

# Strong Edge Colorings of Uniform Graphs

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## Abstract

For a graph  $G = (V(G), E(G))$ , a strong edge coloring of  $G$  is an edge coloring in which every color class is an induced matching. The strong chromatic index of  $G$ ,  $\chi_s(G)$ , is the smallest number of colors in a strong edge coloring of  $G$ . The strong chromatic index of the random graph  $G(n, p)$  was considered in [2], [3], [11], and [14]. In this paper, we consider  $\chi_s(G)$  for a related class of graphs  $G$  known as uniform or  $\epsilon$ -regular graphs. In particular, we prove that for  $0 < \epsilon \ll d < 1$ , all  $(d, \epsilon)$ -regular bipartite graphs  $G = (U \cup V, E)$  with  $|U| = |V| \geq n_0(d, \epsilon)$  satisfy  $\chi_s(G) \leq \zeta(\epsilon)\Delta(G)^2$ , where  $\zeta(\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$  (this order of magnitude is easily seen to be best possible). Our main tool in proving this statement is a powerful packing result of Pippenger and Spencer [10].

*Key words:* Strong chromatic index, the regularity lemma

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## 1 Introduction

For a finite simple graph  $G = (V(G), E(G))$ , a *strong edge coloring* of  $G$  is an edge coloring in which every color class is an induced matching. The *strong chromatic index* of  $G$ ,  $\chi_s(G)$ , is the minimum number of colors  $k$  in a strong edge coloring of  $G$ . Strong edge colorings are special types of *proper edge colorings*. A proper edge coloring of  $G$  is an edge coloring in which every color class is a (not necessarily induced) matching. As with the proper edge

coloring problem, it is natural to investigate a connection between  $\chi_s(G)$  and  $\Delta(G)$ . However, unlike the proper edge coloring problem (cf. Vizing [13]), no tight bounds for  $\chi_s(G)$  have been established in terms of  $\Delta(G)$ .

The following conjecture of Erdős and Nešetřil stands widely open today.

**Conjecture 1.1 Erdős and Nešetřil, (1984).** *For all graphs  $G$ ,*

$$\chi_s(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2 & \text{if } \Delta(G) \text{ is even,} \\ \frac{5}{4}\Delta(G)^2 - \frac{1}{2}\Delta(G) + \frac{1}{4} & \text{if } \Delta(G) \text{ is odd.} \end{cases}$$

The “blown-up” pentagon  $C_5(m)$  (i.e. each vertex of  $C_5$  is replaced by  $m$  independent vertices and each edge of  $C_5$  is replaced by  $K_{m,m}$ ) easily shows Conjecture 1.1 would be best possible.

Although Conjecture 1.1 is not the focus of this paper, we do mention that, as noted by other authors, Conjecture 1.1 seems substantially difficult. While it is not difficult to see that every graph  $G$  satisfies  $\chi_s(G) \leq 2\Delta(G)^2 - 2\Delta(G) + 1$ , it took sophisticated probabilistic methods to beat the trivial upper bound  $2\Delta(G)^2$ . Indeed, affirmatively answering a question of Erdős and Nešetřil (cf [4]), Molloy and Reed [9] showed that for  $\epsilon = 0.002$ , all graphs  $G$  of sufficiently large maximum degree satisfy  $\chi_s(G) \leq (2 - \epsilon)\Delta(G)^2$ .

In [2], [3], [11], and [14], the strong chromatic index of the random graph  $G(n, p)$  was studied (cf. [6]). In particular, Z. Palka [11] showed that if  $p = \Theta(n^{-1})$ , then *asymptotically almost surely* (cf. [6]),  $\chi_s(G(n, p)) = \Theta(\Delta(G(n, p)))$ . V. Vu [14] more recently showed that for positive  $\delta, \epsilon < 1$ , if  $n^{-1} \log^{1+\delta} n \leq p \leq n^{-\epsilon}$ , then asymptotically almost surely,  $\chi_s(G(n, p)) = \Theta\left(\frac{\Delta(G(n, p))^2}{\ln \Delta(G(n, p))}\right)$ . In [2], the current authors recently extended Vu’s result to the range  $p \geq n^{\epsilon_0}$  for a suitable  $\epsilon_0 > 0$ .

In this paper, we consider an analogous problem of estimating  $\chi_s(G)$  for so-called *pseudo-random* or *uniform* graphs  $G$ . As we define them below, these are graphs obtained from and identified with the well-known Szemerédi Regularity Lemma (cf [8], [12]). Uniform graphs were studied by Alon, Rödl, and Rucinski [1] who estimated the number of perfect matchings of a super-regular pair (see below for definitions) and by Frieze [5] who estimated the number of hamiltonian cycles and perfect matchings in uniform graphs.

For a bipartite graph  $G = (U \cup V, E)$ , let  $U' \subseteq U$  and  $V' \subseteq V$  be two nonempty sets, and let  $E_G(U', V') = \{\{u, v\} \in E(G) : u \in U', v \in V'\}$  and

$e_G(U', V') = |E_G(U', V')|$ . Define the *density* of the graph  $(U' \cup V', E_G)$  by

$$d_G(U', V') = \frac{e_G(U', V')}{|U'| |V'|}.$$

For constant  $d$  and  $\epsilon > 0$ , we say that a bipartite graph  $G = (U \cup V, E)$  is  $(d, \epsilon)$ -regular if for all  $U' \subseteq U$ ,  $|U'| > \epsilon|U|$ , and all  $V' \subseteq V$ ,  $|V'| > \epsilon|V|$ , the following holds,

$$|d - d_G(U', V')| < \epsilon. \quad (1)$$

If  $G = (U \cup V, E)$  is  $(d, \epsilon)$ -regular for some  $0 \leq d \leq 1$ , then  $G$  is called  $\epsilon$ -regular. Bipartite graphs which are  $(d, \epsilon)$ -regular,  $0 < \epsilon \ll d$ , have uniform edge distributions and therefore behave, in some senses, in a “random-like” manner.

Our theorem is stated as follows.

**Theorem 1.2 (Main Theorem)** *For every  $0 < d < 1$  and  $\mu > 0$ , there exist  $\epsilon > 0$  and integer  $n_0$  such that if  $G = (U \cup V, E)$  is a  $(d, \epsilon)$ -regular bipartite graph with  $|U| = |V| \geq n_0$ , then*

$$\chi_s(G) \leq \mu \Delta(G)^2.$$

As any  $(d, \epsilon)$ -regular bipartite graph  $G = (U \cup V, E)$ ,  $|U| = |V| = n$ , satisfies  $\Delta(G) \geq (d - \epsilon)n$ , it suffices to prove Theorem 1.2 in the following form.

**Theorem 1.3** *For every  $0 < d < 1$  and  $\mu > 0$ , there exist  $\epsilon > 0$  and integer  $n_0$  such that if  $G = (U \cup V, E)$  is a  $(d, \epsilon)$ -regular bipartite graph with  $|U| = |V| = n \geq n_0$ , then*

$$\chi_s(G) \leq \mu n^2.$$

The following observation shows that the order of magnitude for the upper bound in Theorem 1.2 is best possible.

**Fact 1** *Let  $0 < d < 1$  be fixed. For all  $\epsilon > 0$  and integers  $n$ , there exists a  $(d, \epsilon)$ -regular bipartite graph  $G_0 = (U \cup V, E)$ ,  $|U| = |V| = N \geq n$ , satisfying*

$$\chi_s(G_0) \geq \frac{\epsilon^2}{2} \Delta(G_0)^2.$$

The observation in Fact 1, in various forms, has been noted by various researchers (e.g. Prof. T. Łuczak [7] and also an anonymous referee). The proof of Fact 1 is easy and we will present it at the end of Section 3.

The rest of the paper is organized as follows. In Section 2, we state some necessary terminology and auxiliary facts. In Section 3, we prove Theorem 1.3 and verify Fact 1.

### 1.1 Acknowledgement

We wish to thank the referees for suggestions which lead to simplified details in this paper.

## 2 Definitions and Facts

In this section, we give some background material we use to prove Theorem 1.2. We begin our discussion with basic notation and considerations. For a graph  $G = (V(G), E(G))$  and a vertex  $v \in V(G)$ , let  $N(v) = \{x \in V(G) : \{v, x\} \in E\}$  and set  $\deg(v) = |N(v)|$ . In all that follows, graphs  $G = (V, E)$  are identified with their edge sets. We use the following graph notation.

**Notation 2.1** For a graph  $G$  and an edge  $e = \{u, v\} \in G$ , set

$$G_e = G[V(G) \setminus (N(u) \cup N(v))]$$

to be the subgraph of  $G$  induced on the set  $V(G) \setminus (N(u) \cup N(v))$ .

For convenience of calculations, we use the convention  $s = (a \pm b)t$  to mean  $t(a - b) \leq s \leq t(a + b)$ .

### 2.1 $(d, \epsilon)$ -regular bipartite graphs

In our proof of Theorem 1.3, we use two standard facts concerning graph regularity.

**Fact 2 (cf. Fact 1.3, [8])** *Let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -regular bipartite graph. Then, all but  $2\epsilon|U|$  vertices  $u \in U$  satisfy*

$$\deg(u) = (d \pm \epsilon)|V|. \tag{2}$$

Note that, by symmetry, we may conclude from Fact 2 that all but  $2\epsilon|V|$  vertices  $v \in V$  satisfy

$$\deg(v) = (d \pm \epsilon)|U|. \quad (3)$$

**Fact 3 (Slicing Lemma (cf. Fact 1.5, [8]))** *Let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -regular bipartite graph. For  $\alpha > \epsilon$ , let  $U' \subseteq U$  and  $V' \subseteq V$  be given where  $|U'| \geq \alpha|U|$  and  $|V'| \geq \alpha|V|$ . Then, the subgraph  $G[U', V']$  of  $G$  induced on  $U' \cup V'$  is  $(d, \epsilon')$ -regular where  $\epsilon' = \epsilon/\alpha$ .*

We now discuss some easy corollaries whose presentations are well-suited for our argument of Theorem 1.3.

### 2.1.1 Easy corollaries of Facts 2 and 3

For our first corollary, we use the following definition and notation.

**Definition 4** Let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -regular bipartite graph. For a fixed edge  $e = \{u, v\}$ ,  $u \in U$ ,  $v \in V$ , we say  $e$  is a *good edge* if  $u$  satisfies (2) and  $v$  satisfies (3). Otherwise, we say  $e$  is a *bad edge*.

Set

$$G^{\text{good}} = \{e \in G : e \text{ is a good edge}\}, \quad G^{\text{bad}} = G \setminus G^{\text{good}}. \quad (4)$$

From Fact 2, we have the following immediate corollary.

**Corollary 5 (few edges are bad)** *Let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -regular bipartite graph. Then,*

$$|G^{\text{bad}}| < 4\epsilon|U||V|.$$

*In particular, as  $|G| = (d \pm \epsilon)|U||V|$ , we see*

$$|G^{\text{good}}| = (d \pm 5\epsilon)|U||V|.$$

We also have the following trivial corollary of the slicing lemma.

**Corollary 6** *Let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -regular bipartite graph and let  $e \in G^{\text{good}}$ . Then, the graph  $G_e$  is  $(d, \epsilon')$ -regular,  $\epsilon' = \epsilon/(d - \epsilon)$ . Moreover, the bipartition of  $G_e$  given by  $(U \setminus N(v)) \cup (V \setminus N(u))$  satisfies*

$$|U \setminus N(v)| = (1 - d \pm \epsilon)|U| = (1 - d)|U| \left(1 \pm \frac{\epsilon}{1 - d}\right), \quad (5)$$

and

$$|V \setminus N(u)| = (1 - d \pm \epsilon)|V| = (1 - d)|V| \left(1 \pm \frac{\epsilon}{1 - d}\right). \quad (6)$$

For our final corollary, we use the following definition (cf. Definition 1.6 of [8]).

**Definition 7 (( $d, \epsilon$ )-super regularity)** Let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -regular bipartite graph. We say that  $G$  is  $(d, \epsilon)$ -*super-regular* if all vertices  $u \in U$  satisfy (2) and all vertices  $v \in V$  satisfy (3).

Observe that if  $G = (U \cup V, E)$  is a  $(d, \epsilon)$ -super regular bipartite graph, then

$$G^{\text{good}} = G. \quad (7)$$

Facts 2 and 3 quickly imply the following corollary.

**Corollary 8** Let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -regular bipartite graph where, say,  $3\epsilon < 1 - d$ , and  $|U| = |V| = n$ . Then  $G$  has a  $(d, \epsilon')$ -super-regular induced bipartite subgraph  $G_0 = G_0[U_0 \cup V_0]$ , where  $\epsilon' = \frac{6\epsilon}{d}$  and  $|U_0| = |V_0| > (1 - 2\epsilon)n$ .

The idea behind proving Corollary 8 is to delete the vertices  $u \in U$  not satisfying (2) and the vertices  $v \in V$  not satisfying (3). The precise details of this proof are very standard and we omit them.

## 2.2 Hypergraph Packings

At the heart of our argument for Theorem 1.2 lies an application of the following strong theorem of Pippenger and Spencer (cf. [10]). Let  $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$  be an  $l$ -uniform hypergraph. For a vertex  $u \in V(\mathcal{H})$ , define the *degree* of the vertex  $u$ ,  $\deg(u)$ , as  $\deg(u) = |\{h \in E(\mathcal{H}) : u \in h\}|$ . Set  $\delta(\mathcal{H})$  to be the minimum degree of any vertex in  $\mathcal{H}$  and set  $\Delta(\mathcal{H})$  to the maximum degree of any vertex in  $\mathcal{H}$ . For a pair of distinct vertices  $u, v \in V(\mathcal{H})$ , set  $\text{codeg}(\{u, v\}) = |\{h \in E(\mathcal{H}) : u, v \in h\}|$  and let

$$\text{codeg}(\mathcal{H}) = \max_{u, v \in V(\mathcal{H}), u \neq v} \text{codeg}(\{u, v\}).$$

Then the theorem of [10] is stated as follows.

**Theorem 2.2 (Pippenger, Spencer, [10])** *For all positive integers  $l$  and positive constants  $\gamma$ , there exists  $\epsilon = \epsilon(l, \gamma)$  so that if  $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$  is*

an  $l$ -uniform hypergraph with minimum degree  $\delta(\mathcal{H})$  satisfying

$$\delta(\mathcal{H}) > (1 - \epsilon)\Delta(\mathcal{H})$$

and

$$\text{codeg}(\mathcal{H}) \leq \epsilon\Delta(\mathcal{H}),$$

then there exists a set  $M \subseteq E(\mathcal{H})$ ,  $h \cap h' = \emptyset$  for every  $h \neq h'$  in  $M$ , which covers all but  $\gamma|V(\mathcal{H})|$  vertices of  $\mathcal{H}$ .

### 3 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. The following theorem, combined with Corollary 8, almost immediately implies Theorem 1.3.

**Theorem 3.1** *For every  $0 < d < 1$  and every  $\zeta > 0$ , there exist  $\epsilon > 0$  and integer  $n_0$  such that if  $G = (U \cup V, E)$  is a  $(d, \epsilon)$ -super-regular bipartite graph with  $|U| = |V| = n \geq n_0$ , then*

$$\chi_s(G) \leq \zeta n^2. \tag{8}$$

In view of Theorem 3.1 and Corollary 8, we may prove Theorem 1.3 by producing a promised strong edge coloring in “two rounds”. Indeed, Corollary 8 guarantees a large  $(d, \epsilon')$ -super-regular induced subgraph  $G_0$  of  $G$ . Theorem 3.1 guarantees  $G_0$  admits strong edge colorings using few colors. Fix one such coloring. As  $G \setminus G_0$  is small, we may greedily color the remaining edges. As the subgraph  $G_0$  of  $G$  is induced, the greedy coloring of  $G \setminus G_0$  does not disturb the strong edge coloring of  $G_0$  guaranteed by Theorem 3.1.

It remains to prove Theorem 3.1. We make preparations to that end in what follows.

#### 3.1 Setting up the argument of Theorem 3.1

For an integer  $k \geq 1$  and graph  $G$ , define  $M_k(G)$  to be the set of all induced matchings of size  $k$  in  $G$ . Set  $m_k(G) = |M_k(G)|$ . For a fixed edge  $e \in G$ , denote by  $M_k(e, G)$  the set of all induced matchings of size  $k$  containing edge  $e$ . Set  $m_k(e, G) = |M_k(e, G)|$ .

We use the following easy identities concerning the parameters  $m_k(G)$ ,  $m_k(e, G)$ ,  $e \in G$ , and  $m_k(G_e)$  (cf. Notation 2.1):

$$m_k(e, G) = m_{k-1}(G_e), \quad (9)$$

$$km_k(G) = \sum_{e \in G} m_k(e, G) = \sum_{e \in G} m_{k-1}(G_e). \quad (10)$$

We proceed to the following lemma.

**Lemma 3.2** *Let  $0 < d < 1$  be given. For every integer  $k \geq 1$ , for every  $\theta > 0$ , there exists  $\epsilon > 0$  so that if  $G = (U \cup V, E)$  is a  $(d, \epsilon)$ -regular bipartite graph,  $|U|, |V| \geq n_0(d, k, \theta, \epsilon)$ , then*

$$m_k(G) = d^k(1-d)^{k^2-k} \frac{|U|^k |V|^k}{k!} (1 \pm \theta). \quad (11)$$

To prove Theorem 3.1, we use the following corollary of Lemma 3.2.

**Corollary 9** *Let  $0 < d < 1$  be given. For every integer  $k \geq 1$ , for every  $\rho > 0$ , there exists  $\epsilon > 0$  so that if  $G = (U \cup V, E)$  is a  $(d, \epsilon)$ -super regular bipartite graph,  $|U| = |V| = n \geq n_0(d, k, \rho, \epsilon)$ , then for all  $e \in G$ ,*

$$m_k(e, G) = d^{k-1}(1-d)^{(k-1)^2-(k-1)} \frac{n^{2(k-1)}}{(k-1)!} (1 \pm \rho). \quad (12)$$

The proof of Corollary 9 follows immediately from Lemma 3.2 and the identity in (9). As these details are more or less a subset of the details of proving Lemma 3.2, we omit the easy verification of Corollary 9.

### 3.2 Proof of Theorem 3.1

Let  $0 < d < 1$  and  $\zeta > 0$  be given. Set  $\gamma = \zeta/2$  and  $l = \lceil \frac{1}{\gamma} \rceil$ . Let  $\epsilon_{2.2} = \epsilon_{2.2}(l, \gamma)$  be that constant guaranteed by Theorem 2.2 for the parameters  $l$  and  $\gamma$ . For  $k = l$  and  $\rho = \frac{\epsilon_{2.2}}{2}$ , let  $\epsilon = \epsilon_9(d, l, \frac{\epsilon_{2.2}}{2})$  be that constant guaranteed by Corollary 9. Let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -super-regular bipartite graph where  $|U| = |V| = n$ . We show that  $\chi_s(G) \leq \zeta n^2$ .

To that end, with  $l = \lceil \frac{1}{\gamma} \rceil$ , define auxiliary  $l$ -uniform hypergraph  $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$  to have vertex set  $V(\mathcal{H}) = G$ , the edge set of  $G$ , and  $E(\mathcal{H}) = M_l(G)$ , the set of all induced matchings in  $G$  of size  $l$ . For  $e \in V(\mathcal{H})$ , note that  $\deg_{\mathcal{H}}(e) = m_l(e, G)$ . With  $\epsilon = \epsilon_9(d, l, \frac{\epsilon_{2.2}}{2})$ , we infer from Corollary 9 that for every  $e \in V(\mathcal{H})$ ,

$$\deg_{\mathcal{H}}(e) = d^{l-1}(1-d)^{(l-1)^2-(l-1)} \left( \frac{n^{2(l-1)}}{(l-1)!} \right) (1 \pm \frac{\epsilon_{2.2}}{2}).$$



In particular, we see

$$d^{l-1}(1-d)^{(l-1)^2-(l-1)} \left( \frac{n^{2(l-1)}}{(l-1)!} \right) \left( 1 - \frac{\epsilon_{2.2}}{2} \right) \leq \delta(\mathcal{H}),$$

$$\Delta(\mathcal{H}) \leq d^{l-1}(1-d)^{(l-1)^2-(l-1)} \left( \frac{n^{2(l-1)}}{(l-1)!} \right) \left( 1 + \frac{\epsilon_{2.2}}{2} \right),$$

and consequently,

$$\delta(\mathcal{H}) \geq \frac{1 - \frac{\epsilon_{2.2}}{2}}{1 + \frac{\epsilon_{2.2}}{2}} \Delta(\mathcal{H}) > (1 - \epsilon_{2.2}) \Delta(\mathcal{H}).$$

Clearly,

$$\text{codeg}(\mathcal{H}) \leq n^{2(l-2)},$$

which with  $n$  sufficiently large satisfies

$$\text{codeg}(\mathcal{H}) < \epsilon_{2.2} \Delta(\mathcal{H}).$$

With  $\epsilon_{2.2} = \epsilon_{2.2}(l, \gamma)$ , we apply Theorem 2.2 to  $\mathcal{H}$  to conclude that there exists a set  $\{h_1, \dots, h_t\} \subset E(\mathcal{H})$ ,  $h_i \cap h_j = \emptyset$  for all  $1 \leq i < j \leq t$ , which covers all but  $\gamma|V(\mathcal{H})|$  vertices  $e \in V(\mathcal{H})$ . Note that  $tl \leq |V(\mathcal{H})| = |E(G)|$  trivially follows.

We now give the strong edge coloring of  $G$  using no more than the maximum number of colors required by Lemma 3.1. The edge classes  $\{h_1, \dots, h_t\}$  constitute  $t$  color classes in our coloring. Let  $X = \bigcup_{1 \leq i \leq t} \{e | e \in h_i\}$ . Then the singleton classes  $\{\{e\} | e \in E(G) \setminus X\}$  constitute the remaining coloring classes in our coloring. Since there are at most  $\gamma|V(\mathcal{H})| = \gamma|E(G)|$  edges in  $E(G) \setminus X$  the number of colors used in the above colorings is at most

$$t + \gamma|E(G)| \leq \frac{|E(G)|}{l} + \gamma|E(G)| \leq 2\gamma n^2,$$

where the last inequality follows from the fact that  $l = \lceil \frac{1}{\gamma} \rceil$ . With  $\gamma = \zeta/2$ , we see that at most  $\zeta n^2$  colors have been used. It is easy to see that the obtained coloring is a strong edge coloring of  $E(G)$ .  $\square$

### 3.3 Proof of Lemma 3.2

Our proof of Lemma 3.2 follows by inducting on the variable  $k$ . The base case  $k = 1$  is trivial as  $m_1(G) = |G| = (d \pm \epsilon)|U||V|$ .

**Induction Hypothesis.**

Assume Lemma 3.2 is true for  $k = k_0 - 1 \geq 1$ . That is to say, for  $k = k_0 - 1$ , for all  $0 < d < 1$  and  $\theta_{k_0-1} > 0$ , there exists  $\epsilon_{k_0-1}^{(3.2)} = \epsilon_{k_0-1}^{(3.2)}(k_0 - 1, d, \theta)$  confirming the conclusion of Lemma 3.2.

### Inductive Step.

Let  $k = k_0$  and let  $0 < d < 1$  and  $\theta > 0$  be given. Establishing the inductive step is easy and we therefore wish not to complicate our presentation with a tedious determination of constants. To that end, with  $k = k_0$ ,  $d$  and  $\theta$  given, we take positive auxiliary constant  $\theta_{k_0-1} = \theta_{k_0-1}(k_0, d, \theta)$  sufficiently small with respect to  $k_0$ ,  $d$  and  $\theta$ . With the constant  $\theta_{k_0-1}$ , let  $\epsilon_{k_0-1} = \epsilon_{k_0-1}^{(3.2)}(k_0 - 1, d, \theta_{k_0-1})$  be the constant guaranteed by the Induction Hypothesis. We take  $0 < \epsilon = \epsilon(k_0, d, \theta, \theta_{k_0-1}) < (1 - d)\epsilon_{k_0-1}$  sufficiently small with respect to all preceeding parameters.

Now, let  $G = (U \cup V, E)$  be a  $(d, \epsilon)$ -regular bipartite graph. Recall from (10) that

$$k_0 m_{k_0}(G) = \sum_{e \in G} m_{k_0-1}(G_e).$$

Recalling  $G = G^{\text{good}} \cup G^{\text{bad}}$  from (4), we further conclude

$$\sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e) \leq k_0 m_{k_0}(G) = \sum_{e \in G^{\text{bad}}} m_{k_0-1}(G_e) + \sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e).$$

We therefore conclude from Corollary 5 that

$$\sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e) \leq k_0 m_{k_0}(G) = 4\epsilon |U|^{k_0} |V|^{k_0} + \sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e)$$

which implies

$$k_0 m_{k_0}(G) = \left(1 \pm \frac{4\epsilon |U|^{k_0} |V|^{k_0}}{\sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e)}\right) \sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e). \quad (13)$$

We estimate  $\sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e)$ . To that end, fix  $e = \{u, v\} \in G^{\text{good}}$ . By Corollary 6, we see that  $G_e$  is  $(d, \epsilon')$ -regular where  $\epsilon' = \epsilon/(1 - d) < \epsilon_{k_0-1}$ . We may therefore apply the Induction Hypothesis to  $G_e$  to conclude

$$m_{k_0-1}(G_e) = d^{k_0-1} (1-d)^{(k_0-1)^2 - (k_0-1)} \frac{|U \setminus N(v)|^{k_0-1} |V \setminus N(u)|^{k_0-1}}{(k_0-1)!} (1 \pm \theta_{k_0-1}).$$

Using Corollary 6, we may bound the sizes  $|U \setminus N(v)|$  and  $|V \setminus N(u)|$  to further conclude

$$m_{k_0-1}(G_e) = d^{k_0-1} (1-d)^{k_0^2 - k_0} \frac{|U|^{k_0-1} |V|^{k_0-1}}{(k_0-1)!} \left(1 \pm \frac{\epsilon}{1-d}\right)^{2k_0-2} (1 \pm \theta_{k_0-1}).$$

With sufficiently small choices of  $\theta_{k_0-1} = \theta_{k_0-1}(k_0, d, \theta)$  and  $\epsilon = \epsilon(k_0, d, \theta, \theta_{k_0-1})$ , we may further conclude

$$m_{k_0-1}(G_e) = d^{k_0-1}(1-d)^{k_0^2-k_0} \frac{|U|^{k_0-1}|V|^{k_0-1}}{(k_0-1)!} (1 \pm \theta)^{1/2}.$$

Consequently,

$$\sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e) = |G^{\text{good}}| d^{k_0-1}(1-d)^{k_0^2-k_0} \frac{|U|^{k_0-1}|V|^{k_0-1}}{(k_0-1)!} (1 \pm \theta)^{1/2}. \quad (14)$$

Applying (14) to (13), we obtain that  $k_0 m_{k_0}(G)$  is equal to

$$\left(1 \pm \frac{4\epsilon |U|^{k_0} |V|^{k_0}}{\sum_{e \in G^{\text{good}}} m_{k_0-1}(G_e)}\right) |G^{\text{good}}| d^{k_0-1}(1-d)^{k_0^2-k_0} \frac{|U|^{k_0-1}|V|^{k_0-1}}{(k_0-1)!} (1 \pm \theta)^{1/2}.$$

Since  $|G^{\text{good}}| = (d \pm 5\epsilon)|U||V|$  from Corollary 5 and since  $\epsilon = \epsilon(k, d, \theta, \theta_{k_0-1})$  is sufficiently small, we finally conclude

$$k_0 m_{k_0}(G) = d^{k_0}(1-d)^{k_0^2-k_0} \frac{|U|^{k_0}|V|^{k_0}}{(k_0-1)!} (1 \pm \theta).$$

This concludes our proof of the Inductive Step, and hence, our proof of Lemma 3.2.  $\square$

### 3.4 Proof of Fact 1

Let  $0 < d < 1$  be given along with  $\epsilon$  and integer  $n$ . We produce a graph  $G_0$  satisfying the conclusion of Fact 1. Indeed, fix disjoint sets  $U$  and  $V$  with  $|U| = |V| = N$  where  $N \geq n$  is a sufficiently large integer. Take any  $(d, \epsilon/2)$ -regular bipartite graph  $G$  on  $U \cup V$ . (the existence of such a graph is easily established by the probabilistic method provided  $N$  is sufficiently large) Now, fix any  $U_0 \subset U$  where  $|U_0| = \frac{\epsilon^2}{2}|U|$ . Define the graph  $G_0$  on  $U \cup V$  by

$$G_0 = G \cup K[U_0, V].$$

In other words,  $G_0$  is obtained from  $G$  by replacing the edges  $G[U_0, V]$  with the complete bipartite graph  $K[U_0, V]$ . Clearly,  $\Delta(G_0) = |V| = N$  and

$$\chi_s(G_0) \geq \frac{\epsilon^2}{2} N^2 = \frac{\epsilon^2}{2} \Delta(G_0)^2.$$

What remains to be shown is that  $G_0$  is  $(d, \epsilon)$ -regular. Indeed, let  $U' \subseteq U$  and  $V' \subseteq V$  be given,  $|U'| > \epsilon|U|$  and  $|V'| > \epsilon|V|$ . Set  $U'_0 = U' \cap U_0$ . Then

$$d_G(U', V') \leq d_{G_0}(U', V') \leq d_G(U', V') + \frac{|U'_0|}{|U'|}.$$

Since  $|U'_0| \leq (\epsilon^2/2)|U|$ ,

$$d_G(U', V') \leq d_{G_0}(U', V') \leq d_G(U'_1, V') + \frac{\epsilon}{2}. \quad (15)$$

As  $|U'| > \frac{\epsilon}{2}|U|$  and  $|V'| > \frac{\epsilon}{2}|V|$ , we see from the  $(d, \epsilon/2)$ -regularity of  $G$  that  $|d_G(U', V') - d| < \frac{\epsilon}{2}$ . Consequently, in (15), we see  $|d_{G_0}(U', V') - d| < \epsilon$ . This proves Fact 1.  $\square$

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