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Discrete Mathematics 281 (2004) 43–66

DISCRETE MATHEMATICS

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New bounds for $n_4(k, d)$ and classification of some optimal codes over GF(4)

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Received 10 December 2002; received in revised form 5 November 2003; accepted 11 November 2003

Abstract

Let $n_4(k, d)$ be the minimum length of a linear $[n, k, d]$ code over GF(4) for given values of k and d. For codes of dimension five, we compute the exact values of $n_4(5, d)$ for 75 previously open cases. Additionally, we show that $n_4(6, 14) = 24$, $n_4(7, 9) = 18$, and $n_4(7, 10) = 20$. Moreover, we classify optimal quaternary codes for some values of n and k.

Keywords: Quaternary linear codes; Bounds on codes; Optimal codes

1. Introduction

A central problem in coding theory is to optimize the parameters of error-correcting codes, in particular, linear codes. A q-ary linear $[n, k, d; q]$ code is a k-dimensional linear subspace of $\mathbb{F}_q^n$ with minimum distance d. Here by $\mathbb{F}_q^n$ we denote the vector space of dimension n over the Galois field $\mathbb{F}_q = \mathrm{GF}(q)$. The minimum distance of a code is the minimum Hamming distance between two vectors, i.e., the number of coordinates in

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[1] The work of I. Bouyukliev and Z. Varbanov was partially supported by the Bulgarian National Science Fund under Contract No. MM-1304/2003.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2003.11.003

which they differ. For linear codes, we consider the problem of optimizing one of the parameters n, k, d for given values of the other two. Two versions are:

Problem 1. Find $d_q(n,k)$, the largest value of d for which there exists an $[n,k,d;\text{q}]$ code.

Problem 2. Find $n_q(k,d)$, the smallest value of n for which there exists an $[n,k,d;q]$ code.

A code which achieves one of these two values is called *optimal*. Note that a code can be optimal in the sense that its minimum distance is maximal for the given length and dimension, or that its length is minimal for the given dimension and minimum distance.

Bounds for $d_q(n,k)$ have been published in Brouwer's tables [7]. In this paper, we concentrate on the second problem. A lower bound on $n_q(k,d)$ is the Griesmer bound [18,32] given by

$$n_q(k,d) \geqslant \sum_{i=0}^{k-1} \lceil d/q^i \rceil =: g_q(k,d). \tag{1}$$

For fixed k and sufficiently large d, the lower bound is achieved, i.e., there is a constant $D_0(k)$ such that $n_q(k,d) = g_q(k,d)$ for $d \geqslant D_0(k)$ [1].

Until now, the following exact values for the function $n_q(k,d)$ have been known: Bouyukliev, et al. completed the problem for $n_2(k,d)$ when $k \leqslant 8$ [5]. Landjev solved the last unknown cases for $q=3$ and $k=5$ [24]. The quaternary case was considered, e.g., in [2,9,14,15,17,21,23,25,26]. Recent results on $n_4(5,d)$ can be found in [25,28].

In this paper, we investigate quaternary linear codes. We found 25 new codes and proved the non-existence of four codes, using different approaches. Our main tools are an algorithm for the construction of codes using their residual codes, the dual transform of linear codes, and a heuristic algorithm. Our results give 75 new exact values for $n_4(5,d)$. Additionally, we present some new results on the classification of optimal linear codes over GF(4) (see also the results related to geometrical constructions [21], near-MDS-codes [11], and the results of [22,29]).

2. Our tools

2.1. Q-EXTENSION

The program Q-EXTENSION contains two main approaches to construct new codes from a given code. The first one is based on puncturing, the second one on shortening. While in general the dimension of a code is unchanged by puncturing, this is not true if all non-zero positions of a codeword are deleted. Let G be a generator matrix of a linear $[n,k,d;q]$ code C. Then the *residual code* $\text{Res}(C,c)$ of C with respect to a codeword c is the code generated by the restriction of G to the columns where c

has a zero entry. A lower bound on the minimum distance of the residual code is given by

Lemma 1 (Dodunekov [13]). *Suppose C is an $[n,k,d]$-code over $GF(q)$ and suppose $c \in C$ has weight w, where $d > w(q-1)/q$. Then* $\mathrm{Res}(C,c)$ *is an $[n-w,k-1,d']$-code with $d' \geqslant d - w + \lceil w/q \rceil$.*

Inverting this operation, we search for an $[n,k,d]$ code on the basis of an $[n-w,k-1,d']$ code (its residual code) or an $[n-i,k,d']$ code (punctured code). Starting from the code $[3,2,2]$, we obtain for example the codes $[8,3,5]$ and then $[28,4,20]$.

The second approach increases both the length and the dimension of the code, i.e., we construct an $[n+i,k+i,d]$ or an $[n+i+1,k+i,d]$ code starting from an $[n,k,d']$ code. If G is a generator matrix for an $[n,k,d']$ code, the generator matrix of the new code is of the form

$$\left(\begin{array}{c|c} * & I_i \\ \hline G & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} * & 1_i \; I_i \\ \hline G & 0 \end{array}\right),$$

where I_i denotes an $i \times i$ identity matrix and by 1_i we denote a column vector with i ones. We take the matrix G in systematic form, thus we can fix k additional columns. More information on this topic can be found in [6].

2.2. Heuristic search for good linear codes

Let $\mathscr{S}_{k,q}$ be the set of all column vectors $a = (a_1, a_2, \ldots, a_k)^t \in \mathbb{F}_q^k$ such that either $a_1 = 1$ or $a_1 = a_2 = \cdots = a_{i-1} = 0$, $a_i = 1$ for some integer i in $\{2, 3, \ldots, k\}$, where $k \geqslant 3$ and $\mathbb{F}_q^k$ is the vector space of dimension k over $\mathbb{F}_q$. Then $\mathscr{S}_{k,q}$ consists of all $(q^k-1)/(q-1)$ normalized non-zero vectors in $\mathbb{F}_q^k$ and the vectors in $\mathscr{S}_{k,q}$ can be regarded as $(q^k-1)/(q-1)$ points in a finite projective geometry $\mathrm{PG}(k-1,q)$.

A linear code is called *projective* if no two columns of a generator matrix are linearly dependent. We consider the columns of the generator matrix of a projective code as a subset of $\mathscr{S}_{k,q}$. For any n-element subset of $\mathscr{S}_{k,q}$ there is a projective $[n,k_0,d;q]$ code ($k_0 \leqslant k$) with a generator matrix containing these elements as columns. On the other hand, for a code C we denote by $T(C)$ the set of the columns of the generator matrix of C, considered as elements of $\mathscr{S}_{k,q}$. Then the problem to construct a projective linear $[n,k,d;q]$ code (if such a code exists) can be stated in the following way:

> Find n elements of $\mathscr{S}_{k,q}$ such that the corresponding $k \times n$ matrix generates an $[n,k,d;q]$ code.

We use the following notations:

(1) Each n-element subset S of $\mathscr{S}_{k,q}$ is called *a solution*.
(2) The substitution of a given element of a solution S by another element of $\mathscr{S}_{k,q} \setminus S$ is called *an elementary transformation*.

(3) *The neighborhood* $N(S)$ of a solution S is the set of all solutions S' that can be obtained from S by an elementary transformation.
(4) We define *the evaluation function* by

$$f(S) = d_{\min}(S)q^k - A_{d_{\min}}(S),$$

where $d_{\min}(S)$ is the minimum distance of the corresponding code and $A_{d_{\min}}(S)$ is the number of codewords of minimum weight in the corresponding code.

The problem to construct an $[n,k,d;q]$ code (if such a code exists) can be reformulated as the following combinatorial optimization problem:

$$\max\{f(S), S \subset \mathscr{S}_{k,q}, |S| = n\}.$$

To find a solution of this problem, we use the following heuristic algorithm (similar local optimization algorithms have been used e.g. in [19,20,30,31]):

```
variables
  S, S': solution;
  Number_of_Descents: integer;
begin
  Get_Initial_Solution(S);
  for Number_of_Descents:= 1 to CONST1 do
    while exists S' in N(S) such that f(S') > f(S) do
      S := S';
    end while;
    if d_min(S) >= d_target then
      Output_Solution(S)
    else
      S:= Add_Noise(S);
    end if;
  end for;
end.
```

The initial solution S can be chosen in several ways

(1) By using a code C' with parameters $[n-l,k,d-\delta;q]$ where $\delta \leqslant l$. In this case l elements from $\mathscr{S}_{k,q}$, not belonging to $T(C')$ are randomly added to $T(C')$.
(2) By using a code C' with parameters $[n+l,k,d+l-1;q]$. In this case l elements of $T(C')$ are randomly deleted.

The function Add_Noise(S) replaces m ($1 \leqslant m \leqslant$ CONST) random elements of S by m other, randomly chosen elements of $\mathscr{S}_{k,q} \setminus S$.

This algorithm can not only be used to construct optimal projective codes. If $n > q^{k-1}$, we just take several copies of the set $\mathscr{S}_{k,q}$. For all codes in this paper constructed by this method, the set $\mathscr{S}_{k,q}$ is taken twice.

Some other codes were found by random search using the computer algebra system MAGMA [3].

2.3. Dual transform

Below we describe a construction by Brouwer and van Eupen [8], which in some cases produces optimal codes. Let $\mathscr{C}$ be a projective $[n,k,d;q]$ code with non-zero weights $w_1, w_2, \ldots, w_s$ and let $\mathscr{D}$ be a subcode of dimension $k-1$. By $B_i(\mathscr{D})$, we denote the number of codewords in $\mathscr{D}$ of weight i divided by $q-1$. Let further $n_{\mathscr{D}}$ be the number of coordinate positions where not all codewords of $\mathscr{D}$ are zero. One has

$$\sum_{i=1}^{s} B_i(\mathscr{D}) = \frac{q^{k-1}-1}{q-1}$$

and

$$\sum_{i=1}^{s} (n_{\mathscr{D}} - w_i)B_i(\mathscr{D}) = n_{\mathscr{D}} \frac{q^{k-2}-1}{q-1}.$$

Since $\mathscr{C}$ is projective, we have $n_{\mathscr{D}} = n-1$ for n subcodes of $\mathscr{C}$ and $n_{\mathscr{D}} = n$ for the remaining $(q^{k-1}-1)/(q-1)-n$ subcodes of dimension $k-1$. It follows that for an arbitrary choice of α and β the sum

$$\sum_{i=1}^{s} (\alpha w_i + \beta)B_i(\mathscr{D}) = \alpha q^{k-2} n_{\mathscr{D}} + \beta \frac{q^{k-1}-1}{q-1}$$

does not depend on $\mathscr{D}$, but only on $n_{\mathscr{D}}$ and hence it takes only two values. We fix α and β in such a way that $\alpha w_i + \beta$ are all non-negative integers. Then consider the multiset $\mathscr{X}$ of n-tuples defined in the following way: for each one-dimensional space $\langle \mathbf{c} \rangle \in \mathscr{C}$ with $\mathrm{wt}(\mathbf{c}) = w$ take $\alpha w + \beta$ copies of the vector $\mathbf{c}$. Fixing a generator matrix of C, every such vector can be considered as a point in the projective space $PG(k-1,q)$. Each hyperplane in $\mathrm{PG}(k-1,q)$ corresponds to a $k-1$ dimensional subspace $\mathscr{D}$ of $\mathscr{C}$ and contains $\sum_{i=1}^{s} (\alpha w_i + \beta)B_i(\mathscr{D})$ points of $\mathscr{X}$.

Let $G_{\mathscr{X}}$ be a $k \times |\mathscr{X}|$ matrix having the vectors of $\mathscr{X}$ as its columns and consider the code $\mathscr{C}_{\mathscr{X}}$ generated by the rows of $G_{\mathscr{X}}$. Obviously, its length is

$$N = |\mathscr{X}| = \sum_{i=1}^{s} (\alpha w_i + \beta) \frac{A_i}{q-1} = \beta \frac{q^{k-1}-1}{q-1} + q^{k-1}\alpha n,$$

where $(A_0, A_1, \ldots, A_n)$ is the weight distribution of $\mathscr{C}$. Its dimension is $K \leqslant k$. It follows that $\mathscr{C}_{\mathscr{X}}$ is a two-weight code of minimum distance

$$D = |\mathscr{X}| - \beta \frac{q^{k-1}-1}{q-1} - q^{k-2}\max(\alpha n, \alpha(n-1)),$$

with non-zero weights

$$v = |\mathscr{X}| - \frac{|\mathscr{X}| - \beta}{q} \quad \text{and} \quad u = v + \alpha q^{k-2}.$$

Generalizations of this construction can be found in [12].

3. New bounds for $n_4(k, d)$

Using the program Q-EXTENSION, we have proved the non-existence of several codes with given parameters and in this way we have found new lower bounds for the function $n_4(k,d)$. Most of the lower bounds in this section are known from [7,28].

Theorem 2. *$n_4(5,17) = 27$ and $n_4(5,21) = 32$.*

Proof. The Griesmer bound yields $n_4(5,17) \geqslant g_4(5,17) = 26$. There exist exactly four non-equivalent $[9,4,5]$ codes over GF(4). Using Q-EXTENSION we prove that none of those codes can be extended to a $[26,5,17;4]$ code, but a code $[27,5,17;4]$ exists. Hence $n_4(5,17) = 27$.

The Griesmer bound yields $n_4(5,21) \geqslant g_4(5,21) = 31$. There exist exactly two inequivalent $[10,4,6;4]$ codes. Using Q-EXTENSION we see that neither of them can be extended to a $[31,5,21;4]$ code, but a code $[32,5,21;4]$ exists. Hence $n_4(5,21) = 32$. □

Theorem 3. *$n_4(6,14) = 24$.*

Proof. The Griesmer bound yields $n_4(6,14) \geqslant g_4(6,14) = 22$. The residue code of a code $[22,6,14;4]$ would be a code $[8,5,4;4]$ which does not exist. Hence $n_4(6,14) \geqslant 23$. There exist exactly 19 optimal $[9,5,4]$ codes over GF(4). Using Q-EXTENSION we obtain that none of these codes can be extended to a $[23,6,14;4]$ code, and hence $n_4(6,14) = 24$, as a code $[24,6,14;4]$ exists. □

Theorem 4. *$n_4(7,9) = 18$ and $n_4(7,10) = 20$.*

Proof. The Griesmer bound yields $n_4(7,9) \geqslant g_4(7,9) = 17$. The residue code of a code $[17,7,9;4]$ would be an MDS code $[8,6,3;4]$. Its dual would be an MDS code $[8,2,7;4]$ which does not exist as $g_4(2,7) = 9$. Hence $n_4(7,9) \geqslant 18$. The dual distance (the minimum distance of the dual code) of an $[18,7,9;4]$ code must be at least 4 because a code with parameters $[15,5,9;4]$ does not exist. There exist exactly 4 optimal $[9,6,3;4]$ codes with dual distance 4 and 13 optimal $[9,6,3;4]$ codes with dual distance $d \geq 5$. None of the codes with dual distance 4 can be extended to an $[18,7,9;4]$ code, but the extension of a code with dual distance 5 yields an $[18,7,9;4]$ code with generator matrix

$$\begin{pmatrix} 130121231010000000 \\ 133203320031200000 \\ 303332320310100000 \\ 131030320010011000 \\ 130001110020230300 \\ 301202120000230010 \\ 302110010030220003 \end{pmatrix}.$$

(Here and throughout the paper we use $\{0,1,2,3\}$ to denote the elements of $\mathbb{F}_4$.) Its weight enumerator is $1+393z^9+666z^{10}+1245z^{11}+2193z^{12}+3315z^{13}+3597z^{14}+2799z^{15}+1554z^{16}+504z^{17}+117z^{18}$.

For $n_4(7,10)$, the Griesmer bound yields $n_4(7,10)\geqslant g_4(7,10)=18$, but via the residue code, we get $n_4(7,10)\geqslant 19$. Using Q-EXTENSION we obtain exactly three non-equivalent $[18,6,10;4]$ codes, but none of these codes can be extended to a $[19,7,10;4]$ code, therefore $n_4(7,10)=20$, as a $[20,7,10;4]$ code exists. □

The results in Theorem 5 are obtained by the heuristic search described in Section 2.2, in combination with previously known lower bounds.

Theorem 5. *$n_4(5,24)=35$, $n_4(5,50)=70$, $n_4(5,74)=102$, $n_4(5,156)=210$, $n_4(5,164)=221$, $n_4(5,168)=226$, $n_4(5,172)=231$, $n_4(5,260)=349$, $n_4(5,264)=354$, $n_4(5,268)=359$, $n_4(5,300)=402$, $n_4(5,316)=423$, $n_4(5,356)=477$, $n_4(5,360)=482$, and $n_4(5,364)=487$.*

Proof. There exist codes with parameters: $[35,5,24]$, $[70,5,50]$, $[102,5,74]$, $[210,5,156]$, $[221,5,164]$, $[226,5,168]$, $[231,5,172]$, $[349,5,260]$, $[354,5,264]$, $[359,5,268]$, $[402,5,300]$, $[423,5,316]$, $[477,5,356]$, $[482,5,360]$, $[487,5,364]$.

The weight enumerators of the codes are shown in Table 1, and generator matrices are given in Appendix A. □

The following results are obtained by the dual transform described in Section 2.3, again in combination with previously known lower bounds.

Theorem 6. *$n_4(5,144)=194$, $n_4(5,160)=215$, $n_4(5,176)=236$, $n_4(5,272)=364$, $n_4(5,288)\leqslant 386$, $n_4(5,304)=407$, and $n_4(5,320)=428$.*

Table 1
Weight enumerators of the codes in Theorem 5

Code	Weight enumerator
$[35,5,24]$	$1+504z^{24}+456z^{28}+63z^{32}$
$[70,5,50]$	$1+462z^{50}+315z^{52}+15z^{56}+210z^{58}+21z^{60}$
$[102,5,74]$	$1+489z^{74}+306z^{76}+15z^{80}+162z^{82}+30z^{84}+21z^{90}$
$[210,5,156]$	$1+870z^{156}+63z^{160}+90z^{172}$,
$[221,5,164]$	$1+768z^{164}+171z^{168}+9z^{172}+75z^{180}$
$[226,5,168]$	$1+825z^{168}+120z^{172}+3z^{176}+75z^{184}$
$[231,5,172]$	$1+885z^{172}+63z^{176}+75z^{188}$
$[349,5,260]$	$1+765z^{260}+168z^{264}+9z^{268}+78z^{276}+3z^{280}$
$[354,5,264]$	$1+819z^{264}+120z^{268}+3z^{272}+81z^{280}$
$[359,5,268]$	$1+879z^{268}+63z^{272}+81z^{284}$
$[402,5,300]$	$1+867z^{300}+57z^{304}+93z^{316}+6z^{320}$
$[423,5,316]$	$1+879z^{316}+63z^{320}+78z^{332}+3z^{348}$
$[477,5,356]$	$1+774z^{356}+153z^{360}+9z^{364}+69z^{372}+18z^{376}$
$[482,5,360]$	$1+825z^{360}+108z^{364}+3z^{368}+75z^{376}+12z^{380}$
$[487,5,364]$	$1+879z^{364}+57z^{368}+81z^{380}+6z^{384}$

Proof. There exist projective [35, 5, 24; 4] and [38, 5, 24; 4] codes with weights 24, 28, 32 and generator matrices

$$\begin{pmatrix} 02332211332200113300112211313000000 \\ 12332200223333220022330023112200000 \\ 32202101131321021103301011022031000 \\ 02110033110022331122330011313000220 \\ 32323021202332213113233001133030201 \end{pmatrix}$$

$$\begin{pmatrix} 20220000110113311111333222331100000000 \\ 22222111003220033333111000113320000000 \\ 20313133101003211122320032233221210000 \\ 20102011210110322200032012030310103330 \\ 00033223023320332123303032011133101302 \end{pmatrix}$$

Their weight enumerators are:

$$1 + 504z^{24} + 456z^{28} + 63z^{32}$$
$$1 + 144z^{24} + 600z^{28} + 279z^{32}$$

Let $\alpha = \frac{1}{4}$ and $\beta = -6$. Using dual transform, we obtain a two-weight codes with parameters [194, 5, 144; 4] and [386, 5, 288; 4], respectively.

There exist projective [30, 5, 20; 4] and [33, 5, 20; 4] codes with weights 20, 24, 28 and generator matrices

$$\begin{pmatrix} 000000000011122222222233333333 \\ 000001223301200111223001112233 \\ 000130231201033013013021221302 \\ 011010110101023333332232222323 \\ 202000011201310313303133023230 \end{pmatrix}$$

$$\begin{pmatrix} 102101123100133302021122330000000 \\ 122110030233311120203300112000000 \\ 003101033001103213002103123221000 \\ 202132120130220302303131202110330 \\ 203101212220003013303003011010102 \end{pmatrix}$$

Their weight enumerators are:

$$1 + 423z^{20} + 555z^{24} + 45z^{28}$$

$$1 + 99z^{20} + 627z^{24} + 297z^{28}$$

Let $\alpha = \frac{1}{4}$ and $\beta = -5$. Using dual transform, we obtain a two-weight codes with parameters $[215, 5, 160; 4]$ and $[407, 5, 304; 4]$, respectively.

There exist projective $[25, 5, 16; 4]$, $[27, 5, 16; 4]$, and $[28, 5, 16; 4]$ codes with weights 16, 20, 24 and generator matrices

$$\begin{pmatrix} 2101232330203322022200000 \\ 3131012123222110201000000 \\ 0301322313001300231131000 \\ 2301010112021100022220220 \\ 1101320020023130332100102 \end{pmatrix} \quad \begin{pmatrix} 120130120233111130013000000 \\ 333122003133301021103000000 \\ 101030033131011031002111000 \\ 022300030133301033032200130 \\ 002333331001301030001130203 \end{pmatrix}$$

$$\begin{pmatrix} 3310312310000031131231000000 \\ 0032320311122100332031000000 \\ 1320301231201300011000321000 \\ 1200012132302100000132030310 \\ 3320111102012220120303310201 \end{pmatrix}.$$

Their weight enumerators are:

$$1 + 345z^{16} + 648z^{20} + 30z^{24}$$

$$1 + 117z^{16} + 720z^{20} + 186z^{24}$$

$$1 + 57z^{16} + 648z^{20} + 318z^{24}$$

Let $\alpha=\frac{1}{4}$ and $\beta=-4$. Using dual transform, we obtain two-weight codes with parameters $[236, 5, 172; 4]$, $[364, 5, 272; 4]$, $[428, 5, 320; 4]$,respectively. $\square$

Remark 7. There exist codes with parameters $[210, 5, 156; 4]$, $[215, 5, 160; 4]$, and $[256, 5, 192; 4]$. Pasting these codes together, we obtain codes with parameters $[466, 5, 348; 4]$ and $[471, 5, 352; 4]$.

Remark 8. Obviously, if a code with parameters $[n, k, d]$ does not exist, then there are no codes with parameters $[n + 1, k, d + 1]$ and $[n + 1, k + 1, d]$. Using Theorems 2 and 3 we obtain

$n_4(5, 18)=28$, $n_4(6, 15)=25$, $n_4(6, 17)=28$, $n_4(6, 18) \geqslant 29$, $n_4(7, 14) \geqslant 25$, $n_4(7, 17) \geqslant 29$, $n_4(8, 14) \geqslant 26$, $n_4(8, 15) \geqslant 27$, $n_4(9, 14) \geqslant 27$, $n_4(9, 15) \geqslant 28$, $n_4(10, 14) \geqslant 28$, and $n_4(10, 15) \geqslant 29$.

All the bounds derived from Theorem 5 and Theorem 6 are summarized in Table 2. Furthermore, the following results have been included:

Remark 9. In [10], Dissett has shown that there exists a two-weight projective code with parameters $[99, 5, 72]$ and weights $w_1 = 72$ and $w_2 = 80$.

Remark 10. Recently in [26,16] some new bounds for $n_4(5, d)$ have been found.

Table 2
Values and bounds comparing the Griesmer bound $g_4(5, d)$ and $n_4(5, d)$. Entries in boldface are given in this paper. For missing entries, $n_4(5, d) = g_4(5, d)$

d	$g_4(5,d)$	$n_4(5,d)$	d	$g_4(5,d)$	$n_4(5,d)$	d	$g_4(5,d)$	$n_4(5,d)$
1	5	5	57	78	79–81	113	153	154–155
2	6	6	58	79	80–82	114	154	155–156
3	7	8	59	80	81–83	115	155	156–157
4	8	9	60	81	82–84	116	156	157–158
5	10	10	61	83	84–85	117	158	159–160
6	11	11	62	84	85–86	118	159	160–161
7	12	13	63	85	87	119	160	161–162
8	13	14	64	86	88	120	161	162–163
9	15	16	65	90	90–91	121	163	164
10	16	17	66	91	91–92	122	164	165
11	17	19	67	92	92–93	123	165	166–167
12	18	20	68	93	93–94	124	166	167–168
13	20	21	69	95	95–96	125	168	169
14	21	22	70	96	96–97	126	169	170
15	22	23	71	97	98	127	170	171
16	23	24	72	98	99	128	171	172
17	26	**27**	73	100	**101**	129	175	175–176
18	27	**28**	74	101	**102**	130	176	176–177
19	28	29	75	102	103–104	131	177	177–178
20	29	30	76	103	104–105	132	178	178–179
21	31	**32**	77	105	106	133	180	180–181
22	32	33	78	106	107	134	181	181–182
23	33	**34**	79	107	108	135	182	182–183
24	34	**35**	80	108	109	136	183	183–184
25	36	37	81	111	111–112	137	185	185–186
26	37	38	82	112	112–113	138	186	186–187
27	38	39	83	113	113–114	139	187	188
28	39	40	84	114	114–115	140	188	189
29	41	42	85	116	117	141	190	**191**
30	42	43	86	117	118	142	191	**192**
31	43	44–45	87	118	119	143	192	**193**
32	44	46	88	119	120	144	193	**194**
33	47	48	89	121	122–123	145	196	197
34	48	49	90	122	123–124	146	197	198
35	49	50	91	123	124–125	147	198	199
36	50	51	92	124	125–126	148	199	200
37	52	53–54	93	126	127–128	149	201	202
38	53	54–55	94	127	128–129	150	202	203
39	54	55–56	95	128	129–130	151	203	204

Table 2 (*continued*)

d	$g_4(5,d)$	$n_4(5,d)$	d	$g_4(5,d)$	$n_4(5,d)$	d	$g_4(5,d)$	$n_4(5,d)$
40	55	56–57	96	129	130–131	152	204	205
41	57	58–59	97	132	133–134	153	206	**207**
42	58	59–60	98	133	134–135	154	207	**208**
43	59	60–61	99	134	135–136	155	208	**209**
44	60	61–62	100	135	136–137	156	209	**210**
45	62	63–64	101	137	138	157	211	**212**
46	63	64–65	102	138	139–140	158	212	**213**
47	64	65–66	103	139	140–141	159	213	**214**
48	65	66–67	104	140	141–142	160	214	**215**
49	68	**69**	105	142	143–144	161	217	218
50	69	**70**	106	143	144–145	162	218	219
51	70	71–72	107	144	145–146	163	219	**220**
52	71	72–73	108	145	146–147	164	220	**221**
53	73	74	109	147	148–149	165	222	**223**
54	74	75	110	148	149–150	166	223	**224**
55	75	76	111	149	150–151	167	224	**225**
56	76	77	112	150	151–152	168	225	**226**
169	227	**228**	257	346	346	313	419	**420**
170	228	**229**	258	347	347	314	420	**421**
171	229	**230**	259	348	**348**	315	421	**422**
172	230	**231**	260	349	**349**	316	422	**423**
173	232	**233**	261	351	351	317	424	**425**
174	233	**234**	262	352	352	318	425	**426**
175	234	**235**	263	353	**353**	319	426	**427**
176	235	**236**	264	354	**354**	320	427	**428**
177	238	239	265	356	**356**	321	431	431–432
178	239	240	266	357	**357**	322	432	432–433
179	240	241	267	358	**358**	323	433	433–434
180	241	242	268	359	**359**	324	434	434–435
181	243	244	269	361	**361**	325	436	436–437
182	244	245	270	362	**362**	326	437	437–438
183	245	246	271	363	**363**	327	438	438–439
184	246	247	272	364	**364**	328	439	439–440
185	248	249	273	367	367–368	329	441	441–442
186	249	250	274	368	368–369	330	442	442–443
187	250	251	275	369	369–370	331	443	443–444
188	251	252	276	370	370–371	332	444	444–445
189	253	253	277	372	372–373	333	446	446–447
190	254	254	278	373	373–374	334	447	447–448
191	255	255	279	374	374–375	335	448	448–449
192	256	256	280	375	375–376	336	449	449–450
193	260	260	281	377	377–378	337	452	452–453
194	261	261	282	378	378–379	338	453	453–454
195	262	262	283	379	379–380	339	454	454–455
196	263	263	284	380	380–381	340	455	455–456
197	265	265	285	382	382–383	341	457	457–458
198	266	266	286	383	383–384	342	458	458–459
199	267	267	287	384	384–**385**	343	459	459–460
200	268	268	288	385	385–**386**	344	460	460–461
201	270	270	289	388	388–389	345	462	462–**463**

Table 2 (*continued*)

d	$g_4(5,d)$	$n_4(5,d)$	d	$g_4(5,d)$	$n_4(5,d)$	d	$g_4(5,d)$	$n_4(5,d)$
202	271	271	290	389	389–390	346	463	463–**464**
203	272	272	291	390	390–391	347	464	**465**
204	273	273	292	391	391–392	348	465	**466**
205	275	276	293	393	393–394	349	467	**468**
206	276	277	294	394	394–395	350	468	**469**
207	277	278	295	395	395–396	351	469	**470**
208	278	279	296	396	396–397	352	470	**471**
209	281	281	297	398	398–**399**	353	473	473–**474**
210	282	282	298	399	399–**400**	354	474	474–**475**
211	283	283	299	400	**401**	355	475	**476**
212	284	284	300	401	**402**	356	476	**477**
213	286	286	301	403	403–**404**	357	478	**479**
214	287	287	302	404	404–**405**	358	479	**480**
215	288	289	303	405	**406**	359	480	**481**
216	289	290	304	406	**407**	360	481	**482**
217	291	292	305	409	410	361	483	**484**
218	292	293	306	410	411	362	484	**485**
219	293	294	307	411	412	363	485	**486**
220	294	295	308	412	413	364	486	**487**
221	296	297	309	414	415	365	488	489
222	297	298	310	415	416	366	489	490
223	298	299	311	416	417	367	490	491
224	299	300	312	417	418	368	491	492

Remark 11. Ward has shown in [33] that there are no codes $[98,5,72;4]$ and $[119,5,88;4]$. Hence $n_4(72,5)=99$ and $n_4(88,5)=120$.

Remark 12. Maruta has shown in [27] that there are no codes $[116,5,85;4]$ and $[187,5,139;4]$. Hence $n_4(85,5)=117$ and $n_4(139,5)=188$.

Remark 13. It turned out that there is an error in the table of [4] which gives $d_4(190,5)=141$. Landjev and Maruta have shown in [25] that there is no code $[190,5,141;4]$. Hence $n_4(141,5)=191$.

Remark 14. All new codes from Theorems 5 and 6 are Hermitian self-orthogonal.

4. Classification results

Finally, we present some classification results for codes $[n,k,d;4]$ with $n \leqslant 35$ and $k \leqslant 6$. First we note that codes with parameters $[16,3,12;4]$, $[20,3,15;4]$, and $[21,3,16;4]$ are McDonald codes, so each of them is unique. Our new classification results have been obtained by Q-EXTENSION. We summarize them together with known results, including references, in Table 3. Note that we only count codes with dual distance greater than one, i.e., there is no position in the code that is constantly zero.

Table 3
Bounds for $d_4(n,k)$ for $k=3,4,5,6$ and the number of inequivalent codes for some optimal codes. Note that we only count codes with dual distance greater than one

$q=4$	$k=3$		$k=4$		$k=5$		$k=6$	
n	d	Number	d	Number	d	Number	d	Number
3	1							
4	2	1	1					
5	3	1	2	1	1			
6	4	1	2	8	2	1	1	
7	4	**7**	3	**10**	2	**11**	2	1
8	5	**3**	4	**16**	3	**13**	2	**16**
9	6	3 [11]	5	**4**	4	**19**	3	**17**
10	6	**45**	6	2 [11]	5	**4**	4	**23**
11	7	**25**	6	841	6	1 [11]	5	1 [11]
12	8	**16**	7	**275**	6	19181	6	1 [11]
13	9	**4**	8	**30**	7	**452**	6	
14	10	**2**	9	**1**	8	**6**	7	**14**
15	11	**1**	10	**1**	8		8	**3**
16	12	1 MD	11	**1**	9		8	
17	12	**12**	12	1 BCH	10		9	
18	13	**2**	12	**20**	10		10	
19	14	**1**	12		11		10	
20	15	1 MD	13		12		11	
21	16	1 MD	14		13		12	
22	16	**6**	15	**15**	14		12–13	
23	16	**498**	16	**3**	15	**2**	**13**	
24	17	**102**	16		16	1 [22]	**14**	
25	18	**27**	17		16		15	
26	19	**5**	18	**48**	**16**		16	
27	20	**2**	19	**2**	**17**		**16**	
28	20	**217**	20	**1**	18		**17**	
29	21	**38**	20		19		17–18	
30	22	**13**	21	**6**	20		18–19	
31	23	**3**	22	**1**	**20**		19–20	
32	24	**2**	22		21		20	
33	24	**79**	23		22		20–21	
34	25	**11**	24		**23**		21–22	
35	26	**5**	24		**24**		22–23	

Additionally, in Appendix A.2 we list all weight enumerators for the cases when there are no more than 12 different enumerators. If inequivalent codes have the same weight enumerator, we also list the number of such codes.

For the new code $[236, 5, 176; 4]$ obtained by dual transform, we have obtained the following partial classification results:

Remark 15. There exist exactly 31 inequivalent codes $[25, 5, 16; 4]$ and weights 16, 20, and 24. Their weight enumerators are $1 + 345y^{16} + 648y^{20} + 30y^{24}$ for 29 codes and

$1+357y^{16}+624y^{20}+42y^{24}$ for the two remaining codes. Using the dual transform, one obtains codes $[236,5,176;4]$ with weight enumerators $1+948y^{176}+75y^{192}$ and $1+951y^{176}+69y^{192}+3y^{208}$, respectively.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments, in particular for providing the entries for the codes with parameters $[11,4,6;4]$ and $[12,5,6;4]$ in Table 3. M. Grassl acknowledges the hospitality of the Mathematical Sciences Research Institute, Berkeley, USA, where part of this work has been performed.

Appendix A

A.1. Generator matrices for some codes

Let $\mathscr{S}_{k,q}$ be the set of all column vectors $a=(a_1,a_2,\ldots,a_k)^t\in\mathbb{F}_q^k$ such that either $a_1=1$ or $a_1=a_2=\cdots=a_{i-1}=0$, $a_i=1$ for some integer i in $\{2,3,\ldots,k\}$, where $k\geqslant 3$. Let the vectors from $\mathscr{S}_{k,q}$ be arranged in lexicographic order. A generator matrix of a projective code can be represented by a binary vector of length $(q^k-1)/(q-1)$ with 1 indicating the presence, and 0 the absence, of a vector of $\mathscr{S}_{k,q}$. This binary vector is then broken into blocks of length 4, each of which is represented by a hexadecimal symbol from $\{\mathtt{0},\mathtt{1},\ldots,\mathtt{9},\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{e},\mathtt{f}\}$. If it is necessary the last block is beforehand completed by zeroes.

Example. If $q=4$ and $k=3$ the set $\mathscr{S}_{k,q}$ looks like this

000001111111111111111

011110000111122223333

101230123012301230123

Then the code [15,3,10;4] with generator matrix

000011111111111

011100111222333

101223012123013

is represented in the following way:

$$1111|0001|1111|0011|1110|1_{(2)}=\mathtt{f1f3e8}_{(16)}.$$

If the code is not projective, e.g., when $n > (q^k - 1)/(q - 1)$, we take several copies of $\mathscr{S}_{k,q}$ and encode the set of columns by the corresponding number of such strings.

[35, 5, 24]
c40204010000030008104400000812200080080000000004004020104002008020880000000160008400420

[70, 5, 50]
c400041200220000090904801825 22c180021121214082c810a21800080321020018420d03085026013088

[102, 5, 74]
d6081c22448212d04a251c13044223058b281bc408108560a0b22ca3220428c0551a40529100480da00040

[194, 5, 144]
002800020337ad51de32f4ac8029b23cf63b88f0a10c59f32d19ea3eda5d3fcf7bd3f2d3b06d99fa78bbf8
00000002000000000800100000000004020000000000001000000800000100020102200200208120100308

[210, 5, 156]
d72f9c2972f7c89f2b2edc9f0833d9dc358df210835e763bf9ebf269b42d923deeb6f8bb32afebf1da1b98
00080020008000000000800000030880000000200000102000000100000008000020000000000000020000000

[211, 5, 154]
cd6c977d979fe59fd2fe6d983e29393560ecc3c3dd9f77e9bb4ebff9b746bbed8d36bc6bfbf76f5d9a4a50

[215, 5, 160]
e7bd750813af99ef3b6bddb3b53bc4e83943e9f3dedb3b51aeb6865ae9f5eb3693983f7d1eb6bd0ba95ab0
0000000002000020000004080210a0000000000000080020000000000281000000000000200000008000000000

[216, 5, 158]
cd6c977d979fe5bfd2fe6d983e293d3560ecc3e3dd9f77e9bb4ebff9b747bbed8d36bc6bfbf76f5d9a4a58

[221, 5, 162]
d56c977db79fe5bfd2fe6d983e293d3562ecc3e3dd9f77e9bb4efff9b747bbed8d3ebc6bfbf76f5d9a51e8

[236, 5, 176]
d7f8f57ff1902f3fcfad6ffbe1c5da65bffaf53c9eebff6fb761cdea61dbf5affdbd43df7636d549ecfb38
000040000000000200004080000000000000000000000000001000000000022000000000080000000000008200

[349, 5, 260]
3177f9effffef0ffbfbffbffb5efe6ffffbdf3fffbfbfffbfffbfbf7d3bffffffffffbffa7f7fbffbedfb8
00000000004220008000800000 8a800280000110030020582ab88000010b984e801c506600531010200c00

[354, 5, 264]
ef7dbefffffffefffbfbffede8ffdffbf7efbb1bebffabbfdbfff5ffdefffffebffffbfb6efffffffef7ff8
022000000080302e0009a02020085002e0ab0000082000001820001000130804b908001022013116004188

[359, 5, 268]
3977f9effffef0ffbfbffbffb7efe6ffffbdf3fffbfbfffffffbfbf7dbbffffffffffbffa7f7fbffbffff8
00000000004220008002800000 8a800280000110030020582ab88000010b984e801c506600531010201c80

[364, 5, 272]
dffefffffdffefeffdfffdfdfcff8f7fffbffbf7fbdfffbbaff7fffff7ff3ffff7d6dfffffe673fefffffff8
120680000300940000518084014 01004c251804d1000020092208d280020040c210120810c000604800828

[386, 5, 288]
25cb19d7bffefbfffcfbf3bbbbffffbfffffbfdffffffffbfffffffe3afffffffbbbbff772fffffff7bdf8
000000040d181930ac43801328e8500480201015684b0b1bd96884402867490ac00080000023bc1c903020

[402, 5, 300]
33ef8bbfff7f7bfffbfdfffb6ffefdffffffbdffffffffffff8fbfb3fffafffffffbffffffffefefbbfef8
000081004a500062e3100000022a4808400b3949512f59afc02a18000002542851370030492700ab81bab8

[407, 5, 304]
b3eabb3dfefdfffff7edfbdfeffffdfbf7dfffffffffffffff7ffb7ffffffffffbbffefffbfffffbfbf7f8
000800350040094911059800ca500808c05188142b31a9eb2022a8010b3b21e1ab3b380080100448c21560

[423, 5, 316]
bbfdfb3ffb7fffffbff7fbffbbffffbffefffb7bfbfffffffffffbffbfffffffffffbffdfffffffffff7b8
0000202010189822883088100218121180a28149a3b8a2738185a80d0abb09bb0136286189a1b271a13220

[428, 5, 320]
fffefffdfbfffff7ffffcfffffeeffffbffffffffdfffffffffffff7bfffdfffbffefff7ff75fffffffff8
18c8b488680064825d9e0221300ee8050108a90c483931982000084481c98649862c4701644112221e6010

[472, 5, 351]
5ffefffffffdfffffbfdfffffffffff7fffbfffffff7fffffffffbfffffffffffffffffefffdffffbbfff0
02b000000185f87b8028b001e07b2ae2c800805029b7e9b7fb75033b897ebba15a05bb507095da5fb88ca0

[477, 5, 355]
5ffffffffffdfffffbfdfffffffffff7fffbfffffff7fffffffffbfffffffffffffffffefffffffffbfff0
02b000000185f87b8028b001e07b2ae2c800805029b7e9b7fb75033b897ebba15a05bbd07095da5fb89ca0

[482, 5, 359]
f7fbffff7ffffffffffffffffffffff7fffffbffffffffffffffffffff7bfffbfffff7fffffffffffffff8
0000008b162adb7f99f1f93022028a5da929d81aa4180e17292b58083042e123f1df02939873af79f2df68

Let

$$G_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 1 & 2 \\ 2 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 3 & 0 & 1 & 3 & 1 & 2 & 2 & 3 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 & 1 & 2 & 1 & 1 & 2 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \end{pmatrix}$$

$$\text{and} \quad G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 1 & 3 \\ 1 & 3 & 0 & 1 & 3 \end{pmatrix}.$$

Then we obtain generator matrices of non-projective codes with parameters $[221, 5, 164]$, $[226, 5, 168]$, $[231, 5, 172]$, $[477, 5, 356]$, $[482, 5, 360]$, $[487, 5, 364]$ concatenating the generator matrices of the codes [211, 5, 154], [216, 5, 158], [221, 5, 162], [472, 5, 351], [477, 5, 355], [482, 5, 359] given above and the matrices G_2, G_2, G_2, G_1, G_1, G_3, respectively.

A.2. Weight enumerators for the classification results

In the following, we list all weight enumerators for codes $[n, k, d; 4]$ with $n \leqslant 34$ and $k \leqslant 6$ for the cases when there are no more than 12 different enumerators. If inequivalent codes have the same weight enumerator, we also list the number of such codes.

[7, 3, 4]
$1 + 9z^4 + 36z^5 + 6z^6 + 12z^7$
$1 + 12z^4 + 27z^5 + 15z^6 + 9z^7$
$1 + 15z^4 + 18z^5 + 24z^6 + 6z^7$ (two codes)
$1 + 15z^4 + 30z^5 + 18z^7$
$1 + 21z^4 + 12z^5 + 18z^6 + 12z^7$
$1 + 21z^4 + 42z^6$

[8, 3, 5]
$1 + 18z^5 + 30z^6 + 6z^7 + 9z^8$
$1 + 21z^5 + 21z^6 + 15z^7 + 6z^8$
$1 + 24z^5 + 12z^6 + 24z^7 + 3z^8$

[13, 3, 9]
$1 + 27z^9 + 21z^{10} + 9z^{11} + 6z^{12}$
$1 + 21z^9 + 36z^{10} + 3z^{12} + 3z^{13}$
$1 + 24z^9 + 27z^{10} + 9z^{11} + 3z^{13}$
$1 + 30z^9 + 12z^{10} + 18z^{11} + 3z^{12}$

[14, 3, 10]
$1 + 33z^{10} + 24z^{11} + 3z^{12} + 3z^{14}$
$1 + 42z^{10} + 21z^{12}$

[15, 3, 11]
$1 + 45z^{11} + 15z^{12} + 3z^{15}$

[17, 3, 12]
$1 + 48z^{12} + 15z^{16}$
$1 + 33z^{12} + 30z^{14}$
$1 + 15z^{12} + 45z^{13} + 3z^{17}$
$1 + 36z^{12} + 24z^{14} + 3z^{16}$
$1 + 12z^{12} + 48z^{13} + 3z^{16}$
$1 + 39z^{12} + 18z^{14} + 6z^{16}$

$1+21z^{12}+24z^{13}+18z^{14}$
$1+27z^{12}+18z^{13}+12z^{14}+6z^{15}$
$1+33z^{12}+12z^{13}+6z^{14}+12z^{15}$
$1+27z^{12}+21z^{13}+12z^{14}+3z^{17}$
$1+18z^{12}+33z^{13}+9z^{14}+3z^{15}$
$1+24z^{12}+24z^{13}+12z^{14}+3z^{16}$

[18, 3, 13]
$1+24z^{13}+36z^{14}+3z^{16}$
$1+27z^{13}+27z^{14}+9z^{15}$

[19, 3, 14]
$1+36z^{14}+24z^{15}+3z^{16}$

[22, 3, 16]
$1+51z^{16}+12z^{20}$
$1+39z^{16}+24z^{18}$
$1+15z^{16}+48z^{17}$
$1+42z^{16}+18z^{18}+3z^{20}$
$1+45z^{16}+15z^{18}+3z^{22}$
$1+27z^{16}+24z^{17}+12z^{18}$

[26, 3, 19]
$1+42z^{19}+12z^{20}+6z^{23}+3z^{24}$
$1+33z^{19}+18z^{20}+3z^{21}+9z^{22}$
$1+36z^{19}+9z^{20}+12z^{21}+6z^{22}$
$1+39z^{19}+15z^{20}+9z^{23}$
$1+30z^{19}+15z^{20}+18z^{21}$

[27, 3, 20]
$1+54z^{20}+9z^{24}$
$1+45z^{20}+18z^{22}$

[30, 3, 22]
$1+36z^{22}+18z^{23}+3z^{24}+6z^{27}$
$1+33z^{22}+21z^{23}+3z^{24}+3z^{26}+3z^{27}$
$1+33z^{22}+24z^{23}+3z^{24}+3z^{30}$
$1+30z^{22}+24z^{23}+3z^{24}+6z^{26}$ (two codes)
$1+33z^{22}+24z^{23}+3z^{26}+3z^{28}$
$1+45z^{22}+15z^{24}+3z^{30}$
$1+42z^{22}+15z^{24}+6z^{26}$
$1+42z^{22}+18z^{24}+3z^{28}$
$1+39z^{22}+21z^{24}+3z^{26}$
$1+30z^{22}+18z^{23}+9z^{24}+6z^{25}$
$1+27z^{22}+27z^{23}+9z^{25}$
$1+36z^{22}+27z^{24}$

[31, 3, 23]

$1 + 45z^{23} + 12z^{24} + 3z^{27} + 3z^{28}$
$1 + 45z^{23} + 15z^{24} + 3z^{31}$
$1 + 42z^{23} + 15z^{24} + 6z^{27}$

[32, 3, 24]

$1 + 57z^{24} + 6z^{28}$
$1 + 60z^{24} + 3z^{32}$

[34, 3, 25]

$1 + 30z^{25} + 12z^{26} + 18z^{27} + 3z^{28}$
$1 + 27z^{25} + 21z^{26} + 9z^{27} + 6z^{28}$
$1 + 21z^{25} + 36z^{26} + 3z^{28} + 3z^{29}$
$1 + 24z^{25} + 27z^{26} + 9z^{27} + 3z^{29}$
$1 + 36z^{25} + 6z^{26} + 12z^{27} + 9z^{28}$
$1 + 24z^{25} + 33z^{26} + 3z^{28} + 3z^{30}$
$1 + 33z^{25} + 12z^{26} + 12z^{27} + 3z^{28} + 3z^{29}$
$1 + 27z^{25} + 24z^{26} + 9z^{27} + 3z^{30}$
$1 + 36z^{25} + 9z^{26} + 12z^{27} + 3z^{28} + 3z^{30}$
$1 + 27z^{25} + 27z^{26} + 6z^{27} + 3z^{31}$
$1 + 24z^{25} + 36z^{26} + 3z^{32}$

[35, 3, 26]

$1 + 36z^{26} + 21z^{27} + 3z^{28} + 3z^{31}$
$1 + 33z^{26} + 24z^{27} + 3z^{28} + 3z^{30}$
$1 + 36z^{26} + 24z^{27} + 3z^{32}$
$1 + 45z^{26} + 15z^{28} + 3z^{30}$
$1 + 42z^{26} + 21z^{28}$

[7, 4, 3]

$1 + 9z^{3} + 69z^{4} + 54z^{5} + 90z^{6} + 33z^{7}$
$1 + 12z^{3} + 57z^{4} + 72z^{5} + 78z^{6} + 36z^{7}$
$1 + 15z^{3} + 45z^{4} + 90z^{5} + 66z^{6} + 39z^{7}$ (two codes)
$1 + 15z^{3} + 57z^{4} + 54z^{5} + 102z^{6} + 27z^{7}$
$1 + 18z^{3} + 45z^{4} + 72z^{5} + 90z^{6} + 30z^{7}$
$1 + 21z^{3} + 21z^{4} + 126z^{5} + 42z^{6} + 45z^{7}$
$1 + 21z^{3} + 33z^{4} + 90z^{5} + 78z^{6} + 33z^{7}$
$1 + 24z^{3} + 33z^{4} + 72z^{5} + 102z^{6} + 24z^{7}$
$1 + 33z^{3} + 33z^{4} + 66z^{5} + 78z^{6} + 45z^{7}$

[8, 4, 4]

$1 + 18z^{4} + 96z^{5} + 24z^{6} + 96z^{7} + 21z^{8}$
$1 + 24z^{4} + 72z^{5} + 60z^{6} + 72z^{7} + 27z^{8}$ (three codes)
$1 + 27z^{4} + 60z^{5} + 78z^{6} + 60z^{7} + 30z^{8}$ (three codes)
$1 + 30z^{4} + 48z^{5} + 96z^{6} + 48z^{7} + 33z^{8}$ (three codes)

$1 + 30z^4 + 60z^5 + 60z^6 + 84z^7 + 21z^8$
$1 + 33z^4 + 48z^5 + 78z^6 + 72z^7 + 24z^8$
$1 + 36z^4 + 36z^5 + 96z^6 + 60z^7 + 27z^8$
$1 + 39z^4 + 36z^5 + 78z^6 + 84z^7 + 18z^8$
$1 + 42z^4 + 168z^6 + 45z^8$
$1 + 54z^4 + 24z^5 + 72z^6 + 72z^7 + 33z^8$

[9, 4, 5]
$1 + 42z^5 + 84z^6 + 36z^7 + 75z^8 + 18z^9$
$1 + 45z^5 + 72z^6 + 54z^7 + 63z^8 + 21z^9$
$1 + 48z^5 + 60z^6 + 72z^7 + 51z^8 + 24z^9$ (two codes)

[14, 4, 9]
$1 + 66z^9 + 99z^{10} + 36z^{11} + 3z^{12} + 42z^{13} + 9z^{14}$

[15, 4, 10]
$1 + 99z^{10} + 90z^{11} + 15z^{12} + 45z^{14} + 6z^{15}$

[16, 4, 11]
$1 + 144z^{11} + 60z^{12} + 48z^{15} + 3z^{16}$

[22, 4, 15]
$1 + 102z^{15} + 51z^{16} + 54z^{17} + 18z^{19} + 12z^{20} + 18z^{21}$
$1 + 111z^{15} + 42z^{16} + 36z^{17} + 18z^{18} + 27z^{19} + 3z^{20} + 18z^{21}$
$1 + 108z^{15} + 45z^{16} + 39z^{17} + 15z^{18} + 30z^{19} + 15z^{21} + 3z^{22}$ (two codes)
$1 + 120z^{15} + 51z^{16} + 72z^{19} + 12z^{20}$ (two codes)
$1 + 96z^{15} + 51z^{16} + 60z^{17} + 24z^{19} + 12z^{20} + 12z^{21}$ (two codes)
$1 + 105z^{15} + 42z^{16} + 42z^{17} + 18z^{18} + 33z^{19} + 3z^{20} + 12z^{21}$ (two codes)
$1 + 102z^{15} + 45z^{16} + 45z^{17} + 15z^{18} + 36z^{19} + 9z^{21} + 3z^{22}$
$1 + 108z^{15} + 63z^{16} + 84z^{19}$
$1 + 105z^{15} + 57z^{16} + 9z^{17} + 36z^{18} + 27z^{19} + 18z^{20} + 3z^{21}$
$1 + 108z^{15} + 39z^{16} + 36z^{17} + 24z^{18} + 36z^{19} + 12z^{21}$
$1 + 90z^{15} + 63z^{16} + 54z^{17} + 30z^{19} + 18z^{21}$

[23, 4, 16]
$1 + 153z^{16} + 54z^{18} + 30z^{20} + 18z^{22}$
$1 + 171z^{16} + 84z^{20}$
$1 + 147z^{16} + 60z^{18} + 36z^{20} + 12z^{22}$

[26, 4, 18]
$1 + 96z^{18} + 69z^{19} + 24z^{20} + 24z^{21} + 12z^{22} + 21z^{23} + 3z^{24} + 6z^{25}$
$1 + 93z^{18} + 72z^{19} + 27z^{20} + 21z^{21} + 15z^{22} + 18z^{23} + 9z^{25}$
$1 + 90z^{18} + 72z^{19} + 27z^{20} + 42z^{21} + 18z^{24} + 6z^{25}$
$1 + 90z^{18} + 81z^{19} + 18z^{20} + 24z^{21} + 18z^{22} + 9z^{23} + 9z^{24} + 6z^{25}$ (two codes)

$1 + 93z^{18} + 75z^{19} + 21z^{20} + 24z^{21} + 15z^{22} + 15z^{23} + 6z^{24} + 6z^{25}$
$1 + 93z^{18} + 84z^{19} + 12z^{20} + 48z^{22} + 12z^{23} + 3z^{24} + 3z^{26}$ (ten codes)
$1 + 96z^{18} + 78z^{19} + 15z^{20} + 45z^{22} + 18z^{23} + 3z^{26}$ (eighteen codes)
$1 + 120z^{18} + 87z^{20} + 33z^{22} + 12z^{24} + 3z^{26}$ (five codes)
$1 + 117z^{18} + 96z^{20} + 24z^{22} + 15z^{24} + 3z^{26}$ (two codes)
$1 + 123z^{18} + 81z^{20} + 33z^{22} + 18z^{24}$ (three codes)
$1 + 120z^{18} + 90z^{20} + 24z^{22} + 21z^{24}$ (four codes)

[27, 4, 19]
$1 + 135z^{19} + 54z^{20} + 54z^{23} + 9z^{24} + 3z^{27}$ (two codes)

[28, 4, 20]
$1 + 189z^{20} + 63z^{24} + 3z^{28}$

[30, 4, 21]
$1 + 93z^{21} + 66z^{22} + 36z^{23} + 33z^{24} + 6z^{26} + 12z^{27} + 6z^{28} + 3z^{29}$
$1 + 84z^{21} + 84z^{22} + 36z^{23} + 15z^{24} + 12z^{25} + 12z^{27} + 12z^{28}$
$1 + 99z^{21} + 42z^{22} + 69z^{23} + 18z^{24} + 21z^{27} + 3z^{28} + 3z^{29}$
$1 + 87z^{21} + 78z^{22} + 36z^{23} + 21z^{24} + 6z^{25} + 6z^{26} + 12z^{27} + 6z^{28} + 3z^{29}$
$1 + 105z^{21} + 36z^{22} + 60z^{23} + 27z^{24} + 24z^{27} + 3z^{29}$
$1 + 96z^{21} + 45z^{22} + 72z^{23} + 15z^{24} + 24z^{27} + 3z^{30}$

[31, 4, 22]
$1 + 141z^{22} + 87z^{24} + 24z^{28} + 3z^{30}$

[7, 5, 2]
$1 + 33z^{2} + 60z^{3} + 195z^{4} + 240z^{5} + 315z^{6} + 180z^{7}$
$1 + 30z^{2} + 75z^{3} + 165z^{4} + 270z^{5} + 300z^{6} + 183z^{7}$
$1 + 21z^{2} + 48z^{3} + 171z^{4} + 312z^{5} + 351z^{6} + 120z^{7}$
$1 + 18z^{2} + 63z^{3} + 141z^{4} + 342z^{5} + 336z^{6} + 123z^{7}$
$1 + 27z^{2} + 30z^{3} + 183z^{4} + 324z^{5} + 333z^{6} + 126z^{7}$
$1 + 6z^{2} + 75z^{3} + 165z^{4} + 318z^{5} + 324z^{6} + 135z^{7}$
$1 + 9z^{2} + 60z^{3} + 195z^{4} + 288z^{5} + 339z^{6} + 132z^{7}$
$1 + 12z^{2} + 57z^{3} + 177z^{4} + 330z^{5} + 306z^{6} + 141z^{7}$
$1 + 15z^{2} + 42z^{3} + 207z^{4} + 300z^{5} + 321z^{6} + 138z^{7}$
$1 + 9z^{2} + 72z^{3} + 147z^{4} + 360z^{5} + 291z^{6} + 144z^{7}$
$1 + 18z^{2} + 39z^{3} + 189z^{4} + 342z^{5} + 288z^{6} + 147z^{7}$

[8, 5, 3]
$1 + 24z^{3} + 102z^{4} + 192z^{5} + 336z^{6} + 264z^{7} + 105z^{8}$ (three codes)
$1 + 27z^{3} + 87z^{4} + 222z^{5} + 306z^{6} + 279z^{7} + 102z^{8}$
$1 + 21z^{3} + 105z^{4} + 210z^{5} + 294z^{6} + 297z^{7} + 96z^{8}$
$1 + 18z^{3} + 120z^{4} + 180z^{5} + 324z^{6} + 282z^{7} + 99z^{8}$
$1 + 24z^{3} + 90z^{4} + 240z^{5} + 264z^{6} + 312z^{7} + 93z^{8}$

$1 + 30z^3 + 72z^4 + 252z^5 + 276z^6 + 294z^7 + 99z^8$
$1 + 30z^3 + 84z^4 + 204z^5 + 348z^6 + 246z^7 + 111z^8$
$1 + 39z^3 + 75z^4 + 198z^5 + 330z^6 + 291z^7 + 90z^8$
$1 + 33z^3 + 69z^4 + 234z^5 + 318z^6 + 261z^7 + 108z^8$
$1 + 42z^3 + 72z^4 + 180z^5 + 372z^6 + 258z^7 + 99z^8$
$1 + 24z^3 + 114z^4 + 144z^5 + 408z^6 + 216z^7 + 117z^8$

$[9, 5, 4]$
$1 + 48z^4 + 138z^5 + 228z^6 + 276z^7 + 267z^8 + 66z^9$ (two codes)
$1 + 54z^4 + 120z^5 + 240z^6 + 288z^7 + 249z^8 + 72z^9$ (four codes)
$1 + 51z^4 + 135z^5 + 210z^6 + 318z^7 + 234z^8 + 75z^9$ (three codes)
$1 + 57z^4 + 117z^5 + 222z^6 + 330z^7 + 216z^8 + 81z^9$ (two codes)
$1 + 45z^4 + 153z^5 + 198z^6 + 306z^7 + 252z^8 + 69z^9$
$1 + 60z^4 + 102z^5 + 252z^6 + 300z^7 + 231z^8 + 78z^9$
$1 + 72z^4 + 90z^5 + 228z^6 + 324z^7 + 243z^8 + 66z^9$
$1 + 63z^4 + 99z^5 + 234z^6 + 342z^7 + 198z^8 + 87z^9$
$1 + 42z^4 + 168z^5 + 168z^6 + 336z^7 + 237z^8 + 72z^9$
$1 + 54z^4 + 132z^5 + 192z^6 + 360z^7 + 201z^8 + 84z^9$
$1 + 78z^4 + 72z^5 + 240z^6 + 336z^7 + 225z^8 + 72z^9$
$1 + 66z^4 + 72z^5 + 312z^6 + 240z^7 + 261z^8 + 72z^9$

$[10, 5, 5]$
$1 + 96z^5 + 150z^6 + 240z^7 + 255z^8 + 240z^9 + 42z^{10}$
$1 + 93z^5 + 165z^6 + 210z^7 + 285z^8 + 225z^9 + 45z^{10}$
$1 + 90z^5 + 180z^6 + 180z^7 + 315z^8 + 210z^9 + 48z^{10}$ (two codes)

$[14, 5, 8]$
$1 + 153z^8 + 144z^9 + 192z^{10} + 192z^{11} + 270z^{12} + 48z^{13} + 24z^{14}$
$1 + 147z^8 + 156z^9 + 198z^{10} + 168z^{11} + 276z^{12} + 60z^{13} + 18z^{14}$
$1 + 189z^8 + 420z^{10} + 378z^{12} + 36z^{14}$ (four codes)

$[23, 5, 15]$
$1 + 342z^{15} + 171z^{16} + 420z^{19} + 84z^{20} + 6z^{23}$ (two codes)

$[14, 6, 7]$
$1 + 168z^7 + 387z^8 + 516z^9 + 750z^{10} + 1152z^{11} + 756z^{12} + 276z^{13} + 90z^{14}$
$1 + 177z^7 + 360z^8 + 525z^9 + 795z^{10} + 1107z^{11} + 747z^{12} + 303z^{13} + 81z^{14}$
$1 + 180z^7 + 354z^8 + 510z^9 + 855z^{10} + 1032z^{11} + 789z^{12} + 294z^{13} + 81z^{14}$
$1 + 192z^7 + 303z^8 + 588z^9 + 810z^{10} + 1032z^{11} + 792z^{12} + 300z^{13} + 78z^{14}$
$1 + 198z^7 + 297z^8 + 546z^9 + 900z^{10} + 1002z^{11} + 726z^{12} + 366z^{13} + 60z^{14}$
$1 + 204z^7 + 249z^8 + 684z^9 + 720z^{10} + 1092z^{11} + 750z^{12} + 324z^{13} + 72z^{14}$
(two codes)
$1 + 210z^7 + 252z^8 + 588z^9 + 945z^{10} + 882z^{11} + 819z^{12} + 336z^{13} + 63z^{14}$
$1 + 216z^7 + 189z^8 + 840z^9 + 420z^{10} + 1512z^{11} + 378z^{12} + 504z^{13} + 36z^{14}$
(six codes)

[15, 6, 8]
$1 + 405z^8 + 1260z^{10} + 1890z^{12} + 540z^{14}$ (three codes)

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