Edge-fault-tolerant strong Menger edge connectivity of bubble-sort star graphs

Jia Guo[1,2*]

[1]School of Software, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, PR China

[2]College of Science, Northwest A&F University, Yangling, Shaanxi 712100, PR China

Abstract

The connectivity and edge connectivity of interconnection network determine the fault tolerance of the network. An interconnection network is usually viewed as a connected graph, where vertex corresponds processor and edge corresponds link between two distinct processors. Given a connected graph G with vertex set $V(G)$ and edge set $E(G)$, if for any two distinct vertices $u, v \in V(G)$, there exist $\min\{d_G(u), d_G(v)\}$ edge-disjoint paths between u and v, then G is strongly Menger edge connected. Let m be an integer with $m \geq 1$. If $G - F_e$ remains strongly Menger edge connected for any $F_e \subseteq E(G)$ with $|F_e| \leq m$, then G is m-edge-fault-tolerant strongly Menger edge connected. If $G - F_e$ is strongly Menger edge connected for any $F_e \subseteq E(G)$ with $|F_e| \leq m$ and $\delta(G - F_e) \geq 2$, then G is m-conditional edge-fault-tolerant strongly Menger edge connected. In this paper, we consider the n-dimensional bubble-sort star graph BS_n. We show that BS_n is $(2n-5)$-edge-fault-tolerant strongly Menger edge connected for $n \geq 3$ and $(6n-17)$-conditional edge-fault-tolerant strongly Menger edge connected for $n \geq 4$. Moreover, we give some examples to show that our results are optimal.

Keywords: fault-tolerance, strong Menger edge connectivity, bubble-sort star graph

1. Introduction

*email: guojia199011@163.com

The connectivity and edge connectivity are two crucial factors for the interconnection networks since they determine the fault tolerance of the networks. An interconnection network can be viewed as a simple connected graph, where vertex corresponds processor and edge corresponds link. In the rest of this paper, we only consider simple connected graphs and we follow the work of [1] for definitions and notations not defined here.

Let $G = (V(G), E(G))$ be a simple connected graph. For a vertex $v \in V(G)$, $N_G(v) = \{u \mid (u, v) \in E(G)\}$ is the set of neighbours of v and $E_G(v) = \{(u, v) \mid (u, v) \in E(G)\}$ is the set of edges that are incident with v. Let $d_G(v) = |N_G(v)|$ be the *degree* of v and $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ be the *minimum degree* of G. If $d_G(v) = k$ for every $v \in V(G)$, then G is *k-regular*. G is bipartite if there exist two vertex subsets V_1, V_2 with $V_1 \cap V_2 = \emptyset$ such that $V(G) = V_1 \cup V_2$ and for each edge $(u, v) \in E(G)$, $|\{u, v\} \cap V_1| = |\{u, v\} \cap V_2| = 1$. It is well known that bipartite graphs contain no odd cycles. Let $F_1, F_2 \subseteq V(G)$ with $F_1 \cap F_2 = \emptyset$, denote $E_G(F_1, F_2) = \{(u, v) \in E(G) \mid u \in F_1,\ v \in F_2\}$. Let $F \subseteq V(G)$ and $F_e \subseteq E(G)$. We use $G - F$ to denote the subgraph of G with vertex set $V(G) - F$ and edge set $E(G) - \{(u, v) \in E(G) \mid \{u, v\} \cap F \neq \emptyset\}$. If $G - F$ is disconnected or has only one vertex, then F is a *vertex cut* of G. We use $G - F_e$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) - F_e$. If $G - F_e$ is disconnected, then F_e is an *edge cut* of G. The *connectivity* (resp. *edge connectivity*) of G, denoted by $\kappa(G)$ (resp. $\lambda(G)$), is the minimum size of F (resp. F_e) such that F (resp. F_e) is a vertex cut (resp. an edge cut) of G. $P_k = uv_2v_3 \cdots v_{k-1}v$ on k distinct vertices $u, v_2, \cdots, v_{k-1}, v$ of G is a *(u, v)-path* if $(u, v_2) \in E(G)$, $(v_{k-1}, v) \in E(G)$, and $(v_i, v_{i+1}) \in E(G)$ for every $i \in \{2, \cdots, k-2\}$. $F \subseteq V(G) - \{u, v\}$ (resp. $F_e \subseteq E(G)$) is an *(u, v)-cut* (resp. *(u, v)-edge-cut*) if $G - F$ (resp. $G - F_e$) has no (u, v)-path. Menger's theorem is a classical theorem about the connectivity and edge connectivity.

Theorem 1.1 [**8**] *Let G be a graph and $u, v \in V(G)$ with $u \neq v$. Then*

(1) the minimum size of an (u, v)-cut equals to the maximum number of disjoint (u, v)-paths for $(u, v) \notin E(G)$;

(2) the minimum size of an (u, v)-edge-cut equals to the maximum number of edge-disjoint (u, v)-paths.

Motivated by Menger's theorem, Oh et al. [9] proposed the strong Menger connectivity (also called the maximal local-connectivity) and Qiao et al. [10] introduced the strong Menger edge connectivity, which are showed in the following definition.

Definition 1.2 *Let G be a connected graph and $u, v \in V(G)$ be any two distinct vertices. Then*

(1) G is strongly Menger connected if there exist $\min\{d_G(u), d_G(v)\}$ *disjoint (u, v)-paths;*

(2) G is strongly Menger edge connected if there exist $\min\{d_G(u), d_G(v)\}$ *edge-disjoint (u, v)-paths.*

Since edge faults may occur in real interconnection networks, the edge-fault-tolerant strong Menger edge connectivity has been proposed.

Definition 1.3 *Let $m \geq 1$ be an integer, G be a connected graph, and $F_e \subseteq E(G)$ be any arbitrary edge subset of G with $|F_e| \leq m$. Then*

(1) G is m-edge-fault-tolerant strongly Menger edge connected if $G - F_e$ is strongly Menger edge connected;

(2) G is m-conditional edge-fault-tolerant strongly Menger edge connected if $G - F_e$ is strongly Menger edge connected for any F_e with $\delta(G - F_e) \geq 2$.

The edge-fault-tolerant strong Menger edge connectivity of many interconnection networks has been studied. For example, Qiao et al. proved that the folded hypercube is $(2n-2)$-conditional edge-fault-tolerant strongly Menger edge connected [10]. Li et al. discussed the edge-fault-tolerant strong Menger edge connectivity of the hypercube-like network [6] and the balanced hypercube [7]. He et al. considered the strong Menger edge connectivity of the regular network [5].

This paper deals with the edge-fault-tolerant strong Menger edge connectivity of the n-dimensional bubble-sort star graph BS_n [3], which gains many nice properties, such as vertex transitive and high degree of regularity. Cai et al. showed that BS_n is $(2n-5)$-fault-tolerant strongly Menger connected [2]. Wang et al. studied the 2-extra diagnosability [11], the 2-good-neighbor diagnosability [12], and the strong connectivity [13] of BS_n. Gu et al. discussed the pessimistic diagnosability of BS_n [4]. Zhao et al. investigated the generalized connectivity of BS_n [14]. Zhu et al. gave an algorithm to determine the h-extra connectivity of BS_n of low dimensions [16]. Zhang et al. considered the structure connectivity and substructure connectivity of BS_n [15].

The remainder of this paper is organized as follows: Section 2 introduces the definition of BS_n and gives some properties of BS_n. In section 3, we demonstrate the edge-fault-tolerant strong Menger edge connectivity of BS_n. In section 4, we discuss the conditional edge-fault-tolerant strong Menger edge connectivity of BS_n. Section 5 concludes this paper.

2. Preliminaries

Let l_1, l_2 be two integers with $1 \leq l_1 \leq l_2$. Set $[l_1, l_2] = \{l \mid l_1 \leq l \leq l_2,\ l \text{ is an integer}\}$.

Now we give the definition of the n-dimensional bubble-sort star graph BS_n.

Definition 2.1 [3] The n-dimensional bubble-sort star graph BS_n has vertex set $V(BS_n)$ and edge set $E(BS_n)$. A vertex $v \in V(BS_n)$ if and only if v is a permutation on $[1, n]$, which is denoted as $v = v_1 v_2 \cdots v_n$. Let $x = x_1 x_2 \cdots x_n \in V(BS_n)$, $y = y_1 y_2 \cdots y_n \in V(BS_n)$ with $x \neq y$. Then $(x, y) \in E(BS_n)$ if and only if there exists an integer k with $k \in [2, n]$ such that $y_{k-1} = x_k$, $y_k = x_{k-1}$, and $y_i = x_i$ for every $i \in [1, n] - \{k-1, k\}$ or $y_1 = x_k$, $y_k = x_1$, and $y_i = x_i$ for every $i \in [2, n] - \{k\}$.

By Definition 2.1, BS_n is a bipartite and $(2n-3)$-regular graph of order $n!$. Fig. 1 illustrates BS_2, BS_3, and BS_4, respectively.

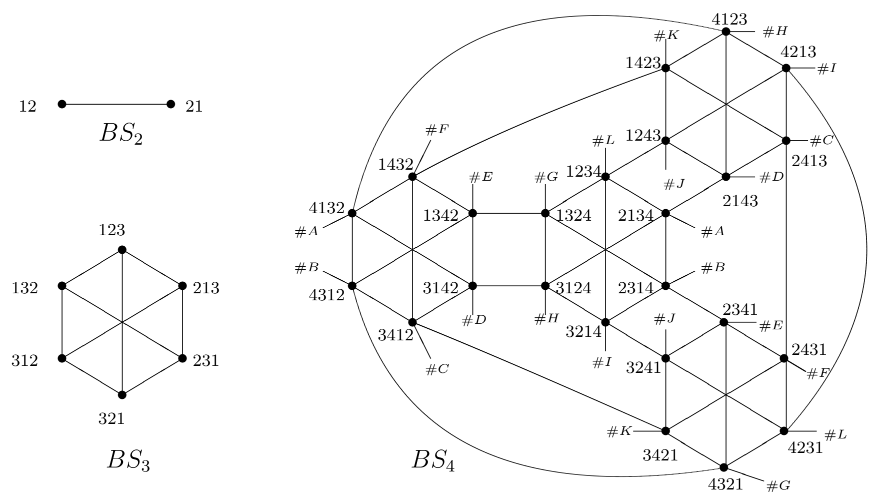

Figure 1: Illustration of BS_n for $n = 2, 3, 4$.

Let integers $j, k \in [1, n]$ with $j \neq k$. Let $x = x_1 x_2 \cdots x_n \in V(BS_n)$ and "$\circ$" be an operation such that $y = y_1 y_2 \cdots y_n = x \circ (j, k)$ if and only if $x_j = y_k$, $x_k = y_j$, and $x_i = y_i$ for every $i \in [1, n] - \{j, k\}$. Thus $(x, y) \in E(BS_n)$ if and only if $y = x \circ (k-1, k)$ or $y = x \circ (1, k)$ for some $k \in [2, n]$. Let $x^- = x \circ (n-1, n)$ and $x^+ = x \circ (1, n)$ for simplicity. Let BS_n^i be the induced subgraph of BS_n by the vertex set $V(BS_n^i) = \{x = x_1 x_2 \cdots x_n \in V(BS_n) \mid x_n = i\}$ for every $i \in [1, n]$. By Definition 2.1, $BS_n^i \cong BS_{n-1}$ for every $i \in [1, n]$. It is obvious that if $x \in V(BS_n^i)$, $x^- \in V(BS_n^j)$, and $x^+ \in V(BS_n^k)$, then i, j, k are three distinct integers in $[1, n]$. Set $E_{i,j}(BS_n) = \{(x, y) \in E(BS_n) \mid x \in V(BS_n^i),\ y \in V(BS_n^j)\}$ for any $i, j \in [1, n]$ with $i \neq j$. For any arbitrary edge set $F_e \subseteq E(BS_n)$, denote $F_e^i = F_e \cap E(BS_n^i)$ for every $i \in [1, n]$ and let $F_e^0 = F_e - \cup_{i=1}^n F_e^i$. For any $L \subseteq [1, n]$, let BS_n^L be the subgraph of BS_n induced by $\cup_{i \in L} V(BS_n^i)$.

Now we give some properties of BS_n.

Lemma 2.2 [2] *Let n be an integer with $n \geq 3$. Then*

(1) $|E_{i,j}(BS_n)| = 2(n-2)!$ for any $i, j \in [1, n]$ with $i \neq j$;

(2) $\{u^+, u^-\} \cap \{v^+, v^-\} = \emptyset$ for any $u, v \in V(BS_n^k)$ ($k \in [1, n]$) with $u \neq v$;

(3) $u^+ \in V(BS_n^{[3,n]})$ or $u^- \in V(BS_n^{[3,n]})$ for any $u \in V(BS_n^{[1,2]})$.

Lemma 2.3 [13] $\lambda(BS_n) = 2n - 3$ *for* $n \geq 3$.

Lemma 2.4 [13] *Let $F_e \subseteq E(BS_n)$ with $|F_e| \leq 4n-9$ for $n \geq 3$. If $BS_n - F_e$ is disconnected, then $BS_n - F_e$ has two components, one of which is an isolated vertex.*

Lemma 2.5 *Let $F_e \subseteq E(BS_3)$ with $|F_e| \leq 4$. If $BS_3 - F_e$ is disconnected, then $BS_3 - F_e$ has two components, one of which is an isolated vertex or an edge.*

Proof. If $|F_e| \leq 3$, then the lemma holds by Lemma 2.4. Now we consider the case that $|F_e| = 4$ and $BS_3 - F_e$ is disconnected. Let $H_1, H_2, \cdots, H_k$ be the k components of $BS_3 - F_e$ with $|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_k)|$ and $k \geq 2$. Since $|V(BS_3)| = 3! = 6$, $3 \geq |V(H_2)| \geq \cdots \geq |V(H_k)|$. If $|V(H_2)| = 3$, then $H_2 = P_3$ as BS_3 is bipartite. Thus $|F_e| \geq 2\times 2+1 = 5 > 4$, a contradiction. Hence $|V(H_2)| \leq 2$. Now we claim that $k = 2$. Suppose, to the contrary, that $k \geq 3$. Note that BS_3 is bipartite. If $|V(H_2)| = |V(H_3)| = 1$, then $|F_e| \geq 2 \times 3 - 1 = 5 > 4$, a contradiction. If $|V(H_2)| = |V(H_3)| = 2$, then $|F_e| \geq 4 \times 2 - 2 = 6 > 4$, a contradiction. If $|V(H_2)| = 2$ and $|V(H_3)| = 1$, then $|F_e| \geq 2 \times 2 + 3 - 1 = 6 > 4$, a contradiction. Thus $k = 2$ and the lemma holds. ■

Lemma 2.6 *Let $F_e \subseteq E(BS_4)$ with $|F_e| \leq 10$. If $BS_4 - F_e$ is disconnected, then $BS_4 - F_e$ has a component H with $|V(H)| \geq 4! - 2$.*

Proof. Suppose that $BS_4 - F_e$ is disconnected. Without loss of generality, we assume $|F_e^1| \geq |F_e^2| \geq |F_e^3| \geq |F_e^4|$. Since $n = 4$, $|E_{i,j}(BS_4)| = 2 \times (4-2)! = 4$ for $i, j \in [1, 4]$ with $i \neq j$ by Lemma 2.2 (1). Since $|F_e| \leq 10$, $|F_e^4| \leq 2$. Hence $BS_4^4 - F_e^4$ is connected by Lemma 2.3. Let H be the component of $BS_4 - F_e$ containing $BS_4^4 - F_e^4$ as a subgraph. Now we will consider the following three cases.

Case 1. $|F_e^1| \geq 5$.

In this case, $|F_e^4| \leq |F_e^3| \leq 2$; otherwise $|F_e| \geq 5+2\times 3 = 11 > 10$, a contradiction. Thus $BS_4^3 - F_e^3$ is connected by Lemma 2.3.

Subcase 1.1. $|F_e^2| \geq 3$.

In this subcase, $|F_e^0| \leq 10 - 5 - 3 = 2$. Since $|E_{3,4}(BS_4) - F_e| \geq |E_{3,4}(BS_4)| - |F_e^0| \geq 4-2 = 2 > 0$, $BS_4^{[3,4]} - F_e$ is a subgraph of H. Since $|F_e^0| \leq 2$, $|V(H)| \geq 4!-2$ by Lemma 2.2 (3).

Subcase 1.2. $|F_e^2| \leq 2$.

In this subcase, $|F_e^0| \leq 10-5=5$ and $BS_4^i - F_e^i$ $(i=2,3,4)$ is connected by Lemma 2.3. We claim that $E_{2,3}(BS_4) - F_e \neq \emptyset$ or $E_{2,4}(BS_4) - F_e \neq \emptyset$; otherwise $|F_e^0| \geq |E_{2,3}(BS_4)| + |E_{2,4}(BS_4)| = 2 \times 4 = 8 > 5$, a contradiction. Without loss of generality, we assume $E_{2,3}(BS_4) - F_e \neq \emptyset$. Similarly, we can get $E_{2,4}(BS_4) - F_e \neq \emptyset$ or $E_{3,4}(BS_4) - F_e \neq \emptyset$. Thus $BS_4^{[2,4]} - F_e$ is a subgraph of H. If $v \in V(BS_4^1)$, then $v^+ \in V(BS_4^{[2,4]})$ and $v^- \in V(BS_4^{[2,4]})$. Since $|F_e^0| \leq 5 < 2 \times 3$, $|V(H)| \geq 4! - 2$ by Lemma 2.2 (2).

Case 2. $3 \leq |F_e^1| \leq 4$.

We will consider the following subcases.

Subcase 2.1. $|F_e^3| \geq 3$.

Since $3 \leq |F_e^3| \leq |F_e^2| \leq |F_e^1| \leq 4$ and $|F_e| \leq 10$, we have $|F_e^3| = |F_e^2| = 3$ and $|F_e^0| \leq 10 - 3 \times 3 = 1$. Hence $BS_4^i - F_e^i$ has a component H_i with $|V(H_i)| \geq 3! - 1$ for $i = 2,3$ by Lemma 2.4. Since $|F_e^1| \leq 4$, $BS_4^1 - F_e^1$ has a component H_1 with $|V(H_1)| \geq 3! - 2$ by Lemma 2.5. Since $|E_{BS_4}(V(H_i), V(BS_4^4)) - F_e| \geq |E_{i,4}(BS_4)| - (3! - |V(H_i)|) - |F_e^0| \geq 4 - 2 - 1 > 0$ for every $i \in [1,3]$, H_i is a subgraph of H. If $BS_4^1 - F_e^1$ is connected, then $|V(H)| \geq 4! - 2$. If $|V(H_1)| \geq 3! - 1$ and $BS_4^2 - F_e^2$ or $BS_4^3 - F_e^3$ is connected, then $|V(H)| \geq 4! - 2$. If $|V(H_1)| \geq 3! - 2$, both $BS_4^2 - F_e^2$ and $BS_4^3 - F_e^3$ are connected, then $|V(H)| \geq 4! - 2$. Hence we just need to consider the following three conditions.

Subcase 2.1.1. $|V(H_1)| = |V(H_2)| = |V(H_3)| = 3! - 1$.

Let $u_i \in V(BS_4^i) - V(H_i)$ for every $i \in [1,3]$. If $u_i \in V(H)$ for some $i \in [1,3]$, then the lemma holds. Now we suppose that $u_i \notin V(H)$ for every $i \in [1,3]$. Note that BS_4 is bipartite. If u_1, u_2, u_3 are three isolated vertices in $BS_4 - F_e$, then $|F_e| \geq 3 \times 5 - 2 = 13 > 10$, a contradiction. If u_1, u_2, u_3 form an edge and an isolated vertex in $BS_4 - F_e$, then $|F_e| \geq 2 \times 4 + 5 - 1 = 12 > 10$, a contradiction. If u_1, u_2, u_3 form a P_3 in $BS_4 - F_e$, then $|F_e| \geq 2 \times 4 + 3 = 11 > 10$, a contradiction.

Subcase 2.1.2. $|V(H_1)| = 3! - 2,\ |V(H_2)| = |V(H_3)| = 3! - 1$.

Let $u_i \in V(BS_4^i) - V(H_i)$ for $i = 2,3$. Let $u_{11}, u_{12} \in V(BS_4^1) - V(H_1)$ with $u_{11} \neq u_{12}$. Hence $|F_e^1| = 4$, $|F_e^0| = 0$, and $(u_{11}, u_{12}) \in E(BS_4^1) - F_e$ by Lemmas 2.4 and 2.5. If $u_{11} \in V(H)$ or $u_{12} \in V(H)$, then the lemma holds. Now we suppose that $u_{11} \notin V(H)$ and $u_{12} \notin V(H)$. Hence $\{u_{11}^+, u_{11}^-\} = \{u_2, u_3\}$ as $|F_e^0| = 0$. Thus $\{u_{12}^+, u_{12}^-\} \subseteq V(H)$ by Lemma 2.2 (2). Since $|F_e^0| = 0$, $u_{12} \in V(H)$, a contradiction.

Subcase 2.1.3. $|V(H_1)| = 3! - 2, |V(H_2)| = 3! - 1, |V(H_3)| = 3!$ *or* $|V(H_1)| = 3! - 2, |V(H_2)| = 3!, |V(H_3)| = 3! - 1$.

Without loss of generality, we assume $|V(H_1)| = 3! - 2, |V(H_2)| = 3! - 1, |V(H_3)| = 3!$. Let $u_{11}, u_{12} \in V(BS_4^1) - V(H_1)$ with $u_{11} \neq u_{12}$ and $u_2 \in V(BS_4^2) - V(H_2)$. Hence $|F_e^1| = 4$, $|F_e^0| = 0$, and $(u_{11}, u_{12}) \in E(BS_4^1) - F_e$ by Lemmas 2.4 and 2.5. Since

$|F_e^0| = 0$, $u_{11}^+ \in V(H)$ or $u_{11}^- \in V(H)$. Hence $u_1 \in V(H)$, the lemma holds.

Subcase 2.2. $|F_e^3| \leq 2$.

In this subcase, $|F_e^0| \leq 10 - 3 = 7$. By Lemma 2.3, $BS_4^3 - F_e^3$ is connected. Now we consider the following three conditions.

Subcase 2.2.1. $|F_e^2| \leq 2$.

$BS_4^2 - F_e^2$ is connected by Lemma 2.3. We claim that $E_{2,3}(BS_4) - F_e \neq \emptyset$ or $E_{2,4}(BS_4) - F_e \neq \emptyset$; otherwise $|F_e^0| \geq |E_{2,3}(BS_4)| + |E_{2,4}(BS_4)| = 2 \times 4 = 8 > 7$, a contradiction. Without loss of generality, we assume $E_{2,3}(BS_4) - F_e \neq \emptyset$. Similarly, we can get $E_{2,4}(BS_4) - F_e \neq \emptyset$ or $E_{3,4}(BS_4) - F_e \neq \emptyset$. Hence $BS_4^{[2,4]} - F_e$ is a subgraph of H. Since $3 \leq |F_e^1| \leq 4$, $BS_4^1 - F_e^1$ has a component H_1 such that $|V(H_1)| \geq 3! - 2$ by Lemma 2.5. Since $\{u^+, u^-\} \subseteq V(BS_4^{[2,4]})$ for every $u \in V(BS_4^1)$, $|E_{BS_4}(V(H_1), V(BS_4^{[2,4]})) - F_e| \geq |E_{1,2}(BS_4)| + |E_{1,3}(BS_4)| + |E_{1,4}(BS_4)| - 2|V(BS_4^1) - V(H_1)| - |F_e^0| \geq 3 \times 4 - 2 \times 2 - 7 > 0$. Thus H_1 is a subgraph of H and the lemma holds.

Subcase 2.2.2. $|F_e^2| = 3$.

In this subcase, we have $|F_e^0| \leq 10 - 3 - 3 = 4$. If $BS_4^2 - F_e^2$ is connected, then the lemma holds by the same argument as that of Subcase 2.2.1.

Now we suppose that $BS_4^2 - F_e^2$ is disconnected. Then by Lemma 2.4, $BS_4^2 - F_e^2$ has a component H_2 such that $|V(H_2)| = 3! - 1$. Let $u_2 \in V(BS_4^2) - V(H_2)$. We claim that $E_{BS_4}(V(H_2), V(BS_4^3)) - F_e \neq \emptyset$ or $E_{BS_4}(V(H_2), V(BS_4^4)) - F_e \neq \emptyset$; otherwise $|F_e^0| \geq |E_{BS_4}(V(H_2), V(BS_4^3))| + |E_{BS_4}(V(H_2), V(BS_4^4))| \geq 4 - 1 + 4 - 1 = 6 > 4$, a contradiction. Without loss of generality, we assume $E_{BS_4}(V(H_2), V(BS_4^3)) - F_e \neq \emptyset$. Similarly, we can get $E_{BS_4}(V(H_2), V(BS_4^4)) - F_e \neq \emptyset$ or $E_{3,4}(BS_4) - F_e \neq \emptyset$. Hence both H_2 and $BS_4^{[3,4]} - F_e$ are subgraphs of H. Since $3 \leq |F_e^1| \leq 4$, $BS_4^1 - F_e^1$ has a component H_1 such that $|V(H_1)| \geq 3! - 2$ by Lemma 2.5. If $|V(H_1)| \geq 3! - 1$, then $|E_{BS_4}(V(H_1), V(BS_4^{[3,4]})) - F_e| \geq |E_{1,3}(BS_4)| + |E_{1,4}(BS_4)| - 2|V(BS_4^1) - V(H)| - |F_e^0| \geq 2 \times 4 - 2 \times 1 - 4 = 2 > 0$, which implies H_1 is a subgraph of H and the lemma holds. Now we consider that $|V(H_1)| = 3! - 2$. Hence $|F_e^1| = 4$ by Lemmas 2.4 and 2.5. Thus $|F_e^0| \leq 10 - 4 - 3 = 3$ and $|E_{BS_4}(V(H_1), V(BS_4^{[3,4]})) - F_e| \geq |E_{1,3}(BS_4)| + |E_{1,4}(BS_4)| - 2|V(BS_4^1) - V(H)| - |F_e^0| \geq 2 \times 4 - 2 \times 2 - 3 = 1 > 0$, which implies H_1 is a subgraph of H. Let $u_{11}, u_{12} \in V(BS_4^1) - V(H_1)$ with $u_{11} \neq u_{12}$. Then the lemma holds by the same argument as that of Subcase 2.1.1.

Subcase 2.2.3. $|F_e^2| = 4$.

Since $|F_e^2| \leq |F_e^1|$, $|F_e^2| = |F_e^1| = 4$ and $|F_e^0| \leq 10 - 4 - 4 = 2$. Since $|E_{3,4}(BS_4) - F_e| \geq |E_{3,4}(BS_4)| - |F_e^0| \geq 4 - 2 = 2 > 0$, $BS_4^{[3,4]} - F_e$ is a subgraph of H. Since $|F_e^0| \leq 2$, the lemma holds by Lemma 2.2 (3).

Case 3. $|F_e^1| \leq 2$.

In this case, $BS_4^i - F_e^i$ $(i = 1, 2, 3, 4)$ is connected by Lemma 2.3. Now we claim that $E_{1,k}(BS_4) - F_e \neq \emptyset$ for some $k \in [2, 4]$; otherwise $|F_e| \geq |E_{1,2}(BS_4)| + |E_{1,3}(BS_4)| + |E_{1,4}(BS_4)| = 3 \times 4 = 12 > 10$, a contradiction. Without loss of generality, we assume $E_{1,2}(BS_4) - F_e \neq \emptyset$. Suppose $E_{1,3}(BS_4) - F_e \neq \emptyset$ or $E_{2,3}(BS_4) - F_e \neq \emptyset$. Thus $BS_4^{[1,3]} - F_e$ is connected. Similarly, we can get $E_{k,4}(BS_4) - F_e \neq \emptyset$ for some $k \in [1, 3]$, which implies $H = BS_4 - F_e$ is connected, a contradiction. Hence $E_{1,3}(BS_4) - F_e = \emptyset$ and $E_{2,3}(BS_4) - F_e = \emptyset$. Thus $|F_e \cap (E_{1,3}(BS_4) \cup E_{2,3}(BS_4))| = 2 \times 4 = 8$. Hence $|E_{k,4}(BS_4) \cap F_e| \leq 10 - 8 = 2$ and $|E_{k,4}(BS_4) - F_e| \geq 4 - 2 = 2 > 0$ for every $k \in [1, 3]$. Hence $H = BS_4 - F_e$ is connected, a contradiction. ∎

Lemma 2.7 *Let $F_e \subseteq E(BS_n)$ with $|F_e| \leq 6n - 14$ for $n \geq 3$. If $BS_n - F_e$ is disconnected, then $BS_n - F_e$ has a component H with $|V(H)| \geq n! - 2$.*

Proof. We prove this lemma by induction on n. For $n = 3, 4$, the result holds by Lemmas 2.5 and 2.6. Assume $n \geq 5$ and $BS_n - F_e$ is disconnected. Without loss of generality, we assume $|F_e^1| \geq |F_e^2| \geq \cdots \geq |F_e^n|$. Since $|F_e| \leq 6n - 14$, $|F_e^n| \leq \cdots \leq |F_e^4| \leq 2n - 6$; otherwise $|F_e| \geq 4(2n - 5) > 6n - 14$ for $n \geq 5$, a contradiction. Hence $BS_n^i - F_e^i$ is connected for every $i \in [4, n]$ by Lemma 2.3. Let H be the component of $BS_n - F_e$ containing $BS_n^n - F_e^n$ as a subgraph. Now we will consider the following four cases.

Case 1. $|F_e^1| \geq 6n - 19$.

In this case, $|F_e^0| \leq (6n - 14) - (6n - 19) = 5$ and $|F_e^3| \leq 2 \leq 2n - 6$ for $n \geq 5$. Hence $BS_n^3 - F_e^3$ is connected by Lemma 2.3. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n - 2)! - 5 > 0$ for $i, j \in [3, n]$ with $i \neq j$ and $n \geq 5$, $BS_n^{[3,n]} - F_e$ is a subgraph of H.

Suppose $BS_n^2 - F_e^2$ is connected. Since $|E_{2,3}(BS_n) - F_e| \geq |E_{2,3}(BS_n)| - |F_e^0| \geq 2(n - 2)! - 5 > 0$ for $n \geq 5$, $BS_n^2 - F_e^2$ is a subgraph of H. Note that $\{u^+, u^-\} \subseteq V(BS_n^{[2,n]})$ for every $u \in V(BS_n^1)$. Since $|F_e^0| \leq 5 < 2 \times 3$, we have $|V(H)| \geq n! - 2$ by Lemma 2.2 (2).

Now we consider that $BS_n^2 - F_e^2$ is disconnected. Then $2n - 5 \leq |F_e^2| \leq 5$, which implies $n = 5$, $|F_e^2| = 5$, and $|F_e^0| = 0$. Since $|F_e^0| = 0$, $H = BS_n - F_e$ is connected by Lemma 2.2 (3), a contradiction.

Case 2. $4n - 12 \leq |F_e^1| \leq 6n - 20$.

In this case, $|F_e^0| \leq (6n - 14) - (4n - 12) = 2n - 2$ and $|F_e^3| \leq 2n - 6$; otherwise $|F_e| \geq 2(2n - 5) + (4n - 12) = 8n - 22 > 6n - 14$ for $n \geq 5$, a contradiction. Thus $BS_n^i - F_e^i$ is connected for every $i \in [3, n]$ by Lemma 2.3. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n - 2)! - (2n - 2) > 0$ for $i, j \in [3, n]$ with $i \neq j$ and $n \geq 5$, $BS_n^{[3,n]} - F_e$ is a subgraph of H.

Suppose $BS_n^2 - F_e^2$ is connected. Since $|E_{2,3}(BS_n) - F_e| \geq |E_{2,3}(BS_n)| - |F_e^0| \geq 2(n-2)! - (2n-2) > 0$ for $n \geq 5$, $BS_n^2 - F_e^2$ is a subgraph of H. Since $4n-12 \leq |F_e^1| \leq 6n-20$, $BS_n^1 - F_e^1$ has a component H_1 with $|V(H_1)| \geq (n-1)! - 2$ by induction hypothesis. Since $|E_{BS_n}(V(H_1), V(BS_n^2)) - F_e| \geq |E_{1,2}(BS_n)| - |V(BS_n^1) - V(H_1)| - |F_e^0| \geq 2(n-2)! - 2 - (2n-2) > 0$ for $n \geq 5$, H_1 is a subgraph of H. Thus $|V(H)| \geq n! - 2$.

Now we consider that $BS_n^2 - F_e^2$ is disconnected. Hence $2n-5 \leq |F_e^2| \leq |F_e^1| \leq 6n-20$ and $|F_e^0| \leq (6n-14) - (4n-12) - (2n-5) = 3$. Since $|F_e^0| \leq 3$, $|V(BS_n) - V(H)| \leq 3$ by Lemma 2.2 (3). If $|V(BS_n) - V(H)| \leq 2$, then the lemma holds. Now we suppose $|V(BS_n) - V(H)| = 3$ and $V(BS_n) - V(H) = \{u_1, u_2, u_3\}$. Note that BS_n is bipartite. If u_1, u_2, u_3 are three isolated vertices in $BS_n - F_e$, then $|F_e| \geq 3(2n-3) - 2 = 6n - 11 > 6n - 14$, a contradiction. If u_1, u_2, u_3 form an edge and an isolated vertex in $BS_n - F_e$, then $|F_e| \geq 2(2n-4) + (2n-3) - 1 = 6n - 12 > 6n - 14$, a contradiction. If u_1, u_2, u_3 form a P_3 in $BS_n - F_e$, then $|F_e| \geq 2(2n-4) + (2n-5) = 6n - 13 > 6n - 14$, a contradiction.

Case 3. $2n - 5 \leq |F_e^1| \leq 4n - 13$.

In this case, $|F_e^0| \leq (6n-14) - (2n-5) = 4n - 9$.

Subcase 3.1. $|F_e^2| \leq 2n - 6$.

In this subcase, $BS_n^i - F_e^i$ is connected for every $i \in [2, n]$ by Lemma 2.3. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n-2)! - (4n-9) > 0$ for $i, j \in [2, n]$ with $i \neq j$ and $n \geq 5$, $BS_n^{[2,n]} - F_e$ is a subgraph of H. Since $2n - 5 \leq |F_e^1| \leq 4n - 13$, $BS_n^1 - F_e^1$ has a component H_1 with $|V(H_1)| \geq (n-1)! - 1$ by Lemma 2.4. Since $|E_{BS_n}(V(H_1), V(BS_n^{[2,3]})) - F_e| \geq |E_{1,2}(BS_n)| + |E_{1,3}(BS_n)| - 2|V(BS_n^1) - V(H_1)| - |F_e^0| \geq 2 \times 2(n-2)! - 2 \times 1 - (4n-9) > 0$ for $n \geq 5$, H_1 is a subgraph of H and $|V(H)| \geq n! - 1$.

Subcase 3.2. $2n - 5 \leq |F_e^2| \leq 4n - 13$.

In this subcase, $|F_e^0| \leq (6n-14) - 2(2n-5) = 2n - 4$. If $|F_e^3| \leq 2n - 6$, then $BS_n^i - F_e^i$ is connected for every $i \in [3, n]$ by Lemma 2.3. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n-2)! - (2n-4) > 0$ for $i, j \in [3, n]$ with $i \neq j$ and $n \geq 5$, $BS_n^{[3,n]} - F_e$ is a subgraph of H. Since $2n - 5 \leq |F_e^2| \leq |F_e^1| \leq 4n - 13$, $BS_n^k - F_e^k$ has a component H_k with $|V(H_k)| \geq (n-1)! - 1$ for $k = 1, 2$ by Lemma 2.4. Since $|E_{BS_n}(V(H_k), V(BS_n^3)) - F_e| \geq |E_{k,3}(BS_n)| - |V(BS_n^k) - V(H_k)| - |F_e^0| \geq 2(n-2)! - 1 - (2n-4) > 0$ for $k \in [1, 2]$ and $n \geq 5$, both H_1 and H_2 are subgraphs of H. Thus $|V(H)| \geq n! - 2$.

Suppose $|F_e^3| \geq 2n-5$. Then $|F_e^0| \leq (6n-14) - 3(2n-5) = 1$. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n-2)! - 1 > 0$ for $i, j \in [4, n]$ with $i \neq j$ and $n \geq 5$, $BS_n^{[4,n]} - F_e$ is a subgraph of H. Since $2n-5 \leq |F_e^3| \leq |F_e^2| \leq |F_e^1| \leq 4n-13$, $BS_n^k - F_e^k$

has a component H_k with $|V(H_k)| \geq (n-1)!-1$ for every $k \in [1,3]$ by Lemma 2.4. Since $|E_{BS_n}(V(H_k), V(BS_n^4)) - F_e| \geq |E_{k,4}(BS_n)| - |V(BS_n^k) - V(H_k)| - |F_e^0| \geq 2(n-2)!-1-1 > 0$ for $k \in [1,3]$ and $n \geq 5$, H_i is a subgraph of H for every $k \in [1,3]$. If $BS_n^k - F_e^k$ is connected for some $k \in [1,3]$, then $|V(H)| \geq n!-2$. Now we consider that $|V(H_1)| = |V(H_2)| = |V(H_3)| = (n-1)!-1$. Let $u_k \in V(BS_n^k) - V(H_k)$ for every $k \in [1,3]$. Then the lemma holds by the same argument as that of Case 2.

Case 4. $|F_e^1| \leq 2n-6$.

In this case, $BS_n^i - F_e^i$ is connected for every $i \in [1,n]$ by Lemma 2.3. We claim that $E_{1,2}(BS_n) - F_e \neq \emptyset$ or $E_{1,3}(BS_n) - F_e \neq \emptyset$; otherwise $|F_e| \geq |E_{1,2}(BS_n)| + |E_{1,3}(BS_n)| = 2 \times 2(n-2)! > 6n-14$ for $n \geq 5$, a contradiction. Without loss of generality, we assume $E_{1,2}(BS_n) - F_e \neq \emptyset$. Similarly, we can get $E_{1,i}(BS_n) - F_e \neq \emptyset$ or $E_{2,i}(BS_n) - F_e \neq \emptyset$ for every $i \in [3,n]$. Thus $H = BS_n - F_e$ is connected, a contradiction. ∎

Lemma 2.8 *Let $F_e \subseteq E(BS_4)$ with $|F_e| \leq 11$. If $BS_4 - F_e$ is disconnected, then $BS_4 - F_e$ has a component H with $|V(H)| \geq 4!-3$.*

Proof. Suppose that $BS_4 - F_e$ is disconnected. Without loss of generality, we assume $|F_e^1| \geq |F_e^2| \geq |F_e^3| \geq |F_e^4|$. Since $n = 4$, $|E_{i,j}(BS_4)| = 2 \times (4-2)! = 4$ for $i, j \in [1,4]$ with $i \neq j$ by Lemma 2.2 (1). Since $|F_e| \leq 11$, $|F_e^4| \leq 2$. Hence $BS_4^4 - F_e^4$ is connected by Lemma 2.3. Let H be the component of $BS_4 - F_e$ containing $BS_4^4 - F_e^4$ as a subgraph. If $|F_e^1| \leq 2$, then the lemma holds by the same argument as that of Case 3 of Lemma 2.6. Hence we just consider the following two cases.

Case 1. $|F_e^1| \geq 5$.

Suppose that $|F_e^3| \geq 3$. Since $|F_e^3| \leq |F_e^2| \leq |F_e^1|$, we have $|F_e^3| = |F_e^2| = 3$, $|F_e^1| = 5$, and $|F_e^0| = 0$. Hence $BS_4^i - F_e^i$ has a component H_i with $|V(H_i)| \geq 3!-1$ for $i = 2,3$ by Lemma 2.4. Since $|E_{BS_4}(V(H_i), V(BS_4^4)) - F_e| \geq |E_{i,4}(BS_4)| - |V(BS_4^i) - V(H_i)| - |F_e^0| \geq 4-1 = 3 > 0$ for $i = 2,3$, both H_2 and H_3 are subgraphs of H. If $BS_4^3 - F_e^3$ is a subgraph of H, then $H = BS_4 - F_e$ is connected by Lemma 2.2 (3), a contradiction. Thus $|V(H_3)| = 3!-1$ and there exists a vertex $u_3 \in V(BS_4^3) - V(H)$. Since $|F_e^0| = 0$ and $u_3 \notin V(H)$, $\{u_3^+, u_3^-\} \subseteq V(BS_4^{[1,2]}) - V(H)$ and $|V(H_2)| = 3!-1$. Let $\{u_3^+, u_3^-\} \cap V(BS_4^i) = u_i$ for $i = 1,2$. Since BS_4 is bipartite and $|V(H_2)| = |V(H_3)| = 3!-1$, $\{u_1^+, u_1^-\} \cap V(H) \neq \emptyset$. Since $|F_e^0| = 0$, $u_1 \in V(H)$, which implies $u_3 \in V(H)$, a contradiction.

Now we suppose that $|F_e^3| \leq 2$. Then $BS_4^3 - F_e^3$ is connected by Lemma 2.3. Hence $|V(H)| \geq 4!-3$ by the same argument as that of Case 1 of Lemma 2.6

Case 2. $3 \leq |F_e^1| \leq 4$.

We will consider the following subcases.

Subcase 2.1. $|F_e^3| \geq 3$.

Since $3 \leq |F_e^3| \leq |F_e^2| \leq |F_e^1| \leq 4$ and $|F_e| \leq 11$, we have $|F_e^3| = 3$. Hence $BS_4^3 - F_e^3$ has a component H_3 such that $|V(H_3)| \geq 3! - 1$ by Lemma 2.4.

Subcase 2.1.1. $|F_e^2| = 4$.

In this subcase, $|F_e^1| = 4$ and $|F_e^0| = 0$. By Lemma 2.5, $BS_4^i - F_e^i$ has a component H_i such that $|V(H_i)| \geq 3! - 2$ for $i = 1, 2$. Since $|E_{BS_4}(V(H_i), V(BS_4^4)) - F_e| \geq |E_{i,4}(BS_4)| - (3! - |V(H_i)|) - |F_e^0| \geq 4 - 2 - 0 > 0$ for $i \in [1, 3]$, H_i is a subgraph of H for every $i \in [1, 3]$. If $BS_4^3 - F_e^3$ is a subgraph of H, then $H = BS_4 - F_e$ by Lemma 2.2 (3), a contradiction. Hence $|V(H_3)| = 3! - 1$ and there exists a vertex $u_3 \in V(BS_4^3) - V(H)$. Since $|F_e^0| = 0$ and $u_3 \notin V(H)$, $\{u_3^+, u_3^-\} \subseteq V(BS_4^{[1,2]}) - V(H)$. Let $\{u_3^+, u_3^-\} \cap V(BS_4^i) = u_i$ for $i = 1, 2$. Since BS_4 is bipartite and $|V(H_3)| = 3! - 1$, there exists a vertex $u_2' \in V(BS_4^2) - V(H) - \{u_2\}$ such that $(u_1, u_2') \in E(BS_4)$. Thus $|V(H_2)| = 3! - 2$ and $(u_2, u_2') \in E(BS_4^2) - F_e$ by Lemma 2.5. Similarly, there exists a vertex $u_1' \in V(BS_4^1) - V(H) - \{u_1\}$ such that $(u_1', u_2) \in E(BS_4)$, $|V(H_1)| = 3! - 2$, and $(u_1, u_1') \in E(BS_4^1) - F_e$. Since $|V(H_3)| = 3! - 1$ and BS_4 is bipartite, $\{u_1'^+, u_1'^-\} - \{u_2\} \subseteq V(H)$ by Lemma 2.2 (3). Since $|F_e^0| = 0$, $u_1' \in V(H)$, which implies $u_2 \in V(H)$, a contradiction.

Subcase 2.1.2. $|F_e^2| = 3$.

By Lemma 2.4, $BS_4^2 - F_e^2$ has a component H_2 such that $|V(H_2)| \geq 3! - 1$.

Suppose $|F_e^1| = 3$, then $|F_e^0| \leq 11 - 3 \times 3 = 2$. By Lemma 2.4, $BS_4^1 - F_e^1$ has a component H_1 such that $|V(H_1)| \geq 3! - 1$. Since $|E_{BS_4}(V(H_i), V(BS_4^4)) - F_e| \geq |E_{i,4}(BS_4)| - (3! - |V(H_i)|) - |F_e^0| \geq 4 - 1 - 2 > 0$ for $i \in [1, 3]$, H_i is a subgraph of H for every $i \in [1, 3]$. Thus $|V(H)| \geq 4! - 3$.

Suppose $|F_e^1| = 4$, then $|F_e^0| \leq 11 - 4 - 2 \times 3 = 1$. By Lemma 2.5, $BS_4^1 - F_e^1$ has a component H_1 such that $|V(H_1)| \geq 3! - 2$. Since $|E_{BS_4}(V(H_i), V(BS_4^4)) - F_e| \geq |E_{i,4}(BS_4)| - (3! - |V(H_i)|) - |F_e^0| \geq 4 - 2 - 1 > 0$ for $i \in [1, 3]$, H_i is a subgraph of H for every $i \in [1, 3]$. If $|V(H_1)| \geq 3! - 1$, then $|V(H)| \geq 4! - 3$. If $|V(H_2)| = 3!$ or $|V(H_3)| = 3!$, then $|V(H)| \geq 4! - 3$. Now we consider that $|V(H_1)| = 3! - 2$ and $|V(H_2)| = |V(H_3)| = 3! - 1$. Let $\{u_{11}, u_{12}\} \subseteq V(BS_4^1) - V(H_1)$ with $u_{11} \neq u_{12}$. Then $(u_{11}, u_{12}) \in E(BS_4^1) - F_e$ by Lemma 2.5. If $u_{11} \in V(H)$ or $u_{12} \in V(H)$, then $|V(H)| \geq 4! - 2$. We suppose that $u_{11} \notin V(H)$ and $u_{12} \notin V(H)$. Since BS_4 is bipartite, $|V(H_2)| = |V(H_3)| = 3! - 1$, and $|F_e^0| \leq 1$, there exists a vertex $v \in \{u_{11}^+, u_{11}^-, u_{12}^+, u_{12}^-\} \cap V(H)$ such that $(u_{11}, v) \in E(BS_4) - F_e$ or $(u_{12}, v) \in E(BS_4) - F_e$ by Lemma 2.2 (2), which implies $u_{11} \in V(H)$ and $u_{12} \in V(H)$, a contradiction.

Subcase 2.2. $|F_e^3| \leq 2$.

In this subcase, $|F_e^0| \leq 11 - 3 = 8$. By Lemma 2.3, $BS_4^3 - F_e^3$ is connected. If $|F_e^2| = 4$, then the lemma holds by the same argument as that of Subcase 2.2.3 of

Lemma 2.6. Hence we just consider the following two conditions.

Subcase 2.2.1. $|F_e^2| \leq 2$.

By Lemma 2.3, $BS_4^2 - F_e^2$ is connected.

Suppose $BS_4^{[2,4]} - F_e$ is connected. By Lemma 2.5, $BS_4^1 - F_e^1$ has a component H_1 such that $|V(H_1)| \geq 3!-2$. If $|V(H_1)| \geq 3!-1$, then $|E_{BS_4}(V(H_1), V(BS_4^{[2,4]})) - F_e| \geq |E_{1,2}(BS_4)| + |E_{1,3}(BS_4)| + |E_{1,4}(BS_4)| - 2(3! - |V(H_1)|) - |F_e^0| \geq 3 \times 4 - 2 \times 1 - 8 > 0$. Hence H_1 is a subgraph of H and $|V(H)| \geq 4! - 1$. Now we consider that $|V(H_1)| = 3! - 2$, which implies $|F_e^1| = 4$ by Lemmas 2.4 and 2.5. Thus $|F_e^0| \leq 11 - 4 = 7$ and $|E_{BS_4}(V(H_1), V(BS_4^{[2,4]})) - F_e| \geq |E_{1,2}(BS_4)| + |E_{1,3}(BS_4)| + |E_{1,4}(BS_4)| - 2(3! - |V(H_1)|) - |F_e^0| \geq 3 \times 4 - 2 \times 2 - 7 > 0$. Hence H_1 is a subgraph of H and $|V(H)| \geq 4! - 2$.

Now we suppose that $BS_4^{[2,4]} - F_e$ is disconnected. Without loss of generality, we assume $E_{2,3}(BS_4) - F_e = E_{2,4}(BS_4) - F_e = \emptyset$. Hence $|F_e^0| \geq |E_{2,3}(BS_4)| + |E_{2,4}(BS_4)| = 2 \times 4 = 8$. Since $|F_e| \leq 11$ and $3 \leq |F_e^1| \leq 4$, we have $|F_e^1| = 3$, $|F_e^2| = 0$, and $F_e^0 = E_{2,3}(BS_4) \cup E_{2,4}(BS_4)$. Thus $E_{3,4}(BS_4) - F_e = E_{3,4}(BS_4)$ and $BS_4^{[3,4]} - F_e$ is connected. By Lemma 2.4, $BS_4^1 - F_e^1$ has a component H_1 such that $|V(H_1)| \geq 3!-1$. Since $|E_{BS_4}(V(H_1), V(BS_4^3)) - F_e| \geq |E_{1,3}(BS_4)| - (3! - |V(H_1)|) \geq 4 - 1 > 0$, H_1 is a subgraph of H. Since $|E_{BS_4}(V(H_1), V(BS_4^2)) - F_e| \geq |E_{1,2}(BS_4)| - (3! - |V(H_1)|) \geq 4 - 1 > 0$, $BS_4^2 - F_e^2$ is a subgraph of H. Thus $|V(H)| \geq 4! - 1$.

Subcase 2.2.2. $|F_e^2| = 3$.

In this subcase, we have $|F_e^0| \leq 11 - 3 - 3 = 5$. By Lemma 2.4, $BS_4^2 - F_e^2$ has a component H_2 such that $|V(H_2)| \geq 3! - 1$. By Lemma 2.5, $BS_4^1 - F_e^1$ has a component H_1 such that $|V(H_1)| \geq 3! - 2$.

Suppose $BS_4^{[3,4]} - F_e$ is connected. Since $|E_{BS_4}(V(H_2), V(BS_4^{[3,4]})) - F_e| \geq |E_{2,3}(BS_4)| + |E_{2,4}(BS_4)| - 2(3! - |V(H_2)|) - |F_e^0| \geq 2 \times 4 - 2 \times 1 - 5 > 0$, H_2 is a subgraph of H. Since $|E_{BS_4}(V(H_1), V(BS_4^{[3,4]}) \cup V(H_2)) - F_e| \geq |E_{1,3}(BS_4)| + |E_{1,4}(BS_4)| + |E_{1,2}(BS_4)| - 2(3! - |V(H_1)|) - (3! - |V(H_2)|) - |F_e^0| \geq 3 \times 4 - 2 \times 2 - 1 - 5 > 0$, H_1 is a subgraph of H and $|V(H)| \geq 4! - 3$.

Now we suppose that $BS_4^{[3,4]} - F_e$ is disconnected. Then $|F_e \cap E_{3,4}(BS_4)| = |E_{3,4}(BS_4)| = 4$ and $|F_e^0 - E_{3,4}(BS_4)| \leq 11-3-3-4 = 1$. Since $|E_{BS_4}(V(H_2), V(BS_4^i)) - F_e| \geq |E_{2,i}(BS_4)| - (3! - |V(H_2)|) - |F_e^0 - E_{3,4}(BS_4)| \geq 4 - 1 - 1 > 0$ for $i = 3, 4$, both H_2 and $BS_4^i - F_e^i$ are subgraphs of H. Since $|E_{BS_4}(V(H_1), V(BS_4^3)) - F_e| \geq |E_{1,3}(BS_4)| - (3! - |V(H_1)|) - |F_e^0 - E_{3,4}(BS_4)| \geq 4 - 2 - 1 > 0$, H_1 is a subgraph of H. Thus $|V(H)| \geq 4! - 3$. ∎

Lemma 2.9 *Let $F_e \subseteq E(BS_n)$ with $|F_e| \leq 8n - 21$ for $n \geq 3$. If $BS_n - F_e$ is disconnected, then $BS_n - F_e$ has a component H with $|V(H)| \geq n! - 3$.*

Proof. We prove this lemma by induction on n. For $n = 3, 4$, the result holds by Lemmas 2.4 and 2.8. Assume $n \geq 5$ and $BS_n - F_e$ is disconnected. Without loss of generality, we assume $|F_e^1| \geq |F_e^2| \geq \cdots \geq |F_e^n|$. Since $|F_e| \leq 8n - 21$, $|F_e^n| \leq \cdots \leq |F_e^4| \leq 2n - 6$; otherwise $|F_e| \geq 4(2n - 5) > 8n - 21$ for $n \geq 5$, a contradiction. Hence $BS_n^i - F_e^i$ is connected for every $i \in [4, n]$ by Lemma 2.3. Let H be the component of $BS_n - F_e$ containing $BS_n^n - F_e^n$ as a subgraph. If $|F_e^1| \leq 2n-6$, then the lemma holds by the same argument as that of Case 4 of Lemma 2.7. Now we will consider the following four cases.

Case 1. $|F_e^1| \geq 8n - 28$.

In this case, $|F_e^0| \leq (8n - 21) - (8n - 28) = 7$ and $|F_e^3| \leq 4 \leq 2n - 6$ for $n \geq 5$. Thus $BS_n^i - F_e^i$ is connected for every $i \in [3, n]$ by Lemma 2.3. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n - 2)! - 7 > 0$ for $i, j \in [3, n]$ with $i \neq j$ and $n \geq 5$, $BS_n^{[3,n]} - F_e$ is a subgraph of H.

Suppose $BS_n^2 - F_e^2$ is connected. Since $|E_{2,3}(BS_n) - F_e| \geq |E_{2,3}(BS_n)| - |F_e^0| \geq 2(n - 2)! - 7 > 0$ for $n \geq 5$, $BS_n^2 - F_e^2$ is a subgraph of H. Note that $\{u^+, u^-\} \subseteq V(BS_n^{[2,n]})$ for every $u \in V(BS_n^1)$. Since $|F_e^0| \leq 7 < 2 \times 4$, we have $|V(H)| \geq n! - 3$ by Lemma 2.2 (2).

Now we consider that $BS_n^2 - F_e^2$ is disconnected. Then $2n - 5 \leq |F_e^2| \leq 7$ for $n \geq 5$, which implies $5 \leq |F_e^2| \leq 4n - 13$ and $|F_e^0| \leq (8n - 21) - (8n - 28) - 5 = 2$. Since $|F_e^0| \leq 2$, $|V(H)| \geq n! - 2$ by Lemma 2.2 (3).

Case 2. $6n - 19 \leq |F_e^1| \leq 8n - 29$.

In this case, $|F_e^0| \leq (8n - 21) - (6n - 19) = 2n - 2$ and $|F_e^3| \leq 2n - 6$; otherwise $|F_e| \geq 2(2n - 5) + (6n - 19) = 10n - 29 > 8n - 21$ for $n \geq 5$, a contradiction. Thus $BS_n^i - F_e^i$ is connected for every $i \in [3, n]$ by Lemma 2.3. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n - 2)! - (2n - 2) > 0$ for $i, j \in [3, n]$ with $i \neq j$ and $n \geq 5$, $BS_n^{[3,n]} - F_e$ is a subgraph of H.

Suppose $BS_n^2 - F_e^2$ is connected. Since $|E_{2,3}(BS_n) - F_e| \geq |E_{2,3}(BS_n)| - |F_e^0| \geq 2(n-2)! - (2n-2) > 0$ for $n \geq 5$, $BS_n^2 - F_e^2$ is a subgraph of H. Since $|F_e^1| \leq 8n-29$, $BS_n^1 - F_e^1$ has a component H_1 with $|V(H_1)| \geq (n-1)! - 3$ by induction hypothesis. Since $|E_{BS_n}(V(H_1), V(BS_n^2)) - F_e| \geq |E_{1,2}(BS_n)| - |V(BS_n^1) - V(H_1)| - |F_e^0| \geq 2(n-2)! - 3 - (2n-2) > 0$ for $n \geq 5$, H_1 is a subgraph of H. Hence $|V(H)| \geq n! - 3$.

Now we suppose $BS_n^2 - F_e^2$ is disconnected. Hence $2n - 5 \leq |F_e^2| \leq 2n - 2$ and $|F_e^0| \leq (8n - 21) - (6n - 19) - (2n - 5) = 3$. Since $|F_e^0| \leq 3$, $|V(H)| \geq n! - 3$ by Lemma 2.2 (3).

Case 3. $4n - 12 \leq |F_e^1| \leq 6n - 20$.

In this case, $|F_e^0| \leq (8n - 21) - (4n - 12) = 4n - 9$. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n - 2)! - (4n - 9) > 0$ for $i, j \in [4, n]$ with $i \neq j$ and $n \geq 5$,

$BS_n^{[4,n]} - F_e$ is a subgraph of H. Since $4n-12 \le |F_e^1| \le 6n-20$, $BS_n^1 - F_e^1$ has a component H_1 with $|V(H_1)| \ge (n-1)! - 2$ by Lemma 2.7.

Subcase 3.1. $4n-12 \le |F_e^2| \le 6n-20$.

In this subcase, $|F_e^0| \le (8n-21) - 2(4n-12) = 3$ and $|F_e^3| \le 3 \le 2n-6$ for $n \ge 5$. Hence $BS_n^3 - F_e^3$ is connected by Lemma 2.3. Since $|E_{3,4}(BS_n) - F_e| \ge |E_{3,4}(BS_n)| - |F_e^0| \ge 2(n-2)! - 3 > 0$ for $n \ge 5$, $BS_n^{[3,n]} - F_e$ is a subgraph of H. Since $|F_e^0| \le 3$, $|V(H)| \ge n! - 3$ by Lemma 2.2 (3).

Subcase 3.2. $2n-5 \le |F_e^2| \le 4n-13$.

By Lemma 2.4, $BS_n^2 - F_e^2$ has a component H_2 with $|V(H_2)| \ge (n-1)! - 1$.

Suppose $2n-5 \le |F_e^3| \le 4n-13$. Then $|F_e^0| \le (8n-21)-(4n-12)-2(2n-5) = 1$. Since $|F_e^3| \le 4n-13$, $BS_n^3 - F_e^3$ has a component H_3 with $|V(H_3)| \ge (n-1)! - 1$ by Lemma 2.4. Since $|E_{BS_n}(V(H_i), V(BS_n^4)) - F_e| \ge |E_{i,4}(BS_n)| - (|V(BS_n^i)| - |V(H_i)|) - |F_e^0| \ge 2(n-2)! - 2 - 1 > 0$ for $i \in [1,3]$ and $n \ge 5$, H_i is a subgraph of H for every $i \in [1,3]$. If $|V(H_1)| \ge (n-1)! - 1$, then $|V(H)| \ge n! - 3$. If $|V(H_2)| = (n-1)!$ or $|V(H_3)| = (n-1)!$, then $|V(H)| \ge n! - 3$. Now we suppose that $|V(H_1)| = (n-1)! - 2$ and $|V(H_2)| = |V(H_3)| = (n-1)! - 1$. Let $\{u_{11}, u_{12}\} = V(BS_n^1) - V(H_1)$, $u_2 \in V(BS_n^2) - V(H_2)$, and $u_3 \in V(BS_n^3) - V(H_3)$. Since $|F_e^0| \le 1$, there exists a vertex $v \in (\{u_{11}^+, u_{11}^-, u_{12}^+, u_{12}^-\} - \{u_2, u_3\}) \cap V(H)$ such that $(v, u_{11}) \in E(BS_n) - F_e$ or $(v, u_{12}) \in E(BS_n) - F_e$ by Lemma 2.2 (2). Hence $|V(H)| \ge n! - 3$.

Suppose $|F_e^3| \le 2n-6$. Then $|F_e^0| \le (8n-21) - (4n-12) - (2n-5) = 2n-4$. By Lemma 2,3, $BS_n^3 - F_e^3$ is connected. Since $|E_{3,4}(BS_n) - F_e| \ge |E_{3,4}(BS_n)| - |F_e^0| \ge 2(n-2)! - (2n-4) > 0$ for $n \ge 5$, $BS_n^3 - F_e^3$ is a subgraph of H. Since $|E_{BS_n}(V(H_i), V(BS_n^4)) - F_e| \ge |E_{i,4}(BS_n)| - (|V(BS_n^i)| - |V(H_i)|) - |F_e^0| \ge 2(n-2)! - 2 - (2n-4) > 0$ for $i = 1, 2$ and $n \ge 5$, H_i is a subgraph of H. Hence $|V(H)| \ge n! - 3$.

Subcase 3.3. $|F_e^2| \le 2n-6$.

By Lemma 2.3, $BS_n^i - F_e^i$ is connected for every $i \in [2,n]$. Since $|E_{i,j}(BS_n) - F_e| \ge |E_{i,j}(BS_n)| - |F_e^0| \ge 2(n-2)! - (4n-9) > 0$ for $i, j \in [2,n]$ with $i \ne j$ and $n \ge 5$, $BS_n^{[2,n]} - F_e$ is a subgraph of H. Since $|E_{BS_n}(V(H_1), V(BS_n^{[2,3]})) - F_e| \ge |E_{1,2}(BS_n)| + |E_{1,3}(BS_n)| - 2(|V(BS_n^1)| - |V(H_1)|) - |F_e^0| \ge 2 \times 2(n-2)! - 2 \times 2 - (4n-9) > 0$, H_1 is a subgraph of H and $|V(H)| \ge n! - 2$.

Case 4. $2n-5 \le |F_e^1| \le 4n-13$.

By Lemma 2.4, $BS_n^1 - F_e^1$ has a component H_1 with $|V(H_1)| \ge (n-1)! - 1$.

Subcase 4.1. $|F_e^3| \ge 2n-5$.

In this subcase, $|F_e^0| \le (8n-21) - 3(2n-5) = 2n-6$. Since $|E_{i,j}(BS_n) - F_e| \ge |E_{i,j}(BS_n)| - |F_e^0| \ge 2(n-2)! - (2n-6) > 0$ for $i, j \in [4,n]$ with $i \ne j$ and $n \ge 5$,

$BS_n^{[4,n]} - F_e$ is a subgraph of H. Since $2n-5 \leq |F_e^3| \leq |F_e^2| \leq |F_e^1| \leq 4n-13$, $BS_n^i - F_e^i$ has a component H_i with $|V(H_i)| \geq (n-1)!-1$ for $i=2,3$ by Lemma 2.4. Since $|E_{BS_n}(V(H_i), V(BS_n^4)) - F_e| \geq |E_{i,4}(BS_n)| - (|V(BS_n^i)| - |V(H_i)|) - |F_e^0| \geq 2(n-2)!-1-(2n-6) > 0$ for $i \in [1,3]$ and $n \geq 5$, H_i is a subgraph of H for every $i \in [1,3]$. Thus $|V(H)| \geq n!-3$.

Subcase 4.2. $|F_e^3| \leq 2n-6$ *and* $|F_e^2| \geq 2n-5$.

In this subcase, $|F_e^0| \leq (8n-21)-2(2n-5) = 4n-11$. By Lemma 2.3, $BS_n^3 - F_e^3$ is connected. Since $|E_{i,j}(BS_n) - F_e| \geq |E_{i,j}(BS_n)| - |F_e^0| \geq 2(n-2)!-(4n-11) > 0$ for $i,j \in [3,n]$ with $i \neq j$ and $n \geq 5$, $BS_n^{[3,n]} - F_e$ is a subgraph of H. Since $2n-5 \leq |F_e^2| \leq |F_e^1| \leq 4n-13$, $BS_n^2 - F_e^2$ has a component H_2 with $|V(H_2)| \geq (n-1)!-1$ by Lemma 2.4. Since $|E_{BS_n}(V(H_i), V(BS_n^4)) - F_e| \geq |E_{i,4}(BS_n)| - (|V(BS_n^i)| - |V(H_i)|) - |F_e^0| \geq 2(n-2)!-1-(4n-11) > 0$ for $i \in [1,2]$ and $n \geq 5$, H_i is a subgraph of H for every $i \in [1,2]$. Hence $|V(H)| \geq n!-2$.

Subcase 4.3. $|F_e^2| \leq 2n-6$.

In this subcase, $|F_e^0| \leq (8n-21)-(2n-5) = 6n-16$. By Lemma 2.3, both $BS_n^2 - F_e^2$ and $BS_n^3 - F_e^3$ are connected. We claim $E_{2,3}(BS_n) - F_e \neq \emptyset$ or $E_{2,4}(BS_n) - F_e \neq \emptyset$; otherwise $|F_e| \geq |E_{2,3}(BS_n)| + |E_{2,4}(BS_n)| = 2 \times 2(n-2)! > 8n-21$ for $n \geq 5$, a contradiction. Without loss of generality, we assume $E_{2,3}(BS_n) - F_e \neq \emptyset$. Similarly, we can get $E_{2,i}(BS_n) - F_e \neq \emptyset$ or $E_{3,i}(BS_n) - F_e \neq \emptyset$ for every $i \in [4,n]$. Thus $BS_n^{[2,n]} - F_e$ is a subgraph of H. Since $|E_{BS_n}(V(H_1), V(BS_n^{[2,3]})) - F_e| \geq |E_{1,2}(BS_n)| + |E_{1,3}(BS_n)| - 2(|V(BS_n^1)| - |V(H_1)|) - |F_e^0| \geq 2 \times 2(n-2)! - 2 \times 1 - (6n-16) > 0$, H_1 is a subgraph of H. Hence $|V(H)| \geq n!-1$. ■

3. Edge-fault-tolerant strong Menger edge connectivity of BS_n

We will consider the edge-fault-tolerant strong Menger edge connectivity of BS_n in this section.

Theorem 3.1 *For $n \geq 3$, the bubble-sort star graph BS_n is $(2n-5)$-edge-fault-tolerant strongly Menger edge connected and the bound $2n-5$ is sharp.*

Proof. Let $F_e \subseteq E(BS_n)$ be an arbitrary faulty edge set with $|F_e| \leq 2n-5$. By Lemma 2.3, $BS_n - F_e$ is connected. Let u, v with $u \neq v$ be any two vertices in BS_n and $t = \min\{d_{BS_n - F_e}(u), d_{BS_n - F_e}(v)\}$. By Theorem 1.1, it suffices to show that u and v are connected in $BS_n - F_e - E_f$ for any $E_f \subseteq E(BS_n) - F_e$ with $|E_f| \leq t-1$. Suppose on the contrary, that u and v are disconnected in $BS_n - F_e - E_f$ for some $E_f \subseteq E(BS_n) - F_e$ with $|E_f| \leq t-1$. Since $d_{BS_n - F_e}(u) \leq 2n-3$ and $d_{BS_n - F_e}(v) \leq 2n-3$, $|E_f| \leq 2n-4$. Thus $|F_e \cup E_f| \leq (2n-5)+(2n-4) = 4n-9$. By Lemma 2.4, $BS_n - F_e - E_f$ has a component H with $|V(H)| \geq n!-1$. Since u and v are disconnected in $BS_n - F_e - E_f$, $|V(H)| = n! - 1$ and $|\{u, v\} \cap V(H)| = 1$. Without loss of generality, we assume $u \notin V(H)$ and $v \in V(H)$. Hence $E_{BS_n}(\{u\}, N_{BS_n - F_e}(u)) \subseteq E_f$, which implies $|E_f| \geq d_{BS_n - F_e}(u)$, a contradiction to $|E_f| \leq t - 1 \leq d_{BS_n - F_e}(u) - 1$. Hence BS_n is $(2n-5)$-edge-fault-tolerant strongly Menger edge connected.

Next, we will show the bound $2n - 5$ is sharp. Let $u, u_1 \in V(BS_n)$ with $(u, u_1) \in E(BS_n)$. Let $F_e = E_{BS_n}(u_1) - (u, u_1)$ and $v \in V(BS_n) - N_{BS_n}(u_1) - \{u_1\}$ (see Fig.2). Then $|F_e| = 2n - 4$, $d_{BS_n - F_e}(u) = d_{BS_n - F_e}(v) = 2n - 3$. Obviously, there are at most $2n - 4$ edge-disjoint (u, v)-paths. ■

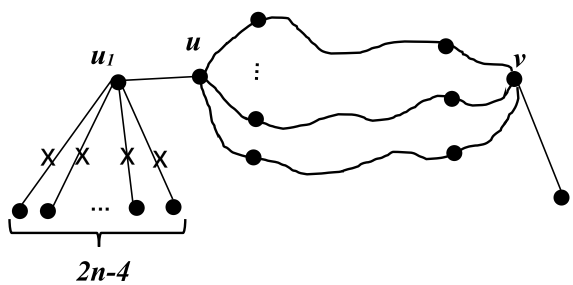

Figure 2: Illustration of Theorem 3.1.

4. Conditional edge-fault-tolerant strong Menger edge connectivity of BS_n

We will consider the conditional edge-fault-tolerant strong Menger edge connectivity of BS_n in this section.

Theorem 4.1 *For $n \geq 4$, the bubble-sort star graph BS_n is $(6n-17)$-conditional edge-fault-tolerant strongly Menger edge connected and the bound $6n - 17$ is sharp.*

Proof. Let $F_e \subseteq E(BS_n)$ be an arbitrary faulty edge set with $|F_e| \leq 6n - 17$ and $\delta(BS_n - F_e) \geq 2$. Since $|F_e| \leq 6n - 17 \leq 6n - 14$ and $\delta(BS_n - F_e) \geq 2$, $BS_n - F_e$ is connected by Lemma 2.7. Let u, v with $u \neq v$ be any two vertices in BS_n and $t = \min\{d_{BS_n - F_e}(u), d_{BS_n - F_e}(v)\}$. By Theorem 1.1, it suffices to show that u and v are connected in $BS_n - F_e - E_f$ for any $E_f \subseteq E(BS_n) - F_e$ with $|E_f| \leq t - 1$. Suppose on the contrary, that u and v are disconnected in $BS_n - F_e - E_f$ for some $E_f \subseteq E(BS_n) - F_e$ with $|E_f| \leq t - 1$. Since $d_{BS_n - F_e}(u) \leq 2n - 3$ and $d_{BS_n - F_e}(v) \leq 2n - 3$, $|E_f| \leq 2n - 4$. Thus $|F_e \cup E_f| \leq (6n - 17) + (2n - 4) = 8n - 21$. By Lemma 2.9, $BS_n - F_e - E_f$ has a component H with $|V(H)| \geq n! - 3$. Since u and v are disconnected in $BS_n - F_e - E_f$, $|\{u, v\} \cap V(H)| \leq 1$. Without loss of generality, we assume $u \notin V(H)$. Let H_1 be the component in $BS_n - F_e - E_f$ containing u. If $d_{H_1}(u) = 0$, then $E_{BS_n}(\{u\}, N_{BS_n - F_e}(u)) \subseteq E_f$, which implies $|E_f| \geq d_{BS_n - F_e}(u)$, a contradiction to $|E_f| \leq t-1 \leq d_{BS_n-F_e}(u)-1$. Suppose that $d_{H_1}(u)=i$ ($i \in [1,2]$). Since BS_n is bipartite, H_1 is a path P_2 or P_3 and there are i vertices in $V(H_1)-\{u\}$ that have degree one in H_1. Since $\delta(BS_n-F_e) \geq 2$, every vertex with degree one in H_1 is incident with at least one edge in E_f. Thus $|E_f| \geq d_{BS_n-F_e}(u)-i+i = d_{BS_n-F_e}(u)$, a contradiction to $|E_f| \leq t-1 \leq d_{BS_n-F_e}(u)-1$. Hence BS_n is $(6n-17)$-conditional edge-fault-tolerant strongly Menger edge connected.

Next, we will show the bound $6n-17$ is sharp. Let $u, u_1, u_2, u_3 \in V(BS_n)$ with $(u,u_1),(u_1,u_2),(u_2,u_3),(u_3,u) \in E(BS_n)$ and $u_{11} \in N_{BS_n}(u_1)-\{u,u_2\}$. Let $F_e = E_{BS_n}(u_1) \cup E_{BS_n}(u_2) \cup E_{BS_n}(u_3) - \{(u,u_1),(u_1,u_2),(u_2,u_3),(u_3,u),(u_1,u_{11})\}$ and $v \in V(BS_n)-N_{BS_n}(u_1) \cup N_{BS_n}(u_2) \cup N_{BS_n}(u_3)$ (see Fig.3). Then $|F_e| = (2n-6)+2(2n-5) = 6n-16$, $d_{BS_n-F_e}(u) = d_{BS_n-F_e}(v) = 2n-3$, and $\delta(BS_n-F_e) \geq 2$ for $n \geq 4$. Obviously, there are at most $2n-4$ edge-disjoint (u,v)-paths. ■

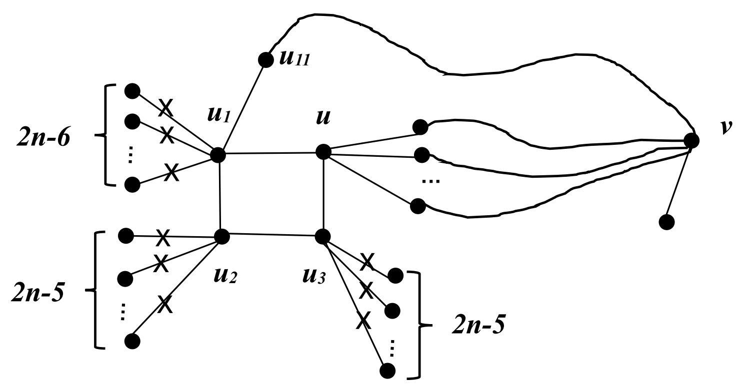

Figure 3: Illustration of Theorem 4.1.

5. Conclusion

In this paper, we study the edge-fault-tolerant strong Menger edge connectivity of n-dimensional bubble-sort star graph BS_n. We show that every pair of distinct vertices u and v in BS_n are connected by $\min\{d_{BS_n-F_e}(u), d_{BS_n-F_e}(v)\}$ edge-disjoint paths in $BS_n - F_e$, where F_e is an arbitrary edge subset of BS_n with $|F_e| \leq 2n-5$. We also show that every pair of distinct vertices u and v in BS_n are connected by $\min\{d_{BS_n-F_e}(u), d_{BS_n-F_e}(v)\}$ edge-disjoint paths in $BS_n - F_e$, where F_e is an arbitrary edge subset of BS_n with $|F_e| \leq 6n-17$ and $\delta(BS_n - F_e) \geq 2$. Moreover, we give two examples to show that our results are optimal. The connectivity and edge connectivity of interconnection network determine the fault tolerance of the network. They are issues worth studying.

Acknowledgements

This research is supported by National Natural Science Foundation of China (No. 11801450), Natural Science Foundation of Shaanxi Province, China (No. 2019JQ-506).

References

[1] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.

[2] H. Cai, H. Liu, M. Lu, Fault-tolerant maximal local-connectivity on Bubble-sort star graphs, Discrete Applied Mathematics 181(2015) 33-40.

[3] Z. T. Chou, C. C. Hsu, J. P. Sheu, Bubblesort star graphs: a new interconnection network, International Conference on Parallel and Distributed Systems (1996) 41-48.

[4] M. M. Gu, R. X. Hao, Y. Q. Feng, The pessimistic diagnosability of bubble-sort star graphs and augmented k-ary n-cubes, International Journal of Computer Mathematics: Computer Systems Theory 1(3-4)(2016) 98-112.

[5] S. He, R. X. Hao, E. Cheng, Strongly Menger-edge-connectedness and strongly Menger-vertex-connectedness of regular networks, Theoretical Computer Science 731(2018) 50-67.

[6] P. Li, M. Xu, Edge-fault-tolerant strong Menger edge connectivity on the class of hypercube-like networks, Discrete Applied Mathematics 259(2019) 145-152.

[7] P. Li, M. Xu, Fault-tolerant strong Menger (edge) connectivity and 3-extra edge-connectivity of balanced hypercubes, Theoretical Computer Science 707(2018) 56-68.

[8] K. Menger, Zur allgemeinen kurventheorie, Fundamenta Mathematicae 10(1)(1927) 96-115.

[9] E. Oh, J. Chen, On strong Menger-connectivity of star graphs, Discrete Applied Mathematics 129(2-3)(2003) 499-511.

[10] Y. Qiao, W. Yang, Edge disjoint paths in hypercubes and folded hypercubes with conditonal faults, Applied Mathematics and Computation 294(2017) 96-101.

[11] S. Wang, Z. Wang, M. Wang, The 2-extra connectivity and 2-extra diagnosability of bubble-sort star graph networks, The Computer Journal 59(12)(2016) 1839-1856.

[12] S. Wang, Z. Wang, M. Wang, The 2-good-neighbor connectivity and 2-good-neighbor diagnosability of bubble-sort star graph networks, Discrete Applied Mathematics 217(2017) 691-706.

[13] S. Wang, M. Wang, The strong connectivity of bubble-sort star graphs, The Computer Journal 62(5)(2018) 715-729.

[14] S. L. Zhao, R. X. Hao, The Generalized Connectivity of Bubble-Sort Star Graphs, International Journal of Foundations of Computer Science 30(05)(2019) 793-809.

[15] G. Zhang, D. Wang, Structure connectivity and substructure connectivity of bubble-sort star graph networks, Applied Mathematics and Computation 363(2019) 124632.

[16] Q. Zhu, J. Zhang, L. L. Li, The h-extra connectivity and h-extra conditional diagnosability of Bubble-sort star graphs, Discrete Applied Mathematics 251(2018) 322-333.