \sqcup (I_\alpha - u_{\ell-1}) \sqcup (I_\beta + u_\ell)$ or $(F + u_{\ell-1} - u_\ell) \sqcup (I'_\alpha - u_{\ell-1}) \sqcup (I'_\beta + u_\ell)$ is an $\mathcal{FII}$-partition of G. If $u_{\ell-1} \in F$, then since neither

$F \sqcup I_\alpha \sqcup I_\beta$ nor $F \sqcup I'_\alpha \sqcup I'_\beta$ is an $\mathcal{FII}$-partition of G, we know $u_2, z_2 \in F$. Now, either $(F - u_1) \sqcup (I_\alpha + u_1) \sqcup I_\beta$ or $(F - u_1) \sqcup (I'_\alpha + u_1) \sqcup I'_\beta$ is an $\mathcal{FII}$-partition of G.

Suppose that $z_\ell \notin F$, and without loss of generality assume $z_\ell \in I_\alpha$. Moreover, we may assume that all neighbors of z_ℓ are in F. Otherwise, it is the case where $z_\ell \in W_4$, which is already covered by the case where $z_\ell \in F$ has neighbors in I_β and I_α. Since neither $F \sqcup I_\alpha \sqcup I_\beta$ nor $F \sqcup I'_\alpha \sqcup I'_\beta$ is an $\mathcal{FII}$-partition of G, $u_{\ell-1} \in F$ and $|\{v_{\ell-1}, u_{\ell-2}\} \cap F| \geq 1$. Now, either $(F - u_\ell) \sqcup I_\alpha \sqcup (I_\beta + u_\ell)$ or $(F - u_\ell) \sqcup I'_\alpha \sqcup (I'_\beta + u_\ell)$ is an $\mathcal{FII}$-partition of G. □

5 Remarks

There is a natural generalization of $\mathcal{FII}$-partitions. For a nonnegative integer k, we say a graph G has an $\mathcal{FI}_k$-partition $F \sqcup I_1 \sqcup \cdots \sqcup I_k$ if $F, I_1, \ldots, I_k$ is a partition of $V(G)$ such that $G[F]$ is a forest and each I_i is a 2-independent set. As explained in the introduction, a graph with an $\mathcal{FI}_k$-partition can be star $(k+3)$-colored. Let h and f be functions such that

$$h(k) = \inf\{\text{mad}(G) : G \text{ has no } \mathcal{FI}_k\text{-partition}\} \qquad f(k) = \inf\{\text{mad}(G) : \chi_s(G) > k\}.$$

Since a forest is star 3-colorable, for an integer k, $h(k+3) \leq f(k)$.

Determining the exact values of $f(k)$ and $h(k)$ is a difficult, yet interesting problem. From [6, 7], we know $f(1) = 1, f(2) = \frac{3}{2}, f(3) = 2$, and $\frac{5}{2} \leq f(4) \leq \frac{18}{7}$. Our main result implies $f(5) \geq \frac{8}{3}$. As stated in [7], determining the exact value of $f(k)$ for $k \geq 4$ remains an intriguing question.

Question 1 ([7]). What is the exact value of $f(k)$ for $k \geq 4$?

The motivation of $\mathcal{FI}_k$-partitions comes from star colorings, but it is interesting in its own right. It is easy to see that a graph G has an $\mathcal{FI}_0$-partition if and only if G is a forest. Since a forest has maximum average degree less than 2, it follows that $h(0) = 2$. Since a graph H with $\text{mad}(H) = \frac{5}{2}$ where H has no $\mathcal{FI}_1$-partition was constructed in [7], we know $h(1) \leq \frac{5}{2}$. Yet, Brandt et al. [6] proved that a graph G with $\text{mad}(G) < \frac{5}{2}$ has an $\mathcal{FI}_1$-partition, so the value of $h(1)$ is determined, namely, $h(1) = \frac{5}{2}$. In this term, our main result is equivalent to $h(2) \geq \frac{8}{3}$. We explicitly ask the question of determining the value of $h(k)$ for $k \geq 2$.

Question 2. What is the exact value of $h(k)$ for $k \geq 2$?

It is tempting to guess $h(k) = \frac{4+k}{2}$, yet we provide a construction that shows $h(2) \leq \frac{46}{17} < 3$.

Construction 5.1. For a positive integer n, let G_{5n} be the graph obtained from a $5n$-cycle $v_0, \ldots, v_{5n-1}$ by attaching two pendent triangles to v_i where $i \pmod 5 \in \{1, 2, 3\}$. It is not hard to see that $\text{mad}(G_{5n}) = \frac{46}{17}$. Now suppose to the contrary that G_{5n} has an $\mathcal{FII}$-partition $F \sqcup I_\alpha \sqcup I_\beta$. By Lemma 1.6, we know that if $i \not\equiv 2 \pmod 5$ then the vertex v_i of G_{5n} is in F. This also forces v_{5j+2} to be in F, which is a contradiction since F is a forest. Hence, G_{5n} has no $\mathcal{FII}$-partition.

As the above infinite family of graphs exhibit $h(2) \leq \frac{46}{17}$, we seek the exact value of $h(2)$.

Question 3. What is the value $h(2)$? In particular, is $h(2) = \frac{46}{17}$?

As layed out in Table 1, a planar graph with girth at least 10 is star 4-colorable [6], which is sharp in the sense that the number of colors cannot be reduced [1]. The main result in this paper implies that a planar graph with girth at least 8 is star 5-colorable. It is also known that there exists a planar graph with girth 7 that is not star 4-colorable [21]. Regarding star 5-colorings, the only remaining case in terms of girth is to determine whether planar graphs with girth 7 are star 5-colorable or not.

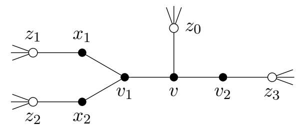


Figure 18: The graphs G_5, G_{10}, and G_{15}.

Question 4. *Does there exist a planar graph with girth 8 that is not star 4-colorable or is every planar graph with girth at least 7 star 5-colorable?*

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