# DP-3-coloring of planar graphs without certain cycles

Mengjiao Rao Tao Wang<sup>\*</sup>

Institute of Applied Mathematics Henan University, Kaifeng, 475004, P. R. China

#### Abstract

DP-coloring is a generalization of list-coloring, which was introduced by Dvořák and Postle. Zhang showed that every planar graph with neither adjacent triangles nor 5-, 6-, 9-cycles is 3-choosable. Liu et al. showed that every planar graph without 4-, 5-, 6- and 9-cycles is DP-3-colorable. In this paper, we show that every planar graph with neither adjacent triangles nor 5-, 6-, 9-cycles is DP-3-colorable, which generalizes these results. Yu et al. gave three Bordeaux-type results by showing that: (i) every planar graph with the distance of triangles at least three and no 4-, 5-cycles is DP-3-colorable; (ii) every planar graph with the distance of triangles at least two and no 4-, 5-, 6-cycles is DP-3-colorable; (iii) every planar graph with the distance of triangles at least two and no 5-, 6-, 7-cycles is DP-3-colorable. We also give two Bordeaux-type results in the last section: (i) every planar graph with neither 5-, 6-, 8-cycles nor triangles at distance less than two is DP-3-colorable; (ii) every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is DP-3-colorable.

## 1 Introduction

All graphs considered in this paper are finite, simple and undirected. A planar graph is a graph that can be embedded into the plane so that its edges meet only at their ends. A plane graph is a particular embedding of a planar graph into the plane. We set a plane graph G = (V, E, F) where V, E, F are the sets of vertices, edges, and faces of G, respectively. A vertex v and a face f are **incident** if  $v \in V(f)$ . Two faces are **adjacent** if they have at least one common edge. Call  $v \in V(G)$  a k-vertex, or a k<sup>+</sup>-vertex, or a k<sup>-</sup>-vertex if its degree is equal to k, or at least k, or at most k, respectively. The notions of a k-face, a k<sup>+</sup>-face and a k<sup>-</sup>-face are similarly defined.

A proper k-coloring of a graph G is a mapping  $f: V(G) \longrightarrow [k]$  such that  $f(u) \neq f(v)$  whenever  $uv \in E(G)$ , where  $[k] = \{1, 2, ..., k\}$ . The smallest integer k such that G has a proper k-coloring is called the **chromatic number** of G, denoted by  $\chi(G)$ . Vizing [21], and independently Erdős, Rubin, and Taylor [5] introduced list-coloring as a generalization of proper coloring. A **list-assignment** L gives each vertex v a list of available colors L(v). A graph G is L-colorable if there is a proper coloring  $\phi$  of G such that  $\phi(v) \in L(v)$  for each  $v \in V(G)$ . A graph G is k-choosable if G is L-colorable for each L with  $|L(v)| \ge k$ . The minimum integer k such that G is k-choosable is called the **list-chromatic number**  $\chi_{\ell}(G)$ .

For ordinary coloring, since every vertex has the same color set [k], the operation of vertex identification is allowed. For list-coloring, the vertices may have different lists, so it is infeasible to identify vertices in general. To overcome this difficulty, Dvořák and Postle [4] introduced DP-coloring under the name "correspondence coloring", showing that every planar graph without cycles of lengths 4 to 8 is 3-choosable.

<sup>\*</sup>Corresponding author: wangtao@henu.edu.cn; iwangtao8@gmail.com

**Definition 1.** Let G be a simple graph and L be a list-assignment for G. For each vertex  $v \in V(G)$ , let  $L_v = \{v\} \times L(v)$ ; for each edge  $uv \in E(G)$ , let  $\mathscr{M}_{uv}$  be a matching between the sets  $L_u$  and  $L_v$ , and let  $\mathscr{M} = \bigcup_{uv \in E(G)} \mathscr{M}_{uv}$ , called the **matching assignment**. The matching assignment is called k-matching assignment if L(v) = [k] for each  $v \in V(G)$ . A cover of G is a graph  $H_{L,\mathscr{M}}$  (simply write H) satisfying the following two conditions:

- (C1) the vertex set of H is the disjoint union of  $L_v$  for all  $v \in V(G)$ ;
- (C2) the edge set of H is the matching assignment  $\mathcal{M}$ .

Note that the matching  $\mathcal{M}_{uv}$  is not required to be a perfect matching between the sets  $L_u$  and  $L_v$ , and possibly it is empty. The induced subgraph  $H[L_v]$  is an independent set for each vertex  $v \in V(G)$ .

**Definition 2.** Let G be a simple graph and H be a cover of G. An  $\mathcal{M}$ -coloring of H is an independent set  $\mathcal{I}$  in H such that  $|\mathcal{I} \cap L_v| = 1$  for each vertex  $v \in V(G)$ . The graph G is **DP**-k-colorable if for any list-assignment  $|L(v)| \ge k$  and any matching assignment  $\mathcal{M}$ , it has an  $\mathcal{M}$ -coloring. The **DP**-chromatic number  $\chi_{\text{DP}}(G)$  of G is the least integer k such that G is DP-k-colorable.

We mainly concentrate on DP-coloring of planar graphs in this paper. Dvořák and Postle [4] noticed that  $\chi_{DP}(G) \leq 5$  if G is a planar graph, and  $\chi_{DP}(G) \leq 3$  if G is a planar graph with girth at least five. Several groups have given sufficient conditions for a planar graph to be DP-3-colorable, which extends the 3-choosability of such graphs.

**Theorem 1.1** (Liu et al. [16]). A planar graph is DP-3-colorable if it satisfies one of the following conditions:

- (1) it contains no 3, 6, 7, 8-cycles.
- (2) it contains no 3, 5, 6-cycles.
- (3) it contains no 4, 5, 6, 9-cycles.
- (4) it contains no 4, 5, 7, 9-cycles.
- (5) the distance of triangles is at least two and it contains no 5, 6, 7-cycles.

**Theorem 1.2** (Liu et al. [15]). If a and b are distinct values from  $\{6, 7, 8\}$ , then every planar graph without 4-, a-, b-, 9-cycles is DP-3-colorable.

Zhang and Wu [32] showed that every planar graph without 4-, 5-, 6- and 9-cycles is 3-choosable. Zhang [28] generalized this result by showing that every planar graph with neither adjacent triangles nor 5-, 6- and 9-cycles is 3-choosable. Liu et al. [16] showed that every planar graph without 4-, 5-, 6- and 9-cycles is DP-3-colorable. In this paper, we first extend these results by showing the following theorem.

Theorem 1.3. Every planar graph with neither adjacent triangles nor 5-, 6- and 9-cycles is DP-3-colorable.

The distance of two triangles T and T' is defined as the value  $\min\{\operatorname{dist}(x, y) : x \in T \text{ and } y \in T'\}$ , where  $\operatorname{dist}(x, y)$  is the distance of the two vertices x and y. In general, we use  $\operatorname{dist}^{\nabla}$  to denote the minimum distance of two triangles in a graph. Yin and Yu [26] gave the following Bordeaux condition for planar graphs to be DP-3-colorable.

**Theorem 1.4** (Yin and Yu [26]). A planar graph is DP-3-colorable if it satisfies one of the following two conditions:

(1) the distance of triangles is at least three and it contains no 4, 5-cycles.

$\operatorname{dist}^{\nabla}$	3	4	5	6	7	8	9	10	list-3-coloring	DP-3-colorable
	X	X	-				-	-	Thomassen, 1995 [20]	Dvořák, Postle, 2018 [4]
	X		X	X					Lam, Shiu, Song, 2005 [8]	Liu et al. 2019 [16]
	X			X	X				Dvořák et al. 2010 [3]	?
	X			X	X		X		Zhang, Xu, 2004 [33]	
	X			X	X	×			Lidický, 2009 [12]	Liu et al. 2019 [16]
	X				X	×			Dvořák et al. 2009 $[2]$	?
		X	×	X	X	×			Dvořák, Postle, 2018 $[4]$	?
		X	X	ø			X		Zhang, Wu, 2005 [32]	Liu et al. 2019 [16]
		X	X		X		X		Zhang, Wu, 2004 [31]	Liu et al. 2019 [16]
		X	X		X			X	Zhang, 2012 [27]	?
		X	X			×	X		Wang, Lu, Chen, 2010 [23]	?
		X		ø	X		X		Wang, Lu, Chen, 2008 [22]	Liu et al. 2019 [15]
		X		X		×	X		Shen, Wang, 2007 [19]	Liu et al. 2019 [15]
		X			X	×	X		Wang, Wu, Shen, 2011 [25]	Liu et al. 2019 [15]
		X	X			×		X	Wang, Wu, 2011 [24]	?
$\geq 3$		X	X						derived from [26]	Yin, Yu, 2019 [26]
$\geq 2$		X	X	X					derived from [26]	Yin, Yu, 2019 [26]
$\geq 2$			X	ø	X				Li, Chen, Wang, 2016 [9]	Liu et al. 2019 [16]
$\geq 2$			X	X		×			Zhang, Sun, 2008 [30]	this paper
$\geq 3$			X	ø				X	Zhang, 2016 [29]	
$\geq 3$		X			X		X		Li, Wang, 2016 [11]	
$\geq 2$		X	X		X				Han, 2009 [6]	this paper

Table 1: List-3-coloring and DP-3-coloring.

(2) the distance of triangles is at least two and it contains no 4, 5, 6-cycles.

Theorem 1.4 implies the following new results on 3-choosability.

Corollary 1.5. A planar graph is 3-choosable if it satisfies one of the following conditions:

- (1) the distance of triangles is at least three and it contains no 4, 5-cycles.
- (2) the distance of triangles is at least two and it contains no 4, 5, 6-cycles.

The following are two Bordeaux-type results on 3-choosability.

**Theorem 1.6** (Zhang and Sun [30]). Every planar graph with neither 5-, 6-, 8-cycles nor triangles at distance less than two is 3-choosable.

**Theorem 1.7** (Han [6]). Every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is 3-choosable.

In the last section, we give two Bordeaux-type results on DP-3-coloring. The first one improves Theorem 1.6 and the second one improves Theorem 1.7.

**Theorem 1.8.** Every planar graph with neither 5-, 6-, 8-cycles nor triangles at distance less than two is DP-3-colorable.

**Theorem 1.9.** Every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is DP-3-colorable.

It is observed that every k-degenerate graph is DP-(k+1)-colorable. Theorem 1.9 can be derived from the following Theorem 1.10.

**Theorem 1.10.** Every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is 2-degenerate.

For more results on DP-coloring of planar graphs, we refer the reader to [1, 7, 10, 13, 14, 17]. For convenience, we collect some results on list-3-coloring and DP-3-coloring in Table 1. If uv is incident with a 7<sup>+</sup>-face and a 4<sup>-</sup>-face, then we say uv controls the 4<sup>-</sup>-face. Similarly, if uv is on a 7<sup>+</sup>-cycle and a 4<sup>-</sup>-cycle, then we say uv controls the 4<sup>-</sup>-cycle. A vertex v on a 7<sup>+</sup>-face f is rich to f if none of the two incident edges on f control a 4<sup>-</sup>-face, semi-rich if exactly one of the two incident edges on f controls a 4<sup>-</sup>-face. A 3-vertex v is weak if v is incident with a 3-face, semi-weak if v is incident with a 4-face, and strong if v is incident with no 4<sup>-</sup>-face.

For a face  $f \in F$ , if all the vertices on f in a cyclic order are  $v_1, v_2, \ldots, v_k$ , then we write  $f = v_1 v_2 \ldots v_k$ , and call f a  $(d(v_1), d(v_2), \ldots, d(v_k))$ -face. A face is called a *k*-regular face if every vertex incident with it is a *k*-vertex. A  $(d_1, d_2, \ldots, d_t)$ -path  $v_1 v_2 \ldots v_t$  on a face g is a set of consecutive vertices along the facial walk of g such that  $d(v_i) = d_i$  and the vertices are different. The notions of  $d^+$  (or  $d^-$ ) are similarly for  $d(v) \ge d$ (or  $d(v) \le d$ ).

# 2 Preliminary

In this short section, some preliminary results are given, and these results can be used separately elsewhere. Liu et al. [16] showed the "nearly (k - 1)-degenerate" subgraph is reducible for DP-k-coloring.

**Lemma 2.1** (Liu et al. [16]). Let  $k \ge 3$ , K be a subgraph of G and G' = G - V(K). If the vertices of K can be ordered as  $v_1, v_2, \ldots, v_t$  such that the following hold:

- (1)  $|V(G') \cap N_G(v_1)| < |V(G') \cap N_G(v_t)|;$
- (2)  $d_G(v_t) \leq k$  and  $v_1 v_t \in E(G)$ ;
- (3) for each  $2 \le i \le t 1$ ,  $v_i$  has at most k 1 neighbors in  $G \{v_{i+1}, v_{i+2}, \dots, v_t\}$ ,

then any DP-k-coloring of G' can be extended to a DP-k-coloring of G.

A graph is **minimal non-DP**-*k*-colorable if it is not DP-*k*-colorable but every subgraph with fewer vertices is DP-*k*-colorable. We give more specific reducible "nearly 2-degenerate" configuration for DP-3-coloring.

**Lemma 2.2.** Suppose that G is a minimal non-DP-3-colorable graph and it has no adjacent 4<sup>-</sup>-cycles. Let  $\mathcal{C}$  be an m-cycle  $v_1v_2\ldots v_m$ , let  $X = \{i : d(v_i) = 4, 1 \le i \le m\}$  and  $E^+ = \{v_iv_{i+1} : i \in X\} \cup \{v_mv_1\}$ . If  $v_m$  is a 3-vertex and  $v_mv_1$  controls a 3-cycle  $v_mv_1u$  or a 4-cycle  $v_mv_1uw$ , then G contains no configuration satisfying all the following conditions:

- (i) every edge e in  $E^+$  controls a 4<sup>-</sup>-cycle  $C_e$ ;
- (ii) all the vertices on C and the other vertices on cycles controlled by  $E^+$  are distinct;
- (iii) every vertex on C is a 4<sup>-</sup>-vertex;

- (iv) every vertex on cycles controlled by  $E^+$  but not on C is a 3-vertex;
- (v) the vertex u has a neighbor neither on  $\mathcal{C}$  nor on the cycles controlled by  $E^+$ .

**Proof.** Suppose to the contrary that there exists such a configuration. For the path  $P = v_1 v_2 \dots v_m$ , replace each edge  $v_i v_{i+1}$  in  $E(P) \cap E^+$  by the other part of the controlled cycle, and append  $v_m u$  (when  $v_m v_1 u$  is the controlled 3-cycle) or  $v_m w u$  (when  $v_m v_1 u w$  is the controlled 4-cycle) at the end. This yields a path starting at  $v_1$  and ending at u. This path trivially corresponds to a sequence of vertices, and the sequence satisfies the condition of Lemma 2.1 with k = 3, a contradiction.

By the definition of minimal non-DP-k-colorable, it is easy to obtain the following lemma.

**Theorem 2.1.** If G is a minimal non-DP-k-colorable graph, then  $\delta(G) \ge k$ .

The following structural result for minimal non-DP-k-colorable graphs is a consequences of Theorem in [18].

**Theorem 2.2.** Let G be a graph and B be a 2-connected induced subgraph of G with  $d_G(v) = k$  for all  $v \in V(B)$ . If G is a minimal non-DP-k-colorable graph, then B is a cycle or a complete graph.

# 3 Proof of Theorem 1.3

Recall that our first main result is the following.

Theorem 1.3. Every planar graph with neither adjacent triangles nor 5-, 6- and 9-cycles is DP-3-colorable.

**Proof.** Let G be a counterexample to the theorem with fewest number of vertices. We may assume that G has been embedded in the plane. Thus, it is a minimal non-DP-3-colorable graph with  $\delta(G) \geq 3$ , and

- (1) G is connected;
- (2) G is a plane graph without adjacent triangles and 5-, 6-, 9-cycles;
- (3) G is not DP-3-colorable;
- (4) any subgraph with fewer vertices is DP-3-colorable.

A poor face is a 10-face incident with ten 3-vertices, adjacent to one 4-face and four 3-faces. A **bad face** is a 10-face incident with ten 3-vertices and adjacent to five 3-faces. A **bad vertex** is a 3-vertex on a bad face. A **bad edge** is an edge on the boundary of a bad face. A **special face** is a (3, 3, 3, 3, 3, 4, 3, 3, 4, 3)-face adjacent to six 3-faces. A **semi-special face** is a (3, 3, 3, 3, 3, 4, 3, 3, 4, 3)-face adjacent to five 3-faces and one 4-face as depicted in Fig. 1d. An illustration of these faces is in Fig. 1.

By Theorem 2.2, we can easy obtain the following structural result.

**Lemma 3.1.** Let f be a 10-face bounded by a cycle in G. If f is incident with ten 3-vertices and it controls a 4<sup>-</sup>-face, then the controlled 4<sup>-</sup>-face is incident with at least one 4<sup>+</sup>-vertex.

By Lemma 3.1 and the definitions of poor faces and bad faces, we have the following consequences.

### Lemma 3.2.

- (i) There are no adjacent poor faces.
- (ii) There are no adjacent bad faces.

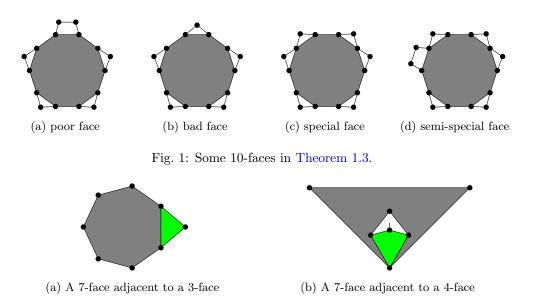


Fig. 2: A 7-face adjacent to a 4<sup>-</sup>-face.

(iii) There are no poor faces adjacent to bad faces.

The following structural results will be frequently used.

#### Lemma 3.3.

- (a) Every 7<sup>-</sup>-cycle is chordless.
- (b) Every 3-cycle is not adjacent to 6<sup>-</sup>-cycle.
- (c) Every 7-face is adjacent to at most one 4<sup>-</sup>-face; the possible situations see Fig. 2. Consequently, there are no bad faces adjacent to 7-faces.
- (d) No 8-face is adjacent to a 3-face; no 9-face is adjacent to a 3-face.
- (e) There are no adjacent  $6^-$ -faces; thus every 3-vertex is adjacent to at most one  $4^-$ -face.

**Proof.** (a) If a 4-cycle has a chord, then there are two adjacent triangles. Note that 5-cycles and 6-cycles are excluded in G. If a 7-cycle has a chord, then there is a 5- or 6-cycle, a contradiction.

(b) Note that 5-cycles and 6-cycles are excluded in G. If a 3-cycle is adjacent to a 3- or 4-cycle, then it contradicts Lemma 3.3(a).

(c) Let f be a 7-face and C be its boundary. (i) Suppose that C is a cycle. If  $w_1w_2w_3w_4$  is on the boundary and  $w_2w_3$  is incident with a 4-face  $u_1w_2w_3u_4$ , then none of  $u_1$  and  $u_4$  is on C because C is chordless and  $\delta(G) \geq 3$ , but C and  $u_1w_2w_3u_4$  form a 9-cycle, a contradiction. Suppose that f is adjacent to two 3-faces uvwand u'v'w' with uv, u'v' on C. If w = w', then there are two adjacent triangles or a 5-cycle, a contradiction; and if  $w \neq w'$ , then there is a 9-cycle, a contradiction. (ii) Suppose that C is not a cycle, and thus it consists of a 3-cycle and a 4-cycle. Hence, f cannot be adjacent to any 3-face by Lemma 3.3(b). If f is adjacent to a 4-face, then it can only be shown in Fig. 2b. Therefore, f is adjacent to at most one 4<sup>-</sup>-face.

(d) If an 8-face is bounded by a cycle, then it cannot be adjacent to a 3-face, otherwise they form a 9-cycle or a 8-cycle with two chords, a contradiction. Suppose that the boundary of an 8-face is not a cycle but it is adjacent to a 3-face. By Lemma 3.3(b), the boundary of the 8-face must contain a 7<sup>+</sup>-cycle, but this is impossible.

Since there is no 9-cycle, the boundary of a 9-face is not a cycle. Suppose the boundary of a 9-face is adjacent to a 3-face. By Lemma 3.3(b), the boundary of the 9-face must contain a 7<sup>+</sup>-cycle, but this is impossible.

(e) Since there is no 6-cycle, the boundary of a 6-face consists of two triangles. It is easy to check that there are no adjacent  $6^-$ -faces.

**Lemma 3.4.** Each  $(3, 3, 3^+, 4^+)$ -face f is adjacent to at most one poor face.

**Proof.** Since every poor face is incident with ten 3-vertices, f can only be adjacent to poor faces via (3, 3)edges. Let  $f = v_1 v_2 v_3 v_4$  with  $d(v_1) = d(v_2) = 3$ ,  $d(v_3) \ge 3$  and  $d(v_4) \ge 4$ . If  $d(v_3) \ge 4$ , then f is incident
with exactly one (3, 3)-edge, and then it is adjacent to at most one poor face. Suppose that  $d(v_3) = 3$  and fis adjacent to two poor faces  $f_1$  and  $f_2$  via  $v_1 v_2$  and  $v_2 v_3$ . Since  $v_2$  is a 3-vertex, the poor face  $f_1$  is adjacent
to the poor face  $f_2$ , but this contradicts Lemma 3.2.

Lemma 3.5. Each bad face is adjacent to at most two special faces.

**Proof.** Let  $f = v_1v_2 \dots v_{10}$  be a bad face and incident with five 3-faces  $v_1v_2u_1, v_3v_4u_3, v_5v_6u_5, v_7v_8u_7, v_9v_{10}u_9$ . Suppose that f is adjacent to  $f_i$  via edge  $v_iv_{i+1}$  for  $1 \le i \le 10$ , where the subscripts are taken modulo 10. Suppose to the contrary that f is adjacent to at least three special faces. Then there exist two special faces  $f_m$  and  $f_n$  such that |m - n| = 2 or 8, where  $\{m, n\} \subset \{2, 4, 6, 8, 10\}$ . Without loss of generality, assume that  $f_2$  and  $f_4$  are the two special faces. By Lemma 2.2 and the definition of special face,  $d(u_1) = d(u_3) = 4$ . Let  $x_3$  and  $x_4$  be the neighbors of  $u_3$  other than  $v_3$  and  $v_4$ . Since  $f_2$  and  $f_4$  are special faces, we have that  $x_3x_4 \in E(G)$  and  $d(x_3) = d(x_4) = 3$ , but this contradicts Lemma 2.2.

**Lemma 3.6.** Suppose that f is a 10<sup>+</sup>-face and it is not a bad face. Let t be the number of incident bad edges, and  $t \ge 1$ . Then  $3t \le d(f)$ . Moreover, if d(f) > 3t, then f is incident with at least (t+1) 4<sup>+</sup>-vertices (repeated vertices are counted as the number of appearance on the boundary).

**Proof.** Suppose that f is adjacent to a bad face through uv. Let x be the neighbor of u on f and y be the neighbor of v on f. Then u and v are bad vertices and the faces controlled by f through xu and vy are all 3-faces. By Lemma 3.1 and the definition of bad face,  $d(x) \ge 4$  and  $d(y) \ge 4$ . It is observed that two bad edges are separated by at least two other edges along the boundary of f, this implies that  $3t \le d(f)$ .

By the above discussion, every bad vertex has a 4<sup>+</sup>-neighbor along the boundary. Since 3t < d(f), there are two bad edges separated by at least two 4<sup>+</sup>-vertices, thus f is incident with at least (t+1) 4<sup>+</sup>-vertices.  $\Box$ 

To prove the theorem, we are going to use discharging method. Define the initial charge function  $\mu(x)$ on  $V \cup F$  to be  $\mu(v) = d(v) - 6$  for  $v \in V$  and  $\mu(f) = 2d(f) - 6$  for  $f \in F$ . By Euler's formula, we have the following equality,

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12.$$

We design suitable discharging rules to change the initial charge function  $\mu(x)$  to the final charge function  $\mu'(x)$  on  $V \cup F$  such that  $\mu'(x) \ge 0$  for all  $x \in V \cup F$ , this leads to a contradiction and completes the proof.

The following are the needed discharging rules.

- **R1** Each 4-face sends  $\frac{1}{2}$  to each incident 3-vertex.
- R2 Each 6-face sends 1 to each incident vertex.
- **R3** Each 7-face sends  $\frac{3}{2}$  to each incident semi-rich 3-vertex, 1 to each other incident vertex.

- **R4** Each 8-face sends  $\frac{5}{4}$  to each incident vertex.
- **R5** Each 9-face sends  $\frac{4}{3}$  to each incident vertex.
- **R6** Suppose that v is a 3-vertex incident with a  $10^+$ -face f and two other faces g and h.
  - (a) If v is incident with three  $5^+$ -faces, then f sends 1 to v.
  - (b) If v is incident with a 4-face, then f sends  $\frac{5}{4}$  to v;
  - (c) If f is a bad face, g is a 3-face and h is not a special face, then f sends  $\frac{4}{3}$  to v and h sends  $\frac{5}{3}$  to v.
  - (d) Otherwise, f sends  $\frac{3}{2}$  to v.
- **R7** Let v be a 4-vertex on a  $10^+$ -face f.
  - (a) If v is a rich vertex or a poor vertex of f, then f sends 1 to v.
  - (b) Otherwise, f sends  $\frac{1}{2}$  to v.
- **R8** Let v be a 5-vertex on a  $10^+$ -face f.
  - (a) If v is incident with two 4<sup>-</sup>-face, then f sends  $\frac{1}{3}$  to v.
  - (b) If v is incident with exactly one 4<sup>-</sup>-face, then f sends  $\frac{1}{4}$  to v.
  - (c) Otherwise, f sends  $\frac{1}{5}$  to v.
- **R9** Each  $(3, 3, 3^+, 4^+)$ -face sends  $\frac{1}{2}$  to adjacent poor face.

**R10** Each  $(3, 4, 3^+, 4^+)$ -face and  $(3, 4, 4^+, 3^+)$ -face send  $\frac{1}{4}$  to each adjacent semi-special face.

It remains to check that the final charge of every element in  $V \cup F$  is nonnegative.

(1) Let v be an arbitrary vertex of G.

By Theorem 2.1, G has no 2<sup>-</sup>-vertices. If v is a 6<sup>+</sup>-vertex, then  $\mu'(v) \ge \mu(v) = d(v) - 6 \ge 0$ . We may assume that  $3 \le d(v) \le 5$ .

Suppose that v is a 3-vertex. By Lemma 3.3(e), v is incident with at most one 4<sup>-</sup>-face. If v is incident with no 4<sup>-</sup>-face, then it receives at least 1 from each incident face, and then  $\mu'(v) \ge 3-6+3\times 1=0$ . If v is incident with a 4-face, then it receives at least  $\frac{5}{4}$  from each incident 7<sup>+</sup>-face, and then  $\mu'(v) \ge 3-6+2\times \frac{5}{4}+\frac{1}{2}=0$ . If v is incident with a 3-face and a 7-face, then the other incident face is not a bad face by Lemma 3.3(c), and then  $\mu'(v) = 3-6+2\times \frac{3}{2}=0$ . By Lemma 3.3(d), if v is incident with a 3-face, then it is not incident with a 3-face. If v is incident with a 3-face and two 10<sup>+</sup>-faces, then  $\mu'(v) \ge 3-6+\min\{\frac{4}{3}+\frac{5}{3},2\times\frac{3}{2}\}=0$ .

Suppose that v is a 4-vertex. By Lemma 3.3(e), v is incident with at most two 4<sup>-</sup>-faces. If v is incident with exactly one 4<sup>-</sup>-face, then  $\mu'(v) \ge 4 - 6 + 2 \times \frac{1}{2} + 1 = 0$ . If v is incident with two 4<sup>-</sup>-faces, then  $\mu'(v) \ge 4 - 6 + 2 \times 1 = 0$ . If v is incident with no 4<sup>-</sup>-face, then  $\mu'(v) \ge 4 - 6 + 4 \times 1 > 0$ .

Suppose that v is a 5-vertex. By Lemma 3.3(e), v is incident with at most two 4<sup>-</sup>-faces. If v is incident with no 4<sup>-</sup>-face, then it receives at least  $\frac{1}{5}$  from each incident 5<sup>+</sup>-face, and  $\mu'(v) \ge 5 - 6 + 5 \times \frac{1}{5} = 0$ . If v is incident with exactly one 4<sup>-</sup>-face, then it receives at least  $\frac{1}{4}$  from each incident 5<sup>+</sup>-face, and  $\mu'(v) \ge 5 - 6 + 4 \times \frac{1}{4} = 0$ . If v is incident with two 4<sup>-</sup>-faces, then it receives at least  $\frac{1}{3}$  from each incident 5<sup>+</sup>-face, and  $\mu'(v) \ge 5 - 6 + 4 \times \frac{1}{4} = 0$ . If v is incident with two 4<sup>-</sup>-faces, then it receives at least  $\frac{1}{3}$  from each incident 5<sup>+</sup>-face, and  $\mu'(v) \ge 5 - 6 + 3 \times \frac{1}{3} = 0$ .

(2) Let f be an arbitrary face in F(G).

If f is a 3-face, then  $\mu'(f) = \mu(f) = 0$ . Suppose that f is a 4-face. If f is incident with four 3-vertices, then  $\mu'(f) = 2 - 4 \times \frac{1}{2} = 0$ . If f is incident with exactly one 4<sup>+</sup>-vertex, then it is adjacent to at most one poor face by Lemma 3.4, and then  $\mu'(f) \ge 2 - 3 \times \frac{1}{2} - \frac{1}{2} = 0$ . If f is a  $(3, 3, 4^+, 4^+)$ -face, then it is adjacent

to at most one poor face and at most two semi-special faces, and then  $\mu'(f) \ge 2 - 2 \times \frac{1}{2} - \frac{1}{2} - 2 \times \frac{1}{4} = 0$ . If f is a  $(3, 4^+, 3, 4^+)$ -face, then it sends at most  $\frac{1}{4}$  to each adjacent face, and  $\mu'(f) \ge 2 - 2 \times \frac{1}{2} - 4 \times \frac{1}{4} = 0$ . If f is incident with exactly three 4<sup>+</sup>-vertices, then it is adjacent to at most two semi-special faces, and  $\mu'(f) \ge 2 - \frac{1}{2} - 2 \times \frac{1}{4} > 0$ . If f is incident with four 4<sup>+</sup>-vertices, then  $\mu'(f) = \mu(f) = 2$ .

If f is a 6-face, then  $\mu'(f) = 6 - 6 \times 1 = 0$ . Suppose that f is a 7-face. By Lemma 3.3(c), f is adjacent to at most one 4<sup>-</sup>-face. If f is adjacent to a 4<sup>-</sup>-face (see Fig. 2), then f is incident with at most two semi-rich 3-vertices, which implies that  $\mu'(f) \ge 8 - 2 \times \frac{3}{2} - 5 \times 1 = 0$ . If f is not adjacent to any 4<sup>-</sup>-face, then f sends 1 to each incident vertex, and  $\mu'(f) = 8 - 7 \times 1 > 0$ . If f is an 8-face, then  $\mu'(f) = 10 - 8 \times \frac{5}{4} = 0$ . If f is a 9-face, then  $\mu'(f) = 12 - 9 \times \frac{4}{3} = 0$ .

Suppose that f is a 10<sup>+</sup>-face. Let t be the number of incident bad edges. Hence, f is incident with exactly 2t bad vertices. By Lemma 3.6, f is incident with at least t 4<sup>+</sup>-vertices. Thus,  $\mu'(f) \ge 2d(f) - 6 - 2t \times \frac{5}{3} - t \times 1 - (d(f) - 3t) \times \frac{3}{2} = \frac{1}{2}d(f) - 6 + \frac{t}{6}$ . If  $d(f) \ge 12$ , then  $\mu'(f) \ge 12 \times \frac{1}{2} - 6 + \frac{t}{6} \ge 0$ . So it suffices to consider 10-faces and 11-faces.

Suppose that f is an 11-face. (i) t = 0. It follows that f is not incident with any bad vertex, and it sends at most  $\frac{3}{2}$  to each incident vertex. If f is incident with a 4<sup>+</sup>-vertex, then  $\mu'(f) \ge 16 - 10 \times \frac{3}{2} - 1 = 0$ . Suppose that f is a 3-regular face. By Lemma 3.3(e), every vertex on f is incident with at most one 4<sup>-</sup>-face. Since d(f) is odd, f must be incident with a rich 3-vertex. This implies that  $\mu'(f) \ge 16 - 10 \times \frac{3}{2} - 1 = 0$ . (ii)  $t \ge 1$ . It follows that f is incident with exactly 2t bad vertices and at least (t+1) 4<sup>+</sup>-vertices, and then  $\mu'(f) \ge 16 - 2t \times \frac{5}{3} - (t+1) \times 1 - (11 - (3t+1)) \times \frac{3}{2} = \frac{t}{6} > 0$ .

Finally we may assume that f is a 10-face. If f is a special face, then  $\mu'(f) = 14 - 8 \times \frac{3}{2} - 2 \times 1 = 0$ . If f is a bad face, then it is adjacent to at most two special faces by Lemma 3.5, which implies that  $\mu'(f) \ge 14 - 4 \times \frac{3}{2} - 6 \times \frac{4}{3} = 0$ .

So we may assume that f is neither a bad face nor a special face. By Lemma 3.6,  $t \leq \lfloor \frac{d(f)}{3} \rfloor = 3$ .

• t = 0. It follows that f is not incident with any bad vertex. Hence, f sends at most  $\frac{3}{2}$  to each incident 3-vertex, at most 1 to each incident 4-vertex, and at most  $\frac{1}{3}$  to each incident 5-vertex. If f is incident with a 5<sup>+</sup>-vertex, then  $\mu'(f) \ge 14 - 9 \times \frac{3}{2} - \frac{1}{3} > 0$ . If f is incident with at least two 4-vertices, then  $\mu'(f) \ge 14 - 8 \times \frac{3}{2} - 2 \times 1 = 0$ .

So we may assume that f is incident with at most one 4-vertex and no 5<sup>+</sup>-vertices. If f is incident with a semi-rich 4-vertex, then  $\mu'(f) \ge 14 - 9 \times \frac{3}{2} - \frac{1}{2} = 0$ . If f is incident with a rich 4-vertex and nine 3-vertices, then at least one of the incident 3-vertices is rich, and then  $\mu'(f) \ge 14 - 1 - 8 \times \frac{3}{2} - 1 = 0$ . If f is incident with a poor 4-vertex, then there exists a rich 3-vertex incident with f, and  $\mu'(f) \ge 14 - 1 - 8 \times \frac{3}{2} - 1 = 0$ .

Suppose that f is incident with ten 3-vertices. If f is adjacent to at most four 4<sup>-</sup>-faces, then  $\mu'(f) \ge 14 - 8 \times \frac{3}{2} - 2 \times 1 = 0$ . If f is adjacent to at least two 4-faces, then  $\mu'(f) \ge 14 - 6 \times \frac{3}{2} - 4 \times \frac{5}{4} = 0$ . If f is adjacent to four 3-faces and one 4-face, then f must be a poor face and the 4-face must be  $(3, 3, 3^+, 4^+)$ -face, and then  $\mu'(f) = 14 - 8 \times \frac{3}{2} - 2 \times \frac{5}{4} + \frac{1}{2} = 0$ . If f is adjacent to five 3-faces, then it is a bad face, so we are done.

• t = 1. It follows that f is incident with exactly two bad vertices and at least two 4<sup>+</sup>-vertices. If f is incident with a rich 3-vertex or at least three 4-vertices, then  $\mu'(f) \ge 14 - 2 \times \frac{5}{3} - 3 \times 1 - 5 \times \frac{3}{2} > 0$ . If f is incident with a 5<sup>+</sup>-vertex, then  $\mu'(f) \ge 14 - 2 \times \frac{5}{3} - 1 - \frac{1}{3} - 6 \times \frac{3}{2} > 0$ . If f is incident with a semi-rich 4-vertex, then  $\mu'(f) \ge 14 - 2 \times \frac{5}{3} - 1 - \frac{1}{2} - 6 \times \frac{3}{2} > 0$ . So we may assume that f is incident with two poor 4-vertices and eight semi-rich 3-vertices. Thus f is a (3, 4, 3, 3, 4, 3, 3, 3, 3)-face  $w_1w_2 \dots w_{10}$ , where  $w_3w_4$  is a bad edge, each of  $w_2w_3$  and  $w_4w_5$  controls a 3-face, each of  $w_1w_2$  and  $w_5w_6$  controls a 4<sup>-</sup>-face. If f controls at least one 4-face by a (3,3)-edge, then  $\mu'(f) \ge 14 - 2 \times \frac{5}{3} - 2 \times 1 - 2 \times \frac{5}{4} - 4 \times \frac{3}{2} > 0$ . So we may further assume that each of  $w_7w_8$  and  $w_9w_{10}$  controls a 3-face. If each of  $w_1w_2$  and  $w_5w_6$  controls a 4-face, then  $\mu'(f) \ge 14 - 2 \times \frac{5}{3} - 2 \times 1 - 2 \times \frac{5}{4} - 4 \times \frac{3}{2} > 0$ . So we may



Fig. 3: Certain 7-faces in Theorem 1.8.

must be a special face, a contradiction. If exactly one of  $w_1w_2$  and  $w_5w_6$  controls a 4-face, then f must be a semi-special face. In this case the controlled 4-face must be a  $(3, 4, 3^+, 4^+)$ -face or  $(3, 4, 4^+, 3^+)$ -face due to Lemma 2.2. Thus  $\mu'(f) \ge 14 - 2 \times \frac{5}{3} - 2 \times 1 - \frac{5}{4} - 5 \times \frac{3}{2} + \frac{1}{4} \ge 0$ .

• t = 2. It follows that f is incident with exactly four bad vertices and at least three 4<sup>+</sup>-vertices. If f is incident with at least four 4<sup>+</sup>-vertices, then  $\mu'(f) \ge 14 - 4 \times \frac{5}{3} - 4 \times 1 - (10 - 4 - 4) \times \frac{3}{2} > 0$ . Thus, f is incident with exactly four bad vertices and exactly three 4<sup>+</sup>-vertices. If there is a semi-rich 4<sup>+</sup>-vertex, then  $\mu'(f) \ge 14 - 4 \times \frac{5}{3} - 2 \times 1 - \frac{1}{2} - 3 \times \frac{3}{2} > 0$ . Therefore, the three 4<sup>+</sup>-vertices are all poor, so there must be a rich 3-vertex. This implies that  $\mu'(f) \ge 14 - 4 \times \frac{5}{3} - 3 \times 1 - 2 \times \frac{3}{2} - 1 > 0$ .

• t = 3. It follows that f is incident with six bad vertices and four 4<sup>+</sup>-vertices. Thus,  $\mu'(f) \ge 14 - 6 \times \frac{5}{3} - 4 \times 1 = 0$ .

### 4 Distance of triangles at least two

In this section, we give two Bordeaux type results on planar graphs with distance of triangles at least two.

### 4.1 Planar graphs without 5-, 6- and 8-cycles

Recall that our second main result is the following.

**Theorem 1.8.** Every planar graph with neither 5-, 6-, 8-cycles nor triangles at distance less than two is DP-3-colorable.

**Proof.** Suppose to the contrary that G is a counterexample with the number of vertices as small as possible. We may assume that G has been embedded in the plane. Thus, G is a minimal non-DP-3-colorable graph.

### Lemma 4.1.

- (a) There are no 5-faces and no 6-faces.
- (b) A 3-face cannot be adjacent to an 8<sup>-</sup>-face.
- (c) There are no adjacent 6<sup>-</sup>-faces.

**Proof.** Since every 5-face is bounded by a 5-cycle, but there is no 5-cycles in G, this implies that there is no 5-faces in G. Since there is no 6-cycles in G, the boundary of every 6-face consists of two triangles, thus the distance of these triangles is zero, a contradiction. Therefore, there is no 5-faces and no 6-faces in G.

It is easy to check that every 7<sup>-</sup>-cycle is chordless. Let f be a 3-face. If f is adjacent to a 4-face g, then they form a 5-cycle with a chord, a contradiction. Suppose that g is a 7-face. Then g may be bounded by a cycle or a closed walk with a cut-vertex. If g is bounded by a cycle and it is adjacent to f, then these two cycles form an 8-cycle with a chord, a contradiction. If the boundary of g contains a cut-vertex, then the boundary consists of a 3-cycle and a 4-cycle, and neither the 3-cycle nor the 4-cycle can be adjacent to the 3-face f. If g is an 8-face, then the boundary of g consists of two 4-cycles, or two triangles and a cut-edge, but no edge on such boundary can be adjacent to the 3-face f.

By the hypothesis and fact that a 3-face cannot be adjacent to an  $8^-$ -face, it suffices to prove that there is no adjacent 4-faces. Since every 4-cycle has no chords, two adjacent 4-faces must form a 6-cycle with a chord, a contradiction.

A 7-face f is **special** if f is incident with six semi-weak 3-vertices and a poor 4-vertex, see Fig. 3a. A 7-face f is **poor** if f is incident with six semi-weak 3-vertices and a strong 3-vertex, see Fig. 3b. Note that a 7-face is not adjacent to any 3-face by Lemma 4.1(b).

**Lemma 4.2.** Each poor 7-face is adjacent to three  $(3, 3, 3^+, 4^+)$ -faces.

**Proof.** Suppose that  $f = w_1 w_2 \dots w_7$  is a poor 7-face and it is adjacent to a 4-face  $g = u_1 w_2 w_3 u_4$ . Since f is incident with seven 3-vertices, it must be bounded by a 7-cycle. Note that every 7-cycle has no chords, we have that  $\{u_1, u_2\} \cap \{w_1, w_2, \dots, w_7\} = \emptyset$ . The subgraph induced by  $\{w_1, w_2, \dots, w_7\} \cup \{u_1, u_2\}$  is 2-connected, and it is neither a complete graph nor a cycle. By Theorem 2.2, g must be incident with a 4<sup>+</sup>-vertex.

Applying Lemma 2.2 to a special 7-face, we get the following result.

**Lemma 4.3.** If f is a special 7-face and it controls a (3,3,3,3)-face, then each of the face controlled by (3,4)-edge has at least two 4<sup>+</sup>-vertices.

Define the initial charge function  $\mu(x)$  on  $V \cup F$  to be  $\mu(v) = d(v) - 6$  for  $v \in V$  and  $\mu(f) = 2d(f) - 6$  for  $f \in F$ . By Euler's formula, we have the following equality,

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12.$$

We give some discharging rules to change the initial charge function  $\mu(x)$  to the final charge function  $\mu'(x)$  on  $V \cup F$  such that  $\mu'(x) \ge 0$  for all  $x \in V \cup F$ , which leads to a contradiction.

The following are the discharging rules.

- **R1** Each 4-face sends  $\frac{1}{2}$  to each incident 3-vertex.
- **R2** Each 7<sup>+</sup>-face sends  $\frac{3}{2}$  to each incident weak 3-vertex,  $\frac{5}{4}$  to each incident semi-weak 3-vertex, 1 to each incident strong 3-vertex.
- **R3** Each 7<sup>+</sup>-face sends 1 to each incident poor 4-vertex,  $\frac{3}{4}$  to each incident semi-rich 4-vertex,  $\frac{1}{2}$  to each incident rich 4-vertex.
- **R4** Each 7<sup>+</sup>-face sends  $\frac{1}{3}$  to each incident 5-vertex.
- **R5** Each  $(3, 3, 3^+, 4^+)$ -face sends  $\frac{1}{4}$  to each adjacent poor 7-face and each adjacent special 7-face through (3, 3)-edge, respectively.
- $\textbf{R6} \ \text{Each} \ (3,4,3^+,4^+) \text{-face and} \ (3,4,4^+,3^+) \text{-face sends} \ \tfrac{1}{4} \ \text{to each adjacent special 7-face through} \ (3,4) \text{-edge.} \ (3$

It remains to check that the final charge of every element in  $V \cup F$  is nonnegative.

(1) Let v be an arbitrary vertex of G.

By Theorem 2.1, G has no 2<sup>-</sup>-vertices. If v is a 6<sup>+</sup>-vertex, then it is not involved in the discharging procedure, hence  $\mu'(v) = \mu(v) = d(v) - 6 \ge 0$ . Next, we may assume that  $3 \le d(v) \le 5$ .

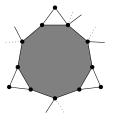


Fig. 4: A 9-face incident with exactly five weak 3-vertices.

Suppose that v is a 3-vertex. If v is incident with no 4<sup>-</sup>-face, then it is incident with three 7<sup>+</sup>-faces, and then  $\mu'(v) = \mu(v) + 3 \times 1 = 0$ . If v is incident with a 3-face, then the other two incident faces are 9<sup>+</sup>-faces by Lemma 4.1(b), and then  $\mu'(v) = \mu(v) + 2 \times \frac{3}{2} = 0$ . If v is incident with a 4-face, then the other two incident faces are 7<sup>+</sup>-faces by Lemma 4.1(c), and then  $\mu'(v) = \mu(v) + 2 \times \frac{3}{2} = 0$ .

Suppose that v is a 4-vertex. By Lemma 4.1(c), v is incident with at most two 4<sup>-</sup>-faces. If v is incident with no 4<sup>-</sup>-face, then it is incident with four 7<sup>+</sup>-faces, and then  $\mu'(v) = \mu(v) + 4 \times \frac{1}{2} = 0$ . If v is incident with exactly one 4<sup>-</sup>-face, then  $\mu'(v) = \mu(v) + 2 \times \frac{3}{4} + \frac{1}{2} = 0$ . If v is incident with exactly two 4<sup>-</sup>-faces, then  $\mu'(v) = \mu(v) + 2 \times \frac{3}{4} + \frac{1}{2} = 0$ . If v is incident with exactly two 4<sup>-</sup>-faces, then  $\mu'(v) = \mu(v) + 2 \times \frac{3}{4} + \frac{1}{2} = 0$ .

Suppose that v is a 5-vertex. By Lemma 4.1(c), v is incident with at most two 4<sup>-</sup>-faces. Therefore, it is incident with at least three 7<sup>+</sup>-faces, and  $\mu'(v) \ge \mu(v) + 3 \times \frac{1}{3} = 0$ .

(2) Let f be an arbitrary face in F(G).

Since the distance of triangles is at least two, each k-face is adjacent to at most  $\lfloor \frac{k}{3} \rfloor$  triangular-faces, thus f contains at most  $2 \times \lfloor \frac{k}{3} \rfloor$  weak 3-vertices. As observed above, G has no 5-face and 6-face. If f is a 3-face, then it is not involved in the discharging procedure, and then  $\mu'(f) = \mu(f) = 0$ .

Suppose that f is a 4-face. If f is incident with four 3-vertices, then  $\mu'(f) = \mu(f) - 4 \times \frac{1}{2} = 0$ . If f is incident with exactly one 4<sup>+</sup>-vertex, then f sends at most  $\frac{1}{4}$  through each incident (3,3)-edge, and then  $\mu'(f) \ge \mu(f) - 3 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$ . If f is incident with at least two 4<sup>+</sup>-vertices, then  $\mu'(f) \ge \mu(f) - 2 \times \frac{1}{2} - 4 \times \frac{1}{4} = 0$ .

If f is an 8-face, then it is not adjacent to any 3-face and it sends at most  $\frac{5}{4}$  to each incident vertex, and then  $\mu'(f) \ge \mu(f) - 8 \times \frac{5}{4} = 0$ .

Suppose that f is a 9-face. Recall that f is incident with at most six weak 3-vertices. If f is incident with exactly six weak 3-vertices, then f sends at most 1 to each other incident vertex, and then  $\mu'(f) \ge \mu(f) - 6 \times \frac{3}{2} - (9 - 6) \times 1 = 0$ . If f is incident with exactly five weak 3-vertices, then f must be adjacent to three 3-faces and one of the six incident vertices on triangles must be a 4<sup>+</sup>-vertex (see Fig. 4), and then  $\mu'(f) \ge \mu(f) - 5 \times \frac{3}{2} - 1 - \frac{5}{4} - 2 \times 1 > 0$ . If f is incident with exactly four weak 3-vertices and at least one 4<sup>+</sup>-vertex, then  $\mu'(f) \ge \mu(f) - 4 \times \frac{3}{2} - 1 - (9 - 4 - 1) \times \frac{5}{4} = 0$ . If f is incident with exactly four weak 3-vertices and at least one 4<sup>+</sup>-vertex, then  $\mu'(f) \ge \mu(f) - 4 \times \frac{3}{2} - 1 - (9 - 4 - 1) \times \frac{5}{4} = 0$ . If f is incident with exactly four weak 3-vertices and at least one 4<sup>+</sup>-vertex, then f is incident with at least one strong 3-vertex and at most four semi-weak 3-vertices, and then  $\mu'(f) \ge \mu(f) - 4 \times \frac{3}{2} - 4 \times \frac{5}{4} - 1 = 0$ . If f is incident with at most three weak 3-vertices, then  $\mu'(f) \ge \mu(f) - 3 \times \frac{3}{2} - (9 - 3) \times \frac{5}{4} = 0$ .

If f is a 10<sup>+</sup>-face, then  $\mu'(f) \ge \mu(f) - 2 \times \lfloor \frac{d(f)}{3} \rfloor \times \frac{3}{2} - \left( d(f) - 2 \times \lfloor \frac{d(f)}{3} \rfloor \right) \times \frac{5}{4} \ge 0.$ 

Suppose that f is a 7-face. By Lemma 4.1(b), f is not incident with any weak 3-vertex. It is observed that f is incident with at most six semi-weak 3-vertices. If there is an incident vertex receives at most  $\frac{1}{2}$  from f, then  $\mu'(f) \ge \mu(f) - \frac{1}{2} - (7-1) \times \frac{5}{4} = 0$ . So we may assume that f is incident with seven 4<sup>-</sup>-vertices and no rich 4-vertex. If f is incident with at most four semi-weak 3-vertices, then  $\mu'(f) \ge \mu(f) - 4 \times \frac{5}{4} - 3 \times 1 = 0$ . So we may further assume that f is incident with at least five semi-weak 3-vertices and at most two 4-vertices. If f is incident with two semi-rich 4-vertices, then  $\mu'(f) = \mu(f) - 2 \times \frac{3}{4} - (7-2) \times \frac{5}{4} > 0$ . If f is incident

with a semi-rich 4-vertex and a poor 4-vertex, then  $\mu'(f) = \mu(f) - \frac{3}{4} - 1 - (7-2) \times \frac{5}{4} = 0$ . It is impossible that f is incident with five semi-weak 3-vertices and two poor 4-vertices.

In the following, assume that f is incident with at most one 4-vertex and at least five semi-weak 3-vertices. If f is incident with a semi-rich 4-vertex, then it is incident with at most five semi-weak 3-vertices, and then  $\mu'(f) \ge \mu(f) - \frac{3}{4} - 5 \times \frac{5}{4} - 1 = 0$ . Suppose that f is incident with a poor 4-vertex, then it must be adjacent to six semi-weak 3-vertices, i.e., f is a special 7-face, see Fig. 3a. If f controls two  $(3, 3, 3^+, 4^+)$ -faces through (3, 3)-edges, then  $\mu'(f) = \mu(f) - 1 - 6 \times \frac{5}{4} + 2 \times \frac{1}{4} = 0$ . Then we may assume that f controls at least one (3, 3, 3, 3)-face. By Lemma 4.3, f controls two 4-faces incident with at least two 4<sup>+</sup>-vertices through (3, 4)-edges, thus  $\mu'(f) = \mu(f) - 1 - 6 \times \frac{5}{4} + 2 \times \frac{1}{4} = 0$ . Finally, we may assume that f is incident with three  $(3, 3, 3^+, 4^+)$ -faces, thus  $\mu'(f) = \mu(f) - 1 - 6 \times \frac{5}{4} + 2 \times \frac{1}{4} = 0$ . Finally, we may assume that f is incident with three  $(3, 3, 3^+, 4^+)$ -faces, thus  $\mu'(f) = \mu(f) - 1 - 6 \times \frac{5}{4} + 3 \times \frac{1}{4} > 0$ .

### 4.2 Planar graphs without 4-, 5- and 7-cycles

The third main result can be derived from the following theorem on degeneracy.

**Theorem 1.10.** Every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is 2-degenerate.

**Proof.** Suppose that G is a planar graph satisfying all the hypothesis but the minimum degree is at least three. Without loss of generality, we may assume that G is connected and it has been embedded in the plane.

#### Lemma 4.4.

- (a) There is no 4-, 5-, 7-faces. Every 6-face is bounded by a 6-cycle.
- (b) A 3-face cannot be adjacent to a 7<sup>-</sup>-face.

**Proof.** (a) Since every 4-face must be bounded by a 4-cycle, but there is no 4-cycles in G, this implies that there is no 4-faces in G. Similarly, there is no 5-faces in G. Since there is no 7-cycles in G, there is no 7-face bounded by a cycle, and then the boundary of every 7-face must consist of a triangle and a 4-cycle, but this contradicts the absence of 4-cycles. If the boundary of a 6-face is not a cycle, then it must consist of two triangles, and the distance of these two triangles is zero, a contradiction. Therefore, every 6-face is bounded by a 6-cycle.

(b) It is easy to check that every  $8^-$ -cycle is chordless. Since there is no two triangles at distance less than two, there is no two adjacent 3-faces. Every 6-face is bounded by a 6-cycle and it is chordless, thus a 3-face cannot be adjacent to a 6-face, for otherwise they form a 7-cycle with a chord, a contradiction.

Define the initial charge function  $\mu(x)$  on  $V \cup F$  to be  $\mu(v) = d(v) - 6$  for  $v \in V$  and  $\mu(f) = 2d(f) - 6$  for  $f \in F$ . By Euler's formula, we have the following equality,

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12.$$

Next, we define some discharging rules to change the initial charge function  $\mu(x)$  to the final charge function  $\mu'(x)$  on  $V \cup F$  such that  $\mu'(x) \ge 0$  for all  $x \in V \cup F$ . This leads to a contradiction, and then we complete the proof.

**R1** Each 6<sup>+</sup>-face sends 1 to each incident strong 3-vertex,  $\frac{1}{2}$  to each incident rich 4-vertex,  $\frac{1}{4}$  to each incident 5-vertex.

**R2** Each 8<sup>+</sup>-face sends  $\frac{3}{2}$  to each incident weak 3-vertex,  $\frac{3}{4}$  to each incident semi-rich 4-vertex.

It remains to check that the final charge of every element in  $V \cup F$  is nonnegative.

• Let v be an arbitrary vertex of G.

If v is a 6<sup>+</sup>-vertex, then it is not involved in the discharging procedure, hence  $\mu'(v) = \mu(v) = d(v) - 6 \ge 0$ . We may assume that  $3 \le d(v) \le 5$ . Since there is no two triangles at distance less than two, every vertex is incident with at most one 3-face.

Suppose that v is a 3-vertex. If v is not incident with any 3-face, then it is incident with three 6<sup>+</sup>-faces, and then  $\mu'(v) = \mu(v) + 3 \times 1 = 0$ . If v is incident with a 3-face, then the other two incident faces are 8<sup>+</sup>-faces by Lemma 4.4(b), and then  $\mu'(v) = \mu(v) + 2 \times \frac{3}{2} = 0$ .

Suppose that v is a 4-vertex. If v is not incident with any 3-face, then it is incident with four 6<sup>+</sup>-faces, and then  $\mu'(v) = \mu(v) + 4 \times \frac{1}{2} = 0$ . If v is incident with a 3-face, then  $\mu'(v) = \mu(v) + 2 \times \frac{3}{4} + \frac{1}{2} = 0$ .

Suppose that v is a 5-vertex. Since v is incident with at most one 3-face, it is incident with at least four 6<sup>+</sup>-faces, so  $\mu'(v) \ge \mu(v) + 4 \times \frac{1}{4} = 0$ .

• Let f be an arbitrary face in F(G).

Note that there is no 4-, 5-, 7-faces. Since every 3-face f is not involved in the discharging procedure, we have that  $\mu'(f) = \mu(f) = 0$ . By Lemma 4.4(b), every 6-face f is adjacent to six 6<sup>+</sup>-faces, thus  $\mu'(f) \ge \mu(f) - 6 \times 1 = 0$ . Suppose that f is a d-face with  $d \ge 8$ . Since the distance of triangles is at least two, we have that f is adjacent to at most  $\lfloor \frac{d}{3} \rfloor$  triangular-faces, thus it is incident with at most  $2 \times \lfloor \frac{d}{3} \rfloor$  weak 3-vertices. Hence,  $\mu'(f) \ge 2d - 6 - 2 \times \lfloor \frac{d}{3} \rfloor \times \frac{3}{2} - (d - 2 \times \lfloor \frac{d}{3} \rfloor) \times 1 = d - 6 - \lfloor \frac{d}{3} \rfloor \ge 0$ .

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