# Nordhaus-Gaddum type inequality for the fractional matching number of a graph 

Ting Yang, Xiying Yuan*<br>Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China


#### Abstract

The fractional matching number of a graph $G$, written as $\alpha^{\prime}(G)$, is the maximum size of a fractional matching of $G$. The following sharp lower bounds for a graph $G$ of order $n$ are proved, and all extremal graphs are characterized in this paper. (1) $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n}{2}$ for $n \geq 2$. (2) If $G$ and $\bar{G}$ are non-empty, then for $n \geq 28, \alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n+1}{2}$. (3) If $G$ and $\bar{G}$ have no isolated vertices, then for $n \geq 28, \alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n+4}{2}$.


Keywords: Nordhaus-Gaddum type inequality, Fractional matching number, Fractional Berge's theorem

## 1. Introduction

Throughout this paper, all graphs are simple, undirected and finite. Undefined terminologies and notations can be found in [2]. Let $G=(V(G), E(G))$ be a graph and $\bar{G}$ be its complement. $n$ will always denote the number of vertices of a given graph $G$. For a vertex $v \in V(G)$, its degree $d_{G}(v)$ is the number of edges incident to it in $G$, its neighborhood, denoted by $N(v)$, is the set of vertices, which are adjacent to $v$. An edge set $M$ of $G$ is called a matching if any two edges in $M$ have no common vertices. The matching number of a graph $G$, written $\alpha(G)$, is the number of edges in a maximum matching. As in [13], a fractional matching of a graph $G$ is a function $f: E(G) \longrightarrow[0,1]$ such that $f(v) \leq 1$ for each vertex $v \in V(G)$, where $f(v)$ is the sum of $f(e)$ of edges incident to $v$. The fractional matching number of $G$, written $\alpha^{\prime}(G)$, is the maximum value of $f(G)$ over all fractional matchings, where $f(G)$ denotes the sum of $f(e)$ of all edges in $G$. A fractional perfect matching of a graph $G$ is a fractional matching $f$ with $f(G)=\alpha^{\prime}(G)=\frac{n}{2}$. Obviously, fractional matching is a generalization of matching. Choi et al. [4] proved the difference and ratio of the fractional matching number and the matching number of graphs and characterized all infinite extremal family of graphs.

[^0]Nordhaus and Gaddum [11] considered lower and upper bounds on the sum and the product of chromatic number $\chi(G)$ of a graph $G$ and its complement $\bar{G}$. They showed that

$$
2 \sqrt{n} \leq \chi(G)+\chi(\bar{G}) \leq n+1
$$

Since then, any bound on the sum or the product of an invariant in a graph $G$ and the same invariant in its complement $\bar{G}$ is called a Nordhaus-Gaddum type inequality. The Nordhaus-Gaddum type inequality of various graph parameters has attracted much attention (see [5][6][7][8][9][12][14]). Aouchiche and Hansen [1] wrote a stimulating survey on this topic, and we refer the reader to that article for additional information. Chartrand and Schuster [3] proved Nordhaus-Gaddum type result for the matching number of a graph. They showed that $\alpha(G)+\alpha(\bar{G}) \geq\left\lfloor\frac{n}{2}\right\rfloor$. Laskar and Auerbach [7] improved the bound by considering that $G$ and $\bar{G}$ contain no isolated vertices. They showed that $\alpha(G)+\alpha(\bar{G}) \geq\left\lfloor\frac{n}{2}\right\rfloor+2$. Later, Lin et al. [10] characterized all extremal graphs which attain the lower bounds of the results of Chartrand et al. and Laskar et al.. Motivated by them, we consider Nordhaus-Gaddum type inequality for the fractional matching number of a graph. First we prove some auxiliary results of fractional matching, which are selfcontained. Then we establish lower bounds on the sum of fractional matching number of a graph $G$ and its complement (see Theorem 4.3 and Theorem 4.6). Moreover, we show those bounds are sharp.

## 2. Auxiliary results of fractional matching

Based on an optimal fractional matching of graph $G$, we present a good partition of $V(G)$. We further characterize some properties of this partition.

Lemma 2.1. ([13]) For any graph $G, 2 \alpha^{\prime}(G)$ is an integer. Moreover, there is a fractional matching $f$ for which

$$
f(G)=\alpha^{\prime}(G)
$$

such that $f(e) \in\left\{0, \frac{1}{2}, 1\right\}$ for every edge $e$ of $G$.
An $f$ is called an optimal fractional matching of graph $G$ in this paper, if we have (1) $f(G)=\alpha^{\prime}(G)$.
(2) $f(e) \in\left\{0, \frac{1}{2}, 1\right\}$ for every edge $e$.
(3) $f$ has the greatest number of edges $e$ with $f(e)=1$.

In this paper, for a graph $G$, given a fractional matching $f$, an unweighted vertex $v$ is a vertex with $f(v)=0$. A full vertex $v$ is a vertex with $f(v w)=1$ for some edge $v w$ and we may call vertex $w$ is the full neighbour of $v$. An $i-e d g e ~ e$ is an edge with $f(e)=i . \frac{1}{2}-$ cycle in a graph $G$ is an odd cycle induced by $\frac{1}{2}$-edges in $G$.

Lemma 2.2. Let $f$ be an optimal fractional matching of graph $G$. Then we have the following:
(1) ([4]) The maximal subgraph induced by the $\frac{1}{2}$-edges is the union of odd cycles.
(2) ([4]) The set of the unweighted vertices is an independent set of $G$. Furthermore, every unweighted vertex is adjacent only to a full vertex.
(3) No $\frac{1}{2}$-cycle has an unweighted vertex as a neighbour.

Proof By (2), every unweighted vertex is adjacent only to a full vertex, while every vertex on $\frac{1}{2}$-cycle is not a full vertex, then no $\frac{1}{2}$-cycle has an unweighted vertex as a neighbour.

Lemma 2.3. ([13]) Suppose that $f$ is a fractional matching of graph $G$. Then $f$ is a fractional perfect matching if and only if $f(v)=1$ for every vertex $v \in V(G)$.

Lemma 2.4. For any graph $G$, we have a partition $V(G)=V_{1} \cup V_{2}$ with $\left|V_{1}\right|=2 \alpha^{\prime}(G)$, and $G\left[V_{1}\right]$ contains a fractional perfect matching and $V_{2}$ is empty or an independent set.

Proof Suppose $f$ is an optimal fractional matching of $G$. If $\alpha^{\prime}(G)=\frac{n}{2}$, we take $V_{1}=V(G)$. If $\alpha^{\prime}(G)<\frac{n}{2}$, set $V_{1}=\{v \in V(G) \mid f(v)>0\}$ and $V_{2}=V(G) \backslash V_{1}$. We will show that $f(v)=1$ for each vertex $v$ in $V_{1}$. Suppose to the contrary that there exists a vertex $v_{0} \in V_{1}$ with $0<f\left(v_{0}\right)<1$, i.e., $f\left(v_{0}\right)=\frac{1}{2}$, say $f\left(v_{0} v_{1}\right)=\frac{1}{2}$ for some vertex $v_{1} \in V_{1}$. By Lemma $2.2(1), v_{0} v_{1}$ lies in a $\frac{1}{2}$-cycle, then there exists a vertex $v_{t}\left(\neq v_{1}\right)$ such that $f\left(v_{0} v_{t}\right)=\frac{1}{2}$. Thus $f\left(v_{0}\right)=1$, which is a contradiction. Thus $G\left[V_{1}\right]$ contains a fractional perfect matching by Lemma 2.3. Since $V_{2}$ is a set of the unweighted vertices, $V_{2}$ is an independent set by Lemma 2.2 (2). Furthermore, we have $\alpha^{\prime}(G)=\alpha^{\prime}\left(G\left[V_{1}\right]\right)=\frac{\left|V_{1}\right|}{2}$.

For a graph $G$ with $\alpha^{\prime}(G)=t$, we write $V(G)=V_{1} \dot{\cup} V_{2}$ according to the results of Lemma 2.4, and let $s$ be the maximum number of independent edges in $\left[V_{1}, V_{2}\right]$. Now we will further decompose $V_{1}=V_{11} \cup \dot{U} V_{12}$ and $V_{2}=V_{21} \cup V_{22}$ such that [ $V_{11}, V_{21}$ ] contains exactly $s$ independent edges. Obviously, $\left|V_{11}\right|=\left|V_{21}\right|=s$ holds. We call $V(G)=\left(V_{11} \cup V_{12}\right) \cup\left(V_{21} \cup V_{22}\right)$ a good partition of $G$ in this paper (see Figure 1).

Lemma 2.5. Let $V(G)=\left(V_{11} \cup V_{12}\right) \cup\left(V_{21} \cup V_{22}\right)$ be a good partition of $G$ with $\left|V_{11}\right|=s$ and $f$ be the corresponding optimal fractional matching of $G$.
(1) If $f(u v)=1$ for some edge $u v$ in $G$, then there is no vertex in $V_{2} \cap N(u) \cap N(v)$.
(2) If $e$ is an edge in $G\left[V_{11}\right]$, then $f(e)=0$.
(3) Each vertex in $V_{11}$ is a full vertex.

Proof (1) Suppose there exists a vertex $w \in V_{2} \cap N(u) \cap N(v)$. Since $w$ is an unweighted vertex, we have $f(u w)=f(v w)=0$. Now set $f^{*}(u w)=f^{*}(v w)=f^{*}(u v)=\frac{1}{2}$ and other assignments remain unchanged. Then $f^{*}(G)=\alpha^{\prime}(G)+\frac{1}{2}$, which is a contradiction.
(2) Suppose $u v$ is an edge in $G\left[V_{11}\right]$ with $f(u v)=a>0$. Let $u x$ and $v y$ be two independent edges in $\left[V_{11}, V_{21}\right]$. Now set $f^{*}(u x)=f^{*}(v y)=1$ and $f^{*}(u z)=f^{*}(v w)=0$


G

Figure 1: A good partition of graph $G$
for any $z \in N(u) \backslash\{x\}$ and $w \in N(v) \backslash\{y\}$ and other assignments remain unchanged. Then $f^{*}(G)-f(G) \geq a$, which is a contradiction to the choice of $f$.
(3) Suppose $u \in V_{11}$ is not a full vertex. Then there exists a vertex $v$ such that $f(u v)=\frac{1}{2}$. Furthermore, $u v$ lies in a $\frac{1}{2}-$ cycle by Lemma 2.2 (1). While $u$ has a neighbour in $V_{21}$, which contradicts to Lemma 2.2 (3).

From Lemma 2.5 (3), we know that each vertex in $V_{11}$ is a full vertex. Denote by $X$ the set of all full neighbours of vertices in $V_{11}$, and then $X \subseteq V_{12}$ by Lemma 2.5 (2). It is obvious that $|X|=\left|V_{11}\right|=s$. Furthermore, we have the following results.

Lemma 2.6. (1) If $|X|=s \geq 2$, then $X$ is an independent set of $G$.
(2) There is no edge between $V_{2}$ and $X$ in $G$.

Proof (1) Suppose there exist two vertices $u, v \in X$ such that $u v \in E(G)$ and $f(g u)=$ $f(h v)=1$ for vertices $g, h$ in $V_{11}$. Let $g w$ and $h y$ be two independent edges in $\left[V_{11}, V_{21}\right]$. Now set $f^{*}(g u)=f^{*}(h v)=0$ and $f^{*}(u v)=f^{*}(g w)=f^{*}(h y)=1$ and other assignments remain unchanged. Then $f^{*}(G)=\alpha^{\prime}(G)+1$, which is a contradiction.
(2) Suppose to the contrary that there exists an edge $v x$ with $v \in X$ and $x \in V_{21}$. Then there exists a vertex $u \in V_{11}$ such that $u v \in E(G)$ and $f(u v)=1$. We have $u x \notin E(G)$ in virtue of Lemma 2.5 (1). Then there exists a vertex say $y \in V_{21}$ such that $u y$ is one of the independent edges in $\left[V_{11}, V_{21}\right]$. Now we may modify $f$ to $f^{*}$. Let $f^{*}(u v)=0, f^{*}(v x)=f^{*}(u y)=1$ and other assignments remain unchanged. Then $f^{*}(G)=\alpha^{\prime}(G)+1$, which is a contradiction. It is obvious that there is no edge between
$V_{22}$ and $X$ in $G$. Thus there is no edge between $V_{2}$ and $X$ in $G$.

## 3. Graphs with small fractional matching number

In virtue of the fractional Berge's theorem, we will characterize some graphs with small fractional matching number in this section.

Lemma 3.1. ([13]) For any graph $G$ of order n, we have

$$
\alpha^{\prime}(G)=\frac{1}{2}\left(n-\max _{S \subseteq V(G)}\{i(G-S)-|S|\}\right),
$$

where $i(G-S)$ denotes the number of isolated vertices of $G-S$.
By the fractional Berge's theorem, we immediately have the following results.
Lemma 3.2. (1) For a graph $G$ of order $n, \alpha^{\prime}(G)=1$ if and only if $G \cong K_{1, k} \cup(n-$ $1-k) K_{1}$, where $k \geq 1$.
(2) For a graph $G$ of order $n, \alpha^{\prime}(G)=\frac{3}{2}$ if and only if $G \cong C_{3} \cup(n-3) K_{1}$.

Let $K_{2}(p, q ; \ell)(p \geq q)$ be the graph obtained by attaching $p$ pendent edges at one vertex of $K_{2}$ called $u v, q$ pendent edges at the other vertex of $K_{2}$ and having $\ell$ vertices in $N(u) \cap N(v)$.

Lemma 3.3. For a graph $G$ of order $n, \alpha^{\prime}(G)=2$ if and only if one of the following situations occurs:
(1) $2 K_{2} \cup(n-4) K_{1} \subseteq G \subseteq K_{4} \cup(n-4) K_{1}$.
(2) $2 K_{2} \cup(n-4) K_{1} \subseteq G \subseteq K_{2}(0,0 ; n-2)$.

Proof By considering the fact that if $G_{1} \subseteq G_{2}$, then $\alpha^{\prime}\left(G_{1}\right) \leq \alpha^{\prime}\left(G_{2}\right)$ and $\alpha^{\prime}\left(K_{4} \cup(n-\right.$ 4) $\left.K_{1}\right)=\alpha^{\prime}\left(K_{2}(0,0 ; n-2)\right)=\alpha^{\prime}\left(2 K_{2} \cup(n-4) K_{1}\right)=2$, the sufficiency part is correct.

To show the necessity part, by Lemma 3.1, suppose $S \subseteq V(G)$ such that $n-4=$ $i(G-S)-|S|$. Since $i(G-S) \leq n-|S|$, it follows that $|S| \leq 2$.

If $|S|=0$, then $i(G)=n-4$. $\alpha^{\prime}(G)=2$ implies $G$ contains two independent edges. Thus we have $2 K_{2} \cup(n-4) K_{1} \subseteq G \subseteq K_{4} \cup(n-4) K_{1}$.

If $|S|=1$, then $i(G-S)=n-3$, and $G-S$ contains a subgraph $F=K_{2} . \alpha^{\prime}(G)=2$ implies that there is an edge between $S$ and $(V(G) \backslash\{S \cup V(F)\})$. Thus $2 K_{2} \cup(n-4) K_{1}$ is a subgraph of $G$. Since there are at most $n-3$ edges in $[S, V(G) \backslash\{S \cup V(F)\}]$ and there are at most two edges in $[S, V(F)], G$ is a subgraph of $K_{2}(n-3,0 ; 1)$. Thus $2 K_{2} \cup(n-4) K_{1} \subseteq G \subseteq K_{2}(n-3,0 ; 1)$.

If $|S|=2$, then $i(G-S)=n-2 . \quad \alpha^{\prime}(G)=2$ implies $G$ contains two independent edges in $[S, V(G) \backslash S]$. Thus we have $2 K_{2} \cup(n-4) K_{1} \subseteq G$. Since there are at most $2(n-2)$ edges in $[S, V(G) \backslash S]$ and there is at most one edge in $S, G$ is a subgraph of $K_{2}(0,0 ; n-2)$. Thus $2 K_{2} \cup(n-4) K_{1} \subseteq G \subseteq K_{2}(0,0 ; n-2)$. Noting that $K_{2}(n-3,0 ; 1) \subseteq K_{2}(0,0 ; n-2)$, we have $2 K_{2} \cup(n-4) K_{1} \subseteq G \subseteq K_{2}(0,0 ; n-2)$.

Lemma 3.4. Let $H$ be the graph of order $n$ obtained by attaching $n-4$ pendent edges at one vertex of $K_{4}$. For a graph $G$ of order $n, \alpha^{\prime}(G)=\frac{5}{2}$ if and only if one of the following situations occurs:
(1) $C_{5} \cup(n-5) K_{1} \subseteq G \subseteq K_{5} \cup(n-5) K_{1}$.
(2) $C_{3} \cup K_{2} \cup(n-5) K_{1} \subseteq G \subseteq K_{5} \cup(n-5) K_{1}$.
(3) $C_{3} \cup K_{2} \cup(n-5) K_{1} \subseteq G \subseteq H$.

Proof The sufficiency part is obvious. To show the necessity part, by Lemma 3.1, suppose $S \subseteq V(G)$ such that $n-5=i(G-S)-|S|$. Since $i(G-S) \leq n-|S|$, it follows that $|S| \leq \frac{5}{2}$. Furthermore, $|S|=2$ does not occur. Otherwise we have $i(G-S)=n-3$. While we have $G-S=(n-3) K_{1} \cup K_{1}$, which implies that $i(G-S)=(n-2)$.

If $|S|=0$, then $i(G)=n-5 . \alpha^{\prime}(G)=\frac{5}{2}$ implies $G$ contains an odd cycle. If $G$ contains $C_{3}$ as a subgraph, then we have $C_{3} \cup K_{2} \cup(n-5) K_{1} \subseteq G \subseteq K_{5} \cup(n-5) K_{1}$. If $G$ contains $C_{5}$ as a subgraph, then we have $C_{5} \cup(n-5) K_{1} \subseteq G \subseteq K_{5} \cup(n-5) K_{1}$.

If $|S|=1$, then $i(G-S)=n-4 . \quad \alpha^{\prime}(G)=\frac{5}{2}$ implies that $G-S$ contains $C_{3}$ as a subgraph, and there exists at least one edge in $\left[S, V(G) \backslash\left\{S \cup V\left(C_{3}\right)\right\}\right]$. Thus $C_{3} \cup K_{2} \cup(n-5) K_{1}$ is a subgraph of $G$. Furthermore, we have $G \subseteq H$.

Based on the results of Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have the following results.

Lemma 3.5. Let $G$ be a graph of order $n$. Then the following statements hold.
(1) If $\alpha^{\prime}(G)=1$ and $n \geq 4$, then $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n+1}{2}$.
(2) If $\alpha^{\prime}(G)=\frac{3}{2}$ and $n \geq 6$, then $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G})=\frac{n+3}{2}$.
(3) If $\alpha^{\prime}(G)=2$ and $G$ is not isomorphic to $K_{2}(0,0 ; \ell)$ for $\ell \geq 2$, then $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq$ $\frac{n+3}{2}$ for $n \geq 8$.
(4) If $\alpha^{\prime}(G)=2$ and $G$ is isomorphic to $K_{2}(0,0 ; \ell)$ for $\ell \geq 2$, then $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G})=\frac{n+2}{2}$.
(5) If $\alpha^{\prime}(G)=\frac{5}{2}$ and $n \geq 7$, then $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n}{2}+2$.

Moreover, the equalities in (1), (3) and (5) hold if and only if $G$ contains exactly one vertex with degree $n-1$.

## 4. Nordhaus-Gaddum-type bounds for the fractional matching number

In this section, we will prove the Nordhaus-Gaddum-type bounds for the fractional matching number (see Theorem 4.3 and Theorem 4.6).

In a graph $G$ with $\alpha^{\prime}(G)=t$, we try to find a collection $E$ of some independent edges in $\bar{G}$ and assign 1 to each edge of $E$, and then find a long cycle $C$ (having no common vertices with $E$ ) and assign $\frac{1}{2}$ to each edge of $C$. By this way, we get a lower bound of $\alpha^{\prime}(\bar{G})$.

Lemma 4.1. Let $V(G)=\left(V_{11} \cup V_{12}\right) \cup\left(V_{21} \cup V_{22}\right)$ be a good partition of graph $G$ of order $n$ with $\alpha^{\prime}(G)=t \leq \frac{n}{4}$ and $\left|V_{11}\right|=s$. Then the following statements hold.
(1) $\alpha^{\prime}(\bar{G}) \geq \frac{n-s}{2} \geq \frac{n-t}{2}$.
(2) If both $G$ and $\bar{G}$ contain no isolated vertices and $s \geq 1$, then $\alpha^{\prime}(\bar{G}) \geq \frac{n-s+1}{2}$.
(3) If both $G$ and $\bar{G}$ contain no isolated vertices and $s=t \geq 3$, then $\alpha^{\prime}(\bar{G}) \geq \frac{n-s+2}{2}$.

Proof Since $V(G)=\left(V_{11} \cup V_{12}\right) \cup\left(V_{21} \cup V_{22}\right)$ is a good partition, $\bar{G}\left[V_{2}\right]$ is a clique, and each vertex in $V_{12}$ is adjacent to each vertex in $V_{22}$ in $\bar{G}$. The assumption $n \geq 4 t$ insures $\left|V_{22}\right|=n-2 t-s \geq 2 t-s$.
(1) If $n-2 t-(2 t-s)=0$, that is $s=0$ and $\left|V_{12}\right|=\left|V_{22}\right|=\frac{n}{2}=2 t$, then we may assign number 1 to $2 t$ independent edges in $\bar{G}\left[V_{12}, V_{22}\right]$ for a fractional matching of $\bar{G}$. We have $\alpha^{\prime}(\bar{G}) \geq 2 t=\frac{n}{2}$.

If $n-2 t-(2 t-s)=[(n-2 t-s)-(2 t-s)]+s=1$, then $s \leq 1$. There exist $u \in V_{12}$ and $v, w \in V_{2}$ forming a cycle for a fractional matching of $\bar{G}$. We may assign number 1 to $2 t-s-1$ independent edges in $\bar{G}\left[V_{12} \backslash\{u\}, V_{2} \backslash\{v, w\}\right]$, and $\frac{1}{2}$ to edges of $C_{3}$ induced by vertices $u, v$ and $w$. Thus we have $\alpha^{\prime}(\bar{G}) \geq 2 t-s-1+\frac{3}{2}=\frac{n-s}{2}$.

If $n-2 t-(2 t-s)=2$, then we may choose vertices $u, v \in V_{2}$ and assign number 1 to edge $u v$ in $\bar{G}$ and $2 t-s$ independent edges in $\bar{G}\left[V_{12}, V_{2} \backslash\{u, v\}\right]$ for a fractional matching of $\bar{G}$. Thus we have $\alpha^{\prime}(\bar{G}) \geq 2 t-s+1=\frac{n-s}{2}$.

When $n-2 t-(2 t-s) \geq 3$, we may assign number 1 to $2 t-s$ independent edges in $\bar{G}\left[V_{12}, V_{22}\right]$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(2 t-s)$ in $\bar{G}\left[V_{2}\right]$ for a fractional matching of $\bar{G}$. Then we have

$$
\alpha^{\prime}(\bar{G}) \geq 2 t-s+\frac{n-2 t-(2 t-s)}{2}=\frac{n-s}{2} .
$$

Since $s \leq t$, we have $\alpha^{\prime}(\bar{G}) \geq \frac{n-s}{2} \geq \frac{n-t}{2}$.
(2) Now we suppose $s \geq 1$ and $\bar{G}$ contains no isolated vertices. Let $u$ be a vertex in $V_{11}$ and $v$ be one of its neighbours in $\bar{G}$. First we suppose $v \in V_{12}$. Now we define a fractional matching of $\bar{G}$. Let $\bar{f}(u v)=1$ in $\bar{G}$. Noting that $n-2 t-(2 t-s-1) \geq 2$. If $n-2 t-(2 t-s-1)=2$, that is $s=1$ and $\left|V_{12}\right|=\left|V_{22}\right|=2 t-1=\frac{n}{2}-1$, then we may choose vertices $v_{1} \in V_{22}$ and $v_{2} \in V_{21}$ and assign number 1 to edge $v_{1} v_{2}$ in $\bar{G}$ and $2 t-s-1$ independent edges in $\bar{G}\left[V_{12} \backslash\{v\}, V_{22} \backslash\left\{v_{1}\right\}\right]$. Thus we have $\alpha^{\prime}(\bar{G}) \geq 2+2 t-s-1=\frac{n-s+1}{2}$.

When $n-2 t-(2 t-s-1) \geq 3$, we may assign number 1 to $2 t-s-1$ independent edges in $\bar{G}\left[V_{12} \backslash\{v\}, V_{22}\right]$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(2 t-s-1)$ in $\bar{G}\left[V_{2}\right]$ for a fractional matching of $\bar{G}$. Then we have

$$
\alpha^{\prime}(\bar{G}) \geq 1+(2 t-s-1)+\frac{n-2 t-(2 t-s-1)}{2}=\frac{n-s+1}{2} .
$$

Now suppose $v$ lies in $V_{11}$ or $V_{2}$. If $v \in V_{11}$ or $v \in V_{21}$, then $s \geq 2$. When $v \in V_{22}$, noting $G$ contains no isolated vertices, all possible neighbours of $v$ are in $V_{11}$ and $u v \notin E(G)$, then $\left|V_{11}\right|=s \geq 2$. Hence, we have $|X|=\left|V_{11}\right| \geq 2$ and there is an edge $v_{1} v_{2}$ in $\bar{G}[X]$ by Lemma 2.6 (1). Noting that $n-2 t-(2 t-s-2)-1 \geq 1+s \geq 3$. For a fractional matching of $\bar{G}$, we may assign number 1 to the edges $v_{1} v_{2}, u v$ and $2 t-s-2$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{v_{1}, v_{2}\right\}, V_{22} \backslash\{v\}\right]$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(2 t-s-2)-1$ in $\bar{G}\left[V_{2}\right]$. Thus we have

$$
\alpha^{\prime}(\bar{G}) \geq 1+1+(2 t-s-2)+\frac{n-2 t-(2 t-s-2)-1}{2}=\frac{n-s+1}{2} .
$$

(3) Now we suppose that $s=t \geq 3$ and $\bar{G}$ contains no isolated vertices. The assumption $s=t$ implies $X=V_{12}$. By Lemma 2.6, we obtain that $\bar{G}\left[V_{12}\right]=K_{s}$ and each vertex in $V_{12}$ is adjacent to each vertex in $V_{2}$ in $\bar{G}$. Let $u$ be a vertex in $V_{11}$ and $v$ be one of its neighbours in $\bar{G}$. Noting that $n-2 t-(2 t-s) \geq s \geq 3$. Now we define a fractional matching $\bar{f}$ of $\bar{G}$. We suppose $v \in V_{11}$. First let $\bar{f}(u v)=1$ in $\bar{G}$ and we may assign number 1 to $2 t-s$ independent edges in $\bar{G}\left[V_{12}, V_{22}\right]$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(2 t-s)$ in $\bar{G}\left[V_{2}\right]$. Thus we have

$$
\alpha^{\prime}(\bar{G}) \geq 1+2 t-s+\frac{n-2 t-(2 t-s)}{2}=\frac{n-s+2}{2}
$$

If $v \in V_{2}$, then there exists a vertex say $w \neq u \in V_{11}$ such that $w v \in E(G)$. Since $d_{G}(w) \leq n-2$, there exists a vertex $w^{\prime}$ such that $w w^{\prime} \notin E(G)$. If $w^{\prime} \in V_{11}$, it is the case discussed above. If $w^{\prime} \in V_{12}$, then let $\bar{f}(u v)=\bar{f}\left(w w^{\prime}\right)=1$ in $\bar{G}$ and we may assign number 1 to $2 t-s-1$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{w^{\prime}\right\}, V_{2} \backslash\{v\}\right]$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(2 t-s-1)-1$ in $\bar{G}\left[V_{2}\right]$. Thus we have

$$
\alpha^{\prime}(\bar{G}) \geq 2+(2 t-s-1)+\frac{n-2 t-(2 t-s-1)-1}{2}=\frac{n-s+2}{2} .
$$

If $w^{\prime} \in V_{2}$, then let $\bar{f}(u v)=\bar{f}\left(w w^{\prime}\right)=1$ in $\bar{G}$. Since $\bar{G}\left[V_{12}\right]=K_{s}$, there exist $x_{1}, x_{2} \in V_{12}$ such that $x_{1} x_{2} \in E(\bar{G})$. Let $\bar{f}\left(x_{1} x_{2}\right)=1$ in $\bar{G}$ and we may assign number 1 to $2 t-s-2$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{x_{1}, x_{2}\right\}, V_{2} \backslash\left\{v, w^{\prime}\right\}\right]$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-2-(2 t-s-2)$ in $\bar{G}\left[V_{2}\right]$. Thus we have

$$
\alpha^{\prime}(\bar{G}) \geq 3+2 t-s-2+\frac{n-2 t-2-(2 t-s-2)}{2}=\frac{n-s+2}{2} .
$$

When $v \in V_{11}$ or $v \in V_{2}$, we obtain the desired inequality. So in the following, we may assume that $G\left[V_{11}\right]$ is a clique and each vertex in $V_{11}$ is adjacent to each vertex in $V_{2}$ in $G$. We suppose $v$ lies in $V_{12}$. From the definition of set $X\left(=V_{12}\right)$, we have $v^{\prime} v \in E(G)$ for some $v^{\prime} \in V_{11}$. Since $v^{\prime}$ is not an isolated vertex of $\bar{G}$, we may assume $v^{\prime} v^{\prime \prime} \in E(\bar{G})$ for some vertex $v^{\prime \prime} \in V_{12}$. Now we construct a fractional matching $\bar{f}$ of $\bar{G}$ with $\bar{f}(u v)=\bar{f}\left(v^{\prime} v^{\prime \prime}\right)=1$ in $\bar{G}$ and we may assign number 1 to $2 t-s-2$ independent edges in $\left[V_{12} \backslash\left\{v, v^{\prime \prime}\right\}, V_{22}\right]$ in $\bar{G}$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(2 t-s-2)$ in $\bar{G}\left[V_{2}\right]$. Thus we have

$$
\alpha^{\prime}(\bar{G}) \geq 2+2 t-s-2+\frac{n-2 t-(2 t-s-2)}{2}=\frac{n-s+2}{2} .
$$

This completes the proof.

Lemma 4.2. (1) If $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2}$ when $n \equiv 0,1(\bmod 4)$ or $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+\frac{3}{2}$ when $n \equiv 2,3(\bmod 4)$, then $\alpha^{\prime}(\bar{G}) \geq \frac{n}{4}+3$ for $n \geq 28$.
(2) If $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+1$, then $\alpha^{\prime}(\bar{G}) \geq \frac{n}{4}+3$ for $n \geq 28$.

Proof Recall $V(G)=\left(V_{11} \cup V_{12}\right) \cup\left(V_{21} \cup V_{22}\right),\left|V_{11}\right|=s$ and $\alpha^{\prime}(G)=t$. We will prove $\alpha^{\prime}(\bar{G}) \geq \frac{n-t}{2}$. When $s \leq 1$, noting that $\left|V_{1}\right|=2 \alpha^{\prime}(G)>\frac{n}{2}$ and $\left|V_{12}\right|=2 t-s>$ $n-2 t-s=\left|V_{22}\right|$. Then we may assign number 1 to $n-2 t-s$ independent edges in $\bar{G}\left[V_{12}, V_{22}\right]$ for a fractional matching of graph $\bar{G}$. Thus

$$
\alpha^{\prime}(\bar{G}) \geq n-2 t-s \geq \frac{n-t}{2}
$$

for $n \geq 20$.
When $s \geq 2$, there exist two vertices $v_{1}, v_{2} \in X$ such that $v_{1} v_{2} \notin E(G)$ by Lemma 2.6 (1). We will define a fractional matching $\bar{f}$ of $\bar{G}$. Noting that $\left|V_{22}\right| \geq 3$.
(1) If $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2}$ when $n \equiv 0,1(\bmod 4)$ or $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+\frac{3}{2}$ when $n \equiv 2,3(\bmod 4)$, then $s<t$ and there exists a $\frac{1}{2}$-cycle in $V_{12} \backslash X$. By Lemma 2.2 (3), from the $\frac{1}{2}$-cycle of $G$, we may take two different vertices $v^{\prime}, v^{\prime \prime}$. Let $\bar{f}\left(v^{\prime} x^{\prime}\right)=\bar{f}\left(v^{\prime \prime} x^{\prime \prime}\right)=1$ in $\bar{G}$ for $\left\{x^{\prime}, x^{\prime \prime}\right\} \subseteq V_{21}$. There are $s-2$ independent edges in $\bar{G}\left[X \backslash\left\{v_{1}, v_{2}\right\}, V_{21} \backslash\left\{x^{\prime}, x^{\prime \prime}\right\}\right]$ by Lemma 2.6 (2) and we may assign number 1 to them. Noting that $\left|V_{1}\right|-s-2 \leq\left|V_{2}\right|$, we may assign number 1 to $2 t-2 s-2$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{X, v^{\prime}, v^{\prime \prime}\right\}, V_{22}\right]$. When $n-2 t-(2 t-s-2)=0$, we may assign number 1 to edge $v_{1} v_{2}$ in $\bar{G}$. Thus $\alpha^{\prime}(\bar{G}) \geq 1+2 t-s-2=2 t-s-1=\frac{n-s}{2}>\frac{n-t}{2}$.

If $n-2 t-(2 t-s-2)=1$, that is $n=4 t-s-1$ and $\left|V_{22}\right|=2 t-2 s-1$, then we may choose $w \in V_{22}$ and assign number $\frac{1}{2}$ to edges of $C_{3}$ induced by $v_{1}, v_{2}$ and $w$ in $\bar{G}$, and 1 to $2 t-2 s-2$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{X, v^{\prime}, v^{\prime \prime}\right\}, V_{22}\right]$. Thus we have $\alpha^{\prime}(\bar{G}) \geq \frac{3}{2}+2 t-s-2=2 t-s-\frac{1}{2}=\frac{n-s}{2}>\frac{n-t}{2}$.

If $n-2 t-(2 t-s-2)=2$, then we may choose $w, w^{\prime} \in V_{22}$ and assign number 1 to $2 t-2 s-2$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{X, v^{\prime}, v^{\prime \prime}\right\}, V_{22} \backslash\left\{w, w^{\prime}\right\}\right]$. Let $\bar{f}\left(v_{1} v_{2}\right)=$ $\bar{f}\left(w w^{\prime}\right)=1$ in $\bar{G}$. Thus we have $\alpha^{\prime}(\bar{G}) \geq 2+2 t-s-2=2 t-s=\frac{n-s}{2}>\frac{n-t}{2}$.

When $n-2 t-(2 t-s-2) \geq 3$, we may assign number $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(2 t-s-2)$ in $\bar{G}\left[V_{22}\right]$. Let $\bar{f}\left(v_{1} v_{2}\right)=1$ in $\bar{G}$. Thus

$$
\alpha^{\prime}(\bar{G}) \geq 1+2 t-s-2+\frac{n-2 t-(2 t-s-2)}{2}=\frac{n-s}{2}>\frac{n-t}{2} .
$$

(2) When $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+1$ and $s=t$, by Lemma $2.6(2)$, $u v \in E(\bar{G})$ for each vertex $u \in V_{12}$ and $v \in V_{2}$. Since $\left|V_{22}\right| \geq 3$, we may assign number 1 to $s-2$ independent edges in $\left[V_{12} \backslash\left\{v_{1}, v_{2}\right\}, V_{21}\right]$ in $\bar{G}$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(s-2)$ in $\bar{G}\left[V_{2}\right]$. For a fractional matching $\bar{f}$ of $\bar{G}$, let $\bar{f}\left(v_{1} v_{2}\right)=1$ in $\bar{G}$. Thus we have

$$
\alpha^{\prime}(\bar{G}) \geq 1+s-2+\frac{n-2 t-(s-2)}{2}=\frac{n-s}{2}=\frac{n-t}{2} .
$$

When $s<t$ and there exists a $\frac{1}{2}$-cycle in $V_{12} \backslash X$, it can be proved similar to the above in (1). Assume that there is no $\frac{1}{2}-$ cycle in $V_{12} \backslash X$ and there are $p$ edges assigned number 1 in $E\left[V_{12}\right]$, where $p \geq 1$ and $p$ is an integer. Since $f(v)=1$ for every $v \in V_{1}$, $p=0$ is equivalent to $s=t$ or $G\left[V_{12} \backslash X\right]$ is a collection of disjoint $\frac{1}{2}$-cycles, which has been discussed above. If $p=1$, that is $t=s+1$ and $\left|V_{12}\right|=2 t-s=s+2$, then
there exactly exists an edge say $w w_{1}$ assigned number 1 in $G\left[V_{12} \backslash X\right]$. Without loss of generality, there exists a vertex $x$ in $V_{21}$ such that $x w \in E(\bar{G})$ by Lemma 2.5 (1). Since $\left|V_{22}\right| \geq 3$, let $\bar{f}\left(v_{1} v_{2}\right)=\bar{f}(x w)=\bar{f}\left(y_{1} w_{1}\right)=1$ in $\bar{G}, y_{1} \in V_{22}$. Then for each vertex $v \in V_{12} \backslash\left\{v_{1}, v_{2}, w, w_{1}\right\}$ and each vertex $u \in V_{2} \backslash\left\{x, y_{1}\right\}$, we have $u v \in E(\bar{G})$. We may assign number 1 to $s-2$ independent edges in $\left[V_{12} \backslash\left\{v_{1}, v_{2}, w, w_{1}\right\}, V_{21} \backslash\{x\}\right]$, and $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-s$ in $\bar{G}\left[V_{2}\right]$. Thus we have

$$
\alpha^{\prime}(\bar{G}) \geq 1+1+1+(s-2)+\frac{n-2 t-s}{2}=\frac{n-s}{2}>\frac{n-t}{2} .
$$

If $p \geq 2$, then there exist two edges $w_{1} w_{2}$ and $w_{3} w_{4}$ assigned number 1 in $G\left[V_{12} \backslash X\right]$. Without loss of generality, there exist two vertices $x_{1}, x_{2}$ in $V_{21}$ such that $x_{1} w_{1}, x_{2} w_{3} \in$ $E(\bar{G})$ and let $\bar{f}\left(x_{1} w_{1}\right)=\bar{f}\left(x_{2} w_{3}\right)=1$ in $\bar{G}$. In $\bar{G}$, there are $s-2$ independent edges in $\left[X \backslash\left\{v_{1}, v_{2}\right\}, V_{21} \backslash\left\{x_{1}, x_{2}\right\}\right]$ by Lemma 2.6 (2) and we may assign number 1 to them. Noting that $n-2 t-s \geq 2 t-2 s-2$, we may assign number 1 to $2 t-2 s-2$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{X \cup\left\{w_{1}, w_{3}\right\}\right\}, V_{22}\right]$. If $n-2 t-(2 t-s-2)=0$, that is $n=4 t-s-2$ and $\left|V_{22}\right|=2 t-2 s-2$, we may assign number 1 to edge $v_{1} v_{2}$ in $\bar{G}$. Thus we have $\alpha^{\prime}(\bar{G}) \geq 1+2 t-s-2=2 t-s-1=\frac{n-s}{2}>\frac{n-t}{2}$.

If $n-2 t-(2 t-s-2)=1$, that is $n=4 t-s-1$ and $\left|V_{22}\right|=2 t-2 s-1$, then we may choose $y \in V_{22}$ and assign number $\frac{1}{2}$ to edges of $C_{3}$ induced by $v_{1}, v_{2}$ and $y$ in $\bar{G}$, and 1 to $2 t-2 s-2$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{X \cup\left\{w_{1}, w_{3}\right\}\right\}, V_{22} \backslash\{y\}\right]$. Thus we have $\alpha^{\prime}(\bar{G}) \geq \frac{3}{2}+2 t-s-2=2 t-s-\frac{1}{2}=\frac{n-s}{2}>\frac{n-t}{2}$.

If $n-2 t-(2 t-s-2)=2$, that is $n=4 t-s$ and $\left|V_{22}\right|=2 t-2 s$, then we may choose $y_{1}, y_{2} \in V_{22}$ and assign number 1 to edges $v_{1} v_{2}$ and $y_{1} y_{2}$ in $\bar{G}$, and 1 to $2 t-2 s-2$ independent edges in $\bar{G}\left[V_{12} \backslash\left\{X \cup\left\{w_{1}, w_{3}\right\}\right\}, V_{22} \backslash\left\{y_{1}, y_{2}\right\}\right]$. Thus we have $\alpha^{\prime}(\bar{G}) \geq 1+2 t-s-2+1=2 t-s=\frac{n-s}{2}>\frac{n-t}{2}$.

When $n-2 t-(2 t-s-2) \geq 3$, we may assign number $\frac{1}{2}$ to each edge of a cycle with length $n-2 t-(2 t-s-2)$ in $\bar{G}\left[V_{2}\right]$, and 1 to edge $v_{1} v_{2}$ in $\bar{G}$. Thus we have

$$
\alpha^{\prime}(\bar{G}) \geq 1+2 t-s-2+\frac{n-2 t-(2 t-s-2)}{2}=\frac{n-s}{2}>\frac{n-t}{2} .
$$

Therefore, $\alpha^{\prime}(\bar{G}) \geq \frac{n-t}{2} \geq \frac{n}{4}+3$ for $n \geq 28$. This completes the proof.

Without loss of generality, we may assume that $\alpha^{\prime}(G) \leq \alpha^{\prime}(\bar{G})$ as follows.

Theorem 4.3. Let $G$ be a graph of order $n \geq 28$. If both $G$ and $\bar{G}$ are not empty, then

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n+1}{2}
$$

with equality holds if and only if $G \cong K_{1, n-1}$.
Proof Since $G$ is not empty, $\alpha^{\prime}(G) \geq 1$. By Lemma 3.5 (1)-(5), when $1 \leq \alpha^{\prime}(G)<3$, $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n+1}{2}$. The equality holds if and only if $G \cong K_{1, n-1}$ by Lemma 3.5 (1) and Lemma 3.2 (1).

When $3 \leq \alpha^{\prime}(G) \leq \frac{n}{4}$, by Lemma 4.1 (1), we have

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \alpha^{\prime}(G)+\frac{n-\alpha^{\prime}(G)}{2}=\frac{n+\alpha^{\prime}(G)}{2} \geq \frac{n+3}{2}>\frac{n+1}{2} .
$$

Then we only need to consider the case that $\frac{n}{4}<\alpha^{\prime}(G) \leq \frac{n}{2}$.
If $n \equiv 0,1(\bmod 4)$, then $\frac{n+1}{4} \leq\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2} \leq \alpha^{\prime}(G) \leq \frac{n}{2}$. So we have

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq 2 \alpha^{\prime}(G) \geq 2\left(\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2}\right) \geq \frac{n+1}{2} .
$$

The equality holds if and only if $n \equiv 1(\bmod 4)$ and $\alpha^{\prime}(G)=\alpha^{\prime}(\bar{G})=\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2}$, which is a contradiction to Lemma 4.2 (1) in virtue of $\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2}<\frac{n}{4}+3$.

If $n \equiv 2,3(\bmod 4), \frac{n}{4}<\left\lfloor\frac{n}{4}\right\rfloor+1 \leq \alpha^{\prime}(G) \leq \frac{n}{2}$. When $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+1$, by Lemma $4.2(2), \alpha^{\prime}(\bar{G}) \geq \frac{n}{4}+3$. Thus $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq\left\lfloor\frac{n}{4}\right\rfloor+1+\frac{n}{4}+3 \geq \frac{n}{2}+3>\frac{n+1}{2}$. When $\frac{n+2}{4}<\left\lfloor\frac{n}{4}\right\rfloor+\frac{3}{2} \leq \alpha^{\prime}(G) \leq \frac{n}{2}$, we have

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq 2 \alpha^{\prime}(G) \geq 2\left(\left\lfloor\frac{n}{4}\right\rfloor+\frac{3}{2}\right)>\frac{n+2}{2}>\frac{n+1}{2} .
$$

Therefore, $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G})>\frac{n+1}{2}$. This completes the proof.
The following result is implied by Theorem 4.3.
Lemma 4.4. For a graph of order $n, n \geq 2$, we have

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n}{2}
$$

with equality holds if and only if $G$ is an empty graph or $G$ is a complete graph.
The following result is deduced by Lemma 3.3, Lemma 3.4 and Lemma 3.5.
Lemma 4.5. If $\alpha^{\prime}(G)=2$ or $\alpha^{\prime}(G)=\frac{5}{2}, \bar{G}$ contains no isolated vertices, then $\alpha^{\prime}(\bar{G})=$ $\frac{n}{2}$ for $n \geq 10$.

Theorem 4.6. If both $G$ and $\bar{G}$ contain no isolated vertices and $n \geq 28$. Then

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n+4}{2}
$$

and the equality holds if and only if $G \cong K_{2}(p, q ; \ell)$ or $K_{1, m} \cup K_{1, n-2-m} \subseteq G \subseteq K_{2, n-2}$, where $p, q, \ell, m$ are non-negative integers, $q \geq 1$ and $1 \leq p, m \leq n-3$.

Proof Since both $G$ and $\bar{G}$ contain no isolated vertices, by Lemma 3.2, we have $\alpha^{\prime}(G)$ and $\alpha^{\prime}(\bar{G}) \geq 2$. When $s=0$, since $G$ contains no isolated vertices, $V_{2}$ is empty and $G\left[V_{1}\right]$ contains a fractional perfect matching by Lemma 2.4. The assumption $\alpha^{\prime}(G) \leq \alpha^{\prime}(\bar{G})$ implies $\alpha^{\prime}(\bar{G})=\alpha^{\prime}(G)=\frac{n}{2}$. Thus $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G})=\frac{n}{2}+\frac{n}{2}=n>\frac{n+4}{2}$.

When $\alpha^{\prime}(G)=2$ or $\frac{5}{2}$, by Lemma 4.5, $\alpha^{\prime}(\bar{G})=\frac{n}{2}$. It follows that $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{n+4}{2}$. The equality holds when $\alpha^{\prime}(G)=2$.

When $\alpha^{\prime}(G)=3$, if $s=1$ or 2 , we have $\alpha^{\prime}(\bar{G}) \geq \frac{n-s+1}{2}$ by Lemma 4.1 (2). It follows that $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq 3+\frac{n-s+1}{2}>\frac{n+4}{2}$. If $s=t=3$, we have $\alpha^{\prime}(\bar{G}) \geq \frac{n-3}{2}+1=\frac{n-1}{2}$ by Lemma 4.1 (3), and then $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq 3+\frac{n-1}{2}=\frac{n+5}{2}>\frac{n+4}{2}$.

When $\alpha^{\prime}(G)=\frac{7}{2}$, then $1 \leq s \leq 3$. By Lemma 4.1 (2), we have $\alpha^{\prime}(\bar{G}) \geq \frac{n-s+1}{2}$. It follows that $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq \frac{7}{2}+\frac{n-s+1}{2} \geq \frac{7}{2}+\frac{n-3+1}{2}=\frac{n+5}{2}>\frac{n+4}{2}$.

When $4 \leq \alpha^{\prime}(G)=t \leq \frac{n}{4}$, since $s \geq 1, \alpha^{\prime}(\bar{G}) \geq \frac{n-s+1}{2}$ by Lemma 4.1 (2). Thus

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq t+\frac{n-s+1}{2} \geq t+\frac{n-t+1}{2}=\frac{n+t+1}{2} \geq \frac{n+5}{2}>\frac{n+4}{2} .
$$

Then we only need to consider the case that $\frac{n}{4}<\alpha^{\prime}(G) \leq \frac{n}{2} \leq \alpha^{\prime}(\bar{G})$.
If $n \equiv 0,1(\bmod 4)$, then $\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2} \leq \alpha^{\prime}(G) \leq \frac{n}{2}$. When $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2}$ or $\left\lfloor\frac{n}{4}\right\rfloor+1$, by Lemma $4.2, \alpha^{\prime}(\bar{G}) \geq \frac{n}{4}+3$. Thus

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq\left\lfloor\frac{n}{4}\right\rfloor+\frac{1}{2}+\frac{n}{4}+3>\frac{n}{2}+3>\frac{n+4}{2} .
$$

When $\alpha^{\prime}(G) \geq\left\lfloor\frac{n}{4}\right\rfloor+\frac{3}{2}>\frac{n+4}{4}$, we obtain $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq 2 \alpha^{\prime}(G) \geq 2\left\lfloor\frac{n}{4}\right\rfloor+3>\frac{n+4}{2}$.
If $n \equiv 2,3(\bmod 4),\left\lfloor\frac{n}{4}\right\rfloor+1 \leq \alpha^{\prime}(G) \leq \frac{n}{2}$. When $\alpha^{\prime}(G)=\left\lfloor\frac{n}{4}\right\rfloor+1$ or $\left\lfloor\frac{n}{4}\right\rfloor+\frac{3}{2}$, by Lemma 4.2, $\alpha^{\prime}(\bar{G}) \geq \frac{n}{4}+3$. Thus we have $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq\left\lfloor\frac{n}{4}\right\rfloor+1+\frac{n}{4}+3>\frac{n}{2}+3>\frac{n+4}{2}$. When $\alpha^{\prime}(G) \geq\left\lfloor\frac{n}{4}\right\rfloor+2>\frac{n+4}{4}$, we have

$$
\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G}) \geq 2 \alpha^{\prime}(G) \geq 2\left(\left\lfloor\frac{n}{4}\right\rfloor+2\right)>\frac{n+4}{2} .
$$

Therefore, $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G})>\frac{n+4}{2}$.
If $\alpha^{\prime}(G)+\alpha^{\prime}(\bar{G})=\frac{n+4}{2}$, then $\alpha^{\prime}(G)=2$. By Lemma 3.3 and both $G$ and $\bar{G}$ contain no isolated vertices, we have $G \cong K_{2}(p, q ; \ell)$ or $K_{1, m} \cup K_{1, n-2-m} \subseteq G \subseteq K_{2, n-2}$, where $p, q, \ell, m$ are non-negative integers, $q \geq 1$ and $1 \leq p, m \leq n-3$. Therefore, we complete the proof.

## References

[1] M. Aouchiche, P. Hansen, A survey of Nordhaus-Gaddum type relations. Discrete Appl. Math. 161(2013): 466-546.
[2] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, Elsevier North Holland, New York, 1976.
[3] G. Chartrand, S. Schuster, On the independence number of complementary graphs. Trans NY Acad Sci Ser II. 36(1974): 247-251.
[4] I. Choi, J. Kim, S. O, The difference and ratio of the fractional matching number and the matching number of graphs, Discrete Math. 339(4)(2016): 1382-1386.
[5] W. Goddard, M. A. Henning, H. C. Swart, Some Nordhaus-Gaddum-type results, J. Graph Theory. 16(3) (1992): 221-231.
[6] K. Huang, K. Lih, Nordhaus-Gaddum-type relations of three graph coloring parameters. Discrete Appl. Math. 162(2014): 404-408.
[7] R. Laskar, B. Auerbach, On complementary graphs with no isolated vertices. Discrete Math. 24(1978): 113-118.
[8] D. Li, B. Wu, X. Yang, X. An, Nordhaus-Gaddum-type theorem for Wiener index of graphs when decomposing into three parts. Discrete Appl. Math. 159(2011): 1594-1600.
[9] X. Li, Y. Mao, Nordhaus-Gaddum-type results for the generalized edge-connectivi--ty of graphs. Discrete Appl. Math. 185(2015): 102-112.
[10] H. Lin, J. Shu, B. Wu, Nordhaus-Gaddum type result for the matching number of a graph. J Comb Optim. 34(2017): 916-930.
[11] E. Nordhaus, J. Gaddum, On complementary graphs, Am. Math. Mon. 63 (1956): 175-177.
[12] E. Shan, C. Dang, L. Kang, A note on Nordhaus-Gaddum inequalities for domination. Discrete Appl. Math. 136(2004): 83-85.
[13] E. R. Scheinerman, D. H. Ullman, Fractional Graph Theory: A Rational Approach to the Theory of Graphs, Wiley\&Sons, 2008.
[14] G. Su, L. Xiong, Y. Sun, D. Li, Nordhaus-Gaddum-type inequality for the hyperWiener index of graphs when decomposing into three parts. Theoretical Computer Science. 471(2013): 74-83.


[^0]:    *This work was supported by the National Nature Science Foundation of China (Nos. 11871040).
    *Corresponding author. Email: xiyingyuan@shu.edu.cn (Xiying Yuan).
    Email address: yangting_2019@shu.edu.cn (Ting Yang)

