Nordhaus-Gaddum type inequality for the fractional matching number of a graph

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Abstract

The fractional matching number of a graph G, written as $\alpha'(G)$, is the maximum size of a fractional matching of G. The following sharp lower bounds for a graph G of order n are proved, and all extremal graphs are characterized in this paper.

- (1) $\alpha'(G) + \alpha'(\overline{G}) \ge \frac{n}{2}$ for $n \ge 2$.
- (2) If G and \overline{G} are non-empty, then for $n \geq 28$, $\alpha'(G) + \alpha'(\overline{G}) \geq \frac{n+1}{2}$.
- (3) If G and \overline{G} have no isolated vertices, then for $n \geq 28$, $\alpha'(G) + \alpha'(\overline{G}) \geq \frac{n+4}{2}$.

Keywords: Nordhaus-Gaddum type inequality, Fractional matching number, Fractional Berge's theorem

1. Introduction

Throughout this paper, all graphs are simple, undirected and finite. Undefined terminologies and notations can be found in [2]. Let G = (V(G), E(G)) be a graph and \overline{G} be its complement. n will always denote the number of vertices of a given graph G. For a vertex $v \in V(G)$, its degree $d_G(v)$ is the number of edges incident to it in G, its neighborhood, denoted by N(v), is the set of vertices, which are adjacent to v. An edge set M of G is called a matching if any two edges in M have no common vertices. The matching number of a graph G, written $\alpha(G)$, is the number of edges in a maximum matching. As in [13], a fractional matching of a graph G is a function $f: E(G) \longrightarrow [0,1]$ such that $f(v) \leq 1$ for each vertex $v \in V(G)$, where f(v) is the sum of f(e) of edges incident to v. The fractional matching number of G, written $\alpha'(G)$, is the maximum value of f(G) over all fractional matchings, where f(G) denotes the sum of f(e) of all edges in G. A fractional perfect matching of a graph G is a fractional matching f with $f(G) = \alpha'(G) = \frac{n}{2}$. Obviously, fractional matching is a generalization of matching. Choi et al. [4] proved the difference and ratio of the fractional matching number and the matching number of graphs and characterized all infinite extremal family of graphs.

^{*}This work was supported by the National Nature Science Foundation of China (Nos. 11871040).

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Nordhaus and Gaddum [11] considered lower and upper bounds on the sum and the product of chromatic number $\chi(G)$ of a graph G and its complement \overline{G} . They showed that

$$2\sqrt{n} \le \chi(G) + \chi(\overline{G}) \le n+1.$$

Since then, any bound on the sum or the product of an invariant in a graph G and the same invariant in its complement \overline{G} is called a Nordhaus-Gaddum type inequality. The Nordhaus-Gaddum type inequality of various graph parameters has attracted much attention (see [5][6][7][8][9][12][14]). Anouchiche and Hansen [1] wrote a stimulating survey on this topic, and we refer the reader to that article for additional information. Chartrand and Schuster [3] proved Nordhaus-Gaddum type result for the matching number of a graph. They showed that $\alpha(G) + \alpha(\overline{G}) \geq \lfloor \frac{n}{2} \rfloor$. Laskar and Auerbach [7] improved the bound by considering that G and \overline{G} contain no isolated vertices. They showed that $\alpha(G) + \alpha(\overline{G}) \geq \lfloor \frac{n}{2} \rfloor + 2$. Later, Lin et al. [10] characterized all extremal graphs which attain the lower bounds of the results of Chartrand et al. and Laskar et al.. Motivated by them, we consider Nordhaus-Gaddum type inequality for the fractional matching number of a graph. First we prove some auxiliary results of fractional matching, which are selfcontained. Then we establish lower bounds on the sum of fractional matching number of a graph G and its complement (see Theorem 4.3 and Theorem 4.6). Moreover, we show those bounds are sharp.

2. Auxiliary results of fractional matching

Based on an optimal fractional matching of graph G, we present a good partition of V(G). We further characterize some properties of this partition.

Lemma 2.1. ([13]) For any graph G, $2\alpha'(G)$ is an integer. Moreover, there is a fractional matching f for which

$$f(G) = \alpha'(G)$$

such that $f(e) \in \{0, \frac{1}{2}, 1\}$ for every edge e of G.

An f is called an optimal fractional matching of graph G in this paper, if we have (1) $f(G) = \alpha'(G)$.

- (2) $f(e) \in \{0, \frac{1}{2}, 1\}$ for every edge e.
- (3) f has the greatest number of edges e with f(e) = 1.

In this paper, for a graph G, given a fractional matching f, an unweighted vertex v is a vertex with f(v) = 0. A full vertex v is a vertex with f(vw) = 1 for some edge vw and we may call vertex w is the full neighbour of v. An i-edge e is an edge with f(e) = i. $\frac{1}{2}$ -cycle in a graph G is an odd cycle induced by $\frac{1}{2}$ -edges in G.

Lemma 2.2. Let f be an optimal fractional matching of graph G. Then we have the following:

- (1) ([4]) The maximal subgraph induced by the $\frac{1}{2}$ -edges is the union of odd cycles.
- (2) ([4]) The set of the unweighted vertices is an independent set of G. Furthermore, every unweighted vertex is adjacent only to a full vertex.
- (3) No $\frac{1}{2}$ -cycle has an unweighted vertex as a neighbour.

Proof By (2), every unweighted vertex is adjacent only to a full vertex, while every vertex on $\frac{1}{2}$ -cycle is not a full vertex, then no $\frac{1}{2}$ -cycle has an unweighted vertex as a neighbour.

Lemma 2.3. ([13]) Suppose that f is a fractional matching of graph G. Then f is a fractional perfect matching if and only if f(v) = 1 for every vertex $v \in V(G)$.

Lemma 2.4. For any graph G, we have a partition $V(G) = V_1 \dot{\cup} V_2$ with $|V_1| = 2\alpha'(G)$, and $G[V_1]$ contains a fractional perfect matching and V_2 is empty or an independent set.

Proof Suppose f is an optimal fractional matching of G. If $\alpha'(G) = \frac{n}{2}$, we take $V_1 = V(G)$. If $\alpha'(G) < \frac{n}{2}$, set $V_1 = \{v \in V(G) | f(v) > 0\}$ and $V_2 = V(G) \setminus V_1$. We will show that f(v) = 1 for each vertex v in V_1 . Suppose to the contrary that there exists a vertex $v_0 \in V_1$ with $0 < f(v_0) < 1$, i.e., $f(v_0) = \frac{1}{2}$, say $f(v_0v_1) = \frac{1}{2}$ for some vertex $v_1 \in V_1$. By Lemma 2.2 (1), v_0v_1 lies in a $\frac{1}{2}$ -cycle, then there exists a vertex $v_t(\neq v_1)$ such that $f(v_0v_t) = \frac{1}{2}$. Thus $f(v_0) = 1$, which is a contradiction. Thus $G[V_1]$ contains a fractional perfect matching by Lemma 2.3. Since V_2 is a set of the unweighted vertices, V_2 is an independent set by Lemma 2.2 (2). Furthermore, we have $\alpha'(G) = \alpha'(G[V_1]) = \frac{|V_1|}{2}$.

For a graph G with $\alpha'(G) = t$, we write $V(G) = V_1 \dot{\cup} V_2$ according to the results of Lemma 2.4, and let s be the maximum number of independent edges in $[V_1, V_2]$. Now we will further decompose $V_1 = V_{11} \dot{\cup} V_{12}$ and $V_2 = V_{21} \dot{\cup} V_{22}$ such that $[V_{11}, V_{21}]$ contains exactly s independent edges. Obviously, $|V_{11}| = |V_{21}| = s$ holds. We call $V(G) = (V_{11} \cup V_{12}) \cup (V_{21} \cup V_{22})$ a good partition of G in this paper (see Figure 1).

Lemma 2.5. Let $V(G) = (V_{11} \cup V_{12}) \cup (V_{21} \cup V_{22})$ be a good partition of G with $|V_{11}| = s$ and f be the corresponding optimal fractional matching of G.

- (1) If f(uv) = 1 for some edge uv in G, then there is no vertex in $V_2 \cap N(u) \cap N(v)$.
- (2) If e is an edge in $G[V_{11}]$, then f(e) = 0.
- (3) Each vertex in V_{11} is a full vertex.

Proof (1) Suppose there exists a vertex $w \in V_2 \cap N(u) \cap N(v)$. Since w is an unweighted vertex, we have f(uw) = f(vw) = 0. Now set $f^*(uw) = f^*(vw) = f^*(uv) = \frac{1}{2}$ and other assignments remain unchanged. Then $f^*(G) = \alpha'(G) + \frac{1}{2}$, which is a contradiction.

(2) Suppose uv is an edge in $G[V_{11}]$ with f(uv) = a > 0. Let ux and vy be two independent edges in $[V_{11}, V_{21}]$. Now set $f^*(ux) = f^*(vy) = 1$ and $f^*(uz) = f^*(vw) = 0$

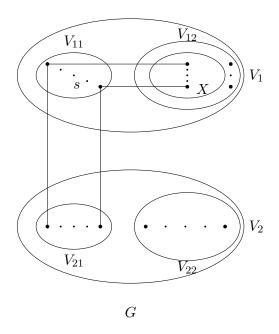


Figure 1: A good partition of graph G

for any $z \in N(u) \setminus \{x\}$ and $w \in N(v) \setminus \{y\}$ and other assignments remain unchanged. Then $f^*(G) - f(G) \ge a$, which is a contradiction to the choice of f.

(3) Suppose $u \in V_{11}$ is not a full vertex. Then there exists a vertex v such that $f(uv) = \frac{1}{2}$. Furthermore, uv lies in a $\frac{1}{2}$ -cycle by Lemma 2.2 (1). While u has a neighbour in V_{21} , which contradicts to Lemma 2.2 (3).

From Lemma 2.5 (3), we know that each vertex in V_{11} is a full vertex. Denote by X the set of all full neighbours of vertices in V_{11} , and then $X \subseteq V_{12}$ by Lemma 2.5 (2). It is obvious that $|X| = |V_{11}| = s$. Furthermore, we have the following results.

Lemma 2.6. (1) If $|X| = s \ge 2$, then X is an independent set of G. (2) There is no edge between V_2 and X in G.

Proof (1) Suppose there exist two vertices $u, v \in X$ such that $uv \in E(G)$ and f(gu) = f(hv) = 1 for vertices g, h in V_{11} . Let gw and hy be two independent edges in $[V_{11}, V_{21}]$. Now set $f^*(gu) = f^*(hv) = 0$ and $f^*(uv) = f^*(gw) = f^*(hy) = 1$ and other assignments remain unchanged. Then $f^*(G) = \alpha'(G) + 1$, which is a contradiction.

(2) Suppose to the contrary that there exists an edge vx with $v \in X$ and $x \in V_{21}$. Then there exists a vertex $u \in V_{11}$ such that $uv \in E(G)$ and f(uv) = 1. We have $ux \notin E(G)$ in virtue of Lemma 2.5 (1). Then there exists a vertex say $y \in V_{21}$ such that uy is one of the independent edges in $[V_{11}, V_{21}]$. Now we may modify f to f^* . Let $f^*(uv) = 0$, $f^*(vx) = f^*(uy) = 1$ and other assignments remain unchanged. Then $f^*(G) = \alpha'(G) + 1$, which is a contradiction. It is obvious that there is no edge between

3. Graphs with small fractional matching number

In virtue of the fractional Berge's theorem, we will characterize some graphs with small fractional matching number in this section.

Lemma 3.1. ([13]) For any graph G of order n, we have

$$\alpha'(G) = \frac{1}{2} \left(n - \max_{S \subseteq V(G)} \left\{ i \left(G - S \right) - |S| \right\} \right),$$

where i(G-S) denotes the number of isolated vertices of G-S.

By the fractional Berge's theorem, we immediately have the following results.

Lemma 3.2. (1) For a graph G of order n, $\alpha'(G) = 1$ if and only if $G \cong K_{1,k} \cup (n - 1 - k)K_1$, where $k \ge 1$.

(2) For a graph G of order n, $\alpha'(G) = \frac{3}{2}$ if and only if $G \cong C_3 \cup (n-3)K_1$.

Let $K_2(p,q;\ell)(p \geq q)$ be the graph obtained by attaching p pendent edges at one vertex of K_2 called uv, q pendent edges at the other vertex of K_2 and having ℓ vertices in $N(u) \cap N(v)$.

Lemma 3.3. For a graph G of order n, $\alpha'(G) = 2$ if and only if one of the following situations occurs:

- (1) $2K_2 \cup (n-4)K_1 \subseteq G \subseteq K_4 \cup (n-4)K_1$.
- (2) $2K_2 \cup (n-4)K_1 \subseteq G \subseteq K_2(0,0;n-2)$.

Proof By considering the fact that if $G_1 \subseteq G_2$, then $\alpha'(G_1) \le \alpha'(G_2)$ and $\alpha'(K_4 \cup (n-4)K_1) = \alpha'(K_2(0,0;n-2)) = \alpha'(2K_2 \cup (n-4)K_1) = 2$, the sufficiency part is correct.

To show the necessity part, by Lemma 3.1, suppose $S \subseteq V(G)$ such that n-4 = i(G-S) - |S|. Since $i(G-S) \le n - |S|$, it follows that $|S| \le 2$.

If |S| = 0, then i(G) = n - 4. $\alpha'(G) = 2$ implies G contains two independent edges. Thus we have $2K_2 \cup (n-4)K_1 \subseteq G \subseteq K_4 \cup (n-4)K_1$.

If |S| = 1, then i(G - S) = n - 3, and G - S contains a subgraph $F = K_2$. $\alpha'(G) = 2$ implies that there is an edge between S and $(V(G) \setminus \{S \cup V(F)\})$. Thus $2K_2 \cup (n - 4)K_1$ is a subgraph of G. Since there are at most n - 3 edges in $[S, V(G) \setminus \{S \cup V(F)\}]$ and there are at most two edges in [S, V(F)], G is a subgraph of $K_2(n - 3, 0; 1)$. Thus $2K_2 \cup (n - 4)K_1 \subseteq G \subseteq K_2(n - 3, 0; 1)$.

If |S| = 2, then i(G - S) = n - 2. $\alpha'(G) = 2$ implies G contains two independent edges in $[S, V(G) \setminus S]$. Thus we have $2K_2 \cup (n-4)K_1 \subseteq G$. Since there are at most 2(n-2) edges in $[S, V(G) \setminus S]$ and there is at most one edge in S, G is a subgraph of $K_2(0,0;n-2)$. Thus $2K_2 \cup (n-4)K_1 \subseteq G \subseteq K_2(0,0;n-2)$. Noting that $K_2(n-3,0;1) \subseteq K_2(0,0;n-2)$, we have $2K_2 \cup (n-4)K_1 \subseteq G \subseteq K_2(0,0;n-2)$. \square

Lemma 3.4. Let H be the graph of order n obtained by attaching n-4 pendent edges at one vertex of K_4 . For a graph G of order n, $\alpha'(G) = \frac{5}{2}$ if and only if one of the following situations occurs:

- (1) $C_5 \cup (n-5)K_1 \subseteq G \subseteq K_5 \cup (n-5)K_1$.
- (2) $C_3 \cup K_2 \cup (n-5)K_1 \subseteq G \subseteq K_5 \cup (n-5)K_1$.
- $(3) C_3 \cup K_2 \cup (n-5)K_1 \subseteq G \subseteq H.$

Proof The sufficiency part is obvious. To show the necessity part, by Lemma 3.1, suppose $S \subseteq V(G)$ such that n-5=i(G-S)-|S|. Since $i(G-S) \le n-|S|$, it follows that $|S| \le \frac{5}{2}$. Furthermore, |S| = 2 does not occur. Otherwise we have i(G-S) = n-3. While we have $G-S = (n-3)K_1 \cup K_1$, which implies that i(G-S) = (n-2).

If |S|=0, then i(G)=n-5. $\alpha'(G)=\frac{5}{2}$ implies G contains an odd cycle. If G contains C_3 as a subgraph, then we have $C_3 \cup K_2 \cup (n-5)K_1 \subseteq G \subseteq K_5 \cup (n-5)K_1$. If G contains C_5 as a subgraph, then we have $C_5 \cup (n-5)K_1 \subseteq G \subseteq K_5 \cup (n-5)K_1$.

If |S|=1, then i(G-S)=n-4. $\alpha'(G)=\frac{5}{2}$ implies that G-S contains C_3 as a subgraph, and there exists at least one edge in $[S,V(G)\setminus\{S\cup V(C_3)\}]$. Thus $C_3\cup K_2\cup (n-5)K_1$ is a subgraph of G. Furthermore, we have $G\subseteq H$.

Based on the results of Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have the following results.

Lemma 3.5. Let G be a graph of order n. Then the following statements hold.

- (1) If $\alpha'(G) = 1$ and $n \ge 4$, then $\alpha'(G) + \alpha'(\overline{G}) \ge \frac{n+1}{2}$.
- (2) If $\alpha'(G) = \frac{3}{2}$ and $n \ge 6$, then $\alpha'(G) + \alpha'(\overline{G}) = \frac{n+3}{2}$.
- (3) If $\alpha'(G) = 2$ and G is not isomorphic to $K_2(0,0;\ell)$ for $\ell \geq 2$, then $\alpha'(G) + \alpha'(\overline{G}) \geq \frac{n+3}{2}$ for $n \geq 8$.
- (4) If $\alpha'(G) = 2$ and G is isomorphic to $K_2(0,0;\ell)$ for $\ell \geq 2$, then $\alpha'(G) + \alpha'(\overline{G}) = \frac{n+2}{2}$.
- (5) If $\alpha'(G) = \frac{5}{2}$ and $n \ge 7$, then $\alpha'(G) + \alpha'(\overline{G}) \ge \frac{n}{2} + 2$.

Moreover, the equalities in (1), (3) and (5) hold if and only if G contains exactly one vertex with degree n-1.

4. Nordhaus-Gaddum-type bounds for the fractional matching number

In this section, we will prove the Nordhaus-Gaddum-type bounds for the fractional matching number (see Theorem 4.3 and Theorem 4.6).

In a graph G with $\alpha'(G) = t$, we try to find a collection E of some independent edges in \overline{G} and assign 1 to each edge of E, and then find a long cycle C (having no common vertices with E) and assign $\frac{1}{2}$ to each edge of C. By this way, we get a lower bound of $\alpha'(\overline{G})$.

Lemma 4.1. Let $V(G) = (V_{11} \cup V_{12}) \cup (V_{21} \cup V_{22})$ be a good partition of graph G of order n with $\alpha'(G) = t \leq \frac{n}{4}$ and $|V_{11}| = s$. Then the following statements hold. (1) $\alpha'(\overline{G}) \geq \frac{n-s}{2} \geq \frac{n-t}{2}$.

- (2) If both G and \overline{G} contain no isolated vertices and $s \geq 1$, then $\alpha'(\overline{G}) \geq \frac{n-s+1}{2}$.
- (3) If both G and \overline{G} contain no isolated vertices and $s=t\geq 3$, then $\alpha'(\overline{G})\geq \frac{n-s+2}{2}$.

Proof Since $V(G) = (V_{11} \cup V_{12}) \cup (V_{21} \cup V_{22})$ is a good partition, $\overline{G}[V_2]$ is a clique, and each vertex in V_{12} is adjacent to each vertex in V_{22} in \overline{G} . The assumption $n \geq 4t$ insures $|V_{22}| = n - 2t - s \geq 2t - s$.

(1) If n-2t-(2t-s)=0, that is s=0 and $|V_{12}|=|V_{22}|=\frac{n}{2}=2t$, then we may assign number 1 to 2t independent edges in $\overline{G}[V_{12},V_{22}]$ for a fractional matching of \overline{G} . We have $\alpha'(\overline{G}) \geq 2t = \frac{n}{2}$.

If n-2t-(2t-s)=[(n-2t-s)-(2t-s)]+s=1, then $s\leq 1$. There exist $u\in V_{12}$ and $v,w\in V_2$ forming a cycle for a fractional matching of \overline{G} . We may assign number 1 to 2t-s-1 independent edges in $\overline{G}[V_{12}\setminus\{u\},V_2\setminus\{v,w\}]$, and $\frac{1}{2}$ to edges of C_3 induced by vertices u,v and w. Thus we have $\alpha'(\overline{G})\geq 2t-s-1+\frac{3}{2}=\frac{n-s}{2}$.

If n-2t-(2t-s)=2, then we may choose vertices $u,v\in V_2$ and assign number 1 to edge uv in \overline{G} and 2t-s independent edges in $\overline{G}[V_{12},V_2\setminus\{u,v\}]$ for a fractional matching of \overline{G} . Thus we have $\alpha'(\overline{G})\geq 2t-s+1=\frac{n-s}{2}$.

When $n-2t-(2t-s) \geq 3$, we may assign number 1 to 2t-s independent edges in $\overline{G}[V_{12},V_{22}]$, and $\frac{1}{2}$ to each edge of a cycle with length n-2t-(2t-s) in $\overline{G}[V_2]$ for a fractional matching of \overline{G} . Then we have

$$\alpha'(\overline{G}) \ge 2t - s + \frac{n - 2t - (2t - s)}{2} = \frac{n - s}{2}.$$

Since $s \le t$, we have $\alpha'(\overline{G}) \ge \frac{n-s}{2} \ge \frac{n-t}{2}$.

(2) Now we suppose $s \geq 1$ and \overline{G} contains no isolated vertices. Let u be a vertex in V_{11} and v be one of its neighbours in \overline{G} . First we suppose $v \in V_{12}$. Now we define a fractional matching of \overline{G} . Let $\overline{f}(uv) = 1$ in \overline{G} . Noting that $n - 2t - (2t - s - 1) \geq 2$. If n - 2t - (2t - s - 1) = 2, that is s = 1 and $|V_{12}| = |V_{22}| = 2t - 1 = \frac{n}{2} - 1$, then we may choose vertices $v_1 \in V_{22}$ and $v_2 \in V_{21}$ and assign number 1 to edge v_1v_2 in \overline{G} and 2t - s - 1 independent edges in $\overline{G}[V_{12} \setminus \{v\}, V_{22} \setminus \{v_1\}]$. Thus we have $\alpha'(\overline{G}) \geq 2 + 2t - s - 1 = \frac{n - s + 1}{2}$.

When $n-2t-(2t-s-1) \geq 3$, we may assign number 1 to 2t-s-1 independent edges in $\overline{G}[V_{12} \setminus \{v\}, V_{22}]$, and $\frac{1}{2}$ to each edge of a cycle with length n-2t-(2t-s-1) in $\overline{G}[V_2]$ for a fractional matching of \overline{G} . Then we have

$$\alpha'(\overline{G}) \ge 1 + (2t - s - 1) + \frac{n - 2t - (2t - s - 1)}{2} = \frac{n - s + 1}{2}.$$

Now suppose v lies in V_{11} or V_2 . If $v \in V_{11}$ or $v \in V_{21}$, then $s \geq 2$. When $v \in V_{22}$, noting G contains no isolated vertices, all possible neighbours of v are in V_{11} and $uv \notin E(G)$, then $|V_{11}| = s \geq 2$. Hence, we have $|X| = |V_{11}| \geq 2$ and there is an edge v_1v_2 in $\overline{G}[X]$ by Lemma 2.6 (1). Noting that $n - 2t - (2t - s - 2) - 1 \geq 1 + s \geq 3$. For a fractional matching of \overline{G} , we may assign number 1 to the edges v_1v_2 , v_2 and $v_1v_2 = v_2 = 1$ independent edges in $\overline{G}[V_{12} \setminus \{v_1, v_2\}, V_{22} \setminus \{v\}]$, and $v_1v_2 = 1$ to each edge of a cycle with length $v_1v_2 = 1$ in $v_2v_1 = 1$ in $v_1v_2 = 1$. Thus we have

$$\alpha'(\overline{G}) \ge 1 + 1 + (2t - s - 2) + \frac{n - 2t - (2t - s - 2) - 1}{2} = \frac{n - s + 1}{2}.$$

(3) Now we suppose that $s=t\geq 3$ and \overline{G} contains no isolated vertices. The assumption s=t implies $X=V_{12}$. By Lemma 2.6, we obtain that $\overline{G}[V_{12}]=K_s$ and each vertex in V_{12} is adjacent to each vertex in V_2 in \overline{G} . Let u be a vertex in V_{11} and v be one of its neighbours in \overline{G} . Noting that $n-2t-(2t-s)\geq s\geq 3$. Now we define a fractional matching \overline{f} of \overline{G} . We suppose $v\in V_{11}$. First let $\overline{f}(uv)=1$ in \overline{G} and we may assign number 1 to 2t-s independent edges in $\overline{G}[V_{12},V_{22}]$, and $\frac{1}{2}$ to each edge of a cycle with length n-2t-(2t-s) in $\overline{G}[V_2]$. Thus we have

$$\alpha'(\overline{G}) \ge 1 + 2t - s + \frac{n - 2t - (2t - s)}{2} = \frac{n - s + 2}{2}.$$

If $v \in V_2$, then there exists a vertex say $w \neq u \in V_{11}$ such that $wv \in E(G)$. Since $d_G(w) \leq n-2$, there exists a vertex w' such that $ww' \notin E(G)$. If $w' \in V_{11}$, it is the case discussed above. If $w' \in V_{12}$, then let $\bar{f}(uv) = \bar{f}(ww') = 1$ in \overline{G} and we may assign number 1 to 2t-s-1 independent edges in $\overline{G}[V_{12} \setminus \{w'\}, V_2 \setminus \{v\}]$, and $\frac{1}{2}$ to each edge of a cycle with length n-2t-(2t-s-1)-1 in $\overline{G}[V_2]$. Thus we have

$$\alpha'(\overline{G}) \ge 2 + (2t - s - 1) + \frac{n - 2t - (2t - s - 1) - 1}{2} = \frac{n - s + 2}{2}.$$

If $w' \in V_2$, then let $\bar{f}(uv) = \bar{f}(ww') = 1$ in \overline{G} . Since $\overline{G}[V_{12}] = K_s$, there exist $x_1, x_2 \in V_{12}$ such that $x_1x_2 \in E(\overline{G})$. Let $\bar{f}(x_1x_2) = 1$ in \overline{G} and we may assign number 1 to 2t - s - 2 independent edges in $\overline{G}[V_{12} \setminus \{x_1, x_2\}, V_2 \setminus \{v, w'\}]$, and $\frac{1}{2}$ to each edge of a cycle with length n - 2t - 2 - (2t - s - 2) in $\overline{G}[V_2]$. Thus we have

$$\alpha'(\overline{G}) \ge 3 + 2t - s - 2 + \frac{n - 2t - 2 - (2t - s - 2)}{2} = \frac{n - s + 2}{2}.$$

When $v \in V_{11}$ or $v \in V_2$, we obtain the desired inequality. So in the following, we may assume that $G[V_{11}]$ is a clique and each vertex in V_{11} is adjacent to each vertex in V_2 in G. We suppose v lies in V_{12} . From the definition of set $X(=V_{12})$, we have $v'v \in E(G)$ for some $v' \in V_{11}$. Since v' is not an isolated vertex of \overline{G} , we may assume $v'v'' \in E(\overline{G})$ for some vertex $v'' \in V_{12}$. Now we construct a fractional matching \overline{f} of \overline{G} with $\overline{f}(uv) = \overline{f}(v'v'') = 1$ in \overline{G} and we may assign number 1 to 2t - s - 2 independent edges in $[V_{12} \setminus \{v, v''\}, V_{22}]$ in \overline{G} , and $\frac{1}{2}$ to each edge of a cycle with length n-2t-(2t-s-2) in $\overline{G}[V_2]$. Thus we have

$$\alpha'(\overline{G}) \ge 2 + 2t - s - 2 + \frac{n - 2t - (2t - s - 2)}{2} = \frac{n - s + 2}{2}.$$

This completes the proof.

Lemma 4.2. (1) If $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + \frac{1}{2}$ when $n \equiv 0, 1 \pmod{4}$ or $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + \frac{3}{2}$ when $n \equiv 2, 3 \pmod{4}$, then $\alpha'(\overline{G}) \geq \frac{n}{4} + 3$ for $n \geq 28$. (2) If $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + 1$, then $\alpha'(\overline{G}) \geq \frac{n}{4} + 3$ for $n \geq 28$. **Proof** Recall $V(G) = (V_{11} \cup V_{12}) \cup (V_{21} \cup V_{22})$, $|V_{11}| = s$ and $\alpha'(G) = t$. We will prove $\alpha'(\overline{G}) \geq \frac{n-t}{2}$. When $s \leq 1$, noting that $|V_1| = 2\alpha'(G) > \frac{n}{2}$ and $|V_{12}| = 2t - s > n - 2t - s = |V_{22}|$. Then we may assign number 1 to n - 2t - s independent edges in $\overline{G}[V_{12}, V_{22}]$ for a fractional matching of graph \overline{G} . Thus

$$\alpha'(\overline{G}) \ge n - 2t - s \ge \frac{n - t}{2}$$

for $n \geq 20$.

When $s \geq 2$, there exist two vertices $v_1, v_2 \in X$ such that $v_1v_2 \notin E(G)$ by Lemma 2.6 (1). We will define a fractional matching \overline{f} of \overline{G} . Noting that $|V_{22}| \geq 3$.

(1) If $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + \frac{1}{2}$ when $n \equiv 0, 1 \pmod{4}$ or $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + \frac{3}{2}$ when $n \equiv 2, 3 \pmod{4}$, then s < t and there exists a $\frac{1}{2}$ -cycle in $V_{12} \setminus X$. By Lemma 2.2 (3), from the $\frac{1}{2}$ -cycle of G, we may take two different vertices v', v''. Let $\bar{f}(v'x') = \bar{f}(v''x'') = 1$ in \overline{G} for $\{x', x''\} \subseteq V_{21}$. There are s - 2 independent edges in $\overline{G}[X \setminus \{v_1, v_2\}, V_{21} \setminus \{x', x''\}]$ by Lemma 2.6 (2) and we may assign number 1 to them. Noting that $|V_1| - s - 2 \le |V_2|$, we may assign number 1 to 2t - 2s - 2 independent edges in $\overline{G}[V_{12} \setminus \{X, v', v''\}, V_{22}]$. When n - 2t - (2t - s - 2) = 0, we may assign number 1 to edge v_1v_2 in \overline{G} . Thus $\alpha'(\overline{G}) \ge 1 + 2t - s - 2 = 2t - s - 1 = \frac{n-s}{2} > \frac{n-t}{2}$.

If n-2t-(2t-s-2)=1, that is n=4t-s-1 and $|V_{22}|=2t-2s-1$, then we may choose $w\in V_{22}$ and assign number $\frac{1}{2}$ to edges of C_3 induced by v_1,v_2 and w in \overline{G} , and 1 to 2t-2s-2 independent edges in $\overline{G}[V_{12}\setminus\{X,v',v''\},V_{22}]$. Thus we have $\alpha'(\overline{G})\geq \frac{3}{2}+2t-s-2=2t-s-\frac{1}{2}=\frac{n-s}{2}>\frac{n-t}{2}$.

If n-2t-(2t-s-2)=2, then we may choose $w,w'\in V_{22}$ and assign number 1 to 2t-2s-2 independent edges in $\overline{G}[V_{12}\setminus\{X,v',v''\},V_{22}\setminus\{w,w'\}]$. Let $\overline{f}(v_1v_2)=\overline{f}(ww')=1$ in \overline{G} . Thus we have $\alpha'(\overline{G})\geq 2+2t-s-2=2t-s=\frac{n-s}{2}>\frac{n-t}{2}$.

When $n-2t-(2t-s-2) \geq 3$, we may assign number $\frac{1}{2}$ to each edge of a cycle with length n-2t-(2t-s-2) in $\overline{G}[V_{22}]$. Let $\bar{f}(v_1v_2)=1$ in \overline{G} . Thus

$$\alpha'(\overline{G}) \ge 1 + 2t - s - 2 + \frac{n - 2t - (2t - s - 2)}{2} = \frac{n - s}{2} > \frac{n - t}{2}.$$

(2) When $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + 1$ and s = t, by Lemma 2.6 (2), $uv \in E(\overline{G})$ for each vertex $u \in V_{12}$ and $v \in V_2$. Since $|V_{22}| \geq 3$, we may assign number 1 to s - 2 independent edges in $[V_{12} \setminus \{v_1, v_2\}, V_{21}]$ in \overline{G} , and $\frac{1}{2}$ to each edge of a cycle with length n - 2t - (s - 2) in $\overline{G}[V_2]$. For a fractional matching \overline{f} of \overline{G} , let $\overline{f}(v_1v_2) = 1$ in \overline{G} . Thus we have

$$\alpha'(\overline{G}) \ge 1 + s - 2 + \frac{n - 2t - (s - 2)}{2} = \frac{n - s}{2} = \frac{n - t}{2}.$$

When s < t and there exists a $\frac{1}{2}$ -cycle in $V_{12} \setminus X$, it can be proved similar to the above in (1). Assume that there is no $\frac{1}{2}$ -cycle in $V_{12} \setminus X$ and there are p edges assigned number 1 in $E[V_{12}]$, where $p \ge 1$ and p is an integer. Since f(v) = 1 for every $v \in V_1$, p = 0 is equivalent to s = t or $G[V_{12} \setminus X]$ is a collection of disjoint $\frac{1}{2}$ -cycles, which has been discussed above. If p = 1, that is t = s + 1 and $|V_{12}| = 2t - s = s + 2$, then

there exactly exists an edge say ww_1 assigned number 1 in $G[V_{12} \setminus X]$. Without loss of generality, there exists a vertex x in V_{21} such that $xw \in E(\overline{G})$ by Lemma 2.5 (1). Since $|V_{22}| \geq 3$, let $\overline{f}(v_1v_2) = \overline{f}(xw) = \overline{f}(y_1w_1) = 1$ in \overline{G} , $y_1 \in V_{22}$. Then for each vertex $v \in V_{12} \setminus \{v_1, v_2, w, w_1\}$ and each vertex $u \in V_2 \setminus \{x, y_1\}$, we have $uv \in E(\overline{G})$. We may assign number 1 to s-2 independent edges in $[V_{12} \setminus \{v_1, v_2, w, w_1\}, V_{21} \setminus \{x\}]$, and $\frac{1}{2}$ to each edge of a cycle with length n-2t-s in $\overline{G}[V_2]$. Thus we have

$$\alpha'(\overline{G}) \ge 1 + 1 + 1 + (s - 2) + \frac{n - 2t - s}{2} = \frac{n - s}{2} > \frac{n - t}{2}.$$

If $p \geq 2$, then there exist two edges w_1w_2 and w_3w_4 assigned number 1 in $G[V_{12} \setminus X]$. Without loss of generality, there exist two vertices x_1, x_2 in V_{21} such that $x_1w_1, x_2w_3 \in E(\overline{G})$ and let $\overline{f}(x_1w_1) = \overline{f}(x_2w_3) = 1$ in \overline{G} . In \overline{G} , there are s-2 independent edges in $[X \setminus \{v_1, v_2\}, V_{21} \setminus \{x_1, x_2\}]$ by Lemma 2.6 (2) and we may assign number 1 to them. Noting that $n-2t-s \geq 2t-2s-2$, we may assign number 1 to 2t-2s-2 independent edges in $\overline{G}[V_{12} \setminus \{X \cup \{w_1, w_3\}\}, V_{22}]$. If n-2t-(2t-s-2)=0, that is n=4t-s-2 and $|V_{22}|=2t-2s-2$, we may assign number 1 to edge v_1v_2 in \overline{G} . Thus we have $\alpha'(\overline{G}) \geq 1+2t-s-2=2t-s-1=\frac{n-s}{2}>\frac{n-t}{2}$.

If n-2t-(2t-s-2)=1, that is n=4t-s-1 and $|V_{22}|=2t-2s-1$, then we may choose $y\in V_{22}$ and assign number $\frac{1}{2}$ to edges of C_3 induced by v_1,v_2 and y in \overline{G} , and 1 to 2t-2s-2 independent edges in $\overline{G}[V_{12}\setminus\{X\cup\{w_1,w_3\}\},V_{22}\setminus\{y\}]$. Thus we have $\alpha'(\overline{G})\geq \frac{3}{2}+2t-s-2=2t-s-\frac{1}{2}=\frac{n-s}{2}>\frac{n-t}{2}$.

If n-2t-(2t-s-2)=2, that is n=4t-s and $|V_{22}|=2t-2s$, then we may choose $y_1,y_2\in V_{22}$ and assign number 1 to edges v_1v_2 and y_1y_2 in \overline{G} , and 1 to 2t-2s-2 independent edges in $\overline{G}[V_{12}\setminus\{X\cup\{w_1,w_3\}\},V_{22}\setminus\{y_1,y_2\}]$. Thus we have $\alpha'(\overline{G})\geq 1+2t-s-2+1=2t-s=\frac{n-s}{2}>\frac{n-t}{2}$.

When $n-2t-(2t-s-2) \geq 3$, we may assign number $\frac{1}{2}$ to each edge of a cycle with length n-2t-(2t-s-2) in $\overline{G}[V_2]$, and 1 to edge v_1v_2 in \overline{G} . Thus we have

$$\alpha'(\overline{G}) \ge 1 + 2t - s - 2 + \frac{n - 2t - (2t - s - 2)}{2} = \frac{n - s}{2} > \frac{n - t}{2}.$$

Therefore, $\alpha'(\overline{G}) \geq \frac{n-t}{2} \geq \frac{n}{4} + 3$ for $n \geq 28$. This completes the proof.

Without loss of generality, we may assume that $\alpha'(G) \leq \alpha'(\overline{G})$ as follows.

Theorem 4.3. Let G be a graph of order $n \geq 28$. If both G and \overline{G} are not empty, then

$$\alpha'(G) + \alpha'(\overline{G}) \ge \frac{n+1}{2}$$

with equality holds if and only if $G \cong K_{1,n-1}$.

Proof Since G is not empty, $\alpha'(G) \geq 1$. By Lemma 3.5 (1)-(5), when $1 \leq \alpha'(G) < 3$, $\alpha'(G) + \alpha'(\overline{G}) \geq \frac{n+1}{2}$. The equality holds if and only if $G \cong K_{1,n-1}$ by Lemma 3.5 (1) and Lemma 3.2 (1).

When $3 \le \alpha'(G) \le \frac{n}{4}$, by Lemma 4.1 (1), we have

$$\alpha'(G) + \alpha'(\overline{G}) \ge \alpha'(G) + \frac{n - \alpha'(G)}{2} = \frac{n + \alpha'(G)}{2} \ge \frac{n+3}{2} > \frac{n+1}{2}.$$

Then we only need to consider the case that $\frac{n}{4} < \alpha'(G) \leq \frac{n}{2}$.

If $n \equiv 0, 1 \pmod{4}$, then $\frac{n+1}{4} \leq \lfloor \frac{n}{4} \rfloor + \frac{1}{2} \leq \alpha'(G) \leq \frac{n}{2}$. So we have

$$\alpha'(G) + \alpha'(\overline{G}) \ge 2\alpha'(G) \ge 2(\lfloor \frac{n}{4} \rfloor + \frac{1}{2}) \ge \frac{n+1}{2}.$$

The equality holds if and only if $n \equiv 1 \pmod{4}$ and $\alpha'(G) = \alpha'(\overline{G}) = \lfloor \frac{n}{4} \rfloor + \frac{1}{2}$, which is a contradiction to Lemma 4.2 (1) in virtue of $\lfloor \frac{n}{4} \rfloor + \frac{1}{2} < \frac{n}{4} + 3$.

If $n \equiv 2, 3 \pmod{4}$, $\frac{n}{4} < \lfloor \frac{n}{4} \rfloor + 1 \le \alpha'(G) \le \frac{n}{2}$. When $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + 1$, by Lemma 4.2 (2), $\alpha'(\overline{G}) \ge \frac{n}{4} + 3$. Thus $\alpha'(G) + \alpha'(\overline{G}) \ge \lfloor \frac{n}{4} \rfloor + 1 + \frac{n}{4} + 3 \ge \frac{n}{2} + 3 > \frac{n+1}{2}$. When $\frac{n+2}{4} < \lfloor \frac{n}{4} \rfloor + \frac{3}{2} \le \alpha'(G) \le \frac{n}{2}$, we have

$$\alpha'(G) + \alpha'(\overline{G}) \ge 2\alpha'(G) \ge 2(\lfloor \frac{n}{4} \rfloor + \frac{3}{2}) > \frac{n+2}{2} > \frac{n+1}{2}.$$

Therefore, $\alpha'(G) + \alpha'(\overline{G}) > \frac{n+1}{2}$. This completes the proof. \Box The following result is implied by Theorem 4.3.

Lemma 4.4. For a graph of order $n, n \ge 2$, we have

$$\alpha'(G) + \alpha'(\overline{G}) \ge \frac{n}{2}$$

with equality holds if and only if G is an empty graph or G is a complete graph.

The following result is deduced by Lemma 3.3, Lemma 3.4 and Lemma 3.5.

Lemma 4.5. If $\alpha'(G) = 2$ or $\alpha'(G) = \frac{5}{2}$, \overline{G} contains no isolated vertices, then $\alpha'(\overline{G}) = \frac{n}{2}$ for $n \geq 10$.

Theorem 4.6. If both G and \overline{G} contain no isolated vertices and $n \geq 28$. Then

$$\alpha'(G) + \alpha'(\overline{G}) \ge \frac{n+4}{2}$$

and the equality holds if and only if $G \cong K_2(p,q;\ell)$ or $K_{1,m} \cup K_{1,n-2-m} \subseteq G \subseteq K_{2,n-2}$, where p,q,ℓ,m are non-negative integers, $q \ge 1$ and $1 \le p,m \le n-3$.

Proof Since both G and \overline{G} contain no isolated vertices, by Lemma 3.2, we have $\alpha'(G)$ and $\alpha'(\overline{G}) \geq 2$. When s = 0, since G contains no isolated vertices, V_2 is empty and $G[V_1]$ contains a fractional perfect matching by Lemma 2.4. The assumption $\alpha'(G) \leq \alpha'(\overline{G})$ implies $\alpha'(\overline{G}) = \alpha'(G) = \frac{n}{2}$. Thus $\alpha'(G) + \alpha'(\overline{G}) = \frac{n}{2} + \frac{n}{2} = n > \frac{n+4}{2}$.

When $\alpha'(G) = 2$ or $\frac{5}{2}$, by Lemma 4.5, $\alpha'(\overline{G}) = \frac{n}{2}$. It follows that $\alpha'(G) + \alpha'(\overline{G}) \ge \frac{n+4}{2}$. The equality holds when $\alpha'(G) = 2$.

When $\alpha'(G) = 3$, if s = 1 or 2, we have $\alpha'(\overline{G}) \geq \frac{n-s+1}{2}$ by Lemma 4.1 (2). It follows that $\alpha'(G) + \alpha'(\overline{G}) \ge 3 + \frac{n-s+1}{2} > \frac{n+4}{2}$. If s = t = 3, we have $\alpha'(\overline{G}) \ge \frac{n-3}{2} + 1 = \frac{n-1}{2}$ by Lemma 4.1 (3), and then $\alpha'(G) + \alpha'(\overline{G}) \ge 3 + \frac{n-1}{2} = \frac{n+5}{2} > \frac{n+4}{2}$.

When $\alpha'(G) = \frac{7}{2}$, then $1 \le s \le 3$. By Lemma 4.1 (2), we have $\alpha'(\overline{G}) \ge \frac{n-s+1}{2}$. It follows that $\alpha'(G) + \alpha'(\overline{G}) \ge \frac{7}{2} + \frac{n-s+1}{2} \ge \frac{7}{2} + \frac{n-3+1}{2} = \frac{n+5}{2} > \frac{n+4}{2}$. When $4 \le \alpha'(G) = t \le \frac{n}{4}$, since $s \ge 1$, $\alpha'(\overline{G}) \ge \frac{n-s+1}{2}$ by Lemma 4.1 (2). Thus

$$\alpha'(G) + \alpha'(\overline{G}) \ge t + \frac{n-s+1}{2} \ge t + \frac{n-t+1}{2} = \frac{n+t+1}{2} \ge \frac{n+5}{2} > \frac{n+4}{2}.$$

Then we only need to consider the case that $\frac{n}{4} < \alpha'(G) \le \frac{n}{2} \le \alpha'(\overline{G})$.

If $n \equiv 0, 1 \pmod{4}$, then $\lfloor \frac{n}{4} \rfloor + \frac{1}{2} \leq \alpha'(G) \leq \frac{n}{2}$. When $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + \frac{1}{2}$ or $\lfloor \frac{n}{4} \rfloor + 1$, by Lemma 4.2, $\alpha'(\overline{G}) \ge \frac{n}{4} + 3$. Thus

$$\alpha'(G) + \alpha'(\overline{G}) \ge \lfloor \frac{n}{4} \rfloor + \frac{1}{2} + \frac{n}{4} + 3 > \frac{n}{2} + 3 > \frac{n+4}{2}.$$

When $\alpha'(G) \geq \lfloor \frac{n}{4} \rfloor + \frac{3}{2} > \frac{n+4}{4}$, we obtain $\alpha'(G) + \alpha'(\overline{G}) \geq 2\alpha'(G) \geq 2\lfloor \frac{n}{4} \rfloor + 3 > \frac{n+4}{2}$. If $n \equiv 2, 3 \pmod{4}$, $\lfloor \frac{n}{4} \rfloor + 1 \leq \alpha'(G) \leq \frac{n}{2}$. When $\alpha'(G) = \lfloor \frac{n}{4} \rfloor + 1$ or $\lfloor \frac{n}{4} \rfloor + \frac{3}{2}$, by Lemma 4.2, $\alpha'(\overline{G}) \ge \frac{n}{4} + 3$. Thus we have $\alpha'(G) + \alpha'(\overline{G}) \ge \lfloor \frac{n}{4} \rfloor + 1 + \frac{n}{4} + 3 > \frac{n}{2} + 3 > \frac{n+4}{2}$. When $\alpha'(G) \ge \lfloor \frac{n}{4} \rfloor + 2 > \frac{n+4}{4}$, we have

$$\alpha'(G) + \alpha'(\overline{G}) \ge 2\alpha'(G) \ge 2(\lfloor \frac{n}{4} \rfloor + 2) > \frac{n+4}{2}.$$

Therefore, $\alpha'(G) + \alpha'(\overline{G}) > \frac{n+4}{2}$.

If $\alpha'(G) + \alpha'(\overline{G}) = \frac{n+4}{2}$, then $\alpha'(G) = 2$. By Lemma 3.3 and both G and \overline{G} contain no isolated vertices, we have $G \cong K_2(p,q;\ell)$ or $K_{1,m} \cup K_{1,n-2-m} \subseteq G \subseteq K_{2,n-2}$, where p,q,ℓ,m are non-negative integers, $q \ge 1$ and $1 \le p,m \le n-3$. Therefore, we complete the proof.

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