# Local WL Invariance and Hidden Shades of Regularity 

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#### Abstract

The $k$-dimensional Weisfeiler-Leman algorithm is a powerful tool in graph isomorphism testing. For an input graph $G$, the algorithm determines a canonical coloring of $s$-tuples of vertices of $G$ for each $s$ between 1 and $k$. We say that a numerical parameter of $s$-tuples is $k$-WL-invariant if it is determined by the tuple color. As an application of Dvorák's result on $k$-WL-invariance of homomorphism counts, we spot some non-obvious regularity properties of strongly regular graphs and related graph families. For example, if $G$ is a strongly regular graph, then the number of paths of length 6 between vertices $x$ and $y$ in $G$ depends only on whether or not $x$ and $y$ are adjacent (and the length 6 is here optimal). Or, the number of cycles of length 7 passing through a vertex $x$ in $G$ is the same for every $x$ (where the length 7 is also optimal).


## 1 Introduction

The $k$-dimensional Weisfeiler-Leman algorithm ( $k$-WL) is a powerful combinatorial tool for detecting non-isomorphism of two given graphs. Playing a constantly significant role in isomorphism testing, it was used, most prominently, in Babai's quasipolynomial-time isomorphism algorithm [2].

For each $k$-tuple $\bar{x}=\left(x_{1}, \ldots, x_{k}\right)$ of vertices in an input graph $G$, the algorithm computes a canonical color $\mathrm{WL}_{k}(G, \bar{x})$; see the details in Section 2. If the multisets of colors $\left\{\mathrm{WL}_{k}(G, \bar{x}): \bar{x} \in V(G)^{k}\right\}$ and $\left\{\mathrm{WL}_{k}(H, \bar{y}): \bar{y} \in V(H)^{k}\right\}$ are different for two graphs $G$ and $H$, then these graphs are clearly non-isomorphic, and we say that $k$-WL distinguishes them. If $k$-WL does not distinguish $G$ and $H$, we say that these graphs are $k$-WL-equivalent and write $G \equiv_{k \text {-wL }} H$. As proved by Cai, Fürer, and Immerman [6], the $k$-WL-equivalence for any fixed dimension $k$ is strictly coarser than the isomorphism relation on graphs. For $k=2$, an example of two non-isomorphic 2-WL-equivalent graphs is provided by any pair of non-isomorphic strongly regular graphs with the same parameters. The smallest such pair consists

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Figure 1: The Shrikhande graph $S$ drawn on a torus; vertices of the same color form 4 -cycles (some edges are not depicted). Though both $a, a^{\prime}$ and $b, b^{\prime}$ are non-adjacent, the common neighbors of $a$ and $a^{\prime}$ are non-adjacent, while the common neighbors of $b$ and $b^{\prime}$ are adjacent. The automorphism group of $S$ acts transitively on the ordered pairs of non-adjacent vertices of each type [20].
of the Shrikhande and the $4 \times 4$ rook's graphs. The Shrikhande graph, which will occur several times in the sequel, is the Cayley graph of the abelian group $\mathbb{Z}_{4} \times \mathbb{Z}_{4}$ with connecting set $\{(1,0),(0,1),(1,1)\}$. Figure 1 shows a natural drawing of this graph on the torus.

Let $\operatorname{hom}(F, G)$ denote the number of homomorphisms from a graph $F$ to a graph $G$. A characterization of the $k$-WL-equivalence in terms of homomorphism numbers by Dvořák [10] implies that the homomorphism count $\operatorname{hom}(F, \cdot)$ is $k$-WL-invariant for each pattern graph $F$ of treewidth at most $k$, that is, $\operatorname{hom}(F, G)=\operatorname{hom}(F, H)$ whenever $G \equiv_{k \text {-wL }} H$.

Let $\operatorname{sub}(F, G)$ denote the number of subgraphs of $G$ isomorphic to $F$. Lovász [17, Section 5.2.3] showed a close connection between the homomorphism and the subgraph counts, which found many applications in various context. Curticapean, Dell, and Marx [9] used this connection to design efficient algorithms for counting the number of $F$-subgraphs in an input graph $G$. In [1], we addressed WL invariance of the subgraph counts. Define the homomorphism-hereditary treewidth htw $(F)$ of a graph $F$ as the maximum treewidth of the image of $F$ under an edge-surjective homomorphism. Then Dvořák's invariance result for homomorphism counts, combined with the Lovász relationship, implies that $\operatorname{sub}(F, G)=\operatorname{sub}(F, H)$ whenever $G \equiv_{k \text {-wL }} H$ for $k=h t w(F)$.

In fact, Dvořák [10] proves his result in a stronger, local form. To explain what we here mean by locality, we need some additional technical concepts. A graph $F$ with $s$ designated vertices $z_{1}, \ldots, z_{s}$ is referred to as s-labeled. A tree decomposition and the treewidth of $(F, \bar{z})$ are defined as usually with the additional requirement that at least one bag of the tree decomposition must contain all $\left.z_{1}, \ldots, z_{s}\right]^{11}$ A homomorphism from an $s$-labeled graph $\left(F, z_{1}, \ldots, z_{s}\right)$ to an $s$-labeled graph $\left(G, x_{1}, \ldots, x_{s}\right)$ must take $z_{i}$ to $x_{i}$ for every $i \leq s$. Denote the number of such homomorphisms by $\operatorname{hom}(F, \bar{z} ; G, \bar{x})$.

The canonical coloring of the Cartesian power $V(G)^{k}$ produced by $k$-WL deter-

[^1]mines a canonical coloring of $V(G)^{s}$ for each $s$ between 1 and $k$. Specifically, if $s<k$, we define $\mathrm{WL}_{k}\left(G, x_{1}, \ldots, x_{s}\right)=\mathrm{WL}_{k}\left(G, x_{1}, \ldots, x_{s}, \ldots, x_{s}\right)$ just by cloning the last vertex in the $s$-tuple $k-s$ times. Dvořák 10 proves that, if an $s$-labeled graph $(F, \bar{z})$ has treewidth $k$, then even local homomorphism counts hom $(F, \bar{z} ; \cdot)$ are $k$-WL-invariant in the sense that $\operatorname{hom}(F, \bar{z} ; G, \bar{x})=\operatorname{hom}(F, \bar{z} ; H, \bar{y})$ whenever $k$-WL assigns the same color to the $s$-tuples $\bar{x}$ and $\bar{y}$, i.e., $\mathrm{WL}_{k}(G, \bar{x})=\mathrm{WL}_{k}(H, \bar{y})$.

Our first observation is that, like for ordinary unlabeled graphs, this result can be extended to local subgraph counts. Given a pattern graph $F$ with labeled vertices $z_{1}, \ldots, z_{s}$ and a host graph $G$ with labeled vertices $x_{1}, \ldots, x_{s}$, we write $\operatorname{sub}(F, \bar{z} ; G, \bar{x})$ to denote the number of subgraphs $S$ of $G$ with $x_{1}, \ldots, x_{s} \in V(S)$ such that there is an isomorphism from $F$ to $S$ mapping $z_{i}$ to $x_{i}$ for all $i \leq s$. The local subgraph counts $\operatorname{sub}\left(F, z_{1}, \ldots, z_{s} ; \cdot\right)$ are $k$-WL-invariant for $k=h t w(F, \bar{z})$, where the concept of the homomorphism-hereditary treewidth is extended to $s$-labeled graphs in a straightforward way. That is, not only the $k$-WL-equivalence type of $G$ determines the total number of $F$-subgraphs in $G$, but even the color $\mathrm{WL}_{k}\left(G, x_{1}, \ldots, x_{s}\right)$ of each $s$-tuple of vertices determines the number of extensions of this particular tuple to an $F$-subgraph (see Lemma 3.3).

Consider as an example the pattern graph $F=P_{6}$ where $P_{6}$ is a path through 6 vertices $z_{1}, \ldots, z_{6}$. Consider also two host graphs $R$ and $S$ where $R$ is the $4 \times 4$ rook's graph, and $S$ is the Shrikhande graph. Since $h t w\left(P_{6}\right)=2$, the "global" invariance result in [10 implies that $R$ and $S$ contain equally many 6-paths. Indeed, $\operatorname{sub}\left(P_{6}, R\right)=\operatorname{sub}\left(P_{6}, S\right)=20448$.

Moreover, we have $\operatorname{htw}\left(P_{6}, z_{1}, z_{6}\right)=2$; see Figure 2 a . It follows that the count $\operatorname{sub}\left(P_{6}, z_{1}, z_{6} ; G, x, x^{\prime}\right)$ is determined by $\mathrm{WL}_{2}\left(G, x, x^{\prime}\right)$ for every graph $G$ and every pair of vertices $x, x^{\prime}$ in $G$. If $G$ is a strongly regular graph, then $\mathrm{WL}_{2}\left(G, x, x^{\prime}\right)$ depends only on the parameters of $G$ and on whether $x$ and $x^{\prime}$ are equal, adjacent, or non-adjacent. Applied to $G \in\{R, S\}$, this justifies the fact that $\operatorname{sub}\left(P_{6}, z_{1}, z_{6} ; R, x, x^{\prime}\right)=$ $\operatorname{sub}\left(P_{6}, z_{1}, z_{6} ; S, y, y^{\prime}\right)=156$ for every pair of adjacent vertices $x, x^{\prime}$ in $R$ and every pair of adjacent vertices $y, y^{\prime}$ in $S$. If $x$ and $x^{\prime}$ as well as $y$ and $y^{\prime}$ are not adjacent, then $\operatorname{sub}\left(P_{6}, z_{1}, z_{6} ; R, x, x^{\prime}\right)=\operatorname{sub}\left(P_{6}, z_{1}, z_{6} ; S, y, y^{\prime}\right)=180$.

Note that the condition $\operatorname{htw}\left(P_{6}, z_{1}, z_{6}\right)=2$ is essential here. Indeed, the slightly modified pattern $\left(P_{6}, z_{2}, z_{5}\right)$ does not enjoy anymore the above invariance property. For example, for the two vertex pairs $a, a^{\prime}$ and $b, b^{\prime}$ in Figure 1 we have $\operatorname{sub}\left(P_{6}, z_{2}, z_{5} ; S, a, a^{\prime}\right)=244$ while $\operatorname{sub}\left(P_{6}, z_{2}, z_{5} ; S, b, b^{\prime}\right)=246$, even though both pairs are non-adjacent. The difference between the patterns $\left(P_{6}, z_{1}, z_{6}\right)$ and $\left(P_{6}, z_{2}, z_{5}\right)$, is explained by the fact that $h t w\left(P_{6}, z_{2}, z_{5}\right)=3$; see Figure 2 b .

As we have seen, the number of 6 -paths between two vertices is the same for any two adjacent (resp., non-adjacent) vertices in $R$ and in $S$. The same holds true also for 7 -paths and for any strongly regular graph. This general fact follows from the 2-WL-invariance of the subgraph counts for $P_{s}$ with labeled end vertices for $s \leq 7$ and from the fact that, if $G$ is a strongly regular graph, then the color $\mathrm{WL}_{k}\left(G, x, x^{\prime}\right)$ is the same for all adjacent and all non-adjacent pairs $x, x^{\prime}$. Our main purpose in this note is to collect such non-obvious regularity properties, which are not directly implied by the definition of a strongly regular graph; see Theorem


Figure 2: (a) Homomorphic images of $\left(P_{6}, z_{1}, z_{6}\right)$ up to isomorphism and root swapping. (b) $\left(P_{6}, z_{2}, z_{5}\right)$ and its image under the homomorphism $h$ which maps $z_{1}$ to $z_{4}, z_{6}$ to $z_{3}$, and fixes all other vertices. Since $h\left(z_{2}\right)$ and $h\left(z_{5}\right)$ must be in one bag, the homomorphism-hereditary treewidth increases to 3 .
4.1. As a further example of implicit regularity, the number of 7 -cycles containing a specified edge in a strongly regular graph does not depend on the choice of this edge. This is actually true for a larger class of graphs, namely for constituent graphs of association schemes, in particular, for all distance-regular graphs; see Theorem 4.3. Also, the number of 7 -cycles passing through a specified vertex in a strongly regular graph does not depend on the choice of this vertex. The last property also holds true for constituent graphs of association schemes and even for a yet larger class of $\mathrm{WL}_{1,2}$-regular graphs [11]; see Theorem 4.4.

Related work. Godsil 13 proves that, if $G$ is a constituent graph of an association scheme, then the number of spanning trees containing a specified edge of $G$ is the same for every choice of the edge. Note that this "local" result also has a "global" invariance analog. If two graphs $G$ and $H$ are 2-WL-equivalent, then they have the same number of spanning trees. This follows from Kirchhoff's theorem because 2-WL-equivalent graphs are cospectral [12].

## 2 Formal definitions

Graph-theoretic preliminaries. The vertex set of a graph $G$ is denoted by $V(G)$. A graph $G$ is $s$-regular if every vertex of $G$ has exactly $s$ neighbors. An $n$-vertex $s$-regular graph $G$ is strongly regular with parameters $(n, s, \lambda, \mu)$ if every two adjacent vertices of $G$ have $\lambda$ common neighbors and every two non-adjacent vertices have $\mu$ common neighbors. The Shrikhande and the $4 \times 4$ rook's graphs, which were mentioned in Section 1, have parameters (16, 6, 2, 2).

A tree decomposition of a graph $G$ is a tree $T$ and a family $\mathcal{B}=\left\{B_{i}\right\}_{i \in V(T)}$ of sets $B_{i} \subseteq V(G)$, called bags, such that the union of all bags covers all $V(G)$, every edge of $G$ is contained in at least one bag, and we have $B_{i} \cap B_{j} \subseteq B_{l}$ whenever $l$ lies on the path from $i$ to $j$ in $T$. The width of the decomposition is equal to $\max \left|B_{i}\right|-1$. The treewidth of $G$, denoted by $t w(G)$, is the minimum width of a tree decomposition of $G$. Moreover, we define htw $(G)$, the homomorphism-hereditary treewidth of $G$, as the maximum $t w(H)$ over all graphs $H$ such that there is an
edge-surjective homomorphism from $G$ to $H$. We give a simple example for further reference.

Example 2.1. $h t w\left(C_{s}\right)=2$ for $3 \leq s \leq 7$, where $C_{s}$ denotes the cycle graph on $s$ vertices. Indeed, a simple inspection shows that, if $s \leq 7$, then all edge-surjective homomorphic images of $C_{s}$ are outerplanar graphs with one exception for the graph formed by three triangles sharing a common edge, which is an image of $C_{7}$. Another argument, based on a characterization of the class of all graphs with homomorphismhereditary treewidth at most 2 , can be found in [1]. Since this class is closed under taking subgraphs, we also have $h t w\left(P_{s}\right)=2$ for $s \leq 7$.

The Weisfeiler-Leman algorithm. The original version of the algorithm, 2-WL, was described by Weisfeiler and Leman in [21]. For an input graph $G$, this algorithm assigns a color to each pair of vertices $(x, y)$ and then refines the coloring of the Cartesian square $V(G)^{2}$ step by step. Initially, $\mathrm{WL}_{2}^{0}(G, x, y)$ is one of three colors depending on whether $x$ and $y$ are adjacent, non-adjacent, or equal. The coloring after the $i$-th refinement step, $\mathrm{WL}_{2}^{i}(G, x, y)$, is computed as

$$
\begin{equation*}
\mathrm{WL}_{2}^{i}(G, x, y)=\left(\mathrm{WL}_{2}^{i-1}(G, x, y),\left\{\left\{\left(\mathrm{WL}_{2}^{i-1}(G, x, z), \mathrm{WL}_{2}^{i-1}(G, z, y)\right)\right\}\right\}_{z \in V(G)}\right) \tag{1}
\end{equation*}
$$

where $\{\}$ denotes the multiset. In words, 2 -WL traces through all extensions of each pair $(x, y)$ to a triangle $(x, z, y)$, classifies each $(x, z, y)$ according to the current colors of its sides $(x, z)$ and $(z, y)$, and assigns a new color to each $(x, y)$ so that any two $(x, y)$ and $\left(x^{\prime}, y^{\prime}\right)$ become differently colored if they have differently many extensions of some type.

Let $\mathcal{P}^{i}$ denote the color partition of $V(G)^{2}$ after the $i$-th refinement. Since $\mathcal{P}^{i}$ is finer than or equal to $\mathcal{P}^{i-1}$, the color partition stabilizes starting from some step $t \leq n^{2}$, where $n=|V(G)|$, that is, $\mathcal{P}^{t+1}=\mathcal{P}^{t}$ and, hence, $\mathcal{P}^{i}=\mathcal{P}^{t}$ for all $i \geq t$. The algorithm terminates as soon as the stabilization is reached. We denote the final color of a vertex pair $(x, y)$ by $\mathrm{WL}_{2}(G, x, y)$. The following simple observation plays an important role below.

Remark 2.2. The color partition is stable from the very beginning, that is, $\mathcal{P}^{1}=\mathcal{P}^{0}$ and $\mathrm{WL}_{2}(G, x, y)=\mathrm{WL}_{2}^{0}(G, x, y)$, whenever $G$ is strongly regular, and only in this case.

The $k$-dimensional version of the algorithm, $k$-WL, operates similarly with $k$ tuples of vertices and computes a stable color $\mathrm{WL}_{k}\left(G, x_{1}, \ldots, x_{k}\right)$ for each $k$-tuple $x_{1}, \ldots, x_{k}$ of vertices of $G$. This coloring extends to $s$-tuples $\bar{x}=\left(x_{1}, \ldots, x_{s}\right)$ for $s<$ $k$ and $s=k+1$ as follows. If $s<k$, we set $\mathrm{WL}_{k}(G, \bar{x})=\mathrm{WL}_{k}\left(G, x_{1}, \ldots, x_{s}, \ldots, x_{s}\right)$, i.e, extend $\bar{x}$ to a $k$-tuple by cloning the last entry. For $\bar{x}=\left(x_{1}, \ldots, x_{k}, x_{k+1}\right)$, we define $\mathrm{WL}_{k}(G, \bar{x})=\left(\mathrm{WL}_{k}\left(G, \bar{x}_{-1}\right), \ldots, \mathrm{WL}_{k}\left(G, \bar{x}_{-(k+1)}\right)\right.$, where $\bar{x}_{-i}$ is obtained from $\bar{x}$ by removing the entry $x_{i}$. We do not describe $k$-WL for $k \geq 3$ in more detail, as the following logical characterization is sufficient for our purposes.

Logic with counting quantifiers. Cai, Fürer, and Immerman [6] established a close connection between the Weisfeiler-Leman algorithm and first-order logic with counting quantifiers. A counting quantifier $\exists^{m}$ opens a logical formula saying that a graph contains at least $m$ vertices with some property. Let $C^{k}$ denote the set of formulas in the standard first-order logical formalism which can additionally contain counting quantifiers and use occurrences of at most $k$ first-order variables.

Let $G$ be a graph with $s \leq k$ labeled vertices $\left(x_{1}, \ldots, x_{s}\right)$. The $\mathrm{C}^{k}$-type of $\left(G, x_{1}, \ldots, x_{s}\right)$ is the set of all formulas with $s$ free variables in $\mathrm{C}^{k}$ that are true on ( $G, x_{1}, \ldots, x_{s}$ ). Despite a rather abstract definition, the $C^{k}$-types admit an efficient encoding by the colors produced by the $(k-1)$-dimensional Weisfeiler-Leman algorithm.

Lemma 2.3 (Cai, Fürer, and Immerman [6]). Let $k \geq 1$ and $0 \leq s \leq k+1$. Then $\left(G, x_{1}, \ldots, x_{s}\right)$ and $\left(H, y_{1}, \ldots, y_{s}\right)$ are of the same $\mathrm{C}^{k+1}$-type if and only if $\mathrm{WL}_{k}\left(G, x_{1}, \ldots, x_{s}\right)=\mathrm{WL}_{k}\left(H, y_{1}, \ldots, y_{s}\right)$.

For completeness, we state this result also for $k=1$, where 1 -WL stands for the classical degree refinement routine [15].

## 3 Local WL invariance

An $s$-labeled graph $F_{\bar{z}}$ is a graph $F$ with a sequence of $s$ (not necessarily distinct) designated vertices $\bar{z}=\left(z_{1}, \ldots, z_{s}\right)$; cf. [18]. Sometimes we will also use more extensive notation $F_{\bar{z}}=\left(F, z_{1}, \ldots, z_{s}\right)$, as we already did in Sections 112, A homomorphism from an s-labeled graph $F_{\bar{z}}$ to an $s$-labeled graph $G_{\bar{x}}$ is a usual homomorphism from $F$ to $G$ taking $y_{i}$ to $x_{i}$ for every $i \leq s$.

A tree decomposition of an $s$-labeled graph $F_{\bar{z}}$ is a tree decomposition of $F$ where there is a bag containing all of the labeled vertices $z_{1}, \ldots, z_{s}$. The treewidth $\operatorname{tw}\left(F_{\bar{z}}\right)$ and the homomorphism-hereditary treewidth $h t w\left(F_{\bar{z}}\right)$ are defined in the same way as for unlabeled graphs. Note that, if $F_{\bar{z}}$ is a graph with $s$ pairwise distinct labeled vertices, then $h t w\left(F_{\bar{z}}\right)=k$ implies that $s \leq k+1$.

Remark 3.1. As it easily follows from the definition, for every 1-labeled graph $F_{z_{1}}$ we have $t w\left(F_{z_{1}}\right)=t w(F)$. Similarly, for a 2-labeled graph $F_{z_{1}, z_{2}}$ we have $t w\left(F_{z_{1}, z_{2}}\right)=$ $t w(F)$ whenever the labeled vertices $z_{1}$ and $z_{2}$ are adjacent.

For two $s$-labeled graphs, let $\operatorname{hom}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ denote the number of homomorphisms from $F_{\bar{z}}$ to $G_{\bar{x}}$. For each pattern graph $F_{\bar{z}}$, the count $\operatorname{sub}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ is an invariant of a host graph $G_{\bar{x}}$. In general, a function $f$ of a labeled graph is an invariant if $f\left(G_{\bar{x}}\right)=f\left(H_{\bar{y}}\right)$ whenever $G_{\bar{x}}$ and $H_{\bar{y}}$ are isomorphic (note that an isomorphism, like any homomorphism, must respect the labeled vertices). We say that an invariant $f_{1}$ is determined by an invariant $f_{2}$ if $f_{2}\left(G_{\bar{x}}\right)=f_{2}\left(H_{\bar{y}}\right)$ implies $f_{1}\left(G_{\bar{x}}\right)=f_{1}\left(H_{\bar{y}}\right)$.

Lemma 3.2 (Dvořák [10, Lemma 4]). For each s-labeled graph $F_{\bar{z}}$, the homomorphism count $\operatorname{hom}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ is determined by the $\mathrm{C}^{k+1}$-type of $(G, \bar{x})$, where $k=t w\left(F_{\bar{z}}\right)$.

Given a pattern graph $F_{\bar{z}}$ with $s$ pairwise distinct labeled vertices $\bar{z}=\left(z_{1}, \ldots, z_{s}\right)$ and a host graph $G_{\bar{x}}$ with $s$ pairwise distinct labeled vertices $\bar{x}=\left(x_{1}, \ldots, x_{s}\right)$, let $\operatorname{sub}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ denote the number of subgraphs $S$ of $G$ with $x_{1}, \ldots, x_{s} \in V(S)$ such that there is an isomorphism from $F$ to $S$ mapping $z_{i}$ to $x_{i}$ for all $i \leq s$. Using the well-known inductive approach due to Lovász [16] to relate homomorphism and subgraph counts, from Lemma 3.2 we derive a related fact about local invariance of subgraph counts.

Lemma 3.3. For each s-labeled pattern graph $F_{\bar{z}}$, the subgraph count $\operatorname{sub}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ is determined by the $\mathrm{C}^{k+1}$-type of $(G, \bar{x})$, where $k=h t w\left(F_{\bar{y}}\right)$.

Proof. Let $\operatorname{hom}^{*}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ denote the number of injective homomorphisms from $F_{\bar{z}}$ to $G_{\bar{x}}$. Counting injective homomorphisms and subgraphs is essentially equivalent, since

$$
\operatorname{sub}\left(F_{\bar{z}}, G_{\bar{x}}\right)=\frac{\operatorname{hom}^{*}\left(F_{\bar{z}}, G_{\bar{x}}\right)}{\left|\operatorname{Aut}\left(F_{\bar{z}}\right)\right|}
$$

where $\operatorname{Aut}\left(F_{\bar{z}}\right)$ is the automorphism group of $F_{\bar{z}}$ (as any homomorphism, automorphisms have to respect the labeled vertices). Therefore, it is enough to prove that the injective homomorphism count hom $^{*}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ is determined by the $\mathrm{C}^{k+1}$-type of $(G, \bar{x})$.

We use induction on the number of vertices in $F$. The base case, namely $V(F)=$ $\left\{z_{1}\right\}$, is obvious. Suppose that $F$ has at least two vertices.

Call a partition $\alpha$ of $V(F)$ proper if every element of $\alpha$ is an independent set of vertices in $F$. The partition of $V(F)$ into singletons is called discrete. A proper partition $\alpha$ determines the homomorphic image $F / \alpha$ of $F$ where $V(F / \alpha)=\alpha$ and two elements $A$ and $B$ of $\alpha$ are adjacent if between the vertex sets $A$ and $B$ there is at least one edge in $F$. Let $\bar{z} / \alpha=\left(A_{1}, \ldots, A_{s}\right)$ where $A_{i}$ is the element of the partition $\alpha$ containing $z_{i}$. The $s$-labeled graph $F_{\bar{z}} / \alpha$ is defined by $F_{\bar{z}} / \alpha=(F / \alpha)_{\bar{z} / \alpha}$. Note that

$$
\operatorname{hom}\left(F_{\bar{z}}, G_{\bar{x}}\right)=\operatorname{hom}^{*}\left(F_{\bar{z}}, G_{\bar{x}}\right)+\sum_{\alpha} \operatorname{hom}^{*}\left(F_{\bar{z}} / \alpha, G_{\bar{x}}\right)
$$

where the sum goes over the non-discrete proper partitions of $V(F)$. Note also that the number of labeled vertices in each $F_{\bar{z}} / \alpha$ stays $s$, but some of them may become equal to each other. In this case, we have $\operatorname{hom}^{*}\left(F_{\bar{z}} / \alpha, G_{\bar{x}}\right)=0$, and all such $\alpha$ can be excluded from consideration.

Now, $\operatorname{hom}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ is determined by the $\mathrm{C}^{k+1}$-type of $(G, \bar{x})$ by Lemma 3.2 because $t w\left(F_{\bar{z}}\right) \leq h t w\left(F_{\bar{z}}\right)$. Each $\operatorname{hom}^{*}\left(F_{\bar{z}} / \alpha, G_{\bar{x}}\right)$ is determined by the $\mathrm{C}^{k+1}$-type of $(G, \bar{x})$ by the inductive assumption. It follows that $\operatorname{hom}^{*}\left(F_{\bar{z}}, G_{\bar{x}}\right)$ is also determined.

## 4 Implicit regularity properties

In what follows, by s-path (resp. s-cycle) we mean a path graph $P_{s}$ (resp. a cycle graph $C_{s}$ ) on $s$ vertices.

## Theorem 4.1.

1. For each $s \leq 7$, the number of $s$-paths between distinct vertices $x$ and $y$ in a strongly regular graph $G$ depends only on the adjacency of $x$ and $y$ (and on the parameters of $G$ ).
2. For each $3 \leq s \leq 5$, the number of $s$-cycles containing two distinct vertices $x$ and $y$ of a strongly regular graph $G$ depends only on the adjacency of $x$ and $y$ (and on the parameters of $G$ ).

Proof. Let $s \leq 7$. Suppose that the path $P_{s}$ goes through vertices $z_{1}, \ldots, z_{s}$ in this order and, similarly, the cycle $C_{s}$ is formed by vertices $z_{1}, \ldots, z_{s}$ in this cyclic order. For cycles, $h t w\left(C_{s}, z_{1}, z_{s}\right)=h t w\left(C_{s}\right)=2$, where the first equality is due to Remark 3.1 and the second equality is noticed in Example 2.1. Let $h$ be an edgesurjective homomorphism from a 2-labeled path $\left(P_{s}, z_{1}, z_{s}\right)$. If $h\left(z_{1}\right)=h\left(z_{s}\right)$, then the image of $\left(P_{s}, z_{1}, z_{s}\right)$ under $h$ is also a homomorphic image of $\left(C_{s-1}, z_{1}, z_{1}\right)$. If $h\left(z_{1}\right) \neq h\left(z_{s}\right)$, then adding an edge between $h\left(z_{1}\right)$ and $h\left(z_{s}\right)$ does not increase the treewidth of the 2-labeled image graph, which can then be seen as a homomorphic image of $\left(C_{s}, z_{1}, z_{s}\right)$. Therefore, $h t w\left(P_{s}, z_{1}, z_{s}\right) \leq \max \left(h t w\left(C_{s}, z_{1}, z_{s}\right)\right.$, htw $\left.\left(C_{s-1}, z_{1}\right)\right)$. This implies $h t w\left(P_{s}, z_{1}, z_{s}\right) \leq 2$. A straightforward inspection shows also that $\operatorname{htw}\left(C_{s}, z_{1}, z_{i}\right)=2$ for all $i \leq s$ if $s \leq 5$. We conclude by Lemma 3.3 that $\operatorname{sub}\left(P_{s}, z_{1}, z_{s} ; G, x, y\right)$ for $s \leq 7$ and $\operatorname{sub}\left(C_{s}, z_{1}, z_{i} ; G, x, y\right)$ for $s \leq 5$ are determined by the $\mathrm{C}^{3}$-type of $(G, x, y)$ and hence, by Lemma 2.3, by the canonical color $\mathrm{WL}_{2}(G, x, y)$. As noticed in Remark [2.2, the color is determined by the adjacency of $x$ and $y$.

The next result applies to a larger class of graphs. The concept of an association scheme appeared in statistics (Bose [3]) and, also known as a homogeneous coherent configuration (Higman [14]), plays an important role in algebra and combinatorics [7, Chapter 17]. The standard definition and an overview of many important results in this area can be found in [8, 22]. For our purposes, an association scheme can be seen as a complete directed graph with colored edges such that

- all loops $(x, x)$ form a separate color class,
- if two edges $(x, y)$ and $\left(x^{\prime}, y^{\prime}\right)$ are equally colored, then their transposes $(y, x)$ and $\left(y^{\prime}, x^{\prime}\right)$ are equally colored too, and
- the coloring is stable under the 2 -WL refinement.

Each non-loop color class, seen as an undirected graph, forms a constituent graph of the association scheme; this concept first appeared apparently in [13].

A strongly regular graph and its complement can be seen as the two constituent graphs of an association scheme. Moreover, all distance-regular graphs [5] are constituent graphs (in fact, every connected strongly regular graph is distance-regular).

Lemma 4.2. Let $\alpha: V^{2} \rightarrow C$ be a coloring defining an association scheme, and $G$ be a constituent graph of this scheme. Then $\mathrm{WL}_{2}(G, x, y)=\mathrm{WL}_{2}\left(G, x^{\prime}, y^{\prime}\right)$ whenever $\alpha(x, y)=\alpha\left(x^{\prime}, y^{\prime}\right)$.
Proof. Using the induction on $i$, we will prove that the equality $\alpha(x, y)=\alpha\left(x^{\prime}, y^{\prime}\right)$ implies the equality $\mathrm{WL}_{2}^{i}(G, x, y)=\mathrm{WL}_{2}^{i}\left(G, x^{\prime}, y^{\prime}\right)$ for every $i$. In the base case of $i=0$ this is true by the definition of a constituent graph. Suppose that the claim is true for some $i$ for all $x, y, x^{\prime}, y^{\prime} \in V$.

To prove the claim for $i+1$, fix $x, y, x^{\prime}, y^{\prime}$ such that $\alpha(x, y)=\alpha\left(x^{\prime}, y^{\prime}\right)$. By the induction assumption, $\mathrm{WL}_{2}^{i}(G, x, y)=\mathrm{WL}_{2}^{i}\left(G, x^{\prime}, y^{\prime}\right)$. For a pair of colors $p \in C^{2}$, let $T(p)=\{z:(\alpha(x, z), \alpha(z, y))=p\}$ and $T^{\prime}(p)=\left\{z^{\prime}:\left(\alpha\left(x^{\prime}, z^{\prime}\right), \alpha\left(z^{\prime}, y^{\prime}\right)\right)=p\right\}$. Since $\alpha$ defines an association scheme, this coloring is not refinable by 2-WL. This means that

$$
\begin{equation*}
|T(p)|=\left|T^{\prime}(p)\right| \text { for every } p \in C^{2} \tag{2}
\end{equation*}
$$

By the induction assumption, for every $z \in T(p)$ and $z^{\prime} \in T^{\prime}(p)$ we have

$$
\left(\mathrm{WL}_{2}^{i}(G, x, z), \mathrm{WL}_{2}^{i}(G, z, y)\right)=\left(\mathrm{WL}_{2}^{i}\left(G, x^{\prime}, z^{\prime}\right), \mathrm{WL}_{2}^{i}\left(G, z^{\prime}, y^{\prime}\right)\right)
$$

Along with Equality (22), this implies that

$$
\left\{\left\{\left(\mathrm{WL}_{2}^{i}(G, x, z), \mathrm{WL}_{2}^{i}(G, z, y)\right)\right\}\right\}_{z \in V}=\left\{\left\{\left(\mathrm{WL}_{2}^{i}\left(G, x^{\prime}, z^{\prime}\right), \mathrm{WL}_{2}^{i}\left(G, z^{\prime}, y^{\prime}\right)\right)\right\}\right\}_{z^{\prime} \in V}
$$

and, therefore, $\mathrm{WL}_{2}^{i+1}(G, x, y)=\mathrm{WL}_{2}^{i+1}\left(G, x^{\prime}, y^{\prime}\right)$, as desired.
Theorem 4.3. Let $G$ be a constituent graph of an association scheme. If $3 \leq s \leq 7$, then the number of s-cycles containing an edge xy in $G$ does not depend on the choice of $x y$.

Proof. As already noted in the proof of Theorem 4.1, htw $\left(C_{s}, z_{1}, z_{2}\right)=2$, where two labeled vertices $z_{1}, z_{2}$ are consecutive in $C_{s}$. Like in the proof of Theorem 4.1, we conclude that the count $\operatorname{sub}\left(C_{s}, z_{1}, z_{2} ; G, x, y\right)$ is determined by the canonical color $\mathrm{WL}_{2}(G, x, y)$. Let $x^{\prime} y^{\prime}$ be another edge of $G$. By Lemma 4.2, $\mathrm{WL}_{2}(G, x, y)=$ $\mathrm{WL}_{2}\left(G, x^{\prime}, y^{\prime}\right)$ or $\mathrm{WL}_{2}(G, x, y)=\mathrm{WL}_{2}\left(G, y^{\prime}, x^{\prime}\right)$. It remains to notice that $\operatorname{sub}\left(C_{s}, z_{1}\right.$, $\left.z_{2} ; G, x^{\prime}, y^{\prime}\right)=\operatorname{sub}\left(C_{s}, z_{1}, z_{2} ; G, y^{\prime}, x^{\prime}\right)$ just because $C_{s}$ has an automorphism transposing $z_{1}$ and $z_{2}$.

The optimality of Theorems 4.1 and 4.3 is certified by Table 1 .
Theorem 4.4 below applies to a yet larger class of graphs. Following [11], we call a graph $G \mathrm{WL}_{1,2}$-regular if $\mathrm{WL}_{2}(G, x, x)=\mathrm{WL}_{2}\left(G, x^{\prime}, x^{\prime}\right)$ for every two vertices $x$ and $x^{\prime}$ of $G$, that is, 2-WL determines a monochromatic coloring of $V(G)$. Applying Lemma 4.2 in the case $y=x$ and $y^{\prime}=x^{\prime}$, we see that any constituent graph of an association scheme is $\mathrm{WL}_{1,2}$-regular. More $\mathrm{WL}_{1,2}$-regular graphs can be obtained by observing that this graph class is closed under taking graph complements and that, if $G_{1}$ and $G_{2}$ are two $\mathrm{WL}_{1,2}$-regular 2-WL-equivalent graphs, then the disjoint union of $G_{1}$ and $G_{2}$ is also $\mathrm{WL}_{1,2}$-regular. In terms of the theory of coherent configurations [8], a graph $G$ is $\mathrm{WL}_{1,2}$-regular if and only if the coherent closure of $G$ is an association scheme.

| $F_{\bar{y}}$ | $G_{\bar{x}}$ | $\operatorname{sub}\left(F_{\bar{y}} ; G_{\bar{x}}\right)$ |
| :---: | :---: | ---: |
| $\left(P_{8}, x_{1}, x_{8}\right)$ | $\left(S, a, a^{\prime}\right)$ | 2500 |
| $\left(P_{8}, x_{1}, x_{8}\right)$ | $\left(S, b, b^{\prime}\right)$ | 2522 |
| $\left(C_{6}, x_{1}, x_{3}\right)$ | $\left(S, a, a^{\prime}\right)$ | 72 |
| $\left(C_{6}, x_{1}, x_{3}\right)$ | $\left(S, b, b^{\prime}\right)$ | 74 |
| $\left(C_{6}, x_{1}, x_{4}\right)$ | $\left(S, a, a^{\prime}\right)$ | 92 |
| $\left(C_{6}, x_{1}, x_{4}\right)$ | $\left(S, b, b^{\prime}\right)$ | 94 |
| $\left(C_{8}, x_{1}, x_{2}\right)$ | $\left(\bar{S}, a, a^{\prime}\right)$ | 48832 |
| $\left(C_{8}, x_{1}, x_{2}\right)$ | $\left(\bar{S}, b, b^{\prime}\right)$ | 48788 |

Table 1: $S$ and $\bar{S}$ denote the Shrikhande graph and its complement, respectively; $a$, $a^{\prime}, b$, and $b^{\prime}$ are the four vertices in $S$ shown in Figure 1.

## Theorem 4.4.

1. For each $s \leq 7$, the number of s-paths emanating from a vertex $x$ in a $\mathrm{WL}_{1,2^{-}}$ regular graph $G$ is the same for every $x$.
2. For each $3 \leq s \leq 7$, the number of $s$-cycles containing a vertex $x$ in a $\mathrm{WL}_{1,2^{-}}$ regular graph $G$ is the same for every $x$.

Proof. Taking into account Remark 3.1 and Example 2.1, we obtain $h t w\left(P_{s}, z_{1}\right)=$ $h t w\left(P_{s}\right) \leq 2$ and $h t w\left(C_{s}, z_{1}\right)=h t w\left(C_{s}\right)=2$. Like in the proofs of Theorems 4.1 and 4.3, we conclude that the counts $\operatorname{sub}\left(P_{s}, z_{1} ; G, x\right)$ and $\operatorname{sub}\left(C_{s}, z_{1} ; G, x\right)$ are determined by the canonical color $\mathrm{WL}_{2}(G, x, x)$. By the definition of $\mathrm{WL}_{1,2}$-regularity, this color is the same for all vertices of $G$.

Theorem 4.4 is optimal regarding the restriction $s \leq 7$. Indeed, consider the strongly regular graphs with parameters $(25,12,5,6)$ without nontrivial automorphisms. These are two complementary graphs. Specifically, we pick $H=P_{25.02}$ (with graph6 code X\}rU\adeSetTjKWNJEYNR]PL jPBgUGVTkK^YKbipMcxbk'\{DlXF) in the graph database [4], which is based in this part on [19]. The graph $H$ has a vertex $x$ with $\operatorname{sub}\left(P_{8}, z_{1}, H, x\right)=11115444$ and $\operatorname{sub}\left(C_{8}, z_{1}, H, x\right)=5201448$ and a vertex $y$ with $\operatorname{sub}\left(P_{8}, z_{1}, H, x\right)=11115510$ and $\operatorname{sub}\left(C_{8}, z_{1}, H, x\right)=5201580$.

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[^1]:    ${ }^{1}$ Imposing this condition is equivalent to the recursive definition given in 10 .

